

# Mixed Methods for a Stream-Function – Vorticity Formulation of the Axisymmetric Brinkman Equations

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**Abstract** This paper is devoted to the numerical analysis of a family of finite element approximations for the axisymmetric, meridian Brinkman equations written in terms of the stream-function and vorticity. A mixed formulation is introduced involving appropriate weighted Sobolev spaces, where well-posedness is derived by means of the Babuška–Brezzi theory. We introduce a suitable Galerkin discretization based on continuous piecewise polynomials of degree  $k \geq 1$  for all the unknowns, where its solvability is established using the same framework as the continuous problem. Optimal a priori error estimates are derived, which are robust with respect to the fluid viscosity, and valid also in the pure Darcy limit. A few numerical examples are presented to illustrate the convergence and performance of the proposed schemes.

**Keywords** Brinkman equations · Stream-function and vorticity formulation · Axisymmetric domains · Finite element method · Stability analysis · Error estimates

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## 1 Introduction

We consider the Brinkman equations, describing viscous flow within axisymmetric domains composed by a porous matrix of low-concentration fixed particles. In particular, we are interested in retrieving the fluid vorticity and the stream-function as principal unknowns of the system. Salient features in formulations of this kind include that: the stream-function is one of the most useful tools in flow visualization; only two *scalar* unknowns are necessary to resolve the flow patterns (in the three-dimensional case the stream-function is a vector field, and these formulations are no longer substantially advantageous over classical velocity–pressure ones); and discrete solutions lead to exactly divergence-free computed velocities. In addition, a manipulation of the equations permits to eliminate the pressure from the formulation. However, if pressure profiles are required, they can be recovered via a generalized Poisson problem with a datum coming from the stream-function solution (in a similar spirit to the decoupled methods recently proposed in [7, 22] for Brinkman equations in Cartesian coordinates). Let us point out that the symmetry of both the solution and the domain may not be common assumptions in classical porous media models employed in reservoir simulation. However, other types of applications (such as oil filters design, carbon dioxide sequestration, seepage, large arteries simulation, or the study of heat transfer enhancement and plug nozzle flow fields [27, 28, 30–32, 35]) do involve axisymmetric porous materials, where viscous properties of the flow are also of interest.

From the viewpoint of the mathematical analysis of the problem at hand, the radial configuration implies that suitable weighted Sobolev spaces are needed (see e.g. the monograph [15]). Regarding discretization, the relevant literature contains numerous studies involving different numerical methods for axisymmetric (viscous or non-viscous) flows (see e.g. [3, 5, 7, 8, 10, 14, 16, 20, 23, 33] and the references therein). More precisely, in the recent contribution [4] the authors propose a spectral method for a stream-function vorticity formulation of the Stokes equations, where the cylindrical symmetry reduces a three-dimensional problem to a bidimensional one. In contrast, here we follow some ideas from [6] (where a stabilized mixed method to resolve axisymmetric velocity–vorticity–pressure Brinkman flows is proposed), and adapt these techniques to a mixed formulation written in terms of stream-function and vorticity. We note that, as a consequence of the zero-order term and the variable permeability tensor, in this case the problem cannot be decoupled as in [4]. Moreover, the analysis of existence and uniqueness of continuous and discrete solutions is established using standard arguments for saddle-point problems (see [21]), and we propose a finite element discretization based on piecewise polynomials of order  $k \geq 1$  for all scalar fields, defined on triangular meshes. This method represents (in its lowest-order version,  $k = 1$ ) only 6 degrees of freedom per element, decoupled from a pressure solve (approximated in axisymmetric  $H^1$ -conforming spaces and having three degrees of freedom per element), thus being a very competitive scheme, for instance, less expensive than the mixed method recently introduced in [6]. Our optimal order error estimates are derived from the continuous dependence on the data and an appropriate Céa estimate, and we stress that these bounds are established with constants independent of the viscosity  $\nu$  and, in particular, valid in the Darcy limit  $\nu \rightarrow 0$ . Moreover, a duality argument allows us to improve the order of convergence of the vorticity and the stream-function approximations in  $L^2$ -norm.

The remainder of this paper is structured as follows. Section 2 collects the relevant formulations of the Brinkman problem, for velocity and pressure in Cartesian coordinates, its

reduction to the axisymmetric case, and a stream-function–vorticity form. The weak formulation, along with some preliminary results are also presented. In Sect. 3, we prove the unique solvability and stability properties of the proposed formulation. In Sect. 4, we introduce the finite element discretization of our variational formulation, for which we prove a discrete inf-sup condition uniformly with respect to the fluid viscosity  $\nu$  and the mesh parameter  $h$ ; moreover, we establish optimal error estimates. Some illustrative numerical tests are postponed to Sect. 5. We close with a few remarks and perspectives in Sect. 6.

## 2 Formulations of the Linear Brinkman Equations in Different Coordinates

### 2.1 Cartesian Coordinates

The linear Brinkman equations govern the motion of an incompressible viscous fluid within a porous medium. The system is

$$\check{\mathbf{K}}^{-1}\check{\mathbf{u}} - \nu\Delta\check{\mathbf{u}} + \nabla\check{p} = \check{\mathbf{f}} \quad \text{in } \check{\Omega}, \tag{2.1a}$$

$$\operatorname{div}\check{\mathbf{u}} = 0 \quad \text{in } \check{\Omega}, \tag{2.1b}$$

$$\check{\mathbf{u}} \cdot \check{\mathbf{n}} = 0 \quad \text{on } \partial\check{\Omega}, \tag{2.1c}$$

$$\operatorname{curl}\check{\mathbf{u}} \times \check{\mathbf{n}} = 0 \quad \text{on } \partial\check{\Omega}, \tag{2.1d}$$

where  $\check{\Omega} \subset \mathbb{R}^3$  is a given spatial domain. The sought quantities are the local volume-average velocity  $\check{\mathbf{u}}$  and the pressure field  $\check{p}$ . The permeability  $\check{\mathbf{K}}$  is a symmetric and positive definite tensor, and without loss of generality we can restrict ourselves to the isotropic case where the inverse permeability distribution can be represented by a scalar function  $\check{\sigma}$ , i.e.  $\check{\mathbf{K}}^{-1} = \check{\sigma}\mathbf{I}$ . The inverse permeability has  $L^\infty(\check{\Omega})$  regularity, with  $\check{\sigma}_{\min} \leq \check{\sigma}(x, y, z) \leq \check{\sigma}_{\max}$  a.e. in  $\check{\Omega}$ . For simplicity, we assume a positive fluid viscosity  $0 < \nu \leq \nu_{\max}$ . Here  $\check{\sigma}_{\min}$ ,  $\check{\sigma}_{\max}$ , and  $\nu_{\max}$  are positive constants.

### 2.2 Axisymmetric Case

Under axial symmetry of the domain (that is,  $\check{\Omega}$  is invariant by rotation around an axis, assumed aligned with  $r = 0$ ), the forcing term, and the inverse permeability, we can replace them by  $\Omega$ ,  $\mathbf{f}$ , and  $\sigma$ , respectively, with  $0 < \sigma_{\min} \leq \sigma(r, z) \leq \sigma_{\max}$  a.e. in  $\Omega$ , and system (2.1a)–(2.1d) can be recast as a problem involving only the meridional domain  $\Omega$  (assumed a polygon) and written in terms of radial and axial velocities  $u_r, u_z$ , and pressure  $p$ . In general, we will denote with  $\check{\mathbf{v}} = (\check{v}_r, \check{v}_\theta, \check{v}_z)$  a vector associated to the three-dimensional domain  $\check{\Omega}$ , whereas under meridian axisymmetry, we have  $\check{v}_\theta = 0$  and its axisymmetric restriction will be denoted by  $\mathbf{v} = (v_r, v_z)$ . We also suppose that the boundary of the meridian domain is disjointedly partitioned into  $\Gamma$ , and  $\Gamma_{\text{sym}}$ , the latter being a segment of the boundary intersecting with the rotation axis  $r = 0$ . In the general case (where  $\check{v}_\theta$  does not necessarily vanish), the full problem can be decoupled in such a way that  $\check{v}_\theta$  is the solution of an independent axisymmetric Laplace problem. Further details can be found in e.g. [15, 18].

Moreover, if we introduce a vorticity field, scaled with respect to viscosity,  $\omega = \sqrt{\nu} \operatorname{rot} \mathbf{u}$ , we arrive at the following problem

$$\sigma \mathbf{u} + \sqrt{\nu} \operatorname{curl}_{\mathbf{a}} \omega + \nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{2.2a}$$

$$\omega - \sqrt{v} \operatorname{rot} \mathbf{u} = 0 \quad \text{in } \Omega, \tag{2.2b}$$

$$\operatorname{div}_a \mathbf{u} = 0 \quad \text{in } \Omega, \tag{2.2c}$$

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma, \tag{2.2d}$$

$$\omega = 0 \quad \text{on } \Gamma, \tag{2.2e}$$

where the vorticity boundary condition on  $\Gamma$  comes from combining (2.1c) and (2.1d) with the definition of vorticity. The axisymmetric counterparts of the usual differential operators acting on vectors and scalars employed herein read

$$\begin{aligned} \operatorname{div}_a \mathbf{v} &:= \partial_r v_r + r^{-1} v_r + \partial_z v_z, & \operatorname{rot} \mathbf{v} &:= \partial_r v_z - \partial_z v_r, \\ \operatorname{curl}_a \varphi &:= \left( \partial_z \varphi, -\partial_r \varphi - \frac{1}{r} \varphi \right)^t. \end{aligned}$$

As mentioned above, the Brinkman model is valid for porous materials consisting of an array of low concentration fixed particles (see e.g. [9]), which can be encountered in some industrial applications. The axisymmetry of the domain and data in these problems would justify formulations of the form (2.2a)–(2.2e).

### 2.3 Axisymmetric Stream-Function–Vorticity Formulation

Next, we realize that the incompressibility condition (2.2c) is equivalent to the existence of a scalar stream-function  $\psi$  satisfying  $\mathbf{u} = \operatorname{curl}_a \psi$ , with  $\psi = 0$  on  $\Gamma$  (cf. Lemma 1 and [4, 24]). Now we take the rot operator on both sides of (2.2a) and employ the relation between the stream-function and the velocity to recast system (2.2) in the form

$$\operatorname{rot}(\sigma \operatorname{curl}_a \psi + \sqrt{v} \operatorname{curl}_a \omega) = \operatorname{rot} f \quad \text{in } \Omega, \tag{2.3a}$$

$$\omega - \sqrt{v} \operatorname{rot}(\operatorname{curl}_a \psi) = 0 \quad \text{in } \Omega, \tag{2.3b}$$

$$\psi = 0 \quad \text{on } \Gamma, \tag{2.3c}$$

$$\omega = 0 \quad \text{on } \Gamma. \tag{2.3d}$$

### 2.4 Recurrent Notation and Auxiliary Results

Before stating a weak form to (2.3), we recall some standard definitions of weighted Sobolev spaces and involved norms (see further details in e.g. [26]). Let  $L^\alpha_\alpha(\Omega)$  denote the weighted Lebesgue space of all measurable functions  $\varphi$  defined in  $\Omega$  for which

$$\|\varphi\|_{L^\alpha_\alpha(\Omega)}^p := \int_\Omega |\varphi|^p r^\alpha \, dr dz < \infty.$$

The subspace  $L^2_{1,0}(\Omega)$  of  $L^2_1(\Omega)$  contains functions  $q$  with zero weighted integral  $(q, 1)_{r,\Omega} = 0$ , where

$$(s, t)_{r,\Omega} := \int_\Omega s t r \, dr dz,$$

for all sufficiently regular functions  $s, t$ . The weighted Sobolev space  $H^k_1(\Omega)$  consists of all functions in  $L^2_1(\Omega)$  whose derivatives up to order  $k$  are also in  $L^2_1(\Omega)$ . This space is provided with semi-norms and norms defined in the standard way; in particular,

$$|\varphi|_{H^1_1(\Omega)}^2 := \int_\Omega (|\partial_r \varphi|^2 + |\partial_z \varphi|^2) r \, dr dz,$$

is a norm onto the Hilbert space  $H_1^1(\Omega) \cap L_{1,0}^2(\Omega)$ . Furthermore, the space  $\tilde{H}_1^1(\Omega) := H_1^1(\Omega) \cap L_{-1}^2(\Omega)$  is endowed with the following norm and semi-norm, respectively (the former being  $\nu$ -dependent):

$$\begin{aligned} \|\varphi\|_{\tilde{H}_1^1(\Omega)} &:= \left( \|\varphi\|_{L_1^2(\Omega)}^2 + \nu |\varphi|_{H_1^1(\Omega)}^2 + \nu \|\varphi\|_{L_{-1}^2(\Omega)}^2 \right)^{1/2}, \\ \|\varphi\|_{\tilde{H}_1^1(\Omega)} &:= \left( |\varphi|_{H_1^1(\Omega)}^2 + \|\varphi\|_{L_{-1}^2(\Omega)}^2 \right)^{1/2}. \end{aligned} \tag{2.4}$$

We will also require the following weighted scalar and vectorial functional spaces:

$$\begin{aligned} H_{1,\diamond}^1(\Omega) &:= \{ \varphi \in H_1^1(\Omega); \varphi = 0 \text{ on } \Gamma \}, \\ \tilde{H}_{1,\diamond}^1(\Omega) &:= \{ \varphi \in \tilde{H}_1^1(\Omega); \varphi = 0 \text{ on } \Gamma \}, \\ \mathbf{H}(\text{div}_a, \Omega) &:= \{ \mathbf{v} \in L_1^2(\Omega)^2; \text{div}_a \mathbf{v} \in L_1^2(\Omega) \}, \\ \mathbf{H}(\mathbf{curl}_a, \Omega) &:= \{ \varphi \in L_1^2(\Omega); \mathbf{curl}_a \varphi \in L_1^2(\Omega)^2 \}, \\ \mathbf{H}(\text{rot}, \Omega) &:= \{ \mathbf{v} \in L_1^2(\Omega)^2; \text{rot } \mathbf{v} \in L_1^2(\Omega) \}. \end{aligned}$$

We observe that as a consequence of [25, Proposition 2.1], the entity in (2.4) is a norm in  $\tilde{H}_{1,\diamond}^1(\Omega)$ . In addition, the spaces  $\mathbf{H}(\text{div}_a, \Omega)$  and  $\mathbf{H}(\mathbf{curl}_a, \Omega)$  are endowed respectively by the norms:

$$\begin{aligned} \|\mathbf{v}\|_{\mathbf{H}(\text{div}_a, \Omega)} &:= \left( \|\mathbf{v}\|_{L_1^2(\Omega)^2}^2 + \|\text{div}_a \mathbf{v}\|_{L_1^2(\Omega)}^2 \right)^{1/2}, \\ \|\varphi\|_{\mathbf{H}(\mathbf{curl}_a, \Omega)} &:= \left( \|\varphi\|_{L_1^2(\Omega)}^2 + \nu \|\mathbf{curl}_a \varphi\|_{L_1^2(\Omega)^2}^2 \right)^{1/2}, \\ \|\varphi\|_{\mathbf{H}(\mathbf{curl}_a, \Omega)} &:= \|\mathbf{curl}_a \varphi\|_{L_1^2(\Omega)^2}. \end{aligned}$$

Moreover, it holds that

$$\|\varphi\|_{\tilde{H}_1^1(\Omega)} \leq \|\varphi\|_{\mathbf{H}(\mathbf{curl}_a, \Omega)} \leq \sqrt{2} \|\varphi\|_{\tilde{H}_1^1(\Omega)} \quad \forall \varphi \in \tilde{H}_1^1(\Omega), \tag{2.5}$$

$$\|\varphi\|_{\tilde{H}_1^1(\Omega)} \leq \|\varphi\|_{\mathbf{H}(\mathbf{curl}_a, \Omega)} \leq \sqrt{2} \|\varphi\|_{\tilde{H}_{1,\diamond}^1(\Omega)} \quad \forall \varphi \in \tilde{H}_{1,\diamond}^1(\Omega). \tag{2.6}$$

The following result will be instrumental in the sequel (see [1, 2, 4]).

**Lemma 1** *Let  $\Omega$  be simply connected. For any  $s > 1$ , if  $\mathbf{v} \in [\tilde{H}_{1,\diamond}^1(\Omega) \cap H_1^s(\Omega)]^2$  satisfies  $\text{div}_a \mathbf{v} = 0$ , and  $\mathbf{v} \cdot \mathbf{n} = 0$  on  $\Gamma$ , then there exists a unique potential  $\varphi \in H_1^{s+1}(\Omega)$  such that  $\mathbf{v} = \mathbf{curl}_a \varphi$ , and  $\varphi = 0$  on  $\Gamma$ .*

On the other hand, let  $H_1^{1/2}(\Gamma)$  be the trace space associated to  $H_1^1(\Omega)$ , and notice that the normal trace operator on  $\Gamma$  is defined by  $\mathbf{v} \mapsto \mathbf{v} \cdot \mathbf{n}|_\Gamma$ , and it is continuous from  $\mathbf{H}(\text{div}_a, \Omega)$  into the dual space of  $H_1^{1/2}(\Gamma)$ . We next recall the following Green identities (cf. [5, 17]).

**Lemma 2** *For any  $\mathbf{v} \in \mathbf{H}(\text{div}_a, \Omega)$  and  $q \in H_1^1(\Omega)$ , the following Green formula holds*

$$(\text{div}_a \mathbf{v}, q)_{r,\Omega} + (\mathbf{v}, \nabla q)_{r,\Omega} = (\mathbf{v} \cdot \mathbf{n}, q)_{r,\Gamma}.$$

**Lemma 3** *For any  $\mathbf{v} \in \mathbf{H}(\text{rot}, \Omega)$  and  $\varphi \in \tilde{H}_1^1(\Omega)$ , we have the following Green formula*

$$(\mathbf{v}, \mathbf{curl}_a \varphi)_{r,\Omega} - (\varphi, \text{rot } \mathbf{v})_{r,\Omega} = (\mathbf{v} \cdot \mathbf{t}, \varphi)_{r,\Gamma}.$$

### 2.5 The Variational Formulation

Now we test system (2.3a)–(2.3b) against  $\varphi \in \tilde{H}^1_{1,\diamond}(\Omega)$ , and  $\theta \in \tilde{H}^1_{1,\diamond}(\Omega)$ , respectively, to obtain

$$\begin{aligned} (\text{rot}[\sigma \mathbf{curl}_a \psi + \sqrt{v} \mathbf{curl}_a \omega], \varphi)_{r,\Omega} &= (\text{rot } \mathbf{f}, \varphi)_{r,\Omega}, \\ (\omega, \theta)_{r,\Omega} - (\sqrt{v} \text{rot}[\mathbf{curl}_a \psi], \theta)_{r,\Omega} &= 0. \end{aligned}$$

Then, combining Lemmas 2 and 3 with a direct application of the boundary conditions, yields the following variational problem: Find  $(\psi, \omega) \in \tilde{H}^1_{1,\diamond}(\Omega) \times \tilde{H}^1_{1,\diamond}(\Omega)$  such that

$$\begin{aligned} a(\psi, \varphi) + b(\varphi, \omega) &= F(\varphi) \quad \forall \varphi \in \tilde{H}^1_{1,\diamond}(\Omega), \\ b(\psi, \theta) - d(\omega, \theta) &= 0 \quad \forall \theta \in \tilde{H}^1_{1,\diamond}(\Omega), \end{aligned} \tag{2.7}$$

where the involved bilinear forms and linear functional are

$$\begin{aligned} a(\psi, \varphi) &:= (\sigma \mathbf{curl}_a \psi, \mathbf{curl}_a \varphi)_{r,\Omega}, \quad b(\varphi, \omega) := (\sqrt{v} \mathbf{curl}_a \omega, \mathbf{curl}_a \varphi)_{r,\Omega}, \\ d(\omega, \theta) &:= (\omega, \theta)_{r,\Omega}, \quad F(\varphi) := (\mathbf{f}, \mathbf{curl}_a \varphi)_{r,\Omega}. \end{aligned}$$

*Remark 1* The discussion about possible shortcomings of the boundary treatment (2.3c), (2.3d) and the associated issues in representing no-slip velocity conditions or other wall laws is not part of the goals of this paper. We refer the interested reader to [7, 11, 13, 34]. However, we do stress that imposition of *tangential* velocities poses no difficulty in our framework. For instance, if we want to set  $\mathbf{u} \cdot \mathbf{t} = u^t$  with a known  $u^t$  on  $\Gamma_t \subset \Gamma$ , then Lemma 3 suggests that the adequate test space for the vorticity field would be

$$\tilde{H}^1_{1,t}(\Omega) := \{ \theta \in \tilde{H}^1_1(\Omega); \theta = 0 \text{ on } \Gamma \setminus \Gamma_t \}.$$

Also from Lemma 3, it follows that a non-homogeneous term

$$H(\theta) := \langle \sqrt{v} u^t, \theta \rangle_{r,\Gamma_t} \quad \forall \theta \in \tilde{H}^1_{1,t}(\Omega),$$

should appear as right hand side in the second equation of (2.7).

### 3 Well-Posedness of the Continuous Problem

In this section, we prove that the continuous variational formulation (2.7) is uniquely solvable. With this aim, we recall the following abstract result (see e.g. [21, Theorem 1.3]):

**Theorem 1** *Let  $(\mathcal{X}, \langle \cdot, \cdot \rangle_{\mathcal{X}})$  be a Hilbert space. Let  $\mathcal{A} : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$  be a bounded symmetric bilinear form, and let  $\mathcal{G} : \mathcal{X} \rightarrow \mathbb{R}$  be a bounded functional. Assume that there exists  $\beta > 0$  such that*

$$\sup_{\substack{y \in \mathcal{X} \\ y \neq 0}} \frac{\mathcal{A}(x, y)}{\|y\|_{\mathcal{X}}} \geq \beta \|x\|_{\mathcal{X}} \quad \forall x \in \mathcal{X}. \tag{3.1}$$

*Then, there exists a unique  $x \in \mathcal{X}$ , such that*

$$\mathcal{A}(x, y) = \mathcal{G}(y) \quad \forall y \in \mathcal{X}. \tag{3.2}$$

*Moreover, there exists  $C > 0$ , independent of the solution, such that*

$$\|x\|_{\mathcal{X}} \leq C \|\mathcal{G}\|_{\mathcal{X}'}$$

**Theorem 2** *The variational problem (2.7) admits a unique solution  $(\psi, \omega) \in \tilde{H}_{1,\diamond}^1(\Omega) \times \tilde{H}_{1,\diamond}^1(\Omega)$ . Moreover, there exists  $C > 0$  independent of  $\nu$  such that*

$$\|\psi\|_{\tilde{H}_{1,\diamond}^1(\Omega)} + \|\omega\|_{\tilde{H}_{1,\diamond}^1(\Omega)} \leq C \|f\|_{L_1^2(\Omega)^2}. \tag{3.3}$$

*Proof* First, we define  $\mathcal{X} := \tilde{H}_{1,\diamond}^1(\Omega) \times \tilde{H}_{1,\diamond}^1(\Omega)$  (endowed with the corresponding product norm:  $\|\cdot\|_{\tilde{H}_{1,\diamond}^1(\Omega)}$  and  $\|\cdot\|_{\tilde{H}_{1,\diamond}^1(\Omega)}$ , respectively) and the following bilinear form and linear functional:

$$\mathcal{A}((\psi, \omega), (\varphi, \theta)) := a(\psi, \varphi) + b(\varphi, \omega) + b(\psi, \theta) - d(\omega, \theta), \quad \mathcal{G}((\varphi, \theta)) := F(\varphi).$$

To continue, it suffices to verify the hypotheses of Theorem 1. First, we note that the linear functional  $\mathcal{G}(\cdot)$  is bounded and as a consequence of the boundedness of  $a(\cdot, \cdot)$ ,  $b(\cdot, \cdot)$ , and  $d(\cdot, \cdot)$ , one has that the bilinear form  $\mathcal{A}(\cdot, \cdot)$  is also bounded with constants independent of  $\nu$ .

The next step consists in proving that the bilinear form  $\mathcal{A}(\cdot, \cdot)$  satisfies the inf-sup condition (3.1). With this aim, we have that for any  $(\psi, \omega) \in \mathcal{X}$ , we define

$$\tilde{\varphi} := (\psi + \hat{c}\sqrt{\nu}\omega) \in \tilde{H}_{1,\diamond}^1(\Omega), \quad \text{and} \quad \tilde{\theta} := -\omega \in \tilde{H}_{1,\diamond}^1(\Omega),$$

where  $\hat{c}$  is a positive constant which will be specified later. Therefore, from the definition of bilinear form  $\mathcal{A}(\cdot, \cdot)$  we obtain

$$\begin{aligned} \mathcal{A}((\psi, \omega), (\tilde{\varphi}, \tilde{\theta})) &= (\sigma \operatorname{curl}_{\mathbf{a}} \psi, \operatorname{curl}_{\mathbf{a}} \tilde{\varphi})_{r,\Omega} + (\sqrt{\nu} \operatorname{curl}_{\mathbf{a}} \omega, \operatorname{curl}_{\mathbf{a}} \tilde{\varphi})_{r,\Omega} \\ &\quad + (\sqrt{\nu} \operatorname{curl}_{\mathbf{a}} \tilde{\theta}, \operatorname{curl}_{\mathbf{a}} \psi)_{r,\Omega} - (\omega, \tilde{\theta})_{r,\Omega} \\ &\geq \sigma_{\min} \|\operatorname{curl}_{\mathbf{a}} \psi\|_{L_1^2(\Omega)^2}^2 + \hat{c}(\sqrt{\nu}\sigma \operatorname{curl}_{\mathbf{a}} \psi, \operatorname{curl}_{\mathbf{a}} \omega)_{r,\Omega} \\ &\quad + \hat{c}\nu \|\operatorname{curl}_{\mathbf{a}} \omega\|_{L_1^2(\Omega)^2}^2 + (\sqrt{\nu} \operatorname{curl}_{\mathbf{a}} \psi, \operatorname{curl}_{\mathbf{a}} \omega)_{r,\Omega} \\ &\quad - (\sqrt{\nu} \operatorname{curl}_{\mathbf{a}} \psi, \operatorname{curl}_{\mathbf{a}} \omega)_{r,\Omega} + \|\omega\|_{L_1^2(\Omega)}^2 \\ &\geq \sigma_{\min} \|\operatorname{curl}_{\mathbf{a}} \psi\|_{L_1^2(\Omega)^2}^2 - \frac{\hat{c}^2 \sigma_{\max}^2 \nu}{2\sigma_{\min}} \|\operatorname{curl}_{\mathbf{a}} \omega\|_{L_1^2(\Omega)^2}^2 \\ &\quad - \frac{\sigma_{\min}}{2} \|\operatorname{curl}_{\mathbf{a}} \psi\|_{L_1^2(\Omega)^2}^2 + \hat{c}\nu \|\operatorname{curl}_{\mathbf{a}} \omega\|_{L_1^2(\Omega)^2}^2 + \|\omega\|_{L_1^2(\Omega)}^2 \\ &= \frac{\sigma_{\min}}{2} \|\operatorname{curl}_{\mathbf{a}} \psi\|_{L_1^2(\Omega)^2}^2 + \hat{c} \left(1 - \frac{\hat{c}\sigma_{\max}^2}{2\sigma_{\min}}\right) \nu \|\operatorname{curl}_{\mathbf{a}} \omega\|_{L_1^2(\Omega)^2}^2 \\ &\quad + \|\omega\|_{L_1^2(\Omega)}^2, \end{aligned}$$

and choosing  $\hat{c} = \frac{\sigma_{\min}}{\sigma_{\max}^2}$ , we can assert that

$$\mathcal{A}((\psi, \omega), (\tilde{\varphi}, \tilde{\theta})) \geq C \|(\psi, \omega)\|_{\mathcal{X}}^2,$$

with  $C$  independent of  $\nu$ , where we have used (2.5) and (2.6) to derive the last inequality. On the other hand, using the definition of  $\|\cdot\|_{\tilde{H}_{1,\diamond}^1(\Omega)}$  and the fact that  $\nu \leq \nu_{\max}$ , we obtain that

$$\|\tilde{\theta}\|_{\tilde{H}_{1,\diamond}^1(\Omega)} = \|\omega\|_{\tilde{H}_{1,\diamond}^1(\Omega)} \quad \text{and} \quad \|\tilde{\varphi}\|_{\tilde{H}_{1,\diamond}^1(\Omega)} \leq C \left( \|\psi\|_{\tilde{H}_{1,\diamond}^1(\Omega)} + \|\omega\|_{\tilde{H}_{1,\diamond}^1(\Omega)} \right), \tag{3.4}$$

with  $C > 0$  independent of  $\nu$  and consequently

$$\sup_{\substack{(\varphi, \theta) \in \mathcal{X} \\ (\varphi, \theta) \neq 0}} \frac{\mathcal{A}((\psi, \omega), (\varphi, \theta))}{\|(\varphi, \theta)\|_{\mathcal{X}}} \geq \frac{\mathcal{A}((\psi, \omega), (\tilde{\varphi}, \tilde{\theta}))}{\|(\tilde{\varphi}, \tilde{\theta})\|_{\mathcal{X}}} \geq C \|(\psi, \omega)\|_{\mathcal{X}} \quad \forall (\psi, \omega) \in \mathcal{X},$$

which gives (3.3). □

*Remark 2* Vorticity and stream-function are available after solving (2.7). On the other hand, as a consequence of the Lax-Milgram Theorem, the pressure can be computed as the unique solution of the following problem: Find  $p \in H^1_1(\Omega) \cap L^2_{1,0}(\Omega)$  such that

$$(\nabla p, \nabla q)_{r,\Omega} = G^\psi(q) := (\mathbf{f} - \sigma \mathbf{curl}_a \psi, \nabla q)_{r,\Omega} \quad \forall q \in H^1_1(\Omega) \cap L^2_{1,0}(\Omega). \quad (3.5)$$

This problem has been obtained by testing (2.2a) against  $\nabla q$  for a generic  $q \in H^1_1(\Omega) \cap L^2_{1,0}(\Omega)$ , and using integration by parts in combination with the relation between the stream-function and the velocity, and the boundary condition (2.3d). Moreover, the following continuous dependence holds: there exists  $C > 0$  independent of  $\nu$  such that

$$\|p\|_{H^1_1(\Omega) \cap L^2_{1,0}(\Omega)} \leq C \|\mathbf{f}\|_{L^2_1(\Omega)^2}.$$

Notice that, according to Remark 1, if tangential velocity is imposed on  $\Gamma_t$ , or if non-homogeneous Dirichlet data are set for the vorticity, then  $G^\psi(q)$  should be replaced by  $G^{\psi,\omega}(q) = (\mathbf{f} - \sigma \mathbf{curl}_a \psi - \sqrt{\nu} \mathbf{curl}_a \omega, \nabla q)_{r,\Omega}$  in (3.5). Analogously for the discrete problem (4.4).

### 4 Mixed Finite Element Approximation

In this section, we construct discrete schemes associated to (2.7) and (3.5), define explicit finite element subspaces yielding its unique solvability, derive a priori error estimates and provide the rate of convergence of the methods.

#### 4.1 Statement of the Galerkin Scheme

Let  $\{\mathcal{T}_h\}_{h>0}$  be a regular family of triangulations of  $\Omega$  by triangles  $T$  with mesh size  $h$ . For  $S \subset \bar{\Omega}$ , we denote by  $\mathbb{P}_k(S)$ ,  $k \in \mathbb{N}$ , the set of polynomials of degree  $\leq k$ . For any  $k \geq 1$ , we adopt the subspaces

$$Z_h := \{\varphi_h \in \tilde{H}^1_{1,\diamond}(\Omega) : \varphi_h|_T \in \mathbb{P}_k(T) \quad \forall T \in \mathcal{T}_h\}, \quad (4.1)$$

$$Q_h := \{q_h \in H^1_1(\Omega) : q_h|_T \in \mathbb{P}_k(T) \quad \forall T \in \mathcal{T}_h\} \cap L^2_{1,0}(\Omega). \quad (4.2)$$

Notice that functions in the discrete trial and test stream-function space  $Z_h$  vanish along the axis of symmetry, due to the properties of  $\tilde{H}^1_{1,\diamond}(\Omega)$ .

Then, the finite element discretization for (2.7) reads: Find  $(\psi_h, \omega_h) \in Z_h \times Z_h$  such that

$$\begin{aligned} a(\psi_h, \varphi_h) + b(\varphi_h, \omega_h) &= F(\varphi_h) & \forall \varphi_h \in Z_h, \\ b(\psi_h, \theta_h) - d(\omega_h, \theta_h) &= 0 & \forall \theta_h \in Z_h. \end{aligned} \quad (4.3)$$

In turn, the discrete counterpart of (3.5) is: Find  $p_h \in Q_h$  such that

$$(\nabla p_h, \nabla q_h)_{r,\Omega} = G^{\psi_h}(q_h) := (\mathbf{f} - \sigma \mathbf{curl}_a \psi_h, \nabla q_h)_{r,\Omega} \quad \forall q_h \in Q_h. \quad (4.4)$$

#### 4.2 Solvability and Stability Analysis

We now establish discrete counterparts of Theorem 2 and Remark 2, which will yield the solvability and stability of problems (4.3) and (4.4). First we state a discrete version of Theorem 1.

**Theorem 3** Let  $(\mathcal{X}, \langle \cdot, \cdot \rangle_{\mathcal{X}})$  be a Hilbert space and let  $\{\mathcal{X}_h\}_{h>0}$  be a sequence of finite-dimensional subspaces of  $\mathcal{X}$ . Let  $\mathcal{A} : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$  be a bounded symmetric bilinear form, and  $\mathcal{G} : \mathcal{X} \rightarrow \mathbb{R}$  a bounded functional. Assume that there exists  $\bar{\beta}_h > 0$  such that

$$\sup_{\substack{y_h \in \mathcal{X}_h \\ y \neq 0}} \frac{\mathcal{A}(x_h, y_h)}{\|y_h\|_{\mathcal{X}}} \geq \bar{\beta}_h \|x_h\|_{\mathcal{X}} \quad \forall x_h \in \mathcal{X}_h. \tag{4.5}$$

Then, there exists a unique  $x_h \in \mathcal{X}_h$ , such that

$$\mathcal{A}(x_h, y_h) = \mathcal{G}(y_h) \quad \forall y_h \in \mathcal{X}_h. \tag{4.6}$$

Moreover, there exist  $C_1, C_2 > 0$ , independent of the solution, such that

$$\|x_h\|_{\mathcal{X}} \leq C_1 \|\mathcal{G}\|_{\mathcal{X}'_h} \|x'_h\|_{\mathcal{X}'_h}, \quad \text{and} \quad \|x - x_h\|_{\mathcal{X}} \leq C_2 \inf_{y_h \in \mathcal{X}_h} \|x - y_h\|_{\mathcal{X}},$$

where  $x \in \mathcal{X}$  is the unique solution of continuous problem (3.2).

*Proof* The proof follows from Theorem 1, and from the discrete inf-sup condition for  $\mathcal{A}(\cdot, \cdot)$ . □

The unique solvability and convergence of the discrete problem (4.3) are stated next.

**Theorem 4** Let  $k \geq 1$  be an integer and let  $Z_h$  be given by (4.1). Then, there exists a unique  $(\psi_h, \omega_h) \in Z_h \times Z_h$  solution of discrete problem (4.3). Moreover, there exist constants  $\hat{C}_1, \hat{C}_2 > 0$  independent of  $h$  and  $\nu$ , such that

$$\|\psi_h\|_{\tilde{H}^1_1(\Omega)} + \|\omega_h\|_{\tilde{H}^1_1(\Omega)} \leq \hat{C}_1 \|f\|_{L^2(\Omega)^2}, \tag{4.7}$$

and

$$\begin{aligned} & \|\psi - \psi_h\|_{\tilde{H}^1_1(\Omega)} + \|\omega - \omega_h\|_{\tilde{H}^1_1(\Omega)} \\ & \leq \hat{C}_2 \inf_{(\varphi_h, \theta_h) \in Z_h \times Z_h} \left( \|\psi - \varphi_h\|_{\tilde{H}^1_1(\Omega)} + \|\omega - \theta_h\|_{\tilde{H}^1_1(\Omega)} \right), \end{aligned} \tag{4.8}$$

where  $(\psi, \omega) \in \tilde{H}^1_1(\Omega) \times \tilde{H}^1_1(\Omega)$  is the unique solution to variational problem (2.7).

*Proof* We define  $\mathcal{X}_h := Z_h \times Z_h$  and we consider  $\mathcal{A}(\cdot, \cdot)$  and  $\mathcal{G}(\cdot)$  as in the proof of Theorem 2. The next step consists in proving that the bilinear form  $\mathcal{A}(\cdot, \cdot)$  satisfies the discrete inf-sup condition (4.5). In fact, given  $(\psi_h, \omega_h) \in \mathcal{X}_h$ , we define

$$\tilde{\theta}_h := -\omega_h \in Z_h, \quad \text{and} \quad \tilde{\varphi}_h := \left( \psi_h + \frac{\sigma_{\min}}{\sigma_{\max}^2} \sqrt{\nu} \omega_h \right) \in Z_h.$$

Thus, we have that  $\|\tilde{\theta}_h\|_{\tilde{H}^1_1(\Omega)} = \|\omega_h\|_{\tilde{H}^1_1(\Omega)}$  and

$$\|\tilde{\varphi}_h\|_{\tilde{H}^1_1(\Omega)} \leq \|\psi_h\|_{\tilde{H}^1_1(\Omega)} + c\sqrt{\nu} \|\omega_h\|_{\tilde{H}^1_1(\Omega)} \leq C \left( \|\psi_h\|_{\tilde{H}^1_1(\Omega)} + \|\omega_h\|_{\tilde{H}^1_1(\Omega)} \right),$$

where we have used the definition of  $\|\cdot\|_{\tilde{H}^1_1(\Omega)}$  and the fact that  $\nu \leq \nu_{\max}$ ; thus, the constant  $C$  is independent of  $h$  and  $\nu$ . Moreover, we have that

$$\mathcal{A}((\psi_h, \omega_h), (\tilde{\varphi}_h, \tilde{\theta}_h)) \geq \hat{C} \|(\psi_h, \omega_h)\|_{\mathcal{X}}^2,$$

with  $\hat{C}$  independent of  $\nu$  and  $h$ . Then, we get

$$\begin{aligned} \sup_{\substack{(\varphi_h, \theta_h) \in \mathcal{X}_h \\ (\varphi_h, \theta_h) \neq 0}} \frac{\mathcal{A}((\psi_h, \omega_h), (\varphi_h, \theta_h))}{\|(\varphi_h, \theta_h)\|_{\mathcal{X}}} &\geq \frac{\mathcal{A}((\psi_h, \omega_h), (\tilde{\varphi}_h, \tilde{\theta}_h))}{\|(\tilde{\varphi}_h, \tilde{\theta}_h)\|_{\mathcal{X}}} \\ &\geq \tilde{C} \|(\psi_h, \omega_h)\|_{\mathcal{X}} \quad \forall (\psi_h, \omega_h) \in \mathcal{X}_h, \end{aligned}$$

with  $\tilde{C}$  independent of  $\nu$  and  $h$ . The proof is complete. □

We now establish the stability and approximation property for the discrete pressure.

**Theorem 5** *Let  $k \geq 1$  be an integer and let  $Q_h$  be given by (4.2). Then, there exists a unique solution  $p_h \in Q_h$  to discrete problem (4.4) and there exists a constant  $C > 0$  such that:*

$$\|p_h\|_{H^1_1(\Omega) \cap L^2_{1,0}(\Omega)} \leq C \|f\|_{L^2_1(\Omega)}.$$

Moreover, there exists a constant  $\hat{C} > 0$  such that

$$\begin{aligned} \|p - p_h\|_{H^1_1(\Omega) \cap L^2_{1,0}(\Omega)} &\leq \hat{C} \left( \inf_{q_h \in Q_h} \|p - q_h\|_{H^1_1(\Omega) \cap L^2_{1,0}(\Omega)} \right. \\ &\quad \left. + \inf_{\varphi_h, \theta_h \in Z_h} \left( \|\psi - \varphi_h\|_{\tilde{H}^1_1(\Omega)} + \|\omega - \theta_h\|_{\tilde{H}^1_1(\Omega)} \right) \right), \end{aligned} \tag{4.9}$$

where  $C$  and  $\hat{C}$  are independent of  $\nu$  and  $h$ , and  $p \in H^1_1(\Omega) \cap L^2_{1,0}(\Omega)$  is the unique solution of problem (3.5).

*Proof* On the one hand, the well posedness of problem (4.4) follows from the Lax-Milgram Theorem. On the other hand, from the well-known first Strang Lemma, we have that

$$\begin{aligned} \|p - p_h\|_{H^1_1(\Omega) \cap L^2_{1,0}(\Omega)} &\leq C \left\{ \inf_{q_h \in Q_h} \|p - q_h\|_{H^1_1(\Omega) \cap L^2_{1,0}(\Omega)} \right. \\ &\quad \left. + \sup_{q_h \in Q_h} \frac{G^{\psi_h}(q_h) - G^\psi(q_h)}{\|q_h\|_{H^1_1(\Omega) \cap L^2_{1,0}(\Omega)}} \right\}. \end{aligned}$$

To estimate the second term on the right-hand side above, we use the definition of  $G^\psi$  (cf. (3.5)) and  $G^{\psi_h}$  (cf. (4.4)) to obtain

$$\sup_{q_h \in Q_h} \frac{G^{\psi_h}(q_h) - G^\psi(q_h)}{\|q_h\|_{H^1_1(\Omega) \cap L^2_{1,0}(\Omega)}} \leq C \|\mathbf{curl}_a(\psi - \psi_h)\|_{L^2_1(\Omega)^2} \leq C \|\psi - \psi_h\|_{\tilde{H}^1_1(\Omega)},$$

where in the last inequality we have used (2.6). Therefore, the proof follows from (4.8). □

### 4.3 Convergence Analysis

According to Theorems 4 and 5, it only remains to prove that  $\psi$ ,  $\omega$  and  $p$  can be conveniently approximated by functions in  $Z_h$  and  $Q_h$ , respectively. With this purpose, we introduce the Lagrange interpolation operator  $\Pi_h : \tilde{H}^1_1(\Omega) \cap H^2_1(\Omega) \rightarrow Z_h$ . Moreover, [29, Lemma 6.3] implies that the following estimate holds (see also [14, Proposition 3]).

**Lemma 4** *There exists  $C > 0$ , independent of  $h$ , such that for all  $\varphi \in H^{k+1}_1(\Omega)$  :*

$$\|\varphi - \Pi_h \varphi\|_{\tilde{H}^1_1(\Omega)} \leq Ch^k \|\varphi\|_{H^{k+1}_1(\Omega)}.$$

We now turn to the statement of convergence properties of the discrete problem (4.3).

**Theorem 6** Let  $k \geq 1$  be an integer and let  $Z_h$  and  $Q_h$  be given by (4.1) and (4.2), respectively. Let  $(\psi, \omega) \in \tilde{H}_{1,\diamond}^1(\Omega) \times \tilde{H}_{1,\diamond}^1(\Omega)$  and  $p \in H_1^1(\Omega) \cap L_{1,0}^2(\Omega)$  be the unique solutions to the continuous problems (2.7) and (3.5), and  $(\psi_h, \omega_h) \in Z_h \times Z_h$  and  $p_h \in Q_h$  be the unique solutions to the discrete problems (4.3) and (4.4), respectively. Assume that  $\psi \in H_1^{k+1}(\Omega)$ ,  $\omega \in H_1^{k+1}(\Omega)$ , and  $p \in H_1^{k+1}(\Omega)$ . Then, the following error estimates hold

$$\begin{aligned} \|\psi - \psi_h\|_{\tilde{H}_1^1(\Omega)} + \|\omega - \omega_h\|_{\tilde{H}_1^1(\Omega)} &\leq C_1 h^k \left( \|\psi\|_{H_1^{k+1}(\Omega)} + \|\omega\|_{H_1^{k+1}(\Omega)} \right), \\ \|p - p_h\|_{H_1^1(\Omega) \cap L_{1,0}^2(\Omega)} &\leq C_2 h^k \left( \|p\|_{H_1^{k+1}(\Omega)} + \|\psi\|_{H_1^{k+1}(\Omega)} + \|\omega\|_{H_1^{k+1}(\Omega)} \right), \end{aligned}$$

where  $C_1$  and  $C_2$  are positive constants independent of  $v$  and  $h$ .

*Proof* The proof follows from estimates (4.8), (4.9) and error estimates from Lemma 4.  $\square$

A natural consequence of this result is that the vorticity and stream-function approximations also converge in the  $L_1^2(\Omega)$ -norm with an order  $O(h^k)$ :

$$\|\omega - \omega_h\|_{L_1^2(\Omega)} = O(h^k), \quad \text{and} \quad \|\psi - \psi_h\|_{L_1^2(\Omega)} = O(h^k).$$

Such an estimate can be improved by one order of convergence, as show by the following theorem.

**Theorem 7** Under the assumptions of Theorem 6, there exists  $C > 0$  independent of  $h$  and  $v$  such that

$$\|\omega - \omega_h\|_{L_1^2(\Omega)} \leq Ch^{k+1} \left( \|\psi\|_{H_1^{k+1}(\Omega)} + \|\omega\|_{H_1^{k+1}(\Omega)} \right), \tag{4.10}$$

$$\|\psi - \psi_h\|_{L_1^2(\Omega)} \leq Ch^{k+1} \left( \|\psi\|_{H_1^{k+1}(\Omega)} + \|\omega\|_{H_1^{k+1}(\Omega)} \right). \tag{4.11}$$

*Proof* The core of the proof is based on a duality argument. We first establish (4.10). Let us consider the following well-posed problem: Given  $g \in L_1^2(\Omega)$ , find  $(\rho, \xi) \in \tilde{H}_{1,\diamond}^1(\Omega) \times \tilde{H}_{1,\diamond}^1(\Omega)$  such that

$$\begin{aligned} a(\varphi, \rho) + b(\varphi, \xi) &= 0 \quad \forall \varphi \in \tilde{H}_{1,\diamond}^1(\Omega), \\ b(\rho, \theta) - d(\theta, \xi) &= G(\theta) \quad \forall \theta \in \tilde{H}_{1,\diamond}^1(\Omega), \end{aligned} \tag{4.12}$$

where  $G(\theta) := (g, \theta)_{r,\Omega}$ . In order to improve the present estimates, we will require the following regularity:  $\rho \in H_1^2(\Omega)$ ,  $\xi \in H_1^2(\Omega)$ . Moreover, we also assume that there exists a constant  $C > 0$ , independent of  $v$  and  $g$  such that

$$\|\rho\|_{H_1^2(\Omega)} + \|\xi\|_{H_1^2(\Omega)} \leq C \|g\|_{L_1^2(\Omega)}. \tag{4.13}$$

Next, choosing  $(\varphi, \theta) = (\psi - \psi_h, \omega - \omega_h)$  in (4.12), we obtain

$$G(\omega - \omega_h) = b(\rho, \omega - \omega_h) - d(\omega - \omega_h, \xi), \tag{4.14}$$

$$a(\psi - \psi_h, \rho) + b(\psi - \psi_h, \xi) = 0. \tag{4.15}$$

Moreover, from (2.7) and (4.3) we have that:

$$\begin{aligned} b(\psi - \psi_h, \xi_h) - d(\omega - \omega_h, \xi_h) &= 0, \\ a(\psi - \psi_h, \rho_h) + b(\rho_h, \omega - \omega_h) &= 0. \end{aligned}$$

Thus, subtracting the above equations and (4.15) from (4.14), we obtain

$$\begin{aligned} G(\omega - \omega_h) &= b(\rho, \omega - \omega_h) - d(\omega - \omega_h, \xi) - b(\psi - \psi_h, \xi_h) + d(\omega - \omega_h, \xi_h) \\ &\quad - a(\psi - \psi_h, \rho_h) - b(\rho_h, \omega - \omega_h) + a(\psi - \psi_h, \rho) + b(\psi - \psi_h, \xi) \\ &= b(\rho - \rho_h, \omega - \omega_h) - d(\omega - \omega_h, \xi - \xi_h) + b(\psi - \psi_h, \xi - \xi_h) \\ &\quad + a(\psi - \psi_h, \rho - \rho_h), \end{aligned}$$

for all  $(\rho_h, \xi_h) \in Z_h \times Z_h$ . Hence,

$$\begin{aligned} |G(\omega - \omega_h)| &\leq C(\|\rho - \rho_h\|_{\tilde{H}_1^1(\Omega)} \|\omega - \omega_h\|_{\tilde{H}_1^1(\Omega)} + \|\omega - \omega_h\|_{L_1^2(\Omega)} \|\xi - \xi_h\|_{L_1^2(\Omega)} \\ &\quad + \|\psi - \psi_h\|_{\tilde{H}_1^1(\Omega)} \|\xi - \xi_h\|_{\tilde{H}_1^1(\Omega)} + \|\psi - \psi_h\|_{\tilde{H}_1^1(\Omega)} \|\rho - \rho_h\|_{\tilde{H}_1^1(\Omega)}) \\ &\leq Ch^k(\|\psi\|_{H_1^{k+1}(\Omega)} + \|\omega\|_{H_1^{k+1}(\Omega)})(\|\rho - \rho_h\|_{\tilde{H}_1^1(\Omega)} + \|\xi - \xi_h\|_{\tilde{H}_1^1(\Omega)}), \end{aligned}$$

for all  $(\rho_h, \xi_h) \in Z_h \times Z_h$ , where in the last inequality we have utilized Theorem 6. Taking in particular  $(\rho_h, \xi_h)$  as the Lagrange interpolants of  $(\rho, \xi)$  (see Lemma 4), and then using the additional regularity result (4.13) in the above estimate, we obtain:

$$|G(\omega - \omega_h)| \leq Ch^{k+1} \left( \|\psi\|_{H_1^{k+1}(\Omega)} + \|\omega\|_{H_1^{k+1}(\Omega)} \right) \|g\|_{L_1^2(\Omega)}.$$

Thus, from the estimate above and the definition by duality of  $\|\cdot\|_{L_1^2(\Omega)}$ , we arrive at

$$\|\omega - \omega_h\|_{L_1^2(\Omega)} = \sup_{g \in L_1^2(\Omega)} \frac{(g, \omega - \omega_h)_{r,\Omega}}{\|g\|_{L_1^2(\Omega)}} \leq Ch^{k+1} \left( \|\psi\|_{H_1^{k+1}(\Omega)} + \|\omega\|_{H_1^{k+1}(\Omega)} \right),$$

where the constant  $C$  is independent of  $h$  and  $v$ .

Finally, (4.11) follows from the same arguments given before for (4.10), but instead of dual problem (4.12), we consider the following one:

$$\begin{aligned} a(\varphi, \rho) + b(\varphi, \xi) &= G(\varphi) \quad \forall \varphi \in \tilde{H}_{1,\diamond}^1(\Omega), \\ b(\rho, \theta) - d(\theta, \xi) &= 0 \quad \forall \theta \in \tilde{H}_{1,\diamond}^1(\Omega), \end{aligned}$$

where in this case  $G(\varphi) := (g, \varphi)_{r,\Omega}$ . □

*Remark 3* We observe that since  $\mathbf{u} = \mathbf{curl}_a \psi$ , the velocity can be readily recovered from the main unknowns of the underlying problem. More precisely, if  $(\psi_h, \omega_h) \in Z_h \times Z_h$  is the unique solution of (4.3), then  $\mathbf{u}_h := \mathbf{curl}_a \psi_h$  approximates the velocity with the same order of the proposed method. This result is summarized as follows.

**Corollary 1** *Assume that the hypotheses of Theorem 6 hold. Then, for  $\mathbf{u}_h := \mathbf{curl}_a \psi_h$ , there exists  $C > 0$  (independent of both  $h$  and  $v$ ) such that*

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{H}(\text{div}_a, \Omega)} \leq Ch^k \left( \|\psi\|_{H_1^{k+1}(\Omega)} + \|\omega\|_{H_1^{k+1}(\Omega)} \right).$$

*Proof* We have that

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{H}(\text{div}_a, \Omega)} = \|\mathbf{u} - \mathbf{u}_h\|_{L_1^2(\Omega)^2} = \|\mathbf{curl}_a \psi - \mathbf{curl}_a \psi_h\|_{L_1^2(\Omega)^2} \leq \|\psi - \psi_h\|_{\tilde{H}_1^1(\Omega)},$$

where in the last inequality we have used (2.6). Thus, the result follows from Theorem 6. □

**Table 1** Example 1: errors and convergence rates ( $r$ ) associated to the piecewise polynomial approximation of stream-function, vorticity and pressure, and postprocessed velocity

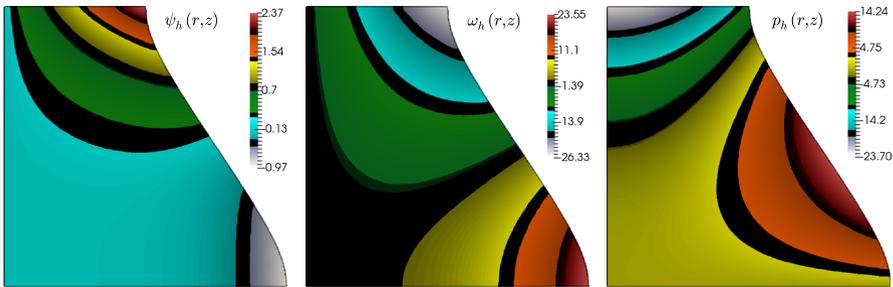
$h$	$\ \psi - \psi_h\ _1$	$r$	$\ \psi - \psi_h\ _0$	$r$	$\ \omega - \omega_h\ _1$	$r$	$\ \omega - \omega_h\ _0$	$r$	$\ p - p_h\ _1$	$r$	$\ \mathbf{u} - \mathbf{u}_h\ _{\text{div}}$	$r$
Approximation with $k = 1$												
0.413	2.149	–	0.185	–	0.438	–	0.054	–	6.461	–	0.734	–
0.350	1.175	0.932	0.084	1.277	0.370	0.845	0.037	1.554	5.605	0.848	0.695	1.227
0.229	0.602	1.572	0.021	2.246	0.172	1.802	0.009	2.265	3.413	1.168	0.376	1.134
0.122	0.352	0.853	0.007	1.730	0.102	0.821	0.003	1.724	1.900	0.929	0.222	0.976
0.078	0.213	1.141	0.002	2.318	0.061	1.149	0.001	2.353	1.173	1.094	0.135	1.047
0.053	0.141	1.044	0.001	2.092	0.041	1.044	5.2e-4	2.012	0.785	1.030	0.090	1.092
0.038	0.100	1.050	6.1e-4	2.084	0.029	1.033	3.1e-4	2.131	0.549	1.098	0.065	0.997
0.028	0.075	1.011	3.3e-4	2.013	0.022	1.029	1.5e-4	2.089	0.411	1.010	0.048	1.074
0.022	0.058	0.983	1.9e-4	1.970	0.016	0.986	8.3e-5	1.983	0.317	0.974	0.036	0.945
Approximation with $k = 2$												
0.413	0.167	–	0.009	–	0.099	–	0.014	–	0.689	–	0.043	–
0.350	0.059	1.743	0.003	2.014	0.041	1.243	0.002	2.516	0.341	1.797	0.030	1.840
0.229	0.019	2.608	5.2e-4	3.066	0.011	2.023	3.2e-4	3.117	0.116	2.134	0.012	1.883
0.122	0.006	1.831	1.9e-4	2.951	0.003	1.976	6.6e-5	2.867	0.035	1.982	0.004	1.890
0.078	0.002	2.259	2.2e-5	3.157	0.001	2.251	1.5e-5	3.186	0.013	2.123	0.002	1.887
0.053	0.001	2.069	7.1e-6	2.993	5.8e-4	2.038	4.5e-6	2.981	0.005	2.084	0.001	1.890
0.038	5.0e-4	2.163	2.4e-6	3.130	3.0e-4	2.107	1.6e-6	3.193	0.003	2.146	4.3e-4	1.908
0.028	3.0e-4	1.996	1.0e-6	2.970	1.8e-4	2.065	6.8e-7	3.098	0.002	2.018	1.2e-4	1.935
0.022	2.0e-4	1.981	4.8e-7	2.972	1.0e-4	1.992	3.0e-7	2.963	0.001	1.991	7.6e-5	1.678
Approximation with $k = 3$												
0.413	0.025	–	0.001	–	0.005	–	3.4e-4	–	0.004	–	0.006	–
0.350	0.010	3.288	5.3e-4	4.158	0.001	3.385	1.3e-4	3.623	0.001	2.843	0.004	2.436
0.229	0.001	3.543	4.3e-5	4.893	4.3e-4	3.247	1.3e-5	4.033	3.8e-4	2.987	8.9e-4	2.989
0.122	1.1e-4	3.526	2.3e-6	4.550	1.4e-4	2.955	1.2e-6	3.975	3.6e-5	3.143	2.4e-4	2.836
0.078	2.5e-5	3.670	3.1e-7	4.528	1.2e-5	3.104	1.7e-7	4.050	6.9e-6	3.166	3.0e-5	2.955
0.053	4.9e-6	3.532	3.6e-8	4.431	3.8e-6	2.984	3.4e-8	4.077	2.2e-6	3.240	7.9e-6	3.056
0.038	1.3e-6	3.430	6.3e-9	4.409	1.3e-6	3.204	8.7e-9	4.195	3.8e-7	3.440	2.8e-6	3.061
0.028	4.0e-7	3.679	1.6e-9	4.530	5.4e-7	3.136	2.6e-9	4.191	1.0e-7	3.154	1.0e-6	2.976
0.022	1.3e-7	3.387	4.4e-10	4.406	2.5e-7	2.914	1.0e-9	3.960	4.1e-8	3.161	4.3e-7	2.985

### 5 Numerical Results

In our first example we test the convergence of the proposed scheme when applied to the axisymmetric version of the classical colliding flow problem (see e.g. [19, Sect. 5.1] for the Cartesian case). The analytic solution is given as follows

$$\psi(r, z) = 5rz^4 - r^5, \quad \omega(r, z) = 12\sqrt{v}(2r^3 - 5rz^2), \quad p(r, z) = 60r^2z - 24z^3,$$

and it is defined on the meridional domain  $\Omega$  having four sides defined by the symmetry axis (left wall  $r = 0$ ), bottom and top lids ( $z = 0$  and  $z = 1$ , respectively), and the curve characterized by the parametrization  $s \in [0, 1], r = 1 - s/2 + 0.15 \cos(\pi s) \sin(\pi s)$ , and  $z = s - 0.15 \cos(\pi s) \sin(\pi s)$ . We set the model parameters to  $\sigma = 10$  and  $v = 0.1$ . The boundary conditions are non-homogeneous and set according to the interpolant of the exact stream-function and vorticity (and the pressure solve is modified according to Remark 2), whereas the forcing term  $f$  has been manufactured using the momentum equation (2.3a). Errors for vorticity and stream-function were measured in the  $\tilde{H}_1^1(\Omega)$  and  $L_1^2(\Omega)$ -norms (denoted hereafter with subscripts 1 and 0, respectively), while those for the pressure correspond to the  $H_1^1(\Omega) \cap L_{1,0}^2(\Omega)$ -norm (denoted with subscript 1). The convergence history (obtained on a family of successively refined unstructured partitions of  $\Omega$ ) is collected in Table 1, confirming the expected behavior predicted by Theorems 6 and 7. For illustrative purposes we also list the error history for the postprocessed velocity, exhibiting a  $O(h^k)$  convergence, as anticipated by Corollary 1. The approximate solutions obtained using the lowest-order method ( $k = 1$ ) on a coarse mesh are displayed in



**Fig. 1** Example 1: approximated stream-function, vorticity, and pressure distribution for the accuracy assessment test on the axisymmetric domain  $\Omega$

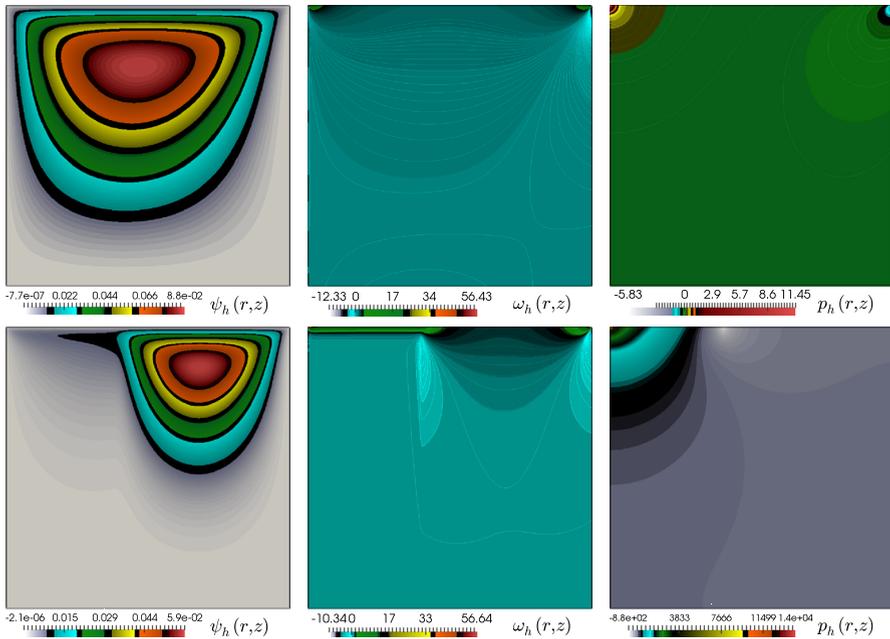
**Table 2** Example 1: errors and stability with respect to the data, for different values of the fluid viscosity

$\nu$	$\frac{\ \psi - \psi_h\ _1}{\ \psi\ _1}$	$\frac{\ \omega - \omega_h\ _1}{\ \omega\ _1}$	$\frac{\ p - p_h\ _1}{\ p\ _1}$	$\frac{\ u - u_h\ _0}{\ u\ _0}$	$\bar{f}$	$\frac{\epsilon}{\bar{f}}$
1.0e-1	1.6e-3	1.4e-3	0.013	0.176	19.794	6.3e-4
1.0e-2	1.2e-3	7.6e-4	0.013	0.161	22.322	5.0e-4
1.0e-3	6.1e-4	2.6e-4	0.013	0.160	22.576	5.4e-4
1.0e-4	5.9e-4	2.3e-4	0.013	0.160	22.601	5.3e-4
1.0e-5	5.7e-4	2.3e-4	0.013	0.160	22.604	5.2e-4
1.0e-6	5.7e-4	2.2e-4	0.013	0.160	22.604	5.2e-4
1.0e-7	5.5e-4	2.1e-4	0.013	0.160	22.604	5.2e-4
1.0e-8	5.5e-4	2.1e-4	0.013	0.160	22.604	5.2e-4
1.0e-9	5.5e-4	2.1e-4	0.013	0.160	22.604	5.2e-4
1.0e-10	5.5e-4	2.1e-4	0.013	0.160	22.604	5.2e-4

Fig. 1. We recall that, by construction, the divergence of the computed velocity is exactly zero.

Following [6, Sect. 4.3] we now illustrate the robustness of the method with respect to the fluid viscosity. We fix a mesh of size  $h = 0.008$ , choose the method with  $k = 1$ , and recompute individual and total errors ( $\epsilon$ ) for different values of  $\nu$ . Table 2 lists the obtained results, showing also a measure of the data  $\bar{f} = \|f\|_0$ . The last column of the table presents the ratio between the total error and the norm of the data, indicating that the method is unaffected by the decrease of the viscosity.

Our next example addresses the well-known lid driven cavity flow. The domain under consideration is the unit square  $\Omega = (0, 1)^2$ , discretized with an unstructured mesh of 80K triangular elements. Following Remark 1, a tangential velocity  $u^t = -1$  is imposed on the top lid of the domain ( $\Gamma_t \subset \Gamma$ ), the symmetry axis  $\Gamma_{\text{sym}}$  is the left wall, and on the remainder of the boundary we set homogeneous Dirichlet data for the stream-function. No boundary conditions are explicitly set for the vorticity. The forcing term is  $f = \mathbf{0}$ , the viscosity is constant  $\nu = 1e - 2$ , and the inverse permeability is, in a first round, constant  $\sigma = 0.1$ . We also test the case where  $\sigma$  is discontinuous across the line  $r = 0.4$ , going from  $\sigma_0 = 0.01$  to  $\sigma_1 = 100$ . Stream-function, vorticity and pressure profiles for both cases are displayed in Fig. 2, where the bottom



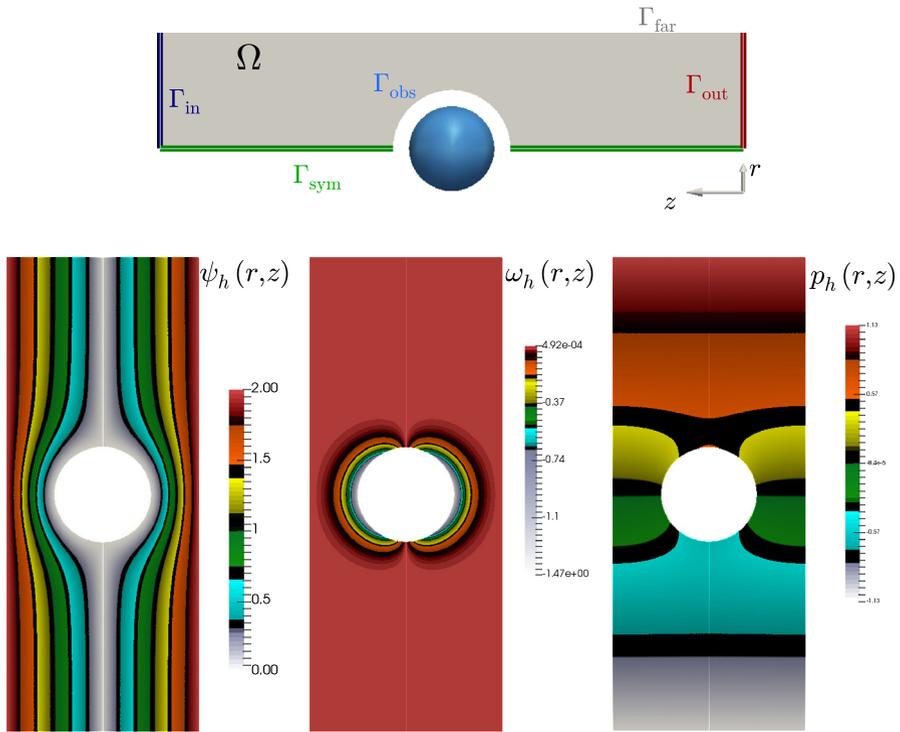
**Fig. 2** Example 2: approximated stream-function, vorticity, and pressure distribution for the lid-driven cavity problem for constant (*top row*) and discontinuous permeability (*bottom panels*)

row shows a clear change of regime between the regions of high and low permeability.

Finally, we perform a simulation of axisymmetric laminar flow past a sphere. The meridional domain configuration is given in the top panel of Fig. 3. The boundary of the meridional  $\Omega$  is decomposed into an inlet boundary (located at  $z = 0$ ), an outlet (at  $z = 10$ ), a “far-field” border (on  $r = 2$ ), the surface of the obstacle (centered at  $r = 0, z = 5$  and with radius 1), and the symmetry axis is located at  $r = 0$ . The domain is discretized into 80K triangular elements and the model parameters are  $\nu = 5e - 3, \sigma = 0.1$ . The boundary conditions are set as follows: on  $\Gamma_{in}$  we set  $\psi = r$ , on  $\Gamma_{far}$  we set  $\psi = \frac{1}{2}r^2$  and  $\omega = 0$ , and on  $\Gamma_{obs}$  we put  $\psi = 0$ . The numerical results are depicted on the reflected domain in Fig. 3, where we observe flow patterns qualitatively agreeing with the expected results (see e.g. [12]).

### 6 Concluding Remarks

In the present paper, we have analyzed a mixed finite element method to approximate a stream-function–vorticity variational formulation for the Brinkman problem in axisymmetric domains, which has been shown to be well-posed using standard arguments for mixed problems. The formulation was discretized by means of continuous piecewise polynomials of degree  $k \geq 1$  for all the unknowns. We proved an  $O(h^k)$  convergence with respect to the mesh size in the natural  $H^1$ -norm, as well as an  $O(h^{k+1})$  order in  $L^2$ -norm by a duality argument, and all estimates are uniform with respect to the fluid viscosity  $\nu$ . Finally, we reported numerical results that confirm the numerical analysis of the proposed method. A distinctive feature of this method is that it allows discrete velocities which are auto-



**Fig. 3** Example 3: meridional domain (where the symmetry axis is  $r = 0$ ) with obstacle and sketch of boundary decomposition (*top panel*); and approximate solutions for stream-function, vorticity, and pressure

matically divergence-free. Extensions of this work include the nonlinear Navier-Stokes equations and coupling with transport problems arising from multiphase flow descriptions.

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