

PERTURBED SADDLE-POINT PROBLEMS IN L^p WITH NON-REGULAR LOADS

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ABSTRACT. In this work, we develop the discrete solvability analysis for perturbed saddle-point problems in Banach spaces with forcing terms regularised by means of a projector constructed using the adjoint of a weighted Clément quasi-interpolation. We take as driving example the linearised Poisson–Boltzmann (an advection–diffusion–reaction problem) in mixed form. We use perturbation arguments on the continuous and discrete levels and then derive a priori estimates that remain valid when the load that appears on the right-hand side of the "second" equation is in H^{-1} . Further, we show a supercloseness result and analyse convergence of an adequate adaptation of Stenberg postprocessing for mixed advection equations with non-regular data. We provide numerical results that illustrate the convergence of the proposed scheme.

1. INTRODUCTION

We are interested in the mixed form of reaction–diffusion equations with singular sources. The main motivation comes from electrochemical flows, in models describing the distribution of electric potential in the direction normal to charged surfaces (see, for instance, [10]). In this case, the continuous problem does not possess the usual regularity for elliptic equations. Moreover, while the discrete problem can be defined, the error in the usual energy norms lacks sense, and the typical error estimates need to be modified accordingly.

Relevant references regarding finite element discretisation include the works for primal formulations in [29, 8, 25, 15, 16] (see also [19] for HHO schemes), as well as saddle-point problems such as Stokes, Darcy, elasticity, and coupled flow–transport treated in, e.g., [4, 1, 5]. More broadly, one way of designing well-defined methods for problems with loads in H^{-1} is to construct quasi-interpolation operators on piecewise polynomial spaces (see, e.g., [22, 28]).

Here, and because of the structure of the model equations, the singular forcing appears as the right-hand side of the constraint equation in the saddle-point problem. Problems of this type have been recently addressed in [20] for the case of mixed Poisson, and also in [21] for first-order least-squares formulations. To handle the singular force, following [20] we replace it by a regularised version using a linear and bounded projection operator. While [20] considers projections onto piecewise constants only, here, we extend their construction to piecewise polynomials. Such a map is constructed based on the adjoint of Clément quasi-interpolators having second-order approximation properties from a piecewise projection on constants. Moreover, to analyse the resulting mixed formulation in terms of the pseudo-potential flux and the potential, we rely on the theory developed in [13], together with the Banach–Nečas–Babuška theorem (cf. [17]). This framework allows us, by means of a global inf–sup condition combined with a smallness assumption on the advecting velocity field, to prove the well-posedness of the problem at the continuous level. For the finite element approximation, we use Raviart–Thomas spaces and, exploiting their inf–sup stability, we can appeal to the discrete perturbed saddle-point theory from [13]. Solutions to the regularised scheme are shown to satisfy a perturbed quasi-optimally result in Banach spaces where the perturbation includes the approximation error of the advection field and the best-approximation of a lower-order term.

We also consider accuracy enhancement by a Stenberg-type postprocessing [30, 11], whose analysis in the present context requires a reformulation of the typical Aubin–Nitsche duality arguments including additional consistency terms which we control with data smallness assumptions and properties of the proposed smoother. The postprocessing technique requires higher regularity to achieve optimal convergence [30].

To the best of the authors' knowledge, the present paper is the first work addressing the analysis of low-regularity forcing terms for advection-diffusion-reaction in the non-Hilbertian setting and in the presence of a lower-diagonal block perturbation.

The contents of the paper have been organised as follows. Section 2 presents the strong form of the modified Poisson–Boltzmann equations including a load regularisation that turns the continuous problem depending on the mesh. We derive a weak formulation in Banach spaces, and establish the well-posedness using Banach–Nečas–Babuška theorem, data smallness assumptions, and the theory of perturbed saddle-point problems. In Section 3 we construct a mixed finite element scheme using conforming Raviart–Thomas elements and show discrete inf-sup conditions by means of a quasi-interpolator. We prove the unique solvability of the discrete problem following a similar structure to the continuous case. Next, Section 4 briefly addresses Céa estimates, and we derive precise a priori error estimates with respect to the original solution of the linearised Poisson–Boltzmann equation. We also present a postprocess that allows us to achieve higher order approximation of the double layer potential (the primal unknown), and show its properties in Section 5. Section 6 has a collection of numerical results that highlight the convergence properties and suitability of the proposed formulation. Finally, in Section 7 we close with a brief discussion and comment on ongoing extensions of this work.

2. CONTINUOUS SETTING

Let us consider a Lipschitz and simply connected bounded domain in \mathbb{R}^n , $n = 2, 3$ with boundary $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}$ disjointly split into two parts where different types of boundary conditions will be considered (we assume that both sub-boundaries have positive measure), and denote by \mathbf{n} the outward unit normal vector on the boundary.

We recall the standard notation for Lebesgue spaces $L^t(\Omega)$, $t \in (1, +\infty)$, with norm $\|\bullet\|_{0,t;\Omega}$, and for Sobolev spaces $H^s(\Omega)$, $s \geq 0$, endowed with the norm $\|\bullet\|_{s,\Omega}$ and seminorm $|\bullet|_{s,\Omega}$. In particular, $H^{1/2}(\partial\Omega)$ stands for the space of traces of functions of $H^1(\Omega)$ and $H^{-1/2}(\partial\Omega)$ denotes its dual (similarly for subsets of the boundary). We recall from [23, Section 2.4.2] the definition of the space $H_{00}^{1/2}(\Gamma_D) = \{\eta \in H^{1/2}(\Gamma_D) : E_{00}(\eta) \in H^{1/2}(\partial\Omega)\}$ supplied with the norm

$$\|\bullet\|_{1/2,00,\Gamma_D} := \|E_{00}(\bullet)\|_{1/2,\partial\Omega},$$

where $E_{00} : H^{1/2}(\Gamma_D) \rightarrow H^{1/2}(\partial\Omega)$ denotes the extension-by-zero operator

$$E_{00}(\eta) = \begin{cases} \eta & \text{on } \Gamma_D, \\ 0 & \text{on } \partial\Omega \setminus \Gamma_D, \end{cases} \quad \forall \eta \in H^{1/2}(\Gamma_D).$$

Furthermore, the restriction of $\psi \in H^{-1/2}(\partial\Omega)$ to Γ_D , defined as

$$\langle \psi|_{\Gamma_D}, \eta \rangle_{\Gamma_D} = \langle \psi, E_{00}(\eta) \rangle_{\partial\Omega} \quad \forall \eta \in H_{00}^{1/2}(\Gamma_D),$$

belongs to the dual space, denoted as $[H_{00}^{1/2}(\Gamma_D)]' =: H_{00}^{-1/2}(\Gamma_D)$. Its norm is

$$\|\psi|_{\Gamma_D}\|_{-1/2,00,\Gamma_D} = \sup_{0 \neq \eta \in H_{00}^{1/2}(\Gamma_D)} \frac{\langle \psi, E_{00}(\eta) \rangle_{\partial\Omega}}{\|E_{00}(\eta)\|_{1/2,\partial\Omega}}. \quad (2.1)$$

Finally, for any pair of normed spaces $(X, \|\bullet\|_X)$ and $(Y, \|\bullet\|_Y)$, we provide the product space $X \times Y$ with the natural norm $\|(x, y)\|_{X \times Y} := \|x\|_X + \|y\|_Y$ for all $(x, y) \in X \times Y$.

For a Dirichlet datum $\psi_D \in H_{00}^{1/2}(\Gamma_D)$, the motivating problem under consideration is the nonlinear Poisson–Boltzmann equation [10],

$$\kappa\psi - \operatorname{div}(\varepsilon\nabla\psi - \mathbf{u}\psi) = g \quad \text{in } \Omega, \quad (2.2a)$$

$$\psi = \psi_D \quad \text{on } \Gamma_D, \quad (2.2b)$$

$$(\varepsilon\nabla\psi - \mathbf{u}\psi) \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_N, \quad (2.2c)$$

including a fixed arbitrary advecting velocity field $\mathbf{u} \in \mathbf{L}^4(\Omega)$. This particular regularity comes from the specific functional structure of the Navier–Stokes equations in mixed form from [14], aimed at coupling with the Poisson–Boltzmann equation as in [3]. The permittivity coefficient ε is heterogeneous and bounded away from zero, the reaction function κ depends nonlinearly on the potential and it is monotone and unbounded, and the right-hand side source g is a sum of Dirac measures concentrated at points, and line sources (when $d = 2$). However, and as a preliminary step, we consider that $\kappa \in L^2(\Omega)$ is heterogeneous but independent of ψ , and that $g \in \tilde{H}^{-1}(\Omega) := (H^1(\Omega))'$.

In the work at hand we are therefore interested in a mixed formulation of the linearised Poisson–Boltzmann equation in terms of the *double layer potential* ψ and the *pseudo potential flux* ζ . This gives

$$\zeta = \varepsilon\nabla\psi - \mathbf{u}\psi \quad \text{in } \Omega, \quad (2.3a)$$

$$\kappa\psi - \operatorname{div} \zeta = g \quad \text{in } \Omega, \quad (2.3b)$$

$$\psi = \psi_D \quad \text{on } \Gamma_D, \quad (2.3c)$$

$$\zeta \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_N. \quad (2.3d)$$

2.1. Analysis of the primal formulation. The weak formulation of (2.2a) reads as follows: Find $\psi \in H_D^1(\Omega)$ such that

$$\int_{\Omega} (\kappa\psi\varphi + \varepsilon\nabla\psi \cdot \nabla\varphi - \mathbf{u}\psi \cdot \nabla\varphi) = \int_{\Omega} g\varphi \quad \forall \varphi \in H_D^1(\Omega), \quad \psi|_{\Gamma_D} = \psi_D. \quad (2.4)$$

Throughout, we make the standing assumption that $\varepsilon \in L^\infty(\Omega)$, $0 < \underline{\varepsilon} \leq \varepsilon \leq \bar{\varepsilon} := \|\varepsilon\|_{0,\infty;\Omega}$, $\kappa \in L^2(\Omega)$, $\kappa \geq 0$ (a.e. in Ω) and that the norm of \mathbf{u} is sufficiently small to guarantee existence and uniqueness of solutions, i.e.,

$$\|\mathbf{u}\|_{0,4;\Omega} < \frac{\underline{\varepsilon}}{C_{\text{Sob}}(1 + c_P^2)^{1/2}}, \quad (2.5)$$

where $c_P > 0$ denotes Poincaré’s constant and $C_{\text{Sob}} > 0$ denotes the continuity constant of the Sobolev embedding $H_D^1(\Omega) \rightarrow L^4(\Omega)$, i.e., the smallest constants such that

$$\|\varphi\|_{0,2;\Omega} \leq c_P \|\nabla\varphi\|_{0,2;\Omega}, \quad \|\varphi\|_{0,4;\Omega} \leq C_{\text{Sob}} \|\varphi\|_{1,2;\Omega}$$

holds for all $\varphi \in H_D^1(\Omega)$. We stress that assumption (2.5) implies that there exists $\alpha \in (0, 1)$ such that

$$\|\mathbf{u}\|_{0,4;\Omega} \leq \alpha \frac{\underline{\varepsilon}}{C_{\text{Sob}}(1 + c_P^2)^{1/2}}.$$

This observation is used in the proof of several results below. In the following, we use notation $H_D^{-1}(\Omega) = (H_D^1(\Omega))'$. And note that any $g \in \tilde{H}^{-1}(\Omega)$ satisfies

$$\|g\|_{H_D^{-1}(\Omega)} := \sup_{0 \neq v \in H_D^1(\Omega)} \frac{\langle g, v \rangle}{\|v\|_{1,2;\Omega}} \leq \sup_{0 \neq v \in H^1(\Omega)} \frac{\langle g, v \rangle}{\|v\|_{1,2;\Omega}} =: \|g\|_{\tilde{H}^{-1}(\Omega)}$$

with $\langle g, v \rangle$ denoting the duality pairing on $\tilde{H}^{-1}(\Omega) \times H^1(\Omega)$.

Using the Sobolev embedding we note that, for $\varphi \in L^{4/3}(\Omega)$,

$$\|\varphi\|_{H_D^{-1}(\Omega)} = \sup_{0 \neq v \in H_D^1(\Omega)} \frac{\langle \varphi, v \rangle}{\|v\|_{1,2;\Omega}} \leq \sup_{0 \neq v \in H_D^1(\Omega)} \frac{\|\varphi\|_{0,4/3;\Omega} \|v\|_{0,4;\Omega}}{\|v\|_{1,2;\Omega}} \leq C_{\text{Sob}} \|\varphi\|_{0,4/3;\Omega}. \quad (2.6)$$

Proposition 2.1. *There exists a constant $C > 0$ such that for any $g \in \tilde{H}^{-1}(\Omega)$ and $\psi_D \in H_{00}^{1/2}(\Gamma_D)$, Problem (2.2) admits a unique solution $\psi \in H_D^1(\Omega)$ and*

$$\|\psi\|_{1,2;\Omega} \leq C(\|\psi_D\|_{1/2,00;\Gamma_D} + \|g\|_{\tilde{H}^{-1}(\Omega)}).$$

Proof. By [27, Theorem 4.10] it remains to show that the bilinear form corresponding to the variational formulation (2.4) is coercive and bounded. We further note that [27] states the bound with $\|g\|_{\tilde{H}^{-1}(\Omega)}$ on the right-hand side. This can be improved to the present bound by following the very same arguments in the proof which are therefore omitted.

Let $\psi, \varphi \in H_D^1(\Omega)$ be given, and note that

$$\int_{\Omega} |\kappa\psi\varphi + \varepsilon \nabla\psi \cdot \nabla\varphi - \mathbf{u}\psi \cdot \nabla\varphi| \leq C(1 + \|\mathbf{u}\|_{0,4;\Omega}) \|\psi\|_{1,2;\Omega} \|\varphi\|_{1,2;\Omega},$$

where $C > 0$ is some generic constant. Further, by our assumption on the norm of \mathbf{u} , there exists $\alpha \in (0, 1)$ such that

$$- \int_{\Omega} |\mathbf{u}\psi \cdot \nabla\psi| \geq -\|\mathbf{u}\|_{0,4;\Omega} \|\psi\|_{0,4;\Omega} \|\nabla\psi\|_{0,2;\Omega} \geq -\alpha\varepsilon \|\nabla\psi\|_{0,2;\Omega}^2.$$

Then, using $\kappa \geq 0$, we arrive at

$$\int_{\Omega} (\kappa\psi^2 + \varepsilon |\nabla\psi|^2 - \mathbf{u}\psi \cdot \nabla\psi) \geq (1 - \alpha)\varepsilon \|\nabla\psi\|_{0,2;\Omega}^2 + \int_{\Omega} \kappa\psi^2 \geq (1 - \alpha)\varepsilon \|\nabla\psi\|_{0,2;\Omega}^2,$$

thus concluding the proof. \square

We remark that the assumption $\kappa \geq 0$ can be replaced by assuming that $\|\kappa\|_{0,2;\Omega}$ is sufficiently small.

2.2. Notation and approximation spaces for mixed formulations. Before we define and analyse the a regularised problem we need to introduce some spaces and notation. In view of the functional structure of the weak formulation associated with (2.10) (dictated by the integrability of each term, in particular the one coming from the last term on the right-hand side of the constitutive equation (2.10a)), we consider the following spaces for the pseudo potential flux and double layer potential, respectively

$$\mathbf{H} := \mathbf{H}_N(\text{div}_{4/3}; \Omega) = \{\boldsymbol{\xi} \in \mathbf{L}^2(\Omega) : \text{div } \boldsymbol{\xi} \in L^{4/3}(\Omega) \text{ and } \boldsymbol{\xi} \cdot \mathbf{n} = 0 \text{ on } \Gamma_N\}, \quad \mathbf{Q} := L^4(\Omega).$$

Let \mathcal{T}_h denote a family of non-degenerate triangular / tetrahedral meshes on Ω and denote by \mathcal{E}_h the set of all facets (edges in 2D) in the mesh. By h_K we denote the diameter of the element K and by h_F we denote the length/area of the facet F . As usual, by h we denote the maximum of the diameters of elements in \mathcal{T}_h . For all meshes we assume that they are sufficiently regular (there exists a uniform positive constant η_1 such that each element K is star-shaped with respect to a ball of radius greater than $\eta_1 h_K$. It is also assumed that there exists $\eta_2 > 0$ such that for each element and every facet $F \in \partial K$, we have that $h_F \geq \eta_2 h_K$, see, e.g., [17]).

By $P_k(K)$ we will denote the scalar space of degree up to k , defined locally on $K \in \mathcal{T}_h$. In addition, we denote by $\mathbf{RT}_k(K) := [P_k(K)]^d \oplus P_k(K) \mathbf{x}$ the local Raviart–Thomas space and by $\mathbf{RT}_k(\mathcal{T}_h)$ its global counterpart.

Spaces \mathbf{H}_h and \mathbf{Q}_h are suitable subspaces of the functional spaces \mathbf{H} and \mathbf{Q} , respectively. For the analysis of the discretised mixed problem below, we require kernel characterisation and discrete inf-sup conditions and so we can simply take, for example, Raviart–Thomas elements

$$\mathbf{H}_h = \mathbf{RT}_k(\mathcal{T}_h) \cap \mathbf{H} = \{\boldsymbol{\xi}_h \in \mathbf{H}_N(\operatorname{div}_{4/3}; \Omega) : \boldsymbol{\xi}_h|_K \in \mathbf{RT}_k(K), \forall K \in \mathcal{T}_h\}, \quad \mathbf{Q}_h = \mathbf{P}_k(\mathcal{T}_h).$$

Note that $\operatorname{div}(\mathbf{H}_h) \subset \mathbf{Q}_h$ which is a property that will be used in the analysis. In particular, we even have that $\operatorname{div}(\mathbf{H}_h) = \mathbf{Q}_h$. We denote by $\mathcal{P}_h^k: L^2(\Omega) \rightarrow \mathbf{Q}_h \subset L^2(\Omega)$ the $L^2(\Omega)$ orthogonal projection.

2.3. The perturbed regularised primal problem. In order to deal with the singular force $g \in \tilde{\mathbf{H}}^{-1}(\Omega)$ we replace it by a regularised one, $\mathcal{Q}_h g$, where $\mathcal{Q}_h: \mathbf{H}_D^{-1}(\Omega) \rightarrow \mathbf{Q}_h$ denotes a bounded linear operator with the properties

$$\mathcal{Q}_h^2 = \mathcal{Q}_h, \quad \|\mathcal{Q}_h\|_{\mathbf{H}_D^{-1}(\Omega) \rightarrow \mathbf{H}_D^{-1}(\Omega)} + \|\mathcal{Q}_h\|_{L^2(\Omega) \rightarrow L^2(\Omega)} \leq C_{\mathcal{Q}_h} < \infty. \quad (2.7)$$

Additionally, \mathbf{u} is replaced by an approximation $\mathbf{u}_h \in L^4(\Omega)$. This yields an auxiliary problem which in its (primal) weak form reads as follows: Find $\psi^* \in \mathbf{H}^1(\Omega)$ such that

$$\int_{\Omega} (\kappa \psi^* \varphi + \varepsilon \nabla \psi^* \cdot \nabla \varphi - \mathbf{u}_h \psi^* \cdot \nabla \varphi) = \int_{\Omega} \mathcal{Q}_h g \varphi \quad \forall \varphi \in \mathbf{H}_D^1(\Omega), \quad \psi^*|_{\Gamma_D} = \psi_D. \quad (2.8)$$

Existence and uniqueness of solutions follows as in Proposition 2.1 assuming that

$$\|\mathbf{u}_h\|_{0,4;\Omega} < \frac{\underline{\varepsilon}}{C_{\text{Sob}}(1 + c_P^2)^{1/2}}. \quad (2.9)$$

Note that if $\mathbf{u}_h \rightarrow \mathbf{u}$ in the sense of $L^4(\Omega)$ when $h \rightarrow 0$, it follows that there exists a sufficiently small h_0 such that (2.9) is satisfied for $h \leq h_0$ provided that assumption (2.5) holds.

Lemma 2.2. *Let ψ and ψ^* denote the unique solutions of Problems (2.4) and (2.8), respectively. Then, there exist constants $C_1 > 0$, $C_2 > 0$ such that*

$$\begin{aligned} \|\psi - \psi^*\|_{1,2;\Omega} &\leq C_1 (\|g\|_{\mathbf{H}_D^{-1}(\Omega)} + \|\psi_D\|_{1/2,0,0;\Gamma_D}) \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega} \\ &\quad + C_2 \left(\min_{\boldsymbol{\tau}_h \in \mathbf{H}_h} \|\boldsymbol{\zeta} - \boldsymbol{\tau}_h\|_{0,2;\Omega} + \min_{\chi_h \in \mathbf{Q}_h} \|\kappa \psi - \chi_h\|_{\mathbf{H}_D^{-1}(\Omega)} \right), \end{aligned}$$

where $\boldsymbol{\zeta} = \varepsilon \nabla \psi - \mathbf{u} \psi$, and, with $\boldsymbol{\zeta}^* = \varepsilon \nabla \psi^* - \mathbf{u}_h \psi^*$,

$$\begin{aligned} \|\psi - \psi^*\|_{0,4;\Omega} + \|\boldsymbol{\zeta} - \boldsymbol{\zeta}^*\|_{0,2;\Omega} &\leq C_1 (\|g\|_{\mathbf{H}_D^{-1}(\Omega)} + \|\psi_D\|_{1/2,0,0;\Gamma_D}) \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega} \\ &\quad + C_2 \left(\min_{\boldsymbol{\tau}_h \in \mathbf{H}_h} \|\boldsymbol{\zeta} - \boldsymbol{\tau}_h\|_{0,2;\Omega} + \min_{\chi_h \in \mathbf{Q}_h} \|\kappa \psi - \chi_h\|_{\mathbf{H}_D^{-1}(\Omega)} \right). \end{aligned}$$

Proof. Let us start by first considering the perturbation in \mathbf{u} . Let $\tilde{\psi}^* \in \mathbf{H}^1(\Omega)$ denote the unique solution to

$$\int_{\Omega} (\kappa \tilde{\psi}^* \varphi + \varepsilon \nabla \tilde{\psi}^* \cdot \nabla \varphi - \mathbf{u}_h \tilde{\psi}^* \cdot \nabla \varphi) = \int_{\Omega} g \varphi \quad \forall \varphi \in \mathbf{H}_D^1(\Omega), \quad \tilde{\psi}^*|_{\Gamma_D} = \psi_D.$$

Set $\psi_d = \psi - \tilde{\psi}^* \in \mathbf{H}_D^1(\Omega)$. Then,

$$\int_{\Omega} (\kappa \psi_d^2 + \varepsilon |\nabla \psi_d|^2) = \int_{\Omega} (\mathbf{u} \psi - \mathbf{u}_h \tilde{\psi}^*) \cdot \nabla \psi_d = \int_{\Omega} (\mathbf{u} - \mathbf{u}_h) \psi \cdot \nabla \psi_d + \int_{\Omega} \mathbf{u}_h \psi_d \cdot \nabla \psi_d.$$

Noting that $\|(\mathbf{u} - \mathbf{u}_h) \psi \cdot \nabla \psi_d\|_{0,1;\Omega} \leq \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega} \|\psi\|_{0,4;\Omega} \|\nabla \psi_d\|_{0,2;\Omega}$ as well as

$$\|\mathbf{u}_h \psi_d \cdot \nabla \psi_d\|_{0,1;\Omega} \leq \|\mathbf{u}_h\|_{0,4;\Omega} \|\psi_d\|_{0,4;\Omega} \|\nabla \psi_d\|_{0,2;\Omega}.$$

Using the continuity of the Sobolev embedding $\mathbf{H}^1(\Omega) \subset L^4(\Omega)$ and assumption (2.9) further yields

$$\|\mathbf{u}_h\|_{0,4;\Omega} \|\psi_d\|_{0,4;\Omega} \leq \alpha \underline{\varepsilon} \|\nabla \psi_d\|_{0,2;\Omega},$$

where $\alpha \in (0, 1)$. Together with $\|\psi\|_{0,4;\Omega} \lesssim \|\psi_D\|_{1/2,00;\Gamma_D} + \|g\|_{\mathbb{H}_D^{-1}(\Omega)}$ (Proposition 2.1)

$$\begin{aligned} \int_{\Omega} (\kappa\psi_d^2 + \varepsilon|\nabla\psi_d|^2) &\leq C(\|u_D\|_{1/2,00;\Gamma_D} + \|g\|_{\mathbb{H}_D^{-1}(\Omega)})\|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega}\|\nabla\psi_d\|_{0,2;\Omega} + \alpha\varepsilon\|\nabla\psi_d\|_{0,2;\Omega}^2 \\ &\leq \delta^{-1}/2C^2(\|u_D\|_{1/2,00;\Gamma_D} + \|g\|_{\mathbb{H}_D^{-1}(\Omega)})^2\|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega}^2 \\ &\quad + \delta/2\|\nabla\psi_d\|_{0,2;\Omega}^2 + \alpha\varepsilon\|\nabla\psi_d\|_{0,2;\Omega}^2 \end{aligned}$$

where in the last estimate we have employed Young's inequality with parameter $\delta > 0$. Subtracting the last two terms on the right-hand side and using that $\kappa \geq 0$, it follows that

$$(1 - \alpha - \delta/2)\|\nabla\psi_d\|_{0,2;\Omega}^2 \leq \delta^{-1}/2C^2(\|u_D\|_{1/2,00;\Gamma_D} + \|g\|_{\mathbb{H}_D^{-1}(\Omega)})^2\|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega}^2.$$

Choosing δ sufficiently small we conclude that

$$\|\psi_d\|_{1,2;\Omega} \lesssim (\|u_D\|_{1/2,00;\Gamma_D} + \|g\|_{\mathbb{H}_D^{-1}(\Omega)})\|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega}.$$

It remains to estimate $\|\tilde{\psi}^* - \psi^*\|_{1,2;\Omega}$. Note that by continuous dependence on data, which can be shown as in Proposition 2.1, it follows that

$$\|\tilde{\psi}^* - \psi^*\|_{1,2;\Omega} \lesssim \|g - \mathcal{Q}_h g\|_{\mathbb{H}_D^{-1}(\Omega)}.$$

Noting that $g = -\operatorname{div}\tilde{\zeta}^* + \kappa\psi^*$, where $\tilde{\zeta}^* = \varepsilon\nabla\tilde{\psi}^* - \mathbf{u}_h\tilde{\psi}^*$, we have that $(1 - \mathcal{Q}_h)g = -(1 - \mathcal{Q}_h)\operatorname{div}\tilde{\zeta}^* + (1 - \mathcal{Q}_h)\kappa\tilde{\psi}^*$. We stress that $(1 - \mathcal{Q}_h)\operatorname{div}\boldsymbol{\tau}_h = 0$ for all $\boldsymbol{\tau}_h \in \mathbf{H}_h$, which follows from the projection property of \mathcal{Q}_h and the fact that $\operatorname{div}(\mathbf{H}_h) \subset \mathcal{Q}_h$, hence, for all $\boldsymbol{\tau}_h \in \mathbf{H}_h$,

$$\begin{aligned} \|(1 - \mathcal{Q}_h)\operatorname{div}\tilde{\zeta}^*\|_{\mathbb{H}_D^{-1}(\Omega)} &= \|(1 - \mathcal{Q}_h)\operatorname{div}(\tilde{\zeta}^* - \boldsymbol{\tau}_h)\|_{\mathbb{H}_D^{-1}(\Omega)} \lesssim \|\operatorname{div}(\tilde{\zeta}^* - \boldsymbol{\tau}_h)\|_{\mathbb{H}_D^{-1}(\Omega)} \\ &= \sup_{0 \neq v \in \mathbb{H}_D^1(\Omega)} \frac{\langle \operatorname{div}(\tilde{\zeta}^* - \boldsymbol{\tau}_h), v \rangle}{\|v\|_{1,2;\Omega}} = \sup_{0 \neq v \in \mathbb{H}_D^1(\Omega)} \frac{-\int_{\Omega} (\tilde{\zeta}^* - \boldsymbol{\tau}_h) \cdot \nabla v}{\|v\|_{1,2;\Omega}} \\ &\leq \|\tilde{\zeta}^* - \boldsymbol{\tau}_h\|_{0,2;\Omega}. \end{aligned}$$

Here, we have used that $\boldsymbol{\tau}_h \cdot \mathbf{n}|_{\Gamma_N} = 0$ for $\boldsymbol{\tau}_h \in \mathbf{H}_h$ and the definition of the conormal derivative, which formally reads $\tilde{\zeta}^* \cdot \mathbf{n}|_{\Gamma_N} = 0$. Further, by the projection and boundedness property of \mathcal{Q}_h we can assert that for any $\chi_h \in \mathcal{Q}_h$

$$\|(1 - \mathcal{Q}_h)(\kappa\tilde{\psi}^*)\|_{\mathbb{H}_D^{-1}(\Omega)} \lesssim \|\kappa\tilde{\psi}^* - \chi_h\|_{\mathbb{H}_D^{-1}(\Omega)}.$$

Combining the latter estimates with the triangle inequality gives

$$\begin{aligned} \|\tilde{\psi}^* - \psi^*\|_{1,2;\Omega} &\lesssim \|g - \mathcal{Q}_h g\|_{\mathbb{H}_D^{-1}(\Omega)} \lesssim \|\tilde{\zeta}^* - \boldsymbol{\tau}_h\|_{0,2;\Omega} + \|\kappa\tilde{\psi}^* - \chi_h\|_{\mathbb{H}_D^{-1}(\Omega)} \\ &\lesssim \|\zeta - \boldsymbol{\tau}_h\|_{0,2;\Omega} + \|\kappa\psi - \chi_h\|_{\mathbb{H}_D^{-1}(\Omega)} + \|\zeta - \tilde{\zeta}^*\|_{0,2;\Omega} + \|\kappa(\psi - \tilde{\psi}^*)\|_{\mathbb{H}_D^{-1}(\Omega)} \\ &\lesssim \|\zeta - \boldsymbol{\tau}_h\|_{0,2;\Omega} + \|\kappa\psi - \chi_h\|_{\mathbb{H}_D^{-1}(\Omega)} + \|\zeta - \tilde{\zeta}^*\|_{0,2;\Omega} + \|\kappa(\psi - \tilde{\psi}^*)\|_{0,4/3;\Omega} \\ &\lesssim \|\zeta - \boldsymbol{\tau}_h\|_{0,2;\Omega} + \|\kappa\psi - \chi_h\|_{\mathbb{H}_D^{-1}(\Omega)} + \|\zeta - \tilde{\zeta}^*\|_{0,2;\Omega} + \|\kappa\|_{0,2;\Omega}\|\psi - \tilde{\psi}^*\|_{1,2;\Omega}. \end{aligned}$$

Employing the very same arguments as before we obtain that

$$\begin{aligned} \|\zeta - \tilde{\zeta}^*\|_{0,2;\Omega} &\lesssim \|\psi - \tilde{\psi}^*\|_{1,2;\Omega} + \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega}\|\psi\|_{0,4;\Omega} + \|\mathbf{u}_h\|_{0,4;\Omega}\|\psi - \tilde{\psi}^*\|_{0,4;\Omega} \\ &\lesssim (\|g\|_{\mathbb{H}_D^{-1}(\Omega)} + \|\psi_D\|_{1/2,00;\Gamma_D})\|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega} + \|\psi - \tilde{\psi}^*\|_{1,2;\Omega}. \end{aligned}$$

Combining this estimate with the previous ones concludes the proof. \square

Remark 2.3. Note that if $\kappa \in L^4(\Omega)$, then, $\kappa\psi \in L^2(\Omega)$. Therefore, duality arguments and the approximation properties of \mathcal{P}_h^k show that

$$\min_{\chi_h \in \mathcal{Q}_h} \|\kappa\psi - \chi_h\|_{\mathbf{H}_D^{-1}(\Omega)} \lesssim h \|(1 - \mathcal{P}_h^k)(\kappa\psi)\|_{0,2;\Omega}.$$

If, in addition, κ is piecewise constant, then $(1 - \mathcal{P}_h^k)(\kappa\psi)|_K = \kappa|_K(1 - \mathcal{P}_h^k)\psi|_K$ for all $K \in \mathcal{T}_h$, and therefore

$$\|(1 - \mathcal{P}_h^k)(\kappa\psi)\|_{0,2;\Omega} \lesssim \min_{v_h \in \mathcal{Q}_h} \|\psi - v_h\|_{0,2;\Omega} \leq \min_{v_h \in \mathcal{Q}_h} \|\psi - v_h\|_{0,4;\Omega}.$$

2.4. Mixed formulation. Let us now rewrite a regularised counterpart of the strong mixed form (2.3):

$$\zeta^* = \varepsilon \nabla \psi^* - \mathbf{u}_h \psi^* \quad \text{in } \Omega, \quad (2.10a)$$

$$\kappa \psi^* - \operatorname{div} \zeta^* = \mathcal{Q}_h g \quad \text{in } \Omega, \quad (2.10b)$$

$$\psi^* = \psi_D \quad \text{on } \Gamma_D, \quad (2.10c)$$

$$\zeta^* \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_N. \quad (2.10d)$$

We remark that the flux and potential (ζ^*, ψ^*) depend upon discretisation—through the dependence on the discrete map \mathcal{Q}_h and on the discrete advection velocity \mathbf{u}_h —but they are not necessarily discrete.

Using appropriate integration by parts formulas in Banach spaces, we obtain a perturbed saddle-point problem: for given $\kappa \in L^2(\Omega)$, $\varepsilon \in L^\infty(\Omega)$, $\mathcal{Q}_h g \in L^{4/3}(\Omega)$, and $\psi_D \in H_{00}^{1/2}(\Gamma_D)$, find $(\zeta^*, \psi^*) \in \mathbf{H} \times \mathbf{Q}$ such that

$$\begin{aligned} a(\zeta^*, \boldsymbol{\xi}) + b(\boldsymbol{\xi}, \psi^*) + d_{\mathbf{u}_h}(\boldsymbol{\xi}, \psi^*) &= F(\boldsymbol{\xi}) \quad \forall \boldsymbol{\xi} \in \mathbf{H}, \\ b(\zeta^*, \varphi) - c(\psi^*, \varphi) &= G(\varphi) \quad \forall \varphi \in \mathbf{Q}, \end{aligned} \quad (2.11)$$

where (again for a fixed \mathbf{u}_h) the bilinear forms $a : \mathbf{H} \times \mathbf{H} \rightarrow \mathbb{R}$, $b : \mathbf{H} \times \mathbf{Q} \rightarrow \mathbb{R}$, $c : \mathbf{Q} \times \mathbf{Q} \rightarrow \mathbb{R}$, $d_{\mathbf{u}_h} : \mathbf{H} \times \mathbf{Q} \rightarrow \mathbb{R}$, and the linear functionals $F : \mathbf{H} \rightarrow \mathbb{R}$ and $G : \mathbf{Q} \rightarrow \mathbb{R}$ are defined as

$$\begin{aligned} a(\zeta^*, \boldsymbol{\xi}) &:= \int_{\Omega} \frac{1}{\varepsilon} \zeta^* \cdot \boldsymbol{\xi}, \quad b(\boldsymbol{\xi}, \varphi) := \int_{\Omega} \varphi \operatorname{div} \boldsymbol{\xi}, \quad c(\psi^*, \varphi) := \int_{\Omega} \kappa \psi^* \varphi, \\ d_{\mathbf{u}_h}(\boldsymbol{\xi}, \varphi) &:= \int_{\Omega} \frac{1}{\varepsilon} \mathbf{u}_h \cdot \boldsymbol{\xi} \varphi, \quad F(\boldsymbol{\xi}) := \langle \boldsymbol{\xi} \cdot \mathbf{n}, \psi_D \rangle, \quad G(\varphi) := - \int_{\Omega} \mathcal{Q}_h g \varphi, \end{aligned} \quad (2.12)$$

where $\langle \bullet, \bullet \rangle$ stands for the duality pairing between $H_{00}^{1/2}(\Gamma_D)$ and $H_{00}^{-1/2}(\Gamma_D)$.

Equivalently, by defining the bilinear forms $A, A_{\mathbf{u}_h} : (\mathbf{H} \times \mathbf{Q}) \times (\mathbf{H} \times \mathbf{Q}) \rightarrow \mathbb{R}$ as

$$A((\zeta^*, \psi^*), (\boldsymbol{\xi}, \varphi)) := a(\zeta^*, \boldsymbol{\xi}) + b(\boldsymbol{\xi}, \psi^*) + b(\zeta^*, \varphi) - c(\psi^*, \varphi), \quad (2.13a)$$

and

$$A_{\mathbf{u}_h}((\zeta^*, \psi^*), (\boldsymbol{\xi}, \varphi)) := A((\zeta^*, \psi^*), (\boldsymbol{\xi}, \varphi)) + d_{\mathbf{u}_h}(\boldsymbol{\xi}, \psi^*), \quad (2.13b)$$

for all $(\zeta^*, \psi^*), (\boldsymbol{\xi}, \varphi) \in \mathbf{H} \times \mathbf{Q}$, we can rewrite (2.11) as find $(\zeta^*, \psi^*) \in \mathbf{H} \times \mathbf{Q}$ such that

$$A_{\mathbf{u}_h}((\zeta^*, \psi^*), (\boldsymbol{\xi}, \varphi)) = F(\boldsymbol{\xi}) + G(\varphi) \quad \forall (\boldsymbol{\xi}, \varphi) \in \mathbf{H} \times \mathbf{Q}. \quad (2.14)$$

The well-posedness analysis follows from [13, Theorem 3.4], along with the Banach–Nečas–Babuška Theorem [17, Theorem 2.6]. In what follows, we set the necessary conditions on the bilinear forms to apply the aforementioned theorems. We start by proving the boundedness of the bilinear forms and functionals, which follows directly from the Cauchy–Schwarz and Hölder inequalities:

$$\begin{aligned} |a(\zeta^*, \boldsymbol{\xi})| &\leq \|a\| \|\zeta^*\|_{\mathbf{H}} \|\boldsymbol{\xi}\|_{\mathbf{H}}, \quad |b(\boldsymbol{\xi}, \varphi)| \leq \|b\| \|\boldsymbol{\xi}\|_{\mathbf{H}} \|\varphi\|_{\mathbf{Q}}, \quad |c(\psi^*, \varphi)| \leq \|c\| \|\psi^*\|_{\mathbf{Q}} \|\varphi\|_{\mathbf{Q}}, \\ |d_{\mathbf{u}_h}(\boldsymbol{\xi}, \varphi)| &\leq \|d\| \|\boldsymbol{\xi}\|_{\mathbf{H}} \|\varphi\|_{\mathbf{Q}}, \quad |F(\boldsymbol{\xi})| \leq \|F\| \|\boldsymbol{\xi}\|_{\mathbf{H}}, \quad |G(\varphi)| \leq \|G\| \|\varphi\|_{\mathbf{Q}}. \end{aligned} \quad (2.15)$$

Here, the bounding constants are

$$\|a\| := \frac{1}{\underline{\varepsilon}}, \quad \|c\| := \|\kappa\|_{0,2;\Omega}, \quad \|d\| := \frac{1}{\underline{\varepsilon}} \|\mathbf{u}_h\|_{0,4;\Omega}, \quad \|F\| := C_T \|\psi_D\|_{1/2,00;\Gamma_D}, \quad \|G\| := \|\mathcal{Q}_h g\|_{0,4/3;\Omega},$$

where C_T denotes the constant from the trace inequality.

The following result establishes the inf-sup condition for the bilinear form $b(\bullet, \bullet)$.

Lemma 2.4. *There exists a positive constant β such that*

$$\sup_{\mathbf{0} \neq \boldsymbol{\xi} \in \mathbf{H}} \frac{b(\boldsymbol{\xi}, \varphi)}{\|\boldsymbol{\xi}\|_{\mathbf{H}}} \geq \beta \|\varphi\|_{\mathbf{Q}} \quad \forall \varphi \in \mathbf{Q}. \quad (2.16)$$

Proof. For a given $\varphi \in \mathbf{Q}$, we set $\widehat{\varphi} := |\varphi|^2 \varphi$. Then, the following holds (cf. [24, Lemma 2.2]):

$$\widehat{\varphi} \in L^{4/3}(\Omega) \quad \text{and} \quad \int_{\Omega} \varphi \widehat{\varphi} = \|\varphi\|_{0,4;\Omega} \|\widehat{\varphi}\|_{0,4/3;\Omega} \quad (2.17)$$

Now, let $z \in H_D^1(\Omega)$ be the unique solution (by the Lax–Milgram lemma) of the auxiliary problem

$$-\Delta z = \widehat{\varphi} \quad \text{in } \Omega, \quad z = 0 \quad \text{on } \Gamma_D, \quad \nabla z \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_N. \quad (2.18)$$

Using Hölder’s inequality, the Sobolev embedding and Poincaré’s inequality we obtain the bound

$$\|\nabla z\|_{0,2;\Omega} \leq C_{\text{Sob}} (1 + c_P^2)^{1/2} \|\widehat{\varphi}\|_{0,4/3;\Omega}. \quad (2.19)$$

Then, we define $\widehat{\boldsymbol{\xi}} := -\nabla z \in \mathbf{L}^2(\Omega)$, and observe that (see (2.18)) $\text{div } \widehat{\boldsymbol{\xi}} = -\Delta z = \widehat{\varphi}$ in Ω , with $\widehat{\boldsymbol{\xi}} \cdot \mathbf{n}|_{\Gamma_N} = 0$. Hence, $\widehat{\boldsymbol{\xi}} \in \mathbf{H}$, and combined with (2.19), this implies

$$\|\widehat{\boldsymbol{\xi}}\|_{\mathbf{H}} \leq (C_{\text{Sob}} (1 + c_P^2)^{1/2} + 1) \|\widehat{\varphi}\|_{0,4/3;\Omega}.$$

Finally, using the definition of $b(\bullet, \bullet)$ (cf. (2.12)), the second part of (2.17), and the last estimate, we obtain

$$\sup_{\mathbf{0} \neq \boldsymbol{\xi} \in \mathbf{H}} \frac{b(\boldsymbol{\xi}, \varphi)}{\|\boldsymbol{\xi}\|_{\mathbf{H}}} \geq \frac{b(\widehat{\boldsymbol{\xi}}, \varphi)}{\|\widehat{\boldsymbol{\xi}}\|_{\mathbf{H}}} \geq \frac{\|\varphi\|_{\mathbf{Q}} \|\widehat{\varphi}\|_{0,4/3;\Omega}}{(C_{\text{Sob}} (1 + c_P^2)^{1/2} + 1) \|\widehat{\varphi}\|_{0,4/3;\Omega}} \geq \beta \|\varphi\|_{\mathbf{Q}},$$

with $\beta := (C_{\text{Sob}} (1 + c_P^2)^{1/2} + 1)^{-1}$. □

Now, let $\mathbf{B} : \mathbf{H} \rightarrow \mathbf{Q}'$ and $\mathbf{B}^t : \mathbf{Q} \rightarrow \mathbf{H}'$ be the bounded linear operators induced by $b(\bullet, \bullet)$

$$\begin{aligned} \mathbf{B}(\boldsymbol{\xi})(\varphi) &:= b(\boldsymbol{\xi}, \varphi) \quad \forall \boldsymbol{\xi} \in \mathbf{H}, \quad \forall \varphi \in \mathbf{Q} \\ \text{and } \mathbf{B}^t(\varphi)(\boldsymbol{\xi}) &:= b(\boldsymbol{\xi}, \varphi) \quad \forall \varphi \in \mathbf{Q}, \quad \forall \boldsymbol{\xi} \in \mathbf{H}, \end{aligned}$$

we have the following kernel characterisations

$$\mathbf{V} := \ker(\mathbf{B}) = \{\boldsymbol{\xi} \in \mathbf{H} : b(\boldsymbol{\xi}, \varphi) = 0 \quad \forall \varphi \in \mathbf{Q}\} = \{\boldsymbol{\xi} \in \mathbf{H} : \text{div } \boldsymbol{\xi} = 0 \quad \text{in } \Omega\}, \quad (2.20a)$$

$$\mathbf{W} := \ker(\mathbf{B}^t) = \{\varphi \in \mathbf{Q} : b(\boldsymbol{\xi}, \varphi) = 0 \quad \forall \boldsymbol{\xi} \in \mathbf{H}\} = \{0\}. \quad (2.20b)$$

On the other hand, it is clear that the bilinear forms $a(\bullet, \bullet)$ and $c(\bullet, \bullet)$ are symmetric. In addition, from the definition of \mathbf{V} and the boundedness of the permittivity, it readily follows that

$$a(\boldsymbol{\xi}, \boldsymbol{\xi}) \geq \alpha_a \|\boldsymbol{\xi}\|_{\mathbf{H}}^2 \quad \forall \boldsymbol{\xi} \in \mathbf{V}, \quad c(\varphi, \varphi) \geq 0 \quad \forall \varphi \in \mathbf{Q}, \quad (2.21)$$

with $\alpha_a := \frac{1}{\underline{\varepsilon}}$. The non-negativity of the bilinear form $c(\bullet, \bullet)$ follows from the assumption $\kappa \geq 0$.

From the bounds of the involved bilinear forms (2.15), the inf-sup condition of the bilinear form $b(\bullet, \bullet)$ in Lemma 2.4, the characterisation of the kernel in (2.20), the ellipticity of $a(\bullet, \bullet)$ on the kernel

of **B** (2.21), and using that $c(\bullet, \bullet)$ is symmetric and positive semidefinite, we have that the hypotheses of [13, Theorem 3.4] are satisfied for $A(\bullet, \bullet)$, implying that there exists a constant $\alpha_{\mathcal{A}} > 0$ such that

$$\sup_{\mathbf{0} \neq (\boldsymbol{\xi}, \varphi) \in \mathbf{H} \times \mathbf{Q}} \frac{A((\boldsymbol{\zeta}^*, \boldsymbol{\psi}^*), (\boldsymbol{\xi}, \varphi))}{\|(\boldsymbol{\xi}, \varphi)\|_{\mathbf{H} \times \mathbf{Q}}} \geq \alpha_{\mathcal{A}} \|(\boldsymbol{\zeta}^*, \boldsymbol{\psi}^*)\|_{\mathbf{H} \times \mathbf{Q}} \quad \forall (\boldsymbol{\zeta}^*, \boldsymbol{\psi}^*) \in \mathbf{H} \times \mathbf{Q}, \quad (2.22)$$

with $\alpha_{\mathcal{A}} := \frac{1}{2\tilde{c}}$, where $\tilde{c} := \max\{c_3, c_4\}$, and

$$\begin{aligned} c_1 &:= \frac{1}{\alpha_a} + \frac{1}{\beta} \left(1 + \frac{\|a\|}{\alpha_a}\right) + \frac{\|c\|}{\beta^2} \left(1 + \frac{\|a\|}{\alpha_a}\right)^2, \\ c_2 &:= \frac{1}{\beta} \left(1 + \frac{\|a\|}{\alpha_a}\right) \left\{1 + \frac{\|a\|}{\beta} + \frac{\|a\|^2 \|c\|}{\beta^3} \left(1 + \frac{\|a\|}{\alpha_a}\right)\right\}, \\ c_3 &:= \frac{1}{\alpha_a} + \frac{1}{\beta} \left(1 + \frac{\|a\|}{\alpha_a}\right) \left\{1 + \left(1 + \frac{\|a\|}{\beta}\right) (2\|c\| \max\{c_1, c_2\})^{1/2}\right\}, \\ c_4 &:= \frac{1}{\beta} \left(1 + \frac{\|a\|}{\alpha_a}\right) \left(1 + \frac{\|a\|}{\beta}\right) \left\{1 + (2\|c\| \max\{c_1, c_2\})^{1/2}\right\}. \end{aligned} \quad (2.23)$$

Moreover, using the bound of the bilinear form $d_{\mathbf{u}_h}(\bullet, \bullet)$ (cf. (2.15)), the definition of the bilinear forms $A(\bullet, \bullet)$ and $A_{\mathbf{u}_h}(\bullet, \bullet)$ (cf. (2.13a) and (2.13b)), and the global inf-sup condition (2.22), we can readily obtain

$$\sup_{\mathbf{0} \neq (\boldsymbol{\xi}, \varphi) \in \mathbf{H} \times \mathbf{Q}} \frac{A_{\mathbf{u}_h}((\boldsymbol{\zeta}^*, \boldsymbol{\psi}^*), (\boldsymbol{\xi}, \varphi))}{\|(\boldsymbol{\xi}, \varphi)\|_{\mathbf{H} \times \mathbf{Q}}} \geq \{\alpha_{\mathcal{A}} - \|a\| \|\mathbf{u}_h\|_{0,4;\Omega}\} \|(\boldsymbol{\zeta}^*, \boldsymbol{\psi}^*)\|_{\mathbf{H} \times \mathbf{Q}} \quad \forall (\boldsymbol{\zeta}^*, \boldsymbol{\psi}^*) \in \mathbf{H} \times \mathbf{Q}. \quad (2.24)$$

Then, assuming that (see also Remark 2.6 below)

$$\|\mathbf{u}_h\|_{0,4;\Omega} \leq \frac{\alpha_{\mathcal{A}}}{2\|a\|}, \quad (2.25)$$

a global inf-sup stability is established for $A_{\mathbf{u}_h}$:

$$\sup_{\mathbf{0} \neq (\boldsymbol{\xi}, \varphi) \in \mathbf{H} \times \mathbf{Q}} \frac{A_{\mathbf{u}_h}((\boldsymbol{\zeta}^*, \boldsymbol{\psi}^*), (\boldsymbol{\xi}, \varphi))}{\|(\boldsymbol{\xi}, \varphi)\|_{\mathbf{H} \times \mathbf{Q}}} \geq \frac{\alpha_{\mathcal{A}}}{2} \|(\boldsymbol{\zeta}^*, \boldsymbol{\psi}^*)\|_{\mathbf{H} \times \mathbf{Q}} \quad \forall (\boldsymbol{\zeta}^*, \boldsymbol{\psi}^*) \in \mathbf{H} \times \mathbf{Q}. \quad (2.26)$$

Similarly, by the symmetry of $A(\bullet, \bullet)$ and under the same assumption on \mathbf{u}_h (2.25), we obtain

$$\sup_{\mathbf{0} \neq (\boldsymbol{\zeta}^*, \boldsymbol{\psi}^*) \in \mathbf{H} \times \mathbf{Q}} \frac{A_{\mathbf{u}_h}((\boldsymbol{\zeta}^*, \boldsymbol{\psi}^*), (\boldsymbol{\xi}, \varphi))}{\|(\boldsymbol{\zeta}^*, \boldsymbol{\psi}^*)\|_{\mathbf{H} \times \mathbf{Q}}} \geq \frac{\alpha_{\mathcal{A}}}{2} \|(\boldsymbol{\xi}, \varphi)\|_{\mathbf{H} \times \mathbf{Q}} \quad \forall (\boldsymbol{\xi}, \varphi) \in \mathbf{H} \times \mathbf{Q}. \quad (2.27)$$

In this way, we are now able to establish the well-posedness of (2.11) (equivalently (2.14)).

Theorem 2.5. *Given $\mathbf{u}_h \in \mathbf{L}^4(\Omega)$ satisfying (2.25), there exists a unique $(\boldsymbol{\zeta}^*, \boldsymbol{\psi}^*) \in \mathbf{H} \times \mathbf{Q}$ solution to (2.11), and there holds*

$$\|\boldsymbol{\zeta}^*\|_{\mathbf{H}} + \|\boldsymbol{\psi}^*\|_{\mathbf{Q}} \leq \frac{2}{\alpha_{\mathcal{A}}} \max\{1, C_T\} \left\{ \|\boldsymbol{\psi}_D\|_{1/2,0,0;\Gamma_D} + \|\mathcal{Q}_h g\|_{0,4/3;\Omega} \right\}. \quad (2.28)$$

Proof. Since (2.26) and (2.27) guarantee that $A_{\mathbf{u}_h}(\bullet, \bullet)$ satisfies the hypotheses of the BNB Theorem [17, Theorem 2.6], the proof becomes a direct application of this theorem. Hence, the estimate (2.28) follows from (2.14) together with the boundedness of $F(\bullet)$ and $G(\bullet)$ (cf. (2.15)). \square

Remark 2.6. *We note here that the mixed formulation (2.10) requires $\|\mathbf{u}_h\| < \frac{\varepsilon}{4C^*}$, whereas the primal problem (2.2) only assumes $\|\mathbf{u}_h\| < \frac{\varepsilon}{C}$. The constant C depends on Ω , as well as the Sobolev and Poincaré constants, while C^* additionally depends on κ , the permittivity bounds, and the inf-sup constant β . Hence, the mixed smallness condition (2.25) is more restrictive.*

3. DISCRETE PROBLEM

Let us denote by \mathbf{M}_h a finite-dimensional subspace of $\mathbf{L}^4(\Omega)$ (for example, $[\mathbf{P}_k(\mathcal{T}_h)]^d$), and consider for this section that the given vector field $\mathbf{u}_h \in \mathbf{M}_h$.

Following the usual approach, we obtain the Galerkin discretisation of (2.10a)-(2.10b) as a perturbed saddle-point problem that looks for $(\boldsymbol{\zeta}_h, \psi_h) \in \mathbf{H}_h \times Q_h$ such that

$$\begin{aligned} a(\boldsymbol{\zeta}_h, \boldsymbol{\xi}_h) + b(\boldsymbol{\xi}_h, \psi_h) + d_{\mathbf{u}_h}(\boldsymbol{\xi}_h, \psi_h) &= F(\boldsymbol{\xi}_h) \quad \forall \boldsymbol{\xi}_h \in \mathbf{H}_h, \\ b(\boldsymbol{\zeta}_h, \varphi_h) - c(\psi_h, \varphi_h) &= G(\varphi_h) \quad \forall \varphi_h \in Q_h, \end{aligned} \quad (3.1)$$

In order to analyse the discrete solvability of (3.1) we use the abstract discrete theory from [13, Theorem 3.5].

We remark that, for our particular choice of Raviart–Thomas elements (and similarly for other finite elements such as the Brezzi–Douglas–Marini (BDM) elements), the following property holds. Since we will later need a more explicit characterisation of the kernel of the discrete divergence operator, denoted by \mathbf{V}_h , this property will play a key role

$$\operatorname{div} \mathbf{H}_h \subseteq Q_h. \quad (3.2)$$

Let us recall some preliminary results. First, we consider the quasi-interpolation operator defined in [18, Theorem 2.2],

$$\Pi_q^{\text{RT}} : \mathbf{H} \rightarrow \mathbf{H}_h,$$

which satisfies the following properties:

- For all $\boldsymbol{\xi}_h \in \mathbf{H}_h$, we have

$$\Pi_q^{\text{RT}}(\boldsymbol{\xi}_h) = \boldsymbol{\xi}_h. \quad (3.3a)$$

- For $1 \leq p \leq \infty$, there exists C_q , independent of h , such that

$$\|\Pi_q^{\text{RT}} \boldsymbol{\xi}\|_{0,p;\Omega} \leq C_q \|\boldsymbol{\xi}\|_{0,p;\Omega}, \quad (3.3b)$$

for all $\boldsymbol{\xi} \in L^p(\Omega)$.

- The following commuting property (cf. [12, Section 5.3]) holds

$$\operatorname{div}(\Pi_q^{\text{RT}} \boldsymbol{\xi}) = \mathcal{P}_h^k(\operatorname{div} \boldsymbol{\xi}) \quad \forall \boldsymbol{\xi} \in \mathbf{H}, \quad (3.4)$$

where $\mathcal{P}_h^k : L^1(\Omega) \rightarrow P_k(\mathcal{T}_h)$ is the usual orthogonal projector with respect to the $L^2(\Omega)$ -inner product, that is, given $w \in L^1(\Omega)$, $\mathcal{P}_h^k(w)$ is the unique element in $P_k(\mathcal{T}_h)$ satisfying

$$\int_{\Omega} \mathcal{P}_h^k(w) q_h = \int_{\Omega} w q_h \quad \forall q_h \in P_k(\mathcal{T}_h).$$

The following result can be found in [24, Eq. (4.4)].

Lemma 3.1. *The $L^2(\Omega)$ -orthogonal projector onto the piecewise polynomial space Q_h is bounded as a mapping from $L^p(\Omega) \rightarrow L^p(\Omega)$ for all $p \in [1, \infty]$.*

In order to analyse the discrete solvability of (3.1) we use [13, Theorem 3.5]. We begin by considering $\mathbf{B}_h : \mathbf{H}_h \rightarrow Q'_h$ and $\mathbf{B}_h^t : Q_h \rightarrow \mathbf{H}'_h$ to be the discrete bounded linear operators induced by $b(\bullet, \bullet)$, that is

$$\begin{aligned} \mathbf{B}_h(\boldsymbol{\xi}_h)(\varphi_h) &:= b(\boldsymbol{\xi}_h, \varphi_h) \quad \forall \boldsymbol{\xi}_h \in \mathbf{H}_h, \quad \forall \varphi_h \in Q_h, \\ \mathbf{B}_h^t(\varphi_h)(\boldsymbol{\xi}_h) &:= b(\boldsymbol{\xi}_h, \varphi_h) \quad \forall \varphi_h \in Q_h, \quad \forall \boldsymbol{\xi}_h \in \mathbf{H}_h. \end{aligned}$$

Using the property (3.2) and proceeding similarly as in the continuous case, we have that the following discrete kernel characterisations hold

$$\mathbf{V}_h := \ker(\mathbf{B}_h) = \{\boldsymbol{\xi}_h \in \mathbf{H}_h : b(\boldsymbol{\xi}_h, \varphi_h) = 0, \quad \forall \varphi_h \in Q_h\} \equiv \{\boldsymbol{\xi}_h \in \mathbf{H}_h : \operatorname{div} \boldsymbol{\xi}_h = 0\}, \quad (3.5a)$$

$$W_h := \ker(\mathbf{B}_h^t) = \{\varphi_h \in Q_h : b(\boldsymbol{\xi}_h, \varphi_h) = 0, \quad \forall \boldsymbol{\xi}_h \in \mathbf{H}_h\} \equiv \{0\}, \quad (3.5b)$$

thus $\mathbf{V}_h \subseteq \mathbf{V}$, and from (2.21), we obtain

$$a(\boldsymbol{\xi}_h, \boldsymbol{\xi}_h) \geq \alpha_a \|\boldsymbol{\xi}_h\|_{\mathbf{H}}^2 \quad \forall \boldsymbol{\xi}_h \in \mathbf{V}_h, \quad c(\varphi_h, \varphi_h) \geq 0 \quad \forall \varphi_h \in Q_h, \quad (3.6)$$

with α_a the same constant as in (2.21).

Lemma 3.2. *There exists a positive constant $\beta_d > 0$, independent of h , such that*

$$\sup_{\mathbf{0} \neq \boldsymbol{\xi}_h \in \mathbf{H}_h} \frac{b(\boldsymbol{\xi}_h, \varphi_h)}{\|\boldsymbol{\xi}_h\|_{\mathbf{H}}} \geq \beta_d \|\varphi_h\|_Q \quad \forall \varphi_h \in Q_h. \quad (3.7)$$

Proof. We first invoke the continuous inf-sup condition (cf. Lemma 2.4) where for a given $\varphi_h \in Q_h \subseteq Q$, we have $\bar{\varphi}_h \in L^{4/3}(\Omega)$ such that there exists $\boldsymbol{\xi} \in \mathbf{H}$, with $\operatorname{div} \boldsymbol{\xi} = \bar{\varphi}_h$ and

$$\|\boldsymbol{\xi}\|_{\mathbf{H}} \leq \frac{1}{\beta} \|\bar{\varphi}_h\|_{0,4/3;\Omega}. \quad (3.8)$$

Next, we take $\bar{\boldsymbol{\xi}}_h \in \mathbf{H}_h$ to be the quasi-interpolation of $\boldsymbol{\xi}$ (i.e. $\Pi_q^{\text{RT}} \boldsymbol{\xi} = \bar{\boldsymbol{\xi}}_h$). Then, using (3.3b) and the definition of the \mathbf{H} -norm, we obtain

$$\|\bar{\boldsymbol{\xi}}_h\|_{0;\Omega} \leq C_q \|\boldsymbol{\xi}\|_{0;\Omega} \leq C_q \|\boldsymbol{\xi}\|_{\mathbf{H}}. \quad (3.9)$$

On the other hand, by applying the commuting property of Π_q^{RT} (cf. (3.4), for $p = 4$), we have

$$\operatorname{div} \bar{\boldsymbol{\xi}}_h = \operatorname{div} \Pi_q^{\text{RT}} \boldsymbol{\xi} = \mathcal{P}_h^k(\operatorname{div} \boldsymbol{\xi}), \quad (3.10)$$

which together with (3.9), (3.8), the continuity of \mathcal{P}_h^k (with $C_{\mathcal{P}} > 0$, independent of h), the definition of the \mathbf{H} -norm, and (3.8), gives

$$\begin{aligned} \|\bar{\boldsymbol{\xi}}_h\|_{\mathbf{H}} &= \|\bar{\boldsymbol{\xi}}_h\|_{0,2;\Omega} + \|\operatorname{div} \bar{\boldsymbol{\xi}}_h\|_{0,4/3;\Omega} \leq \frac{C_q}{\beta} \|\bar{\varphi}_h\|_{0,4/3;\Omega} + \|\mathcal{P}_h^k(\operatorname{div} \boldsymbol{\xi})\|_{0,4/3;\Omega} \\ &\leq \frac{C_q}{\beta} \|\bar{\varphi}_h\|_{0,4/3;\Omega} + C_{\mathcal{P}} \|\operatorname{div} \boldsymbol{\xi}\|_{0,4/3;\Omega} \leq \frac{C_q}{\beta} \|\bar{\varphi}_h\|_{0,4/3;\Omega} + C_{\mathcal{P}} \|\boldsymbol{\xi}\|_{\mathbf{H}} \leq \left(\frac{C_q + C_{\mathcal{P}}}{\beta} \right) \|\bar{\varphi}_h\|_{0,4/3;\Omega}. \end{aligned} \quad (3.11)$$

Finally, according to the definition of $b(\bullet, \bullet)$ and (3.11), we obtain

$$\begin{aligned} \sup_{\mathbf{0} \neq \boldsymbol{\xi}_h \in \mathbf{H}_h} \frac{b(\boldsymbol{\xi}_h, \varphi_h)}{\|\boldsymbol{\xi}_h\|_{\mathbf{H}}} &\geq \frac{b(\bar{\boldsymbol{\xi}}_h, \varphi_h)}{\|\bar{\boldsymbol{\xi}}_h\|_{\mathbf{H}}} = \frac{\int_{\Omega} \varphi_h \operatorname{div} \bar{\boldsymbol{\xi}}_h}{\|\bar{\boldsymbol{\xi}}_h\|_{\mathbf{H}}} = \frac{\int_{\Omega} \varphi_h \mathcal{P}_h^k(\operatorname{div} \boldsymbol{\xi})}{\|\bar{\boldsymbol{\xi}}_h\|_{\mathbf{H}}} = \frac{\int_{\Omega} \varphi_h \mathcal{P}_h^k(\bar{\varphi}_h)}{\|\bar{\boldsymbol{\xi}}_h\|_{\mathbf{H}}} \\ &= \frac{\int_{\Omega} \varphi_h \bar{\varphi}_h}{\|\bar{\boldsymbol{\xi}}_h\|_{\mathbf{H}}} \geq \frac{\beta}{C_q + C_{\mathcal{P}}} \frac{\|\varphi_h\|_Q \|\bar{\varphi}_h\|_{0,4/3;\Omega}}{\|\bar{\varphi}_h\|_{0,4/3;\Omega}} = \frac{\beta}{C_q + C_{\mathcal{P}}} \|\varphi_h\|_Q, \end{aligned}$$

with $\beta_d := \frac{\beta}{C_q + C_{\mathcal{P}}}$. □

Consequently, by applying [13, Theorem 3.5] directly, we obtain that the discrete global inf–sup condition for $A(\bullet, \bullet)$ (cf. (2.13a)) holds with $\alpha_{\mathcal{A},d} > 0$, where the constant depends only on α , β_d , $\|a\|$, and $\|c\|$ (cf.(2.23)). As a result, we have the following discrete analogue of (2.26), that is

$$\sup_{\substack{(\zeta_h, \psi_h) \in \mathbf{H}_h \times \mathbf{Q}_h \\ (\xi_h, \varphi_h) \neq 0}} \frac{A_{\mathbf{u}_h}((\zeta_h, \psi_h), (\xi_h, \varphi_h))}{\|(\xi_h, \varphi_h)\|_{\mathbf{H} \times \mathbf{Q}}} \geq \{\alpha_{\mathcal{A},d} - \|a\| \|\mathbf{u}_h\|_{0,4;\Omega}\} \|(\zeta_h, \psi_h)\|_{\mathbf{H} \times \mathbf{Q}}, \quad (3.12)$$

for all $(\zeta_h, \psi_h) \in \mathbf{H}_h \times \mathbf{Q}_h$. And under the following discrete version of the data assumption

$$\|\mathbf{u}_h\|_{0,4;\Omega} \leq \frac{\alpha_{\mathcal{A},d}}{2\|a\|}, \quad (3.13)$$

we obtain

$$\sup_{0 \neq (\xi_h, \varphi_h) \in \mathbf{H}_h \times \mathbf{Q}_h} \frac{A_{\mathbf{u}_h}((\zeta_h, \psi_h), (\xi_h, \varphi_h))}{\|(\xi_h, \varphi_h)\|_{\mathbf{H} \times \mathbf{Q}}} \geq \frac{\alpha_{\mathcal{A},d}}{2} \|(\zeta_h, \psi_h)\|_{\mathbf{H} \times \mathbf{Q}} \quad \forall (\zeta_h, \psi_h) \in \mathbf{H}_h \times \mathbf{Q}_h. \quad (3.14)$$

As a consequence of the above bound, we are now in a position to establish the solvability of (3.1) (equivalently, the discrete counterpart of (2.14)).

Theorem 3.3. *Given $\mathbf{u}_h \in \mathbf{M}_h \subseteq \mathbf{L}^4(\Omega)$ satisfying $\|\mathbf{u}_h\|_{0,4;\Omega} \leq \frac{\alpha_{\mathcal{A},d}}{2\|a\|}$, there exists a unique $(\zeta_h, \psi_h) \in \mathbf{H}_h \times \mathbf{Q}_h$ solution to (3.1), and there holds*

$$\|\zeta_h\|_{\mathbf{H}} + \|\psi_h\|_{\mathbf{Q}} \leq \frac{2}{\alpha_{\mathcal{A},d}} \max\{1, C_T\} \left\{ \|\psi_D\|_{\frac{1}{2},00;\Gamma_D} + \|\mathcal{Q}_h g\|_{0,\frac{4}{3};\Omega} \right\}. \quad (3.15)$$

Proof. The proof follows from (3.14) and from the fact that, in finite-dimensional linear problems, surjectivity and injectivity are equivalent. In addition, (3.15) is obtained by using arguments similar to those in Theorem 2.5. \square

4. ERROR ESTIMATES

Theorem 4.1. *Let $(\zeta^*, \psi^*), (\zeta_h, \psi_h)$ be the unique solutions of (2.11) and (3.1), respectively. Then, and assuming that (3.13) holds, there exists $\hat{C} > 0$, independent of h , such that*

$$\|\zeta^* - \zeta_h\|_{\mathbf{H}} + \|\psi^* - \psi_h\|_{\mathbf{Q}} \leq \hat{C} \{ \text{dist}(\zeta^*, \mathbf{H}_h) + \text{dist}(\psi^*, \mathbf{Q}_h) \}. \quad (4.1)$$

Proof. The proof follows the same approach as [6, Theorem 4.7]. For a given $(\hat{\xi}_h, \hat{\varphi}_h) \in \mathbf{H}_h \times \mathbf{Q}_h$, we decompose the errors as

$$\begin{aligned} e_{\zeta^*} &:= \eta_{\zeta^*} + \chi_{\zeta^*} = (\zeta^* - \hat{\xi}_h) + (\hat{\xi}_h - \zeta_h), \\ e_{\psi^*} &:= \eta_{\psi^*} + \chi_{\psi^*} = (\psi^* - \hat{\varphi}_h) + (\hat{\varphi}_h - \psi_h). \end{aligned} \quad (4.2)$$

Next, we apply the Galerkin orthogonality property to $A_{\mathbf{u}}(\bullet, \bullet)$ (cf. (2.13b) and (2.14)) and to its corresponding discrete formulation and substitute the decomposition of ψ^* from (4.2), to obtain

$$A((e_{\zeta^*}, e_{\psi^*}), (\xi_h, \varphi_h)) + d_{\mathbf{u}_h}(\xi_h; e_{\psi^*}) = 0. \quad (4.3)$$

Then, using the definition of $A_{\mathbf{u}_h}(\bullet, \bullet)$ in the above equality, together with (4.2), we get

$$A_{\mathbf{u}_h}((\chi_{\zeta^*}, \chi_{\psi^*}), (\xi_h, \varphi_h)) = -A((\eta_{\zeta^*}, \eta_{\psi^*}), (\xi_h, \varphi_h)) - d_{\mathbf{u}_h}(\xi_h, \eta_{\psi^*}).$$

Moreover, by expanding $A(\bullet, \bullet)$ (cf.(2.13a)), we have for all $(\xi_h, \varphi_h) \in \mathbf{H}_h \times \mathbf{Q}_h$

$$A_{\mathbf{u}_h}((\chi_{\zeta^*}, \chi_{\psi^*}), (\xi_h, \varphi_h)) = -a(\eta_{\zeta^*}, \xi_h) - b(\eta_{\zeta^*}, \varphi_h) - b(\xi_h, \eta_{\psi^*}) - c(\eta_{\psi^*}, \varphi_h) - d_{\mathbf{u}_h}(\xi_h, \eta_{\psi^*}).$$

We now bound the right-hand side using the continuity of $a(\bullet, \bullet)$, $b(\bullet, \bullet)$, $c(\bullet, \bullet)$, and $d_{\mathbf{u}_h}(\bullet, \bullet)$ (cf. (2.15)), the smallness assumption on \mathbf{u}_h (3.13), this gives

$$|A_{\mathbf{u}_h}((\boldsymbol{\chi}_{\zeta^*}, \chi_{\psi^*}), (\boldsymbol{\xi}_h, \varphi_h))| \leq \max\{1, \|a\|, \|c\|, \frac{\alpha_{A,d}}{2}\} \|(\boldsymbol{\eta}_{\zeta^*}, \eta_{\psi^*})\|_{\mathbf{H} \times \mathbf{Q}} \|(\boldsymbol{\xi}_h, \varphi_h)\|_{\mathbf{H} \times \mathbf{Q}}.$$

For the left-hand side, we take the supremum over $(\boldsymbol{\xi}_h, \varphi_h)$, and apply the discrete inf-sup condition of $A_{\mathbf{u}_h}(\bullet, \bullet)$ (cf. (3.14)), which gives

$$\|(\boldsymbol{\chi}_{\zeta^*}, \chi_{\psi^*})\| \leq \frac{2}{\alpha_{A,d}} \max\{1, \|a\|, \|c\|, \frac{\alpha_{A,d}}{2}\} \|(\boldsymbol{\eta}_{\zeta^*}, \eta_{\psi^*})\|_{\mathbf{H} \times \mathbf{Q}}. \quad (4.4)$$

Finally, combining (4.2), the triangle inequality, and (4.4), we obtain

$$\|(\mathbf{e}_{\zeta^*}, e_{\psi^*})\| \lesssim \|(\boldsymbol{\eta}_{\zeta^*}, \eta_{\psi^*})\|_{\mathbf{H} \times \mathbf{Q}},$$

and since $(\hat{\boldsymbol{\xi}}_h, \hat{\varphi}_h) \in \mathbf{H}_h \times \mathbf{Q}_h$ is arbitrary, the result follows. \square

We are in a position to state our main result. It is a perturbed quasi-optimality result.

Theorem 4.2. *Let $g \in \tilde{\mathbf{H}}^{-1}(\Omega)$ and $\psi_D \in H_{00}^{1/2}(\Gamma_D)$ be given. Suppose that $\psi \in H_D^1(\Omega)$ solves (2.2) and set $\boldsymbol{\zeta} = \nabla\psi - \mathbf{u}\psi \in L^2(\Omega)$, i.e., $(\boldsymbol{\zeta}, \psi)$ solves (2.3). If $(\boldsymbol{\zeta}_h, \psi_h) \in \mathbf{H}_h \times \mathbf{Q}_h$ is the solution of (3.1), then,*

$$\begin{aligned} & \|\psi - \psi_h\|_{0,4;\Omega} + \|\boldsymbol{\zeta} - \boldsymbol{\zeta}_h\|_{0,2;\Omega} \\ & \lesssim \min_{(v_h, \boldsymbol{\tau}_h) \in \mathbf{H}_h \times \mathbf{Q}_h} (\|\psi - v_h\|_{0,4;\Omega} + \|\boldsymbol{\zeta} - \boldsymbol{\tau}_h\|_{0,2;\Omega}) \\ & \quad + (\|g\|_{\tilde{\mathbf{H}}^{-1}(\Omega)} + \|\psi_D\|_{1/2,00;\Gamma_D}) \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega} + \min_{\chi_h \in \mathbf{Q}_h} \|\kappa\psi - \chi_h\|_{0,4/3;\Omega}. \end{aligned} \quad (4.5)$$

Proof. Let $(\boldsymbol{\zeta}^*, \psi^*) \in \mathbf{H} \times \mathbf{Q}$ denote the unique solution of (2.11). From Theorem 4.1 it follows that

$$\begin{aligned} \|\boldsymbol{\zeta}^* - \boldsymbol{\zeta}_h\|_{0,2;\Omega} + \|\psi^* - \psi_h\|_{0,4;\Omega} & \leq \|\boldsymbol{\zeta}^* - \boldsymbol{\zeta}_h\|_{\mathbf{H}} + \|\psi^* - \psi_h\|_{0,4;\Omega} \\ & \lesssim \|(1 - \Pi_q^{\text{RT}})\boldsymbol{\zeta}^*\|_{\mathbf{H}} + \|\psi^* - \varphi_h\|_{0,4;\Omega} \end{aligned}$$

for all $\varphi_h \in \mathbf{Q}_h$. Since Π_q^{RT} is a projection, using the properties (3.3a)–(3.4) we obtain for any $\boldsymbol{\tau}_h \in \mathbf{H}_h$ that

$$\begin{aligned} \|(1 - \Pi_q^{\text{RT}})\boldsymbol{\zeta}^*\|_{\mathbf{H}} & \leq \|(1 - \Pi_q^{\text{RT}})(\boldsymbol{\zeta}^* - \boldsymbol{\tau}_h)\|_{0,2;\Omega} + \|(1 - \mathcal{P}_h^k) \operatorname{div} \boldsymbol{\zeta}^*\|_{0,4/3;\Omega} \\ & \lesssim \|\boldsymbol{\zeta}^* - \boldsymbol{\tau}_h\|_{0,2;\Omega} + \|(1 - \mathcal{P}_h^k)(\kappa\psi^* - \mathcal{Q}_h g)\|_{0,4/3;\Omega} \\ & = \|\boldsymbol{\zeta}^* - \boldsymbol{\tau}_h\|_{0,2;\Omega} + \|(1 - \mathcal{P}_h^k)(\kappa\psi^*)\|_{0,4/3;\Omega} \\ & \leq \|\boldsymbol{\zeta} - \boldsymbol{\tau}_h\|_{0,2;\Omega} + \|(1 - \mathcal{P}_h^k)(\kappa\psi)\|_{0,4/3;\Omega} \\ & \quad + \|\boldsymbol{\zeta} - \boldsymbol{\zeta}^*\|_{0,2;\Omega} + \|(1 - \mathcal{P}_h^k)(\kappa(\psi - \psi^*))\|_{0,4/3;\Omega} \end{aligned}$$

Using Lemma 3.1 we stress that for any $\chi_h \in \mathbf{Q}_h$,

$$\begin{aligned} \|(1 - \mathcal{P}_h^k)(\kappa\psi)\|_{0,4/3;\Omega} & = \|(1 - \mathcal{P}_h^k)(\kappa\psi - \chi_h)\|_{0,4/3;\Omega} \lesssim \|\kappa\psi - \chi_h\|_{0,4/3;\Omega}, \\ \|(1 - \mathcal{P}_h^k)(\kappa\psi - \kappa\psi^*)\|_{0,4/3;\Omega} & \lesssim \|\kappa(\psi - \psi^*)\|_{0,4/3;\Omega} \lesssim \|\kappa\|_{0,2;\Omega} \|\psi - \psi^*\|_{0,4;\Omega}. \end{aligned}$$

Combining all of the estimates above, together with the triangle inequality, shows that

$$\begin{aligned} \|\boldsymbol{\zeta}^* - \boldsymbol{\zeta}_h\|_{0,2;\Omega} + \|\psi^* - \psi_h\|_{0,4;\Omega} & \lesssim \|\boldsymbol{\zeta} - \boldsymbol{\tau}_h\|_{0,2;\Omega} + \|\psi - \varphi_h\|_{0,4;\Omega} + \|\kappa\psi - \chi_h\|_{0,4/3;\Omega} \\ & \quad + \|\boldsymbol{\zeta} - \boldsymbol{\zeta}^*\|_{0,2;\Omega} + \|\psi - \psi^*\|_{0,4;\Omega}, \end{aligned}$$

for all $(\boldsymbol{\tau}_h, \varphi_h) \in \mathbf{H}_h \times \mathbf{Q}_h$ and $\chi_h \in \mathbf{Q}_h$. Finally, the triangle inequality yields

$$\|\boldsymbol{\zeta} - \boldsymbol{\zeta}_h\|_{0,2;\Omega} + \|\psi - \psi_h\|_{0,4;\Omega} \leq \|\boldsymbol{\zeta} - \boldsymbol{\zeta}^*\|_{0,2;\Omega} + \|\psi - \psi^*\|_{0,4;\Omega} + \|\boldsymbol{\zeta}^* - \boldsymbol{\zeta}_h\|_{0,2;\Omega} + \|\psi^* - \psi_h\|_{0,4;\Omega},$$

and an application of Lemma 2.2 and the estimate (2.6) conclude the proof. \square

5. STENBERG-TYPE POSTPROCESSING

In this section we provide a postprocessing technique that delivers an approximate solution to ψ^* with asymptotic better accuracy than the approximate solution ψ_h . We follow [20, Section 4] for the low-regular forcing case (see also, e.g., [11] for postprocessing of mixed formulations in L^p spaces). The analytical results rely on duality arguments and regularity properties of solutions to the dual problem. For the latter we assume a regularity shift. In order to utilise the additional regularity we require an additional non-standard approximation property for the dual of the projection operator \mathcal{Q}_h . In subsection 5.1 we show how to construct such an operator. Then, we present duality results, the adapted Stenberg postprocessing scheme, and accuracy enhancement results in Subsection 5.2.

5.1. Construction of \mathcal{Q}_h . We first recall from [20, 22] the construction of the weighted Clément quasi-interpolator $J_h : H_D^1(\Omega) \rightarrow P_1(\mathcal{T}_h) \cap H_D^1(\Omega)$. Let V_0 denote the set of interior vertices of \mathcal{T}_h and let V_N denote the set of vertices of \mathcal{T}_h on the boundary $\Gamma \setminus \overline{\Gamma_D}$. We assume that each $z \in V_N$ has a neighbouring interior node $z_0 \in V_0$ in the sense that there exists $K \in \mathcal{T}_h$ with $z, z_0 \in V_K$ where V_K denotes the set of vertices of K . Given the nodal basis $\{\eta_z\}_{z \in V_0 \cup V_N}$ of $P_1(\mathcal{T}_h) \cap H_D^1(\Omega)$, i.e., $\eta_z(z') = \delta_{z,z'}$ for all $z, z' \in V_0 \cup V_N$, the quasi-interpolator J_h is defined by

$$v \mapsto J_h(v) = \sum_{z \in V_0} \left(\int_{\Omega} v \psi_z \right) \eta_z, \quad \text{for } v \in L^2(\Omega).$$

For an interior vertex $z \in V_0$, let \mathcal{T}_z denote the patch of all elements sharing the node $z \in V_0$. For a boundary vertex $z \in V_N$, we set $\mathcal{T}_z = \mathcal{T}_{z_0}$. Further, $\{\psi_z\}_{z \in V_0 \cup V_N}$ is a collection of weight functions given by

$$\psi_z|_K = \begin{cases} \frac{\alpha_{z,K}}{|K|} & K \in \mathcal{T}_z, \\ 0 & \text{otherwise.} \end{cases}$$

For each $z \in V_0 \cup V_N$, the real values $(\alpha_{z,K})_{K \in \mathcal{T}_z}$ are chosen such that $\sum_{K \in \mathcal{T}_z} \alpha_{z,K} s_K = z$, $\sum_{K \in \mathcal{T}_z} \alpha_{z,K} = 1$, where s_K denotes the centre of mass of $K \in \mathcal{T}_h$. Due to the simple geometric observation that not all centre of mass are colinear resp. coplanar, it follows that there exist $(\alpha_{z,K})_{K \in \mathcal{T}_z}$ satisfying the above relations, though they might not be unique, see [20, Remark 10]. If $z \in V_0$ then we can even choose all $\alpha_{z,K}$, $K \in \mathcal{T}_z$ to be non-negative and $\|\psi_z\|_{0,\infty;\Omega} \simeq \frac{1}{|\Omega_z|}$, see [20]. If $z \in V_N$, then some coefficients might be negative whilst others are positive. Anyway, we can always choose them such that $\|\psi_z\|_{0,\infty;\Omega} \simeq \frac{1}{|\Omega_z|}$ holds true as well. Note that, by the above definition, we have $\psi_z \in P_0(\mathcal{T}_h)$.

We also denote by $\eta_K = \prod_{z \in V_K} \eta_z$ an element bubble function. Further, let η_j , $j = 1, \dots, N := \dim(Q_h)$ denote a basis of of $Q_h = P_k(\mathcal{T}_h)$ and define the bubble function space

$$P_k^b(K) = \{\eta_K \eta_j \mid j = 1, \dots, N\} \subset H_0^1(K).$$

Let $\chi_{K,j} \in P_k^b(K)$ ($j = 1, \dots, N$) denote a dual basis in the sense that

$$\int_K \chi_{K,j} \eta_k = \delta_{jk}, \quad j, k = 1, \dots, N.$$

Further, we define the bounded map

$$B_h v = \sum_{K \in \mathcal{T}_h} \sum_{j=1}^N \int_K (v \eta_j) \chi_{K,j}.$$

With these ingredients, the operator \mathcal{Q}_h is defined as follows

$$\mathcal{Q}_h = J_h' + (1 - J_h)' B_h' : H_D^{-1}(\Omega) \rightarrow Q_h = P_k(\mathcal{T}_h) \subset H_D^{-1}(\Omega).$$

We now prove some properties of the map \mathcal{Q}_h . The arguments closely follow the proofs of [20, Theorem 11 and 12] and [22, Theorem 18] and we therefore only focus on essential differences.

Lemma 5.1 (Further properties of \mathcal{Q}_h). *In addition to the properties (2.7), the map \mathcal{Q}_h satisfies*

$$\begin{aligned} \|(1 - \mathcal{Q}'_h)\phi\|_{0,2;\Omega} &\lesssim h\|\nabla\phi\|_{0,2;\Omega} \quad \forall\phi \in \mathbf{H}_D^1(\Omega), \\ \|(1 - \mathcal{Q}'_h)\phi\|_{1,2;\Omega} &\lesssim h^s\|\phi\|_{1+s,2;\Omega} \quad \forall\phi \in \mathbf{H}^{1+s}(\Omega) \cap \mathbf{H}_D^1(\Omega), \quad s \in [0, 1]. \end{aligned}$$

Proof. First, let us verify the projection property. By construction we have that $\int_K B_h(1 - J_h)v\eta_j = \int_K(1 - J_h)v\eta_j$, $j = 1, \dots, N$. Therefore,

$$\int_{\Omega} \mathcal{Q}_h\phi v = \int_{\Omega} \phi(J_h v + B_h(1 - J_h)v) = \int_{\Omega} \phi v \quad \forall\phi \in \mathcal{Q}_h, v \in \mathbf{H}_D^1(\Omega).$$

That is, $\mathcal{Q}_h\phi = \phi$ for all $\phi \in \mathcal{Q}_h$.

Second, we prove boundedness of \mathcal{Q}'_h and therefore \mathcal{Q}_h as an operator from $L^2(\Omega)$ to $L^2(\Omega)$. To do so, note that by standard scaling arguments one proves that $\|B_h v\|_{0,2;K} \lesssim \|v\|_{0,2;K}$ for all $v \in L^2(K)$. Then, with the arguments from [20, Theorem 11] it follows that

$$\|J_h v\|_{0,2;K} \lesssim \|v\|_{0,2;\Omega_K}$$

where $\Omega_K \subset \Omega$ is the domain associated to the patch of K if K does not contain vertices from V_N or the domain associated to the second-order patch of K if K contains vertices of V_N . Combining the latter estimates we conclude that

$$\|\mathcal{Q}'_h v\|_{0,2;K} \lesssim \|v\|_{0,2;\Omega_K} \quad \text{and further} \quad \|\mathcal{Q}'_h v\|_{0,2;\Omega} \lesssim \|v\|_{0,2;\Omega} \quad \forall v \in L^2(\Omega).$$

Third, we stress that by construction J_h preserves affine functions (on patches). Therefore, see [20, Theorem 11], this implies that

$$\|(1 - J_h)v\|_{0,2;K} + h_K\|\nabla(1 - J_h)v\|_{0,2;K} \lesssim h_K\|\nabla v\|_{0,2;\Omega_K}, \quad \forall v \in \mathbf{H}_D^1(\Omega).$$

as well as

$$\|(1 - J_h)v\|_{0,2;K} + h_K\|\nabla(1 - J_h)v\|_{0,2;K} \lesssim h_K^2\|D^2 v\|_{0,2;\Omega_K}, \quad \forall v \in \mathbf{H}^2(\Omega) \cap \mathbf{H}_D^1(\Omega).$$

Together with an inverse inequality we obtain that

$$\begin{aligned} \|\nabla \mathcal{Q}'_h v\|_{0,2;K} &\leq \|\nabla J_h v\|_{0,2;K} + \|\nabla B_h(1 - J_h)v\|_{0,2;K} \lesssim \|\nabla J_h v\|_{0,2;K} + h_K^{-1}\|B_h(1 - J_h)v\|_{0,2;K} \\ &\lesssim \|v\|_{0,2;\Omega_K} + h_K^{-1}\|(1 - J_h)v\|_{0,2;K} \lesssim \|v\|_{0,2;\Omega_K} \quad \forall v \in \mathbf{H}_D^1(\Omega). \end{aligned}$$

We conclude that $\mathcal{Q}'_h : \mathbf{H}_D^1(\Omega) \rightarrow \mathbf{H}_D^1(\Omega)$ is bounded which implies the boundedness of \mathcal{Q}_h in the dual norm.

Finally, to see the stated approximation properties we stress that \mathcal{Q}'_h preserves, such as J_h , affine functions on patches. Therefore, with the very same arguments to show the approximation properties for J_h can be used to show the ones for \mathcal{Q}'_h . The estimate in the non-integer Sobolev spaces follows by interpolation, which in turn finishes the proof. \square

Alternative construction of \mathcal{Q}_h for $k > 0$. For higher polynomial orders ($k > 0$) one can choose J_h to be, e.g., the Scott–Zhang quasi-interpolator into $\mathbf{P}_1(\mathcal{T}_h) \cap \mathbf{H}_D^1(\Omega)$ in the definitions above.

5.2. Adjoint problem and accuracy enhancement. Let us first develop Aubin–Nitsche-type estimates. To that end, consider the adjoint problem

$$\kappa\phi - \operatorname{div} \varepsilon \nabla \phi - \mathbf{u}_h \cdot \nabla \phi = f \quad \text{in } \Omega, \quad (5.1a)$$

$$\psi = 0 \quad \text{on } \Gamma_D, \quad (5.1b)$$

$$\varepsilon \nabla \phi \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_N. \quad (5.1c)$$

We make the following standing assumption throughout this section: Suppose that κ , ε , \mathbf{u}_h , and Ω are such that there exists $0 < s_\Omega \leq 1$, $C_\Omega > 0$ such that for any $f \in L^2(\Omega)$, the unique weak solution of (5.1) satisfies

$$\|\varepsilon \nabla \phi\|_{s_\Omega, 2; \Omega} \lesssim \|\phi\|_{1+s_\Omega, 2; \Omega} \leq C_\Omega \|f\|_{0, 2; \Omega}. \quad (5.2)$$

Lemma 5.2. *Let ψ and ψ^* denote the unique solutions of Problems (2.4) and (2.8), respectively. Then, there exist constants $C_1 > 0$, $C_2 > 0$ such that*

$$\begin{aligned} \|\psi - \psi^*\|_{0, 2; \Omega} &\leq C_1 (\|g\|_{H_D^{-1}(\Omega)} + \|\psi_D\|_{1/2, 0; \Gamma_D}) \|\mathbf{u} - \mathbf{u}_h\|_{0, 4; \Omega} \\ &\quad + C_2 h^{s_\Omega} \left(\min_{\boldsymbol{\tau}_h \in \mathbf{H}_h} \|\boldsymbol{\zeta} - \boldsymbol{\tau}_h\|_{0, 2; \Omega} + \min_{\chi_h \in \mathcal{Q}_h} \|\kappa\psi - \chi_h\|_{H_D^{-1}(\Omega)} \right) \end{aligned}$$

where $\boldsymbol{\zeta} = \varepsilon \nabla \psi - \mathbf{u}\psi$.

Proof. We write $\psi - \psi^* = (\psi - \tilde{\psi}^*) + (\tilde{\psi}^* - \psi^*)$ where $\tilde{\psi}^*$ is defined as in the proof of Lemma 2.2. Recall that in the proof we have shown that

$$\|\psi - \tilde{\psi}^*\|_{1, 2; \Omega} \lesssim (\|g\|_{H_D^{-1}(\Omega)} + \|\psi_D\|_{1/2, 0; \Gamma_D}) \|\mathbf{u} - \mathbf{u}_h\|_{0, 4; \Omega}.$$

By the Poincaré–Friedrichs inequality, the term $\|\psi - \tilde{\psi}^*\|_{0, 2; \Omega}$ can be estimated by the same upper bound.

Consider the solution $\phi \in H_D^1(\Omega)$ of the adjoint problem (5.1) with $f = \tilde{\psi}^* - \psi^*$. Then, integrating by parts twice, we obtain that

$$\begin{aligned} \|\tilde{\psi}^* - \psi^*\|_{0, 2; \Omega}^2 &= \int_{\Omega} (\tilde{\psi}^* - \psi^*) (-\operatorname{div}(\varepsilon \nabla \phi) - \mathbf{u}_h \cdot \nabla \phi + \kappa \phi) = \langle (1 - \mathcal{Q}_h)g, \phi \rangle \\ &= \langle (1 - \mathcal{Q}_h)g, (1 - \mathcal{Q}'_h)\phi \rangle \leq \|(1 - \mathcal{Q}_h)g\|_{H_D^{-1}(\Omega)} \|(1 - \mathcal{Q}'_h)\phi\|_{1, 2; \Omega}. \end{aligned}$$

Here, we have used the projection property $(1 - \mathcal{Q}_h)(1 - \mathcal{Q}_h) = (1 - \mathcal{Q}_h)$. The last term is further estimated using the approximation property of \mathcal{Q}'_h and the elliptic regularity assumption (5.2) leading to

$$\|\tilde{\psi}^* - \psi^*\|_{0, 2; \Omega}^2 \lesssim \|(1 - \mathcal{Q}_h)g\|_{H_D^{-1}(\Omega)} h^{s_\Omega} \|\tilde{\psi}^* - \psi^*\|_{0, 2; \Omega}.$$

Putting all estimates together we conclude that

$$\|\psi - \psi^*\|_{0, 2; \Omega} \leq \|\psi - \tilde{\psi}^*\|_{0, 2; \Omega} + \|\tilde{\psi}^* - \psi^*\|_{0, 2; \Omega} \lesssim \|\mathbf{u} - \mathbf{u}_h\|_{0, 4; \Omega} + h^{s_\Omega} \|(1 - \mathcal{Q}_h)g\|_{H_D^{-1}(\Omega)}.$$

Estimating $\|(1 - \mathcal{Q}_h)g\|_{H_D^{-1}(\Omega)}$ as in the proof of Lemma 2.2 gives

$$\begin{aligned} \|(1 - \mathcal{Q}_h)g\|_{H_D^{-1}(\Omega)} &\lesssim \|\tilde{\boldsymbol{\zeta}}^* - \boldsymbol{\tau}_h\|_{0, 2; \Omega} + \|\kappa\tilde{\psi}^* - \chi_h\|_{H_D^{-1}(\Omega)} \\ &\leq \|\boldsymbol{\zeta} - \boldsymbol{\tau}_h\|_{0, 2; \Omega} + \|\kappa\psi - \chi_h\|_{H_D^{-1}(\Omega)} + \|\boldsymbol{\zeta} - \tilde{\boldsymbol{\zeta}}^*\|_{0, 2; \Omega} + \|\kappa(\psi - \tilde{\psi}^*)\|_{H_D^{-1}(\Omega)} \\ &\lesssim \|\boldsymbol{\zeta} - \boldsymbol{\tau}_h\|_{0, 2; \Omega} + \|\kappa\psi - \chi_h\|_{H_D^{-1}(\Omega)} + \|\boldsymbol{\zeta} - \tilde{\boldsymbol{\zeta}}^*\|_{0, 2; \Omega} + \|\kappa(\psi - \tilde{\psi}^*)\|_{0, 4/3; \Omega} \end{aligned}$$

for all $\boldsymbol{\tau}_h \in \mathbf{H}_h$, $\chi_h \in \mathbf{Q}_h$, and with $\tilde{\boldsymbol{\zeta}}_h = \varepsilon \nabla \tilde{\psi}^* - \mathbf{u}_h \tilde{\psi}^*$. Finally, $\|\kappa(\psi - \tilde{\psi}^*)\|_{0,4/3;\Omega} \leq \|\kappa\|_{0,2;\Omega} \|\psi - \tilde{\psi}^*\|_{0,4;\Omega} \lesssim \|\psi - \tilde{\psi}^*\|_{1,2;\Omega}$. The proof is concluded by noting that as in Lemma 2.2

$$\|\boldsymbol{\zeta} - \tilde{\boldsymbol{\zeta}}^*\|_{0,2;\Omega} + \|\psi - \tilde{\psi}^*\|_{1,2;\Omega} \lesssim \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega}$$

and $h^{s_\Omega} \lesssim 1$. \square

Additionally to the assumption on the regularity shift for the adjoint problem (5.1) we suppose that the solution to the adjoint problem enjoys the approximation estimate

$$\|(1 - \mathcal{P}_h^0)(\kappa\phi - \mathbf{u}_h \cdot \nabla\phi)\|_{0,4/3;\Omega} \leq Ch^{s_\Omega} \|\phi\|_{1+s_\Omega,2;\Omega}. \quad (5.3)$$

To see that the latter estimate is likely to hold, consider for simplicity that κ and \mathbf{u}_h are piecewise constants. Then,

$$\|(1 - \mathcal{P}_h^0)(\kappa\phi - \mathbf{u}_h \cdot \nabla\phi)\|_{0,4/3;\Omega} \leq \|\kappa\|_{0,2;\Omega} \|(1 - \mathcal{P}_h^0)\phi\|_{0,4;\Omega} + \|\mathbf{u}_h\|_{0,4;\Omega} \|(1 - \mathcal{P}_h^0)\nabla\phi\|_{0,2;\Omega}.$$

Employing the regularity shift assumption for the adjoint problem and a Sobolev embedding we conclude that

$$\|(1 - \mathcal{P}_h^0)\phi\|_{0,4;\Omega} + \|(1 - \mathcal{P}_h^0)\nabla\phi\|_{0,2;\Omega} \lesssim h^{s_\Omega} (\|\phi\|_{s_\Omega,4;\Omega} + \|\nabla\phi\|_{s_\Omega,2;\Omega}) \lesssim h^{s_\Omega} \|\phi\|_{1+s_\Omega,2;\Omega}.$$

Lemma 5.3. *Let ψ^* denote the unique solution of Problem (2.8) and let $(\boldsymbol{\zeta}_h, \phi_h) \in \mathbf{H}_h \times \mathbf{Q}_h$ denote the discrete solution of (3.1). Then, there exists constant $C > 0$ such that*

$$\begin{aligned} \|\mathcal{P}_h^0(\psi^* - \psi_h)\|_{0,2;\Omega} &\leq Ch^{s_\Omega} \left(\min_{(v_h, \boldsymbol{\tau}_h) \in \mathbf{H}_h \times \mathbf{Q}_h} (\|\psi - v_h\|_{0,4;\Omega} + \|\boldsymbol{\zeta} - \boldsymbol{\tau}_h\|_{0,2;\Omega}) \right. \\ &\quad \left. + (\|g\|_{\mathbf{H}_D^{-1}(\Omega)} + \|\psi_D\|_{1/2,0;\Gamma_D}) \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega} + \min_{\chi_h \in \mathbf{Q}_h} \|\kappa\psi - \chi_h\|_{0,4/3;\Omega} \right), \end{aligned}$$

where $\boldsymbol{\zeta} = \varepsilon \nabla \psi - \mathbf{u}\psi$.

Proof. Let $\phi \in \mathbf{H}_D^1(\Omega)$ denote the unique solution of (5.1) with

$$f := \mathcal{P}_h^0(\psi^* - \psi_h).$$

Set $\mathbf{r} = -\varepsilon \nabla \phi$ and note that $\operatorname{div} \mathbf{r} \in L^{4/3}(\Omega)$, hence, $\mathbf{r} \in \mathbf{H}$. We have that

$$\begin{aligned} \|f\|_{0,2;\Omega}^2 &= \int_{\Omega} (\psi^* - \psi_h) f = \int_{\Omega} (\psi^* - \psi_h) (\operatorname{div} \mathbf{r} + \frac{1}{\varepsilon} \mathbf{u}_h \cdot \mathbf{r} + \kappa\phi) \\ &= A_{\mathbf{u}_h}((\boldsymbol{\zeta}^* - \boldsymbol{\zeta}_h, \psi^* - \psi_h), (\mathbf{r}, \phi)), \end{aligned}$$

where in the last identity we used that $\varepsilon \nabla \phi + \mathbf{r} = \mathbf{0}$ and, therefore, $\int_{\Omega} (\boldsymbol{\zeta}^* - \boldsymbol{\zeta}_h) (\varepsilon \nabla \phi + \mathbf{r}) = 0$. Employing Galerkin orthogonality we further obtain with continuity of $A_{\mathbf{u}_h}(\cdot, \cdot)$ that

$$\begin{aligned} \|f\|_{0,2;\Omega}^2 &= A_{\mathbf{u}_h}((\boldsymbol{\zeta}^* - \boldsymbol{\zeta}_h, \psi^* - \psi_h), (\mathbf{r} - \mathbf{r}_h, \phi - \phi_h)) \\ &\lesssim \|(\boldsymbol{\zeta}^* - \boldsymbol{\zeta}_h, \psi^* - \psi_h)\|_{\mathbf{H} \times \mathbf{Q}} \|(\mathbf{r} - \mathbf{r}_h, \phi - \phi_h)\|_{\mathbf{H} \times \mathbf{Q}} \end{aligned}$$

for all $(\mathbf{r}_h, \phi_h) \in \mathbf{H}_h \times \mathbf{Q}_h$. For the remainder of the proof we consider $\mathbf{r}_h = \Pi^{\text{RT}} \mathbf{r}$ and $\phi_h = \mathcal{P}_h^0 \phi$. First, observe that

$$\begin{aligned} \|(\boldsymbol{\zeta}^* - \boldsymbol{\zeta}_h, \psi^* - \psi_h)\|_{\mathbf{H} \times \mathbf{Q}} &\lesssim \min_{(v_h, \boldsymbol{\tau}_h) \in \mathbf{H}_h \times \mathbf{Q}_h} (\|\psi - v_h\|_{0,4;\Omega} + \|\boldsymbol{\zeta} - \boldsymbol{\tau}_h\|_{0,2;\Omega}) \\ &\quad + (\|g\|_{\mathbf{H}_D^{-1}(\Omega)} + \|\psi_D\|_{1/2,0;\Gamma_D}) \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega} + \min_{\chi_h \in \mathbf{Q}_h} \|\kappa\psi - \chi_h\|_{0,4/3;\Omega}, \end{aligned}$$

which has already be shown in the proof of Theorem 4.2.

It remains to estimate $\|(\mathbf{r} - \mathbf{r}_h, \phi - \phi_h)\|_{\mathbf{H} \times \mathbf{Q}}$. Clearly, by assumption (5.2) and the Sobolev embedding,

$$\|\mathbf{r} - \mathbf{r}_h\|_{0,2;\Omega} + \|\psi - \psi_h\|_{0,4;\Omega} \lesssim h^{s_\Omega} \|\phi\|_{1+s_\Omega,2;\Omega} \lesssim h^{s_\Omega} \|f\|_{0,2;\Omega}.$$

Note that

$$\operatorname{div}(\mathbf{r} - \mathbf{r}_h) = (1 - \mathcal{P}_h^k) \operatorname{div} \mathbf{r} = (1 - \mathcal{P}_h^k)(f + \mathbf{u}_h \cdot \nabla \phi - \kappa \phi) = (1 - \mathcal{P}_h^k)(\mathbf{u}_h \cdot \nabla \phi - \kappa \phi),$$

with the latter identity following from the fact that f is piecewise constant. Employing assumption (5.3) we conclude that

$$\begin{aligned} \|\operatorname{div}(\mathbf{r} - \mathbf{r}_h)\|_{0,4/3;\Omega} &= \|(1 - \mathcal{P}_h^k)(\mathbf{u}_h \cdot \nabla \phi - \kappa \phi)\|_{0,4/3;\Omega} \\ &\lesssim \|(1 - \mathcal{P}_h^0)(\mathbf{u}_h \cdot \nabla \phi - \kappa \phi)\|_{0,4/3;\Omega} \lesssim h^{s_\Omega} \|f\|_{0,2;\Omega}. \end{aligned}$$

Putting the last two estimates together we infer that

$$\|(\mathbf{r} - \mathbf{r}_h, \phi - \phi_h)\|_{\mathbf{H} \times \mathbf{Q}} \lesssim h^{s_\Omega} \|f\|_{0,2;\Omega}.$$

Overall, we see that

$$\begin{aligned} \|f\|_{0,2;\Omega}^2 &\lesssim \|(\boldsymbol{\zeta}^* - \boldsymbol{\zeta}_h, \psi^* - \psi_h)\|_{\mathbf{H} \times \mathbf{Q}} \|(\mathbf{r} - \mathbf{r}_h, \phi - \phi_h)\|_{\mathbf{H} \times \mathbf{Q}} \\ &\lesssim h^{s_\Omega} \|f\|_{0,2;\Omega} \left(\min_{(v_h, \boldsymbol{\tau}_h) \in \mathbf{H}_h \times \mathbf{Q}_h} (\|\psi - v_h\|_{0,4;\Omega} + \|\boldsymbol{\zeta} - \boldsymbol{\tau}_h\|_{0,2;\Omega}) \right. \\ &\quad \left. + (\|g\|_{\mathbf{H}_D^{-1}(\Omega)} + \|\psi_D\|_{1/2,00;\Gamma_D}) \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega} + \min_{\chi_h \in \mathbf{Q}_h} \|\kappa \psi - \chi_h\|_{0,4/3;\Omega} \right), \end{aligned}$$

which gives the stipulated estimate. \square

Let us now consider a discrete auxiliary local problem on each $K \in \mathcal{T}_h$ (which arises from the pseudo potential flux constitutive equation) similarly to the classical Stenberg postprocess [30], but adapted to our case with advecting velocity. We define $\psi_h^\sharp(K) \in \mathbf{P}_{k+1}(K)$ by

$$\begin{aligned} \int_K \varepsilon \nabla \psi_h^\sharp \cdot \nabla v_h &= \int_K \boldsymbol{\zeta}_h \cdot \nabla v_h + \int_K (\mathbf{u}_h \cdot \nabla v_h) \psi_h \quad \forall v_h \in \mathbf{P}_{k+1}(K), \\ \Pi_K^0 \psi_h^\sharp &= \Pi_K^0 \psi_h. \end{aligned} \tag{5.4}$$

Theorem 5.4. *Under the foregoing assumptions,*

$$\begin{aligned} \|\psi - \psi_h^\sharp\|_{0,2;\Omega} &\leq C (\|g\|_{\mathbf{H}_D^{-1}(\Omega)} + \|\psi_D\|_{1/2,00;\Gamma_D}) \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega} \\ &\quad + Ch^{s_\Omega} \left(\min_{(\boldsymbol{\tau}_h, v_h) \in \mathbf{H}_h \times \mathbf{Q}_h} (\|\psi - v_h\|_{0,4;\Omega} + \|\boldsymbol{\zeta} - \boldsymbol{\tau}_h\|_{0,2;\Omega}) \right. \\ &\quad \left. + (\|g\|_{\mathbf{H}_D^{-1}(\Omega)} + \|\psi_D\|_{1/2,00;\Gamma_D}) \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega} + \min_{\chi_h \in \mathbf{Q}_h} \|\kappa \psi - \chi_h\|_{0,4/3;\Omega} \right) \\ &\quad + Ch \min_{v_h \in \mathbf{P}_{k+1}(\mathcal{T}_h)} \|\nabla(\psi - v_h)\|_{0,2;\Omega}. \end{aligned}$$

Proof. Note that

$$\|\psi - \psi_h^\sharp\|_{0,2;\Omega} \leq \|\psi - \psi^*\|_{0,2;\Omega} + \|\mathcal{P}_h^0(\psi^* - \psi_h^\sharp)\|_{0,2;\Omega} + \|(1 - \mathcal{P}_h^0)(\psi^* - \psi_h^\sharp)\|_{0,2;\Omega}.$$

The first term on the right-hand side is estimated with Lemma 5.2. For the second term we recall that $\mathcal{P}_h^0 \psi_h^\sharp = \mathcal{P}_h^0 \psi_h$ by construction of the postprocessed solution. Then, we employ Lemma 5.3. For the last term, note that $\|(1 - \mathcal{P}_h^0)(\psi^* - \psi_h^\sharp)\|_{0,2;\Omega} \lesssim h \|\nabla(\psi^* - \psi_h^\sharp)\|_{0,2;\Omega}$. Some standard arguments then show that with $\boldsymbol{\zeta}^* = \varepsilon \nabla \psi^* - \mathbf{u}_h \psi^*$

$$\|\nabla(\psi^* - \psi_h^\sharp)\|_{0,2;\Omega} \lesssim \min_{v_h \in \mathbf{P}_{k+1}(\mathcal{T}_h)} \|\nabla(\psi^* - v_h)\|_{0,2;\Omega} + \|\boldsymbol{\zeta}^* - \boldsymbol{\zeta}_h\|_{0,2;\Omega} + \|\mathbf{u}_h\|_{0,4;\Omega} \|\psi^* - \psi_h\|_{0,4;\Omega}.$$

A triangle inequality and uniform boundedness of $\|\mathbf{u}_h\|_{0,4;\Omega}$ further yields

$$\begin{aligned} & \min_{v_h \in \mathbf{P}_{k+1}(\mathcal{T}_h)} \|\nabla(\psi^* - v_h)\|_{0,2;\Omega} + \|\zeta^* - \zeta_h\|_{0,2;\Omega} + \|\mathbf{u}_h\|_{0,4;\Omega} \|\psi^* - \psi_h\|_{0,4;\Omega} \\ & \lesssim \min_{v_h \in \mathbf{P}_{k+1}(\mathcal{T}_h)} \|\nabla(\psi - v_h)\|_{0,2;\Omega} + \|\zeta - \zeta_h\|_{0,2;\Omega} + \|\psi - \psi_h\|_{0,4;\Omega} + \|\zeta - \zeta^*\|_{0,2;\Omega} + \|\psi - \psi^*\|_{0,4;\Omega}. \end{aligned}$$

The last two terms are estimated by Lemma 2.2.

Combining all the results and noting that $h \lesssim h^{s_\Omega}$ concludes the proof. \square

Note that if $s_\Omega = 1$, and (ζ, ψ) is sufficiently smooth, as well as $\|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega} = \mathcal{O}(h^{k+2})$, then,

$$\|\psi - \psi_h^\sharp\|_{0,2;\Omega} = \mathcal{O}(h^{k+2}), \quad (5.5)$$

by the last theorem.

6. NUMERICAL EXPERIMENTS

We present simple computational tests that illustrate the properties of the proposed mixed finite element method as well as the postprocess mechanism. All examples, including the action of the map \mathcal{Q}_h and the local Stenberg-type postprocess, are implemented in the open source finite element library FEniCS [2].

Example 1. First we assess the convergence of the finite element discretisation using manufactured solutions of different regularity on simple domains. We consider on $\Omega = (0, 1)^2$ the smooth and singular solutions to the primal governing PDE

$$\psi_{\text{ex}}(x, y) = \sin(\pi x) \sin(\pi y) \quad \text{and} \quad \psi_{\text{ex}}(x, y) = |x - y|^{\frac{3}{4}} \sin(\pi x) \sin(\pi y), \quad (6.1)$$

and use a smooth and divergence-free vector field as external velocity

$$\mathbf{u}_{\text{ex}}(x, y) = \begin{pmatrix} \cos(\pi x) \sin(\pi y) \\ -\sin(\pi x) \cos(\pi y) \end{pmatrix}.$$

An exact mixed variable ζ_{ex} , an exact source g_{ex} , and exact Dirichlet datum ψ_D and (non-homogeneous) Neumann datum $\zeta \cdot \mathbf{n} = \zeta_N$ are manufactured from these exact primal solutions and considering for this test simply unitary parameters $\varepsilon = \kappa = 1$.

A sequence of successively refined uniform meshes is constructed and on each refinement level we compute $\mathcal{Q}_h g_{\text{ex}}$ and with it we generate approximate solutions $(\zeta_h, \psi_h) \in \mathbf{H}_h \times \mathcal{Q}_h$ with the mixed finite element method (3.1) (using lowest-order Raviart–Thomas spaces), and taking \mathbf{u}_h as the $L^2(\Omega)$ projection of \mathbf{u} onto $\mathbf{M}_h = [\mathbf{P}_1(\mathcal{T}_h)]^2$ (required by the postprocess as indicated in (5.5)). With the obtained discrete pseudo potential flux, at each mesh refinement level we also compute a post-processed double layer potential ψ_h^\sharp from the auxiliary problem (5.4). Errors between the discrete and exact solutions are defined in the following way

$$e_{\text{div}_{4/3}}(\zeta) := \|\zeta_h - \zeta_{\text{ex}}\|_{\text{div}_{4/3},\Omega}, \quad e_{0,4}(\psi) := \|\psi_h - \psi_{\text{ex}}\|_{0,4;\Omega}, \quad e_0(\psi^\sharp) := \|\psi_h^\sharp - \psi_{\text{ex}}\|_{0,2;\Omega},$$

and we compute experimental order of convergence as usual

$$\text{EoC} = \frac{\log[e_{i+1}(\bullet)/e_i(\bullet)]}{\log(h_{i+1}/h_i)},$$

where $e_i(\bullet)$ and h_i denote an individual error and mesh size associated with the i -th refinement level, respectively. We remark that for the second expression in (6.1) the exact flux ζ_{ex} is not regular enough to compute the full $e_{\text{div}_{4/3}}(\zeta)$ error, so in that case we only report $e_0(\zeta) := \|\zeta_h - \zeta_{\text{ex}}\|_{0,2;\Omega}$.

$g \in C^\infty(\Omega)$							
DoFs	h	$e_{\text{div}_{4/3}}(\zeta)$	EoC	$e_{0,4}(\psi)$	EoC	$e_0(\psi^\sharp)$	EoC
24	0.7071	7.28e+00	*	4.89e-01	*	3.49e-01	*
88	0.3536	4.37e+00	0.736	1.79e-01	1.442	5.33e-02	2.703
336	0.1768	1.90e+00	1.205	8.72e-02	1.040	1.05e-02	2.348
1312	0.0884	9.00e-01	1.075	4.36e-02	1.000	2.53e-03	2.050
5184	0.0442	4.80e-01	0.908	2.18e-02	1.000	6.36e-04	1.991
20608	0.0221	2.67e-01	0.843	1.09e-02	1.000	1.60e-04	1.992
82176	0.0110	1.49e-01	0.844	5.45e-03	1.000	4.01e-05	1.996
$g \in H^{-1}(\Omega)$							
DoFs	h	$e_0(\zeta)$	EoC	$e_{0,4}(\psi)$	EoC	$e_0(\psi^\sharp)$	EoC
24	0.7071	7.28e-01	*	1.92e-01	*	1.56e-01	*
88	0.3536	6.90e-01	0.077	9.57e-02	1.005	5.88e-02	1.408
336	0.1768	5.03e-01	0.458	4.48e-02	1.096	1.76e-02	1.738
1312	0.0884	3.92e-01	0.357	2.25e-02	0.992	5.62e-03	1.647
5184	0.0442	3.20e-01	0.296	1.14e-02	0.978	2.14e-03	1.396
20608	0.0221	2.65e-01	0.268	5.79e-03	0.980	9.23e-04	1.212
82176	0.0110	2.22e-01	0.257	2.93e-03	0.982	4.21e-04	1.133

TABLE 6.1. Example 1. Error history for the lowest-order mixed finite element scheme including regularisation through \mathcal{Q}_h and postprocessing, when subjected to a smooth and a rough load that is not in $L^2(\Omega)$ (top and bottom, respectively). Here we use mixed boundary conditions.

The results from the convergence tests are reported in Table 6.1 for mixed boundary conditions—the Neumann boundary is the right edge of the domain whereas the Dirichlet part is the remainder of the boundary. The first part of the table focuses on the smooth case, where we observe optimal convergence for the flux and potential in the natural norms for the mixed regularised problem—we recall that we are only using here the lowest-order case—and we also see the asymptotic second-order convergence of the locally postprocessed solution, as predicted by Theorem 5.4.

The bottom part of both tables shows only the L^2 contribution of the flux error, the potential and postprocessed potential. For these unknowns we see, respectively, approximate rates of convergence of $O(h^{1/4})$, $O(h)$ and $O(h^{5/4})$, which are in complete agreement with the results from [20, Section 5.2]. In Figure 6.1 we show the discrete solutions obtained for both smooth and rough loads, indicating accurate approximations of flux and potential, as well as well-resolved postprocessed potential (with sharper gradients).

Example 2. In this test we examine the behaviour of the postprocess without the regularisation. For that we follow [21, Section 6.3] and use the domain $\Omega = (-1, 1)^2$, the given advection velocity \mathbf{u}_h a projection of

$$\mathbf{u}_{\text{ex}}(x, y) = \begin{pmatrix} \cos(\frac{\pi}{2}x) \sin(\frac{\pi}{2}y) \\ -\sin(\frac{\pi}{2}x) \cos(\frac{\pi}{2}y) \end{pmatrix},$$

and the manufactured potential

$$\psi_{\text{ex}}(x, y) = x|x|^{65/128}(1-x^2)(1-y^2),$$

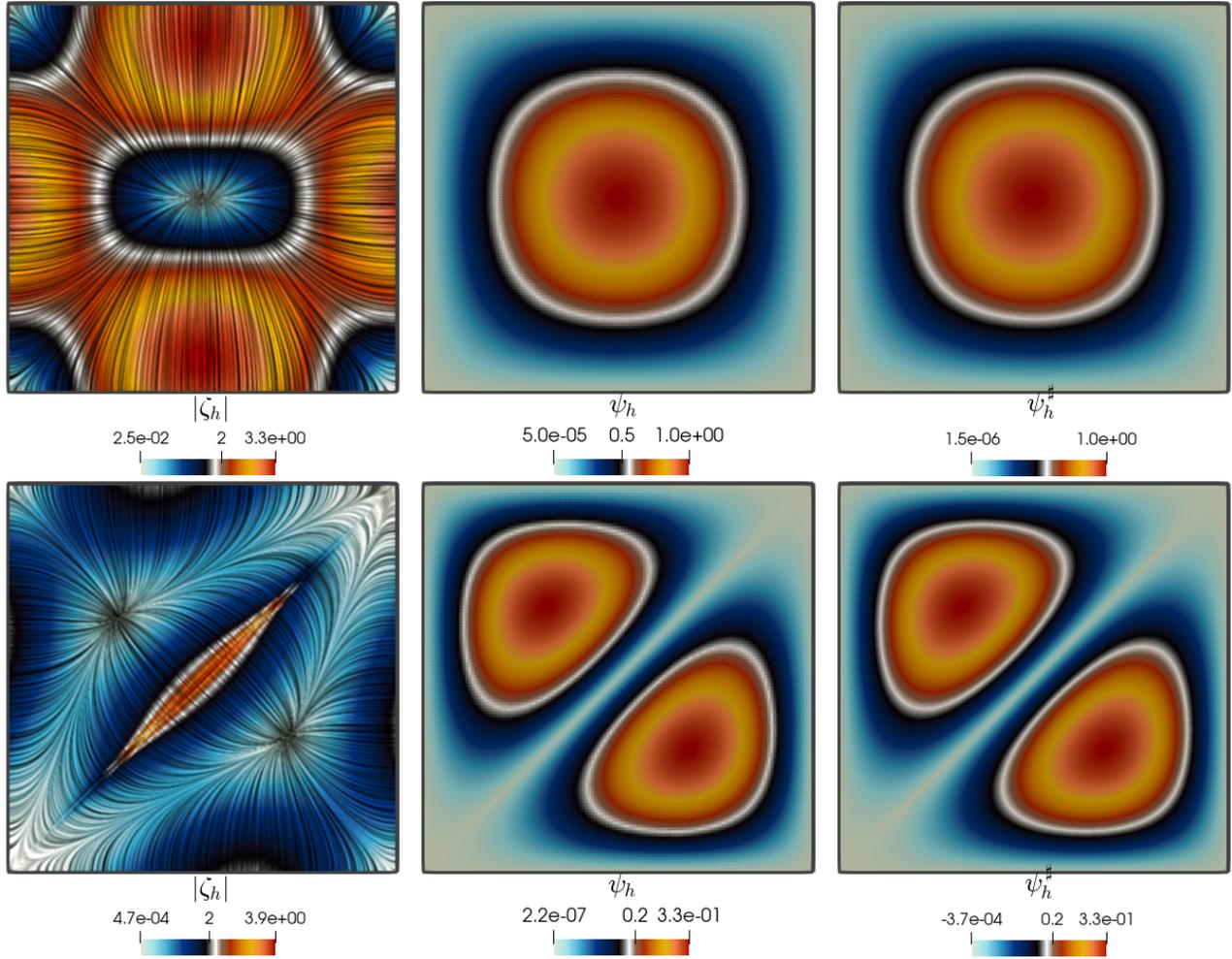


FIGURE 6.1. Example 1. Approximate solutions (line integral contours and magnitude for the pseudo potential flux and double layer potential profile) for the lowest-order mixed finite element scheme, and postprocessed potential. We show results using smooth and rough loadings (top and bottom rows, respectively).

with the now space-dependent coefficients

$$\varepsilon(x, y) = \exp(-xy), \quad \kappa(x, y) = \frac{1}{2} + \sin(xy)^2.$$

With these functions the load is such $g_{\text{ex}} \in L^2(\Omega)$ but $g_{\text{ex}} \notin H^s(\Omega)$ for $s \geq \frac{1}{128}$. This smoothness implies that we can also use g_{ex} directly without \mathcal{Q}_h . We report on the error history of the case with and without \mathcal{Q}_h , which confirms an order of convergence similar to the aforementioned reference: $O(h)$ for the $L^2(\Omega)$ norm of the flux and for the $L^4(\Omega)$ norm of the potential, whereas for the postprocessed solution we obtain approximately $O(h^{5/3})$ and $O(h^2)$ for the methods without and with regularisation \mathcal{Q}_h , respectively. The regularised method is then appropriate to use even if there is no need to regularise the load.

Example 3. Finally, we address the advection diffusion problem with a line Dirac delta function as the source term, which can be relevant not only in the Poisson–Boltzmann context but also in mono-phasic flow in porous media and tissue perfusion [26, 7]. We consider the polygonal domain enclosed by the points $(0, 0)$, $(1, 0)$, $(1.25, 0.75)$, $(0.5, 1.25)$, $(-0.25, 0.75)$ and consider that there is a line fracture γ

DoFs	h	$e_0(\zeta)$	EOC	$e_{0,4}(\psi)$	EOC	$e_0(\psi^\sharp)$	EOC
with load g_{ex}							
24	1.4142	9.39e-01	*	1.59e-01	*	1.65e-01	*
88	0.7071	7.45e-01	0.335	1.36e-01	0.227	1.14e-01	0.540
336	0.3536	4.40e-01	0.758	7.69e-02	0.817	3.65e-02	1.641
1312	0.1768	2.32e-01	0.926	4.04e-02	0.928	1.07e-02	1.774
5184	0.0884	1.18e-01	0.971	2.05e-02	0.983	3.19e-03	1.743
20608	0.0442	5.98e-02	0.983	1.03e-02	0.996	9.99e-04	1.674
82176	0.0221	3.02e-02	0.986	5.13e-03	0.999	3.27e-04	1.610
with load $Q_h g_{\text{ex}}$							
24	1.4142	1.02e+00	*	2.49e-01	*	2.81e-01	*
88	0.7071	7.13e-01	0.517	1.61e-01	0.629	1.56e-01	0.850
336	0.3536	4.20e-01	0.764	8.25e-02	0.967	5.40e-02	1.531
1312	0.1768	2.19e-01	0.941	4.11e-02	1.004	1.55e-02	1.800
5184	0.0884	1.12e-01	0.969	2.06e-02	1.001	4.14e-03	1.904
20608	0.0442	5.68e-02	0.976	1.03e-02	1.001	1.08e-03	1.945
82176	0.0221	2.88e-02	0.980	5.13e-03	1.000	2.77e-04	1.955

TABLE 6.2. Example 2. Error history for the lowest-order mixed finite element scheme subjected to an $L^2(\Omega)$ load and including or not the regularisation through Q_h (top and bottom, respectively). Here we use pure Dirichlet boundary conditions.

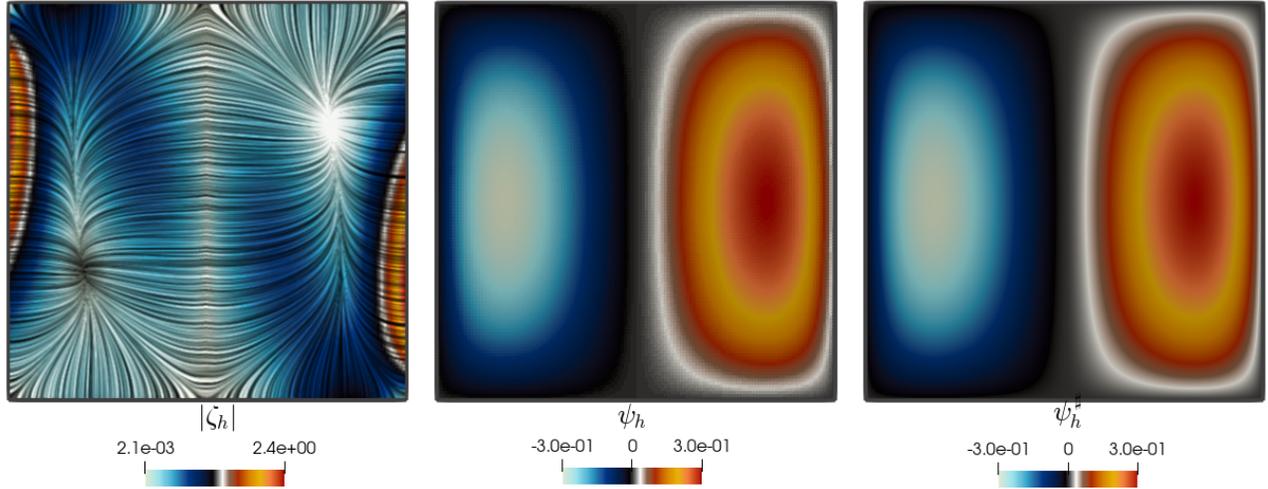


FIGURE 6.2. Example 2. Approximate solutions for the lowest-order mixed finite element scheme and postprocessed potential using the regularisation Q_h .

joining the points $(0.4, 0.25)$ and $(0.6, 0.85)$. We construct a sequence of non-structured triangulations that are conforming with the line fracture, and that are slightly agglomerated near the line. We use a smooth advecting velocity going from left to right with higher magnitude near the domain centre:

$$\eta(x, y) = U_0 \sigma \tanh\left(\frac{y - \frac{1}{2}}{\sigma}\right) (1 + \theta \sin(2\pi x)), \quad \text{and} \quad \mathbf{u}(x, y) = \text{rot } \eta(x, y),$$

and set the parameters $U_0 \in \{0.25, 0.0025\}$, $\sigma = 0.08$, $\theta = 0.12$ and the remaining model coefficients are $\varepsilon = 10^{-3}$ and $\kappa = 1$. We use pure Dirichlet boundary conditions setting $\psi = 0$ on $\partial\Omega$. The source term is δ_γ and in this case its regularity is $H^{-\frac{1}{2}-\epsilon}(\Omega)$ for $\epsilon > 0$.

DoFs	h	$e_0(\zeta)$	EoC	$e_{0,4}(\psi)$	EoC	$e_0(\psi^\#)$	EoC
64	0.7906	8.94e-03	*	2.79e-02	*	1.13e-01	*
179	0.3953	5.87e-03	0.606	1.91e-02	0.543	3.29e-02	1.774
649	0.2146	3.79e-03	0.718	1.16e-02	0.824	1.21e-02	1.631
2778	0.1166	2.28e-03	0.829	5.26e-03	1.293	5.39e-03	1.332
10515	0.0622	1.20e-03	1.026	2.20e-03	1.383	2.23e-03	1.402
40304	0.0325	5.16e-04	1.299	7.64e-04	1.635	7.71e-04	1.640

TABLE 6.3. Example 3. Error history for the lowest-order mixed finite element scheme subjected to a line source, including the regularisation through Q_h . This test is without closed-form manufactured solution, and it uses pure Dirichlet boundary conditions.

Since we do not know the closed-form solution, we use a numerical convergence rate computing errors between approximate solutions and a reference numerical solution (obtained on a finer mesh). The convergence history for this test is reported in Table 6.3. The error decay seems to correspond to the behaviour in Example 2, thanks to the conformity and smoothness of the mesh closer to the line fracture. We show snapshots of the approximate solutions for two advecting intensities U_0 , as well as an example of the unstructured mesh. The top panels in Figure 6.3 illustrate how the higher velocity is able to advect the higher potential produced by the source δ_γ whereas the second row shows flux and potential that remain concentrated near γ .

7. CONCLUSION

We have developed a mixed finite element method for the linearised Poisson–Boltzmann equations assuming an advective velocity in $L^4(\Omega)$. One of the challenges is to derive error estimates when the load is in $H^{-1}(\Omega)$, and this is tackled using a quasi-interpolation operator, here conveniently redefined for $L^p(\Omega)$ spaces and mixed boundary conditions. The continuous and discrete problems were shown to be well-posed and we established quasi-optimality and convergence rates. A postprocess of the double layer potential allows us to get superconvergence.

For us, a natural extension to this work is to perform the full coupling analysis with the Navier–Stokes equations (see, for example, [9, 14, 12, 6]), and to incorporate the unbounded nonlinearity in the reaction function κ as done for the primal coupling in [3]. This requires to add box constraints in the double layer potential space, and to use a fixed-point structure to handle a perturbed saddle-point problem when the nonlinearity is in the lower-diagonal block. We will also address a posteriori error estimates as in [7], and other types of singular forcing terms including Dirac point sources following [25, 16].

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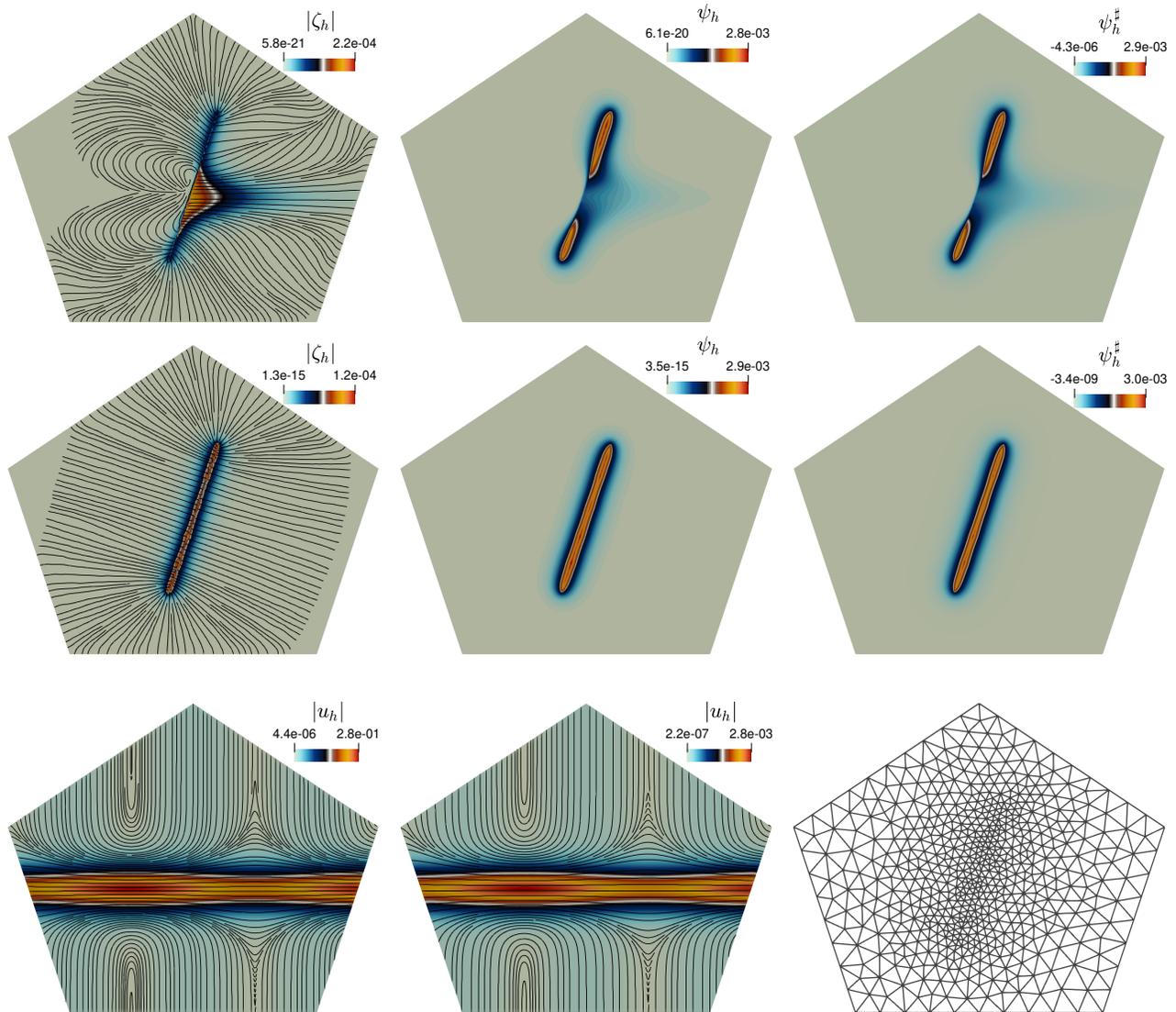


FIGURE 6.3. Example 3. Approximation of pseudo potential flux and streamlines, double layer potential and postprocessed potential for the case of $U_0 = 0.25$ (top row) and $U_0 = 0.0025$ (middle row). The bottom panels show the approximate velocity field and streamlines for the high and low intensity, and a sample coarse mesh after 3 refinements.

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