Stability of a second-order method for phase change in porous media flow

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We analyse the stability of a second-order finite element scheme for the primal formulation of a Brinkman-Boussinesq model where the solidification process influences the drag and the viscosity. The problem is written in terms of velocity, temperature, and pressure, and we produce numerical approximations to the flow observed in heated cavities and near ice sheets.

1 Introduction and governing equations

Many applicative problems deal with the problem of thermal convection, where a dynamic change of phase between solid and liquid is present. Examples are encountered in the melting and solidification of metals, or in ocean and atmospheric phenomena. Phase change has been considered as part of Boussinesq-based models using either an enthalpy-porosity approach, like in the recent contributions [7, 9] where a discontinuity in the drag exerts a large force in the solid phase, or by incorporating jumps in the viscosity, as in [3]. Here we employ a slightly different viscosity-based model that combines the two approaches. This new model, thoroughly discussed in [2], is based on assuming that the solid contains many small particles, just like in porosity-based descriptions of flow in porous media. The jump in the coefficients is here regularised into a smooth transition from one phase to another. model considers the presence of microscopic particles in the solid, which resembles porosity-based models. We choose a transition from fluid to solid having a large gradient, which creates additional numerical challenges.

Apart from proposing a conforming Galerkin method for the space discretisation of the governing equations (momentum, mass, energy, and enthalpy), our contribution focuses also on deriving stability bounds for the discrete solutions. This result generalises other studies, so far focused on the natural or thermal convection of fluids without phase change, see for instance [1, 4–6, 8]. In particular, this note summarises the recent results reported in [2] and presents three new numerical tests.

Let $\Omega \subset \mathbb{R}^2$, be a porous domain saturated with an incompressible viscous fluid with kinematic viscosity ν, thermal expansion coefficient α, and dimensionless specific heat C. The problem of interest consists in solving the conservation of momentum, mass, energy, and enthalpy, written in terms of velocity $\boldsymbol{u}$, pressure p, and temperature θ:

$$\begin{aligned} \partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \frac{1}{\mathrm{Re}}\mathbf{div}\left[2\mu(\theta)\boldsymbol{\varepsilon}(\boldsymbol{u})\right] + \nabla p + \eta(\theta)\boldsymbol{u} = f(\theta)\boldsymbol{k}, \qquad \operatorname{div}\boldsymbol{u} = 0,\\ \partial_t\theta + \boldsymbol{u}\cdot\nabla\theta - C^{-1}\mathrm{Pr}^{-1}\operatorname{div}(\kappa\nabla\theta) + \partial_t s + \boldsymbol{u}\cdot\nabla s = 0, \end{aligned} \tag{1.1}$$

defined in $\Omega\times(0,t_f]$, where $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u}+\nabla\boldsymbol{u}^T)$ is the strain rate, s is the enthalpy (with the property $|\nabla s(\psi)| \le s_2|\nabla\psi|$ $\forall\psi\in\mathbb{R}$), $\boldsymbol{k}$ is the unit vector that points opposite to gravity, κ is the thermal conductivity (uniformly bounded and positive definite with constants κ_1 and κ_0, respectively), η,μ are the temperature-dependent permeability of the porous material and viscosity of the fluid, respectively: $\eta(\theta) := \frac{\eta_s}{2}[\tanh(M_\eta(\theta_f-\theta))+1]$, $\mu(\theta) := \mu_l + \frac{(\mu_s-\mu_l)}{2}[\tanh(M_\mu(\theta_f-\theta))+1]$, where η_s corresponds to the relative size of the imposed force and M_η, M_μ denote both the size of the mushy region; and Re, Pr are the Reynolds and Prandtl numbers, respectively. No-slip velocity is imposed over $\partial\Omega$, and we assume that the boundary admits a splitting between Γ^θ_D and Γ^θ_N, where temperature and normal heat fluxes are prescribed, respectively. The system is initially at rest and isothermal, and so we set $\boldsymbol{u}(0)=\boldsymbol{0}$, $p(0)=0$ and $\theta(0)=\theta_0$ with θ_0 constant.

2 Stability of the fully discrete Galerkin scheme

Let $\{\mathcal{T}_h\}_{h>0}$ be a shape-regular family of partitions of the region $\bar\Omega$, with meshsize h. Finite-dimensional spaces for the approximation of $\boldsymbol{u},p,\theta$ are $\mathbf{V}_h := \{\boldsymbol{v}_h\in\mathbf{H}^1(\Omega) : \boldsymbol{v}_h|_K\in[\mathbb{P}_{k+1}(K)]^2\ \forall K\in\mathcal{T}_h, \text{ and } \boldsymbol{v}_h=\boldsymbol{0} \text{ on } \partial\Omega\}$, $\mathrm{Q}_h := \{q_h\in \mathrm{L}^2(\Omega) : q_h|_K\in\mathbb{P}_k(K)\ \forall K\in\mathcal{T}_h, \text{ and } \int_\Omega q_h = 0\}$, $\mathrm{Z}_h := \{\psi_h\in\mathrm{H}^1(\Omega) : \psi_h|_K\in\mathbb{P}_{k+1}(K)\ \forall K\in\mathcal{T}_h, \text{ and } \psi_h = 0 \text{ on } \Gamma^\theta_D\}$, for $k\ge1$, satisfying a discrete inf-sup condition [2, 5]. For the time discretisation we use a BDF2 method with a uniform timestep Δt. Starting from $\boldsymbol{u}^0_h,\theta^0_h,\boldsymbol{u}^1_h,\theta^1_h$, solve for $n=1,\dots$ the nonlinear system

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$$
\begin{aligned}
&\frac{3}{2\Delta t}(\boldsymbol{u}_h^{n+1},\boldsymbol{v}_h)+c_1(\boldsymbol{u}_h^{n+1};\boldsymbol{u}_h^{n+1},\boldsymbol{v}_h)+\frac{1}{2}(\operatorname{div}\boldsymbol{u}_h^{n+1}\,\boldsymbol{u}_h^{n+1},\boldsymbol{v}_h)+a_1^{\theta_h^{n+1}}(\boldsymbol{u}_h^{n+1},\boldsymbol{v}_h)\\
&\quad+(\eta(\theta_h^{n+1})\boldsymbol{u}_h^{n+1},\boldsymbol{v}_h)+b(\boldsymbol{v}_h,p_h^{n+1})-(f(\theta_h^{n+1})\boldsymbol{k},\boldsymbol{v}_h)=\frac{1}{\Delta t}(2\boldsymbol{u}_h^{n}-\frac{1}{2}\boldsymbol{u}_h^{n-1},\boldsymbol{v}_h)\qquad\forall\boldsymbol{v}_h\in\mathbf{V}_h,\\
&\qquad b(\boldsymbol{u}_h^{n+1},q_h)=0\qquad\forall q_h\in\mathrm{Q}_h,\\
&\frac{3}{2\Delta t}(\theta_h^{n+1}+s_h^{n+1},\psi_h)+c_3(\boldsymbol{u}_h^{n+1};\theta_h^{n+1}+s_h^{n+1},\psi_h)+\frac{1}{2}(\operatorname{div}\boldsymbol{u}_h^{n+1}\,(\theta_h^{n+1}+s_h^{n+1}),\psi_h)\\
&\quad+a_3(\theta_h^{n+1},\psi_h)=\frac{1}{\Delta t}(2[\theta_h^{n}+s_h^{n}]-\frac{1}{2}[\theta_h^{n-1}+s_h^{n-1}],\psi_h)\qquad\forall\psi_h\in\mathrm{Z}_h,
\end{aligned}
\tag{2.1}
$$

where $a_1(\cdot,\cdot)$ is the bilinear form associated to the viscous term, $b(\cdot,\cdot)$ is the usual divergence bilinear form, $a_3(\cdot,\cdot)$ is the temperature diffusion form, and $c_i(\cdot;\cdot,\cdot)$ are the convective bilinear forms.

Theorem 2.1 *Let $(\boldsymbol{u}_h^{n+1},\theta_h^{n+1})\in\mathbf{V}_h\times\mathrm{Z}_h$ be a solution of (2.1) and assume that $\kappa_0>2s_2\kappa_1$. Then,*

$$
\begin{aligned}
&\|\boldsymbol{u}_h^{n+1}\|_{0,\Omega}^2+\|2\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^{n}\|_{0,\Omega}^2+\|\theta_h^{n+1}+s_h^{n+1}\|_{0,\Omega}^2+\|2(\theta_h^{n+1}+s_h^{n+1})-(\theta_h^{n}+s_h^{n})\|_{0,\Omega}^2\\
&+\sum_{m=1}^{n}\left(\|\delta\boldsymbol{u}_h^m\|_{0,\Omega}^2+\|\delta(\theta_h^m+s_h^m)\|_{0,\Omega}^2\right)+\sum_{m=1}^{n}\left(\Delta t\,\|\boldsymbol{u}_h^{m+1}\|_{1,\Omega}^2+\Delta t\,|\theta_h^{m+1}|_{1,\Omega}^2\right)\\
&\leq\; C_1\left(\|\theta_h^1+s_h^1\|_{0,\Omega}^2+\|2(\theta_h^1+s_h^1)-(\theta_h^0+s_h^0)\|_{0,\Omega}^2+\|\boldsymbol{u}_h^1\|_{0,\Omega}^2+\|2\boldsymbol{u}_h^1-\boldsymbol{u}_h^0\|_{0,\Omega}^2\right),
\end{aligned}
\tag{2.2}
$$

where C_1 is a constant independent on h and Δt, and $\delta\boldsymbol{u}_h^m$, $\delta(\theta_h^m+s_h^m)$, are denoted according to [2, eq. (4.6)].

Proof. Taking $\psi_h=4(\theta_h^{n+1}+s_h^{n+1})$ and $\boldsymbol{v}_h=4\boldsymbol{u}_h^{n+1}$ in the third and first equation of (2.1), respectively, and then, applying the algebraic relation given in [2, eq. (4.6)] and Young's inequality, we deduce that

$$
\begin{aligned}
&\|\boldsymbol{u}_h^{n+1}\|_{0,\Omega}^2+\|2\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^{n}\|_{0,\Omega}^2+\|\theta_h^{n+1}+s_h^{n+1}\|_{0,\Omega}^2+\|2(\theta_h^{n+1}+s_h^{n+1})-(\theta_h^{n}+s_h^{n})\|_{0,\Omega}^2+\|\delta\boldsymbol{u}_h^n\|_{0,\Omega}^2\\
&+\|\delta(\theta_h^n+s_h^n)\|_{0,\Omega}^2+c_1\min\left\{\frac{8\mu_1}{\mathrm{Re}},2\eta_1\right\}\Delta t\,\|\boldsymbol{u}_h^{n+1}\|_{1,\Omega}^2+\frac{4(\kappa_0-2s_2\kappa_1)}{C\mathrm{Pr}}\Delta t\,|\theta_h^{n+1}|_{1,\Omega}^2\\
&\leq\; C_2\left(\|\boldsymbol{u}_h^n\|_{0,\Omega}^2+\|2\boldsymbol{u}_h^n-\boldsymbol{u}_h^{n-1}\|_{0,\Omega}^2+\|\theta_h^n+s_h^n\|_{0,\Omega}^2+\|2(\theta_h^n+s_h^n)-(\theta_h^{n-1}+s_h^{n-1})\|_{0,\Omega}^2\right).
\end{aligned}
\tag{2.3}
$$

Finally, by summing over n the estimate (2.3), we obtain (2.2). Further details can be found in [2]. □

3 Three numerical examples

Assuming different coefficients in the enthalpy-viscosity or porosity models, we generate three scenarios. First, a single fluid in a lid driven cavity flow with many solid particles. Secondly, an advection-dominated flow of two fluids of different density and viscosity. The salt distribution drives in turn the flow through the buoyancy term. Lastly, the melting of a circular ice sheet surrounding Antarctica, where the buoyancy acts radially, directing the main flow towards the coastline.

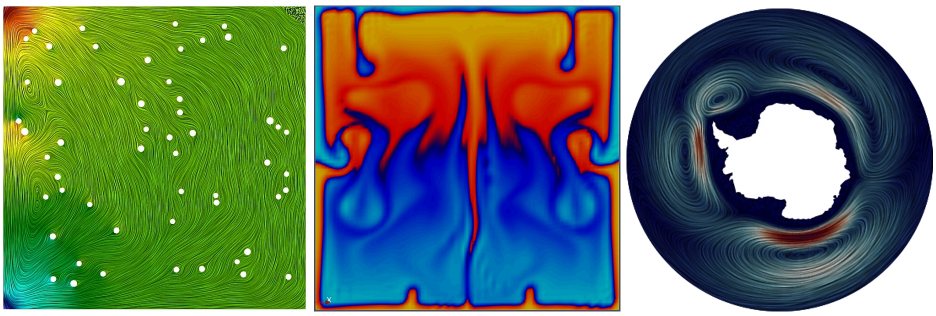

Fig. 1 Approximate pressure in a thermally driven cavity with randomly distributed small obstacles (left). Salt distribution for the Rayleigh-Taylor instability problem (centre), where two approximately immiscible fluids of different densities are in a container. Velocity line integral contours of the melting of ice sheets in Antarctica (right) after an elapsed time.

Acknowledgements This work has been partially supported by Vicerrectoría de Investigación, project 540-B7-233, Universidad de Costa Rica; by CONICYT-Chile through Becas-Chile for foreign students; and by the EPSRC through the research grant EP/R00207X/1.

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