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A Lagrange multiplier-based method for Stokes–generalized poroelasticity interface problems

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We study a mathematical model describing the interaction between a fluid and a poroelastic structure, along with its numerical approximation. The fluid domain is governed by the unsteady incompressible Stokes equations, while the poroelastic region is modeled using the linearized poro-hyperelastic equations. Within this region, the Brinkman equation is employed to describe fluid flow through the porous medium, incorporating inertial effects into the fluid dynamics. A generalized poromechanical framework is adopted to incorporate these inertial effects in accordance with thermodynamic principles. An alternative formulation is used in which the primary variables are the elastic stress and structural velocity. This formulation serves as a mathematical tool to establish the unique solvability of the governing equations, with the existence proof relying on an auxiliary multi-valued parabolic problem. For the numerical approximation, we propose a Lagrange multiplier-based mixed finite element method and demonstrate the well-posedness of both semi-discrete and fully discrete problems. Furthermore, we derive *a priori* error estimates for both discretization schemes. Numerical experiments validate the theoretical convergence rates. Finally, we apply the proposed monolithic scheme to simulate two-dimensional phenomena arising in geophysical flows and brain biomechanics.

Keywords: Coupled poro-hyperelasticity/free-flow problem, saddle-point formulations, error estimates, mixed finite element methods.

1. Introduction

The interaction between a free-flowing fluid and a deformable porous medium poses a challenging multiphysics problem. The complexity arises from the disparate material properties across geometric interfaces, and examples of such challenges widely exist in industrial problems, including groundwater flow in fractured aquifers, oil and gas extraction, and filter design. These processes are also found in biomechanical applications such as perfused living tissues [5], transport of lipids and drugs in blood vessel walls [28, 49], water transport and drug delivery in the brain [45, 47], and addressing ocular diseases like glaucoma [26, 40] or diagnosing fibrosis in the lungs [14].

There is an extensive literature on fluid poroelastic structure interaction (FPSI) problems, which exhibit features of both coupled Stokes–Darcy interfacial flows [30, 35, 39, 51] and fluid structure interaction (FSI)

[13, 32]. In FPSI scenarios, the behavior of the free fluid is described by the Stokes (Navier–Stokes) equations, while the flow within deformable porous media is governed by the Biot system of poroelasticity [15]. This system couples an elasticity equation for the deformation of the porous solid with a Darcy law describing fluid flow, ensuring mass conservation within the pore network. The coupling of the Stokes and Biot regions involves interface conditions ensuring the continuity of normal flux, the Beavers–Joseph–Saffman (BJS) slip condition for tangential velocity, the balance of forces, and the continuity of normal stress. One of the first theoretical studies of the Stokes–Biot model is provided in [43], where well-posedness is demonstrated using semigroup methods. A numerical investigation is presented in [9], employing the variational multiscale finite element method (FEM) and proposing both monolithic and iterative partitioned methods. In [23], a non-iterative operator-splitting method is developed for the coupled Navier–Stokes–Biot model. Additionally, readers are referred to [22], where a loosely coupled partitioned approach is utilized based on Nitsche's method. An analysis of a Lagrange multiplier formulation for imposing normal flux continuity is provided in [4], and for an extension to non-Newtonian fluids, see [2]. A Stokes–Biot model with a total pressure formulation that does not require Lagrange multipliers or a Nitsche parameter for imposing interface conditions is studied in [40]. The well-posedness of a Stokes–Biot system with a multilayered porous medium using Rothe's method is obtained in [16].

More recently, researchers have focused on utilizing the framework of Biot theory with finite strain to develop general poromechanics formulations [27, 50]. A thermodynamically consistent poromechanics formulation was introduced in [24, 27]. In particular, [27] develops a model for the general case of large deformations, illustrating that fluid and solid phases coexist at every point in the computational domain. The nonlinear constitutive behavior and the geometric effects of large deformations were avoided in [24] by considering a linearized version of the aforementioned poromechanic model under the assumption of small deformations. Such a model is known as a linearized poro-hyperelastic or generalized poroelastic model, and its existence and uniqueness are discussed in [12].

In this paper, we present mathematical and numerical analyses for the fully dynamic Stokes–generalized poroelasticity system using a velocity–pressure formulation for Stokes, a displacement formulation for elasticity, and a velocity–pressure formulation for Brinkman. This formulation, which has not been studied in the literature, is attractive due to its relative simplicity, and it primarily follows the approach discussed in [2]. In the porous region, we employ the Brinkman model for fluid flow to ensure mass conservation within the pores, integrating viscous effects into the fluid dynamics in accordance with thermodynamic principles [11]. This model offers a more accurate alternative to the conventional Stokes–Biot model, particularly suitable for thermodynamically consistent scenarios involving variable porosity. We enforce the continuity of the normal velocity on the interface using a Lagrange multiplier. The original weak formulation contains two types of non-coercive terms: one that prevents an energy norm bound for the relative velocity, and another arising from the time derivatives of the displacement. These terms result in a time-dependent system, which introduces analytical challenges. To address these challenges, we reformulate the problem using a mixed elasticity approach, where the elastic stress and structural velocity serve as the primary variables [43]. This leads to a system akin to a degenerate evolution saddle-point problem, which we recast as a parabolic-type system following the methodology in [44]. By applying classical semigroup theory [42], we establish the existence of a solution for the reformulated system, and equivalently for the original formulation, by invoking the invertibility of the stress-strain relation. Subsequently, we return to the original formulation to conduct a stability and error analysis of semi- and fully-discrete FE approximations, employing finite differences in time and FEs in space. We employ a Taylor–Hood FE family for the porous medium to approximate relative velocity, solid displacement, and pressure, ensuring the velocity and displacement approximations are of higher order than the pressure [11]. For the Stokes medium, we use Taylor–Hood elements to approximate fluid velocity and pressure and introduce a conforming Lagrange-multiplier discretization to satisfy discrete inf–sup conditions, ensuring accuracy on nonmatching interface grids as well. Although the convergence rate for both the relative velocity and displacement is suboptimal, this result was expected due to the inability to establish an error

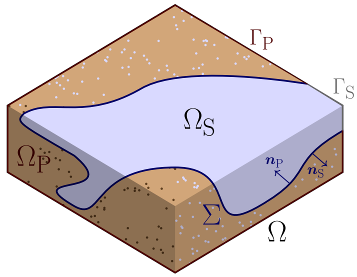


FIG. 2.1. Illustration of typical subdomain configurations, boundaries, and interface in a poromechanic/free flow problem setting.

bound in the energy norm, as mentioned earlier. This work presents a novel contribution to the theoretical and numerical analysis of interface-coupled problems.

Outline of the paper. The rest of the paper is organized as follows. Section 2 introduces the notation, preliminaries, and the mathematical model. Section 3 is devoted to the weak formulation and establishes the well-posedness of the original formulation, along with the corresponding stability bounds. The semi-discrete approximation and its well-posedness analysis are presented in Section 4. Section 5 provides the analysis of the fully discrete scheme. In Section 6, we present numerical experiments to validate the theoretical results on spatio-temporal convergence. Additionally, we simulate: (i) a typical reservoir model using real data, (ii) a scenario with large interface displacements, and (iii) a simplified but physiologically relevant brain biomechanics problem. Finally, we conclude in Section 7 with a summary of our results and directions for future work. In Appendix A, we establish the well-posedness of an alternative formulation; providing the mathematical foundation for the original formulation discussed in Section 3.

2. Multiphysics formulation of the model problem

Notation and preliminaries. Throughout this manuscript, we utilize the classical Sobolev spaces $L^2(\Omega)$ and $H^1(\Omega)$, equipped with their respective norms $\|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{H^1(\Omega)}$. The L^2-inner product is denoted as $(\cdot,\cdot)$, and for any arbitrary Hilbert space H, we represent the duality pairing with its dual space H' as $\langle\cdot,\cdot\rangle_{H',H}$. Also, we use the convention of denoting scalars, vectors, and tensors as a, $\mathbf{a}$, and $\mathbb{A}$, respectively. Finally, we define the Bochner spaces $L^p(0,T;X)$ and $L^\infty(0,T;X)$ for any Banach space X, with norms given by $\left(\int_0^T \|x(s)\|_X^q \,\mathrm{d}s\right)^{1/q}$ and $\sup_{s\in(0,T)} \|x(s)\|_X$, respectively. We consider weak time derivatives in $W^{k,p}(0,T;X)$, defined as $\{x \in L^p(0,T;X) : D^\alpha x \in L^p(0,T;X) \text{ for all } n \in \mathbb{N}, |\alpha| \leq k\}$, where $1 \leq p \leq \infty$. For simplicity, C denotes a generic positive constant independent of the mesh size h but possibly dependent on model parameters. We also use ε for arbitrary constants (with different values in different contexts) arising from Young's inequality. Inequalities with constants independent of h are denoted by $\lesssim$ or $\gtrsim$, omitting the constants.

Governing equations. Let us consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, together with a partition into non-overlapping and connected subdomains Ω_S, Ω_P representing zones occupied by a free fluid region with flow governed by the Stokes equations and a poroelastic material governed by the general thermodynamically consistent linearized poro-hyperelastic system, respectively. The interface between the two subdomains is denoted as $\Sigma = \partial\Omega_S \cap \partial\Omega_P$. The boundary of the domain Ω is separated in terms of the boundaries of two individual subdomains, that is, $\partial\Omega = \Gamma_S \cup \Gamma_P$ (see a sketch in Figure 2.1).

The free fluid region Ω_S is governed by the Stokes equations, with the primary variables being the fluid velocity $\boldsymbol{u}_f^S$ and the fluid pressure p^S:

$$\rho_f \partial_t \boldsymbol{u}_f^S - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}_f^S\left(\boldsymbol{u}_f^S, p^S\right) = \boldsymbol{f}_S \qquad \text{in } \Omega_S \times (0,T], \tag{2.1a}$$

$$\nabla\cdot\boldsymbol{u}_f^{\mathrm{S}}=0 \qquad \text{in } \Omega_{\mathrm{S}}\times(0,T], \tag{2.1b}$$

where $T>0$ is the final time. Here, $\boldsymbol{\varepsilon}(\boldsymbol{u}_f^{\mathrm{S}})=\frac{1}{2}(\nabla\boldsymbol{u}_f^{\mathrm{S}}+(\nabla\boldsymbol{u}_f^{\mathrm{S}})^T)$ denotes the deformation strain tensor; $\boldsymbol{\sigma}_f^{\mathrm{S}}(\boldsymbol{u}_f^{\mathrm{S}},p^{\mathrm{S}})=2\mu_f\boldsymbol{\varepsilon}(\boldsymbol{u}_f^{\mathrm{S}})-p^{\mathrm{S}}\mathbf{I}$, stress tensor; $\boldsymbol{f}_{\mathrm{S}}:(0,T]\to\mathbf{L}^2(\Omega_{\mathrm{S}})$, external load; μ_f, fluid viscosity, and ρ_f, fluid density.

The poroelastic region Ω_{P} is governed by the linearized poro-hyperelastic model (which includes viscoelastic properties), with the primary variables being the absolute fluid velocity $\boldsymbol{u}_f^{\mathrm{P}}$, interstitial pressure p^{P}, solid displacement $\boldsymbol{y}_s^{\mathrm{P}}$, and solid velocity $\boldsymbol{u}_s^{\mathrm{P}}$:

$$\rho_f\phi\partial_t\boldsymbol{u}_f^{\mathrm{P}}-\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}_f^{\mathrm{P}}\left(\boldsymbol{u}_f^{\mathrm{P}},p^{\mathrm{P}}\right)-p^{\mathrm{P}}\nabla\phi+\phi^2\kappa^{-1}\left(\boldsymbol{u}_f^{\mathrm{P}}-\boldsymbol{u}_s^{\mathrm{P}}\right)=2\rho_f\phi\boldsymbol{f}_{\mathrm{P}}+\theta\boldsymbol{u}_f^{\mathrm{P}} \qquad \text{in } \Omega_{\mathrm{P}}\times(0,T], \tag{2.2a}$$

$$(1-\phi)^2K^{-1}\partial_t p^{\mathrm{P}}+\nabla\cdot\left(\phi\boldsymbol{u}_f^{\mathrm{P}}+(1-\phi)\boldsymbol{u}_s^{\mathrm{P}}\right)=\rho_f^{-1}\theta \qquad \text{in } \Omega_{\mathrm{P}}\times(0,T], \tag{2.2b}$$

$$\rho_s(1-\phi)\partial_t\boldsymbol{u}_s^{\mathrm{P}}-\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}_s^{\mathrm{P}}(\boldsymbol{y}_s^{\mathrm{P}},p^{\mathrm{P}})-p^{\mathrm{P}}\nabla(1-\phi)-\frac{\phi^2}{\kappa}\left(\boldsymbol{u}_f^{\mathrm{P}}-\boldsymbol{u}_s^{\mathrm{P}}\right)=\rho_s(1-\phi)\boldsymbol{f}_{\mathrm{P}} \qquad \text{in } \Omega_{\mathrm{P}}\times(0,T], \tag{2.2c}$$

$$\rho_p\boldsymbol{u}_s^{\mathrm{P}}=\rho_p\partial_t\boldsymbol{y}_s^{\mathrm{P}} \qquad \text{in } \Omega_{\mathrm{P}}\times(0,T]. \tag{2.2d}$$

Equation (2.2a) is the conservation of momentum for the fluid (generalized Stokes law with Brinkman effect); (2.2b) represents mass conservation; (2.2c) is the conservation of momentum of the solid phase, and the last one (multiplied by ρ_p to maintain the symmetry of the block system) relates solid displacement and velocity. The relevant parameters are $\phi=\phi(\boldsymbol{x})$, porosity; ρ_f,ρ_s, fluid/solid density; μ_f, fluid viscosity; κ, permeability tensor; $\boldsymbol{f}_{\mathrm{P}}:(0,T]\to\mathbf{L}^2(\Omega_{\mathrm{P}})$, external load; $\theta:(0,T]\to L^2(\Omega_{\mathrm{P}})$, fluid source/sink; K, bulk modulus; and λ_p,μ_p, Lamé parameters. The parameters $\rho_s,\rho_f,\mu_f,\lambda_p,\mu_p$ are assumed to be positive constants. Let us now define stress tensors in the poroelastic sub-domain as

$$\boldsymbol{\sigma}_f^{\mathrm{P}}\left(\boldsymbol{u}_f^{\mathrm{P}},p^{\mathrm{P}}\right):=2\mu_f\phi\boldsymbol{\varepsilon}\left(\boldsymbol{u}_f^{\mathrm{P}}\right)-\phi p^{\mathrm{P}}\mathbf{I}, \tag{2.3a}$$

$$\boldsymbol{\sigma}^{\mathrm{P}}\left(\boldsymbol{y}_s^{\mathrm{P}}\right):=2\mu_p\boldsymbol{\varepsilon}\left(\boldsymbol{y}_s^{\mathrm{P}}\right)+\lambda_p\nabla\cdot\boldsymbol{y}_s^{\mathrm{P}}\mathbf{I}, \tag{2.3b}$$

$$\boldsymbol{\sigma}_s^{\mathrm{P}}\left(\boldsymbol{y}_s^{\mathrm{P}},p^{\mathrm{P}}\right):=\boldsymbol{\sigma}^{\mathrm{P}}-(1-\phi)p^{\mathrm{P}}\mathbf{I}. \tag{2.3c}$$

Remark 2.1. *As discussed in [9], directly using the absolute velocity in the weak formulation results in unbalanced conditions at the interface in linearized poroelastic models. This motivates the use of the relative velocity between the fluid and solid phases $\boldsymbol{u}_r^{\mathrm{P}}=\boldsymbol{u}_f^{\mathrm{P}}-\boldsymbol{u}_s^{\mathrm{P}}$, and also combine (2.2a) and (2.2c) to transform (2.2c) into a total momentum equation.*

Furthermore, we adopt the notation $\boldsymbol{\sigma}_f^{\mathrm{S}}$, $\boldsymbol{\sigma}_f^{\mathrm{P}}$, and $\boldsymbol{\sigma}_s^{\mathrm{P}}$ to denote $\boldsymbol{\sigma}_f^{\mathrm{S}}\left(\boldsymbol{u}_f^{\mathrm{P}},p^{\mathrm{S}}\right)$, $\boldsymbol{\sigma}_f^{\mathrm{P}}\left(\boldsymbol{u}_r^{\mathrm{P}}+\boldsymbol{u}_s^{\mathrm{P}},p^{\mathrm{P}}\right)$, and $\boldsymbol{\sigma}_s^{\mathrm{P}}\left(\boldsymbol{y}_s^{\mathrm{P}},p^{\mathrm{P}}\right)$, respectively. The resulting model is then defined as

$$\rho_f\phi(\partial_t\boldsymbol{u}_r^{\mathrm{P}}+\partial_t\boldsymbol{u}_s^{\mathrm{P}})-\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}_f^{\mathrm{P}}-p^{\mathrm{P}}\nabla\phi+\phi^2\kappa^{-1}\boldsymbol{u}_r^{\mathrm{P}}-\theta(\boldsymbol{u}_s^{\mathrm{P}}+\boldsymbol{u}_r^{\mathrm{P}})=2\rho_f\phi\boldsymbol{f}_{\mathrm{P}} \qquad \text{in } \Omega_{\mathrm{P}}\times(0,T], \tag{2.4a}$$

$$(1-\phi)^2K^{-1}\partial_t p^{\mathrm{P}}+\partial_t\left(\nabla\cdot\boldsymbol{y}_s^{\mathrm{P}}\right)+\nabla\cdot\left(\phi\boldsymbol{u}_r^{\mathrm{P}}\right)=\rho_f^{-1}\theta \qquad \text{in } \Omega_{\mathrm{P}}\times(0,T], \tag{2.4b}$$

$$\rho_f\phi\partial_t\boldsymbol{u}_r^{\mathrm{P}}+\rho_p\partial_t\boldsymbol{u}_s^{\mathrm{P}}-\nabla\cdot\boldsymbol{\sigma}_f^{\mathrm{P}}-\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}_s^{\mathrm{P}}-\theta\boldsymbol{u}_r^{\mathrm{P}}-\theta\boldsymbol{u}_s^{\mathrm{P}}=\rho_p\boldsymbol{f}_{\mathrm{P}}+\rho_f\phi\boldsymbol{f}_{\mathrm{P}} \qquad \text{in } \Omega_{\mathrm{P}}\times(0,T], \tag{2.4c}$$

$$\rho_p\boldsymbol{u}_s^{\mathrm{P}}=\rho_p\partial_t\boldsymbol{y}_s^{\mathrm{P}} \qquad \text{in } \Omega_{\mathrm{P}}\times(0,T], \tag{2.4d}$$

where $\rho_p=\rho_s(1-\phi)+\rho_f\phi$ denotes the density of the saturated porous medium. This system is complemented by the following set of boundary conditions

$$\boldsymbol{u}_f^{\mathrm{S}}=\mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{S}}\times(0,T], \quad \boldsymbol{y}_s^{\mathrm{P}}=\mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{P}}\times(0,T], \quad \boldsymbol{u}_r^{\mathrm{P}}=\mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{P}}\times(0,T].$$

The interface conditions on the fluid–poroelastic interface Σ consist of mass conservation (2.5a), balance of normal stresses (2.5b), and balance of contact forces (2.5c). Conditions (2.5d)–(2.5e) together represent the Beavers–Joseph–Saffman (BJS) slip condition modeling tangential friction, with (2.5d) involving both fluid and solid velocities, and (2.5e) involving only the poroelastic fluid velocity on Σ:

$$\boldsymbol{u}_f^{\mathrm{S}} \cdot \boldsymbol{n}_{\mathrm{S}} + \left(\partial_t \boldsymbol{y}_s^{\mathrm{P}} + \boldsymbol{u}_r^{\mathrm{P}}\right) \cdot \boldsymbol{n}_{\mathrm{P}} = 0 \qquad \text{on} \quad \Sigma \times (0,T], \tag{2.5a}$$

$$-\left(\boldsymbol{\sigma}_f^{\mathrm{S}} \boldsymbol{n}_{\mathrm{S}}\right) \cdot \boldsymbol{n}_{\mathrm{S}} = -\left(\boldsymbol{\sigma}_f^{\mathrm{P}} \boldsymbol{n}_{\mathrm{P}}\right) \cdot \boldsymbol{n}_{\mathrm{P}} \qquad \text{on} \quad \Sigma \times (0,T], \tag{2.5b}$$

$$\boldsymbol{\sigma}_f^{\mathrm{S}} \boldsymbol{n}_{\mathrm{S}} + \boldsymbol{\sigma}_f^{\mathrm{P}} \boldsymbol{n}_{\mathrm{P}} + \boldsymbol{\sigma}_s^{\mathrm{P}} \boldsymbol{n}_{\mathrm{P}} = \mathbf{0} \qquad \text{on} \quad \Sigma \times (0,T], \tag{2.5c}$$

$$-\left(\boldsymbol{\sigma}_f^{\mathrm{S}} \boldsymbol{n}_{\mathrm{S}}\right) \cdot \boldsymbol{\tau}_{f,j} = \mu_f \alpha_{\mathrm{BJS}} \sqrt{Z_j^{-1}} \left(\boldsymbol{u}_f^{\mathrm{S}} - \partial_t \boldsymbol{y}_s^{\mathrm{P}}\right) \cdot \boldsymbol{\tau}_{f,j} \qquad \text{on} \quad \Sigma \times (0,T], \tag{2.5d}$$

$$-\left(\boldsymbol{\sigma}_f^{\mathrm{P}} \boldsymbol{n}_{\mathrm{P}}\right) \cdot \boldsymbol{\tau}_{f,j} = \mu_f \alpha_{\mathrm{BJS}} \sqrt{Z_j^{-1}} \boldsymbol{u}_r^{\mathrm{P}} \cdot \boldsymbol{\tau}_{f,j} \qquad \text{on} \quad \Sigma \times (0,T], \tag{2.5e}$$

where $\boldsymbol{n}_{\mathrm{S}}$ and $\boldsymbol{n}_{\mathrm{P}}$ are the outward unit normal vectors to Ω_{S} and Ω_{P}, respectively, $\boldsymbol{\tau}_{f,j}, 1 \le j \le d-1$, is an orthogonal system of unit tangent vectors on Σ, we denote $Z_j = \left(\kappa \boldsymbol{\tau}_{f,j}\right) \cdot \boldsymbol{\tau}_{f,j}$, and $\alpha_{\mathrm{BJS}} \ge 0$ is an experimentally determined friction coefficient. We further set the initial conditions:

$$\boldsymbol{u}_f^{\mathrm{S}}(\boldsymbol{x},0) = \boldsymbol{u}_{f,0}(\boldsymbol{x}),\ \boldsymbol{u}_r^{\mathrm{P}}(\boldsymbol{x},0) = \boldsymbol{u}_{r,0}(\boldsymbol{x}),\ \boldsymbol{y}_s^{\mathrm{P}}(\boldsymbol{x},0) = \boldsymbol{y}_{s,0}(\boldsymbol{x}),\ \boldsymbol{u}_s^{\mathrm{P}}(\boldsymbol{x},0) = \boldsymbol{u}_{s,0}(\boldsymbol{x}),\ p^{\mathrm{P}}(\boldsymbol{x},0) = p^{\mathrm{P},0}(\boldsymbol{x}). \tag{2.6}$$

3. Weak formulation

We consider the following functional spaces (endowed with the standard norms) as

$$\mathbf{V}_f = \left\{\boldsymbol{u}_f^{\mathrm{S}} \in \mathbf{H}^1(\Omega_{\mathrm{S}}) : \boldsymbol{u}_f^{\mathrm{S}} = \mathbf{0} \text{ on } \Gamma_{\mathrm{S}}\right\}, \quad \mathrm{W}_f = L^2(\Omega_{\mathrm{S}}), \quad \mathbf{V}_r = \left\{\boldsymbol{u}_r^{\mathrm{P}} \in \mathbf{H}^1(\Omega_{\mathrm{P}}) : \boldsymbol{u}_r^{\mathrm{P}} = \mathbf{0} \text{ on } \Gamma_{\mathrm{P}}^{\mathrm{D}}\right\},$$
$$\mathrm{W}_p = L_0^2(\Omega_{\mathrm{P}}), \quad \mathbf{V}_s = \left\{\boldsymbol{y}_s^{\mathrm{P}} \in \mathbf{H}^1(\Omega_{\mathrm{P}}) : \boldsymbol{y}_s^{\mathrm{P}} = \mathbf{0} \text{ on } \Gamma_{\mathrm{P}}\right\}, \quad \mathbf{W}_s = \mathbf{L}^2(\Omega_{\mathrm{P}}),$$

and denote the product space as $\vec{\mathbf{X}} =: \mathbf{V}_f \times \mathbf{V}_r \times \mathbf{V}_s \times \mathbf{W}_s \times \mathrm{W}_f \times \mathrm{W}_p \times \Lambda$.

Let us define, for all $\boldsymbol{u}_f^{\mathrm{S}}, \boldsymbol{v}_f^{\mathrm{S}} \in \mathbf{V}_f, \boldsymbol{u}, \boldsymbol{v} \in \mathbf{H}^1(\Omega_{\mathrm{P}}), \boldsymbol{y}_s^{\mathrm{P}}, \boldsymbol{w}_s^{\mathrm{P}} \in \mathbf{V}_s$, the operators and associated bilinear forms related to the Stokes, Brinkman, and elasticity operators, respectively:

$$\mathscr{A}_f^{\mathrm{S}} : \mathbf{V}_f \to \mathbf{V}_f', \quad \langle \mathscr{A}_f^{\mathrm{S}} \boldsymbol{u}_f^{\mathrm{S}}, \boldsymbol{v}_f^{\mathrm{S}} \rangle = a_f^{\mathrm{S}}\left(\boldsymbol{u}_f^{\mathrm{S}}, \boldsymbol{v}_f^{\mathrm{S}}\right) := \left(2\mu_f \boldsymbol{\varepsilon}(\boldsymbol{u}_f^{\mathrm{S}}), \boldsymbol{\varepsilon}\left(\boldsymbol{v}_f^{\mathrm{S}}\right)\right)_{\Omega_{\mathrm{S}}},$$
$$\mathscr{A}_f^{\mathrm{P}} : \mathbf{H}^1(\Omega_{\mathrm{P}}) \to \mathbf{H}^{-1}(\Omega_{\mathrm{P}}), \quad \langle \mathscr{A}_f^{\mathrm{P}} \boldsymbol{u}, \boldsymbol{v} \rangle = a_f^{\mathrm{P}}(\boldsymbol{u}, \boldsymbol{v}) := \left(2\mu_f \phi \boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v})\right)_{\Omega_{\mathrm{P}}},$$
$$\mathscr{A}_s^{\mathrm{P}} : \mathbf{V}_s \to \mathbf{V}_s', \quad \langle \mathscr{A}_s^{\mathrm{P}} \boldsymbol{y}_s^{\mathrm{P}}, \boldsymbol{w}_s^{\mathrm{P}} \rangle = a_s^{\mathrm{P}}\left(\boldsymbol{y}_s^{\mathrm{P}}, \boldsymbol{w}_s^{\mathrm{P}}\right) := \left(2\mu_p \boldsymbol{\varepsilon}\left(\boldsymbol{y}_s^{\mathrm{P}}\right), \boldsymbol{\varepsilon}\left(\boldsymbol{w}_s^{\mathrm{P}}\right)\right)_{\Omega_{\mathrm{P}}} + \left(\lambda_p \nabla \cdot \boldsymbol{y}_s^{\mathrm{P}}, \nabla \cdot \boldsymbol{w}_s^{\mathrm{P}}\right)_{\Omega_{\mathrm{P}}}.$$

In addition, for all $q^{\mathrm{S}} \in \mathrm{W}_f, q^{\mathrm{P}} \in \mathrm{W}_p, \boldsymbol{v}_f^{\mathrm{S}} \in \mathbf{V}_f, \boldsymbol{w}_s^{\mathrm{P}} \in \boldsymbol{v}_s^{\mathrm{P}}, \boldsymbol{v}_r^{\mathrm{P}} \in \mathbf{V}_r, \boldsymbol{w}, \boldsymbol{\zeta} \in \mathbf{W}_s$, we define

$$\mathscr{B}^{\mathrm{S}} : \mathbf{V}_f \to W_s', \quad \langle \mathscr{B}^{\mathrm{S}} \boldsymbol{v}_f^{\mathrm{S}}, q^{\mathrm{S}} \rangle = b^{\mathrm{S}}(\boldsymbol{v}_f^{\mathrm{S}}, q^{\mathrm{S}}) := -(\nabla \cdot \boldsymbol{v}_f^{\mathrm{S}}, q^{\mathrm{S}})_{\Omega_{\mathrm{S}}},$$
$$\mathscr{B}_s^{\mathrm{P}} : \mathbf{V}_s \to \mathrm{W}_p', \quad \langle \mathscr{B}_s^{\mathrm{P}} \boldsymbol{w}_s^{\mathrm{P}}, q^{\mathrm{P}} \rangle = b_s^{\mathrm{P}}(\boldsymbol{w}_s^{\mathrm{P}}, q^{\mathrm{P}}) := -(\nabla \cdot \boldsymbol{w}_s^{\mathrm{P}}, q^{\mathrm{P}})_{\Omega_{\mathrm{P}}},$$
$$\mathscr{B}_f^{\mathrm{P}} : \mathbf{V}_r \to \mathrm{W}_p', \quad \langle \mathscr{B}_f^{\mathrm{P}} \boldsymbol{v}_r^{\mathrm{P}}, q^{\mathrm{P}} \rangle = b_f^{\mathrm{P}}(\boldsymbol{v}_r^{\mathrm{P}}, q^{\mathrm{P}}) := -(\nabla \cdot (\phi \boldsymbol{v}_r^{\mathrm{P}}), q^{\mathrm{P}})_{\Omega_{\mathrm{P}}},$$
$$\mathscr{M}_\xi : \mathbf{W}_s \to \mathbf{W}_s', \quad \langle \mathscr{M}_\xi \boldsymbol{w}, \boldsymbol{\zeta} \rangle = m_\xi(\boldsymbol{w}, \boldsymbol{\zeta}) := (\xi \boldsymbol{w}, \boldsymbol{\zeta}).$$

Integration by parts in (2.1a), (2.4a) and (2.4c) leads to the interface term

$$I_\Sigma := -\langle \boldsymbol{\sigma}_f^{\mathrm{S}} \boldsymbol{n}_{\mathrm{S}}, \boldsymbol{v}_f^{\mathrm{S}} \rangle_\Sigma - \langle \boldsymbol{\sigma}_f^{\mathrm{P}} \boldsymbol{n}_{\mathrm{P}}, \boldsymbol{w}_s^{\mathrm{P}} \rangle_\Sigma - \langle \boldsymbol{\sigma}_s^{\mathrm{P}} \boldsymbol{n}_{\mathrm{P}}, \boldsymbol{w}_s^{\mathrm{P}} \rangle_\Sigma - \langle \boldsymbol{\sigma}_f^{\mathrm{P}} \boldsymbol{n}_{\mathrm{P}}, \boldsymbol{v}_r^{\mathrm{P}} \rangle_\Sigma.$$

Using the interface condition (2.5b) we set $\lambda := -(\boldsymbol{\sigma}_f^{\mathrm{S}} \boldsymbol{n}_{\mathrm{S}}) \cdot \boldsymbol{n}_{\mathrm{S}} = -(\boldsymbol{\sigma}_f^{\mathrm{P}} \boldsymbol{n}_{\mathrm{P}}) \cdot \boldsymbol{n}_{\mathrm{P}}$ on Σ, which is used as a Lagrange multiplier to impose mass conservation on the interface. Utilizing the BJS conditions (2.5d)-(2.5e) and the balance of stress (2.5b)-(2.5c), we obtain

$$\begin{aligned}
I_\Sigma = &-\int_\Sigma (\boldsymbol{\sigma}_f^{\mathrm{P}}\boldsymbol{n}_{\mathrm{P}})\boldsymbol{n}_{\mathrm{P}}(\boldsymbol{n}_{\mathrm{S}}\cdot\boldsymbol{v}_f^{\mathrm{S}}+\boldsymbol{n}_{\mathrm{P}}\cdot\boldsymbol{v}_r^{\mathrm{P}}+\boldsymbol{n}_{\mathrm{P}}\cdot\boldsymbol{w}_s^{\mathrm{P}})\,\mathrm{d}s-\int_\Sigma(\boldsymbol{\sigma}_f^{\mathrm{P}}\boldsymbol{n}_{\mathrm{P}})\boldsymbol{\tau}_{f,j}(\boldsymbol{\tau}_{f,j}\cdot\boldsymbol{v}_r^{\mathrm{P}})\,\mathrm{d}s\\
&-\int_\Sigma(\boldsymbol{\sigma}_f^{\mathrm{S}}\boldsymbol{n}_{\mathrm{S}})\boldsymbol{\tau}_{f,j}\left(\boldsymbol{v}_f^{\mathrm{S}}-\boldsymbol{w}_s^{\mathrm{P}}\right)\cdot\boldsymbol{\tau}_{f,j}\,\mathrm{d}s,\\
=:&\, a_{\mathrm{BJS}}\left(\boldsymbol{u}_f^{\mathrm{S}},\partial_t\boldsymbol{y}_s^{\mathrm{P}};\boldsymbol{v}_f^{\mathrm{S}},\boldsymbol{w}_s^{\mathrm{P}}\right)+b_{\mathrm{BJS}}\left(\boldsymbol{u}_r^{\mathrm{P}};\boldsymbol{v}_r^{\mathrm{P}}\right)+b_\Gamma\left(\boldsymbol{v}_f^{\mathrm{S}},\boldsymbol{v}_r^{\mathrm{P}},\boldsymbol{w}_s^{\mathrm{P}};\lambda\right)\quad\forall\boldsymbol{u}_f^{\mathrm{S}},\boldsymbol{v}_f^{\mathrm{S}}\in\mathbf{V}_f,\boldsymbol{y}_s^{\mathrm{P}},\boldsymbol{w}_s^{\mathrm{P}}\in\mathbf{V}_s,
\end{aligned}$$

together with the definitions

$$a_{\mathrm{BJS}}\left(\boldsymbol{u}_f^{\mathrm{S}},\partial_t\boldsymbol{y}_s^{\mathrm{P}};\boldsymbol{v}_f^{\mathrm{S}},\boldsymbol{w}_s^{\mathrm{P}}\right):=\sum_{j=1}^{d-1}\int_\Sigma\mu_f\alpha_{\mathrm{BJS}}\sqrt{Z_j^{-1}}\left(\boldsymbol{u}_f^{\mathrm{S}}-\boldsymbol{y}_s^{\mathrm{P}}\right)\cdot\boldsymbol{\tau}_{f,j}\left(\boldsymbol{v}_f^{\mathrm{S}}-\boldsymbol{w}_s^{\mathrm{P}}\right)\cdot\boldsymbol{\tau}_{f,j}\,\mathrm{d}s,$$

$$b_{\mathrm{BJS}}\left(\boldsymbol{u}_r^{\mathrm{P}};\boldsymbol{v}_r^{\mathrm{P}}\right):=\sum_{j=1}^{d-1}\int_\Sigma\mu_f\alpha_{\mathrm{BJS}}\sqrt{Z_j^{-1}}\left(\boldsymbol{u}_r^{\mathrm{P}}\cdot\boldsymbol{\tau}_{f,j}\right)\left(\boldsymbol{v}_r^{\mathrm{P}}\cdot\boldsymbol{\tau}_{f,j}\right)\,\mathrm{d}s,\quad b_\Gamma\left(\boldsymbol{v}_f^{\mathrm{S}},\boldsymbol{v}_r^{\mathrm{P}},\boldsymbol{w}_s^{\mathrm{P}};\lambda\right):=\left\langle\boldsymbol{v}_f^{\mathrm{S}}\cdot\boldsymbol{n}_{\mathrm{S}}+\left(\boldsymbol{v}_r^{\mathrm{P}}+\boldsymbol{w}_s^{\mathrm{P}}\right)\cdot\boldsymbol{n}_{\mathrm{P}},\lambda\right\rangle_\Sigma.$$

We further define

$$\begin{aligned}
|\boldsymbol{u}_f^{\mathrm{S}}-\boldsymbol{y}_s^{\mathrm{P}}|_{\mathrm{BJS}}^2:=a_{\mathrm{BJS}}\left(\boldsymbol{u}_f^{\mathrm{S}},\boldsymbol{y}_s^{\mathrm{P}};\boldsymbol{u}_f^{\mathrm{S}},\boldsymbol{y}_s^{\mathrm{P}}\right)&=\sum_{j=1}^{d-1}\mu_f\alpha_{\mathrm{BJS}}\|Z_j^{-1/4}\left(\boldsymbol{u}_f^{\mathrm{S}}-\boldsymbol{y}_s^{\mathrm{P}}\right)\cdot\boldsymbol{\tau}_{f,j}\|_{0,\Sigma}^2,\\
\left|\boldsymbol{u}_r^{\mathrm{P}}\right|_{\mathrm{BJS}}^2:=b_{\mathrm{BJS}}\left(\boldsymbol{u}_r^{\mathrm{P}};\boldsymbol{v}_r^{\mathrm{P}}\right)&=\sum_{j=1}^{d-1}\mu_f\alpha_{\mathrm{BJS}}\|Z_j^{-1/4}\boldsymbol{u}_r^{\mathrm{P}}\cdot\boldsymbol{\tau}_{f,j}\|_{0,\Sigma}^2.
\end{aligned}$$

Note that for b_Γ to be well-defined, it is necessary that $\lambda\in\Lambda=(\mathbf{V}_r\cdot\boldsymbol{n}_{\mathrm{P}}|_\Sigma)'$, which is the space denoted as $H_{00}^{-1/2}(\Sigma)$ equipped with the norm $\|\cdot\|_{H_{00}^{-1/2}(\Sigma)}$. With the bilinear forms above we define the following operators

$$\begin{aligned}
\mathscr{A}_{fs}^{\mathrm{BJS}}:\mathbf{H}^1(\Omega_{\mathrm{S}})\to\mathbf{H}^{1/2}(\Sigma),&\quad(\mathscr{A}_{fs}^{\mathrm{BJS}}\boldsymbol{u}_f^{\mathrm{S}},\boldsymbol{w}_s^{\mathrm{P}})_\Sigma=a_{\mathrm{BJS}}(\boldsymbol{u}_f^{\mathrm{S}},\mathbf{0};\mathbf{0},\boldsymbol{w}_s^{\mathrm{P}}),\\
\mathscr{A}_{ss}^{\mathrm{BJS}}:\mathbf{H}^1(\Omega_{\mathrm{P}})\to\mathbf{H}^{1/2}(\Sigma),&\quad(\mathscr{A}_{ss}^{\mathrm{BJS}}\boldsymbol{y}_s^{\mathrm{P}},\boldsymbol{w}_s^{\mathrm{P}})_\Sigma=a_{\mathrm{BJS}}(\mathbf{0},\boldsymbol{y}_s^{\mathrm{P}};\mathbf{0},\boldsymbol{w}_s^{\mathrm{P}}),\\
\mathscr{A}_{ff}^{\mathrm{BJS}}:\mathbf{H}^1(\Omega_{\mathrm{S}})\to\mathbf{H}^{1/2}(\Sigma),&\quad(\mathscr{A}_{ff}^{\mathrm{BJS}}\boldsymbol{u}_f^{\mathrm{S}},\boldsymbol{v}_f^{\mathrm{S}})_\Sigma=a_{\mathrm{BJS}}(\boldsymbol{u}_f^{\mathrm{S}},\mathbf{0};\boldsymbol{v}_f^{\mathrm{S}},\mathbf{0}),\\
\mathscr{A}_{rr}^{\mathrm{BJS}}:\mathbf{H}^1(\Omega_{\mathrm{S}})\to\mathbf{H}^{1/2}(\Sigma),&\quad(\mathscr{A}_{rr}^{\mathrm{BJS}}\boldsymbol{u}_r^{\mathrm{P}},\boldsymbol{v}_r^{\mathrm{P}})_\Sigma=b_{\mathrm{BJS}}(\boldsymbol{u}_r^{\mathrm{P}};\boldsymbol{v}_r^{\mathrm{P}}),\\
\mathscr{B}_{f,\Gamma}:\mathbf{V}_f\to H^{1/2}(\Sigma),&\quad\langle\mathscr{B}_{f,\Gamma}\boldsymbol{v}_f^{\mathrm{S}},\mu\rangle_\Sigma=b_\Gamma\left(\boldsymbol{v}_f^{\mathrm{S}},\mathbf{0},\mathbf{0};\mu\right),\\
\mathscr{B}_{p,\Gamma}:\mathbf{V}_r\to H^{1/2}(\Sigma),&\quad\langle\mathscr{B}_{p,\Gamma}\boldsymbol{v}_r^{\mathrm{P}},\mu\rangle_\Sigma=b_\Gamma\left(\mathbf{0},\boldsymbol{v}_r^{\mathrm{P}},\mathbf{0};\mu\right),\\
\mathscr{B}_{s,\Gamma}:\mathbf{V}_s\to H^{1/2}(\Sigma),&\quad\langle\mathscr{B}_{e,\Gamma}\boldsymbol{w}_s^{\mathrm{P}},\mu\rangle_\Sigma=b_\Gamma\left(\mathbf{0},\mathbf{0},\boldsymbol{w}_s^{\mathrm{P}};\mu\right),
\end{aligned}$$

where $(\cdot,\cdot)_\Sigma$ denotes the $L^2(\Sigma)$ inner product of two $H^{1/2}(\Sigma)$ functions [21]. We will use the shorthand notation for trial and test functions $\vec{\boldsymbol{x}}=(\boldsymbol{u}_f^{\mathrm{S}},\boldsymbol{u}_r^{\mathrm{P}},\boldsymbol{y}_s^{\mathrm{P}},\boldsymbol{u}_s^{\mathrm{P}},p^{\mathrm{S}},p^{\mathrm{P}},\lambda)$, $\vec{\boldsymbol{y}}=(\boldsymbol{v}_f^{\mathrm{S}},\boldsymbol{v}_r^{\mathrm{P}},\boldsymbol{w}_s^{\mathrm{P}},\boldsymbol{v}_s^{\mathrm{P}},q^{\mathrm{S}},q^{\mathrm{P}},\mu)$. We define bilinear forms $\mathrm{E},\mathrm{H}:\vec{\mathbf{X}}\times\vec{\mathbf{X}}\to\mathbb{R}$ which contains all terms with and without time derivatives, respectively:

$$\begin{aligned}
\mathrm{E}(\partial_t\vec{\boldsymbol{x}},\vec{\boldsymbol{y}}):=&\,m_{\rho_f}\left(\partial_t\boldsymbol{u}_f^{\mathrm{S}},\boldsymbol{v}_f^{\mathrm{S}}\right)+m_{\rho_f\phi}(\partial_t\boldsymbol{u}_r^{\mathrm{P}},\boldsymbol{v}_r^{\mathrm{P}})+m_{\rho_f\phi}(\partial_t\boldsymbol{u}_s^{\mathrm{P}},\boldsymbol{v}_r^{\mathrm{P}})+m_{\rho_f\phi}(\partial_t\boldsymbol{u}_r^{\mathrm{P}},\boldsymbol{w}_s^{\mathrm{P}})\\
&+m_{\rho_p}(\partial_t\boldsymbol{u}_s^{\mathrm{P}},\boldsymbol{w}_s^{\mathrm{P}})-m_{\rho_p}(\partial_t\boldsymbol{y}_s^{\mathrm{P}},\boldsymbol{v}_s^{\mathrm{P}})+a_f^{\mathrm{P}}(\partial_t\boldsymbol{y}_s^{\mathrm{P}},\boldsymbol{v}_r^{\mathrm{P}})+a_f^{\mathrm{P}}(\partial_t\boldsymbol{y}_s^{\mathrm{P}},\boldsymbol{w}_s^{\mathrm{P}})\\
&+a_{\mathrm{BJS}}\left(0,\partial_t\boldsymbol{y}_s^{\mathrm{P}};\boldsymbol{v}_f^{\mathrm{S}},\boldsymbol{w}_s^{\mathrm{P}}\right)+m_{\frac{(1-\phi)^2}{K}}(\partial_t p^{\mathrm{P}},q^{\mathrm{P}})+b_s^{\mathrm{P}}(q^{\mathrm{P}},\partial_t\boldsymbol{y}_s^{\mathrm{P}})+b_\Gamma\left(0,0,\partial_t\boldsymbol{y}_s^{\mathrm{P}};\mu\right),
\end{aligned}$$

$$
\begin{aligned}
\mathrm{H}(\vec{\boldsymbol{x}},\vec{\boldsymbol{y}}) := {} & a_f^{\mathrm{S}}(\boldsymbol{u}_f^{\mathrm{S}},\boldsymbol{v}_f^{\mathrm{S}}) + a_f^{\mathrm{P}}(\boldsymbol{u}_r^{\mathrm{P}},\boldsymbol{v}_r^{\mathrm{P}}) + a_f^{\mathrm{P}}(\boldsymbol{u}_r^{\mathrm{P}},\boldsymbol{w}_s^{\mathrm{P}}) + a_s^{\mathrm{P}}(\boldsymbol{y}_s^{\mathrm{P}},\boldsymbol{w}_s^{\mathrm{P}}) \\
& - m_\theta(\boldsymbol{u}_r^{\mathrm{P}},\boldsymbol{v}_r^{\mathrm{P}}) - m_\theta(\boldsymbol{u}_s^{\mathrm{P}},\boldsymbol{v}_r^{\mathrm{P}}) - m_\theta(\boldsymbol{u}_r^{\mathrm{P}},\boldsymbol{w}_s^{\mathrm{P}}) - m_\theta(\boldsymbol{u}_s^{\mathrm{P}},\boldsymbol{w}_s^{\mathrm{P}}) \\
& + m_{\phi^2/\kappa}(\boldsymbol{u}_r^{\mathrm{P}},\boldsymbol{v}_r^{\mathrm{P}}) + m_{\rho_p}(\boldsymbol{u}_s^{\mathrm{P}},\boldsymbol{v}_s^{\mathrm{P}}) + b^{\mathrm{S}}(p^{\mathrm{S}},\boldsymbol{v}_f^{\mathrm{S}}) + b_f^{\mathrm{P}}(p^{\mathrm{P}},\boldsymbol{v}_r^{\mathrm{P}}) \\
& + b_s^{\mathrm{P}}(p^{\mathrm{P}},\boldsymbol{w}_s^{\mathrm{P}}) + a_{\mathrm{BJS}}\left(\boldsymbol{u}_f^{\mathrm{S}},0;\boldsymbol{v}_f^{\mathrm{S}},\boldsymbol{w}_s^{\mathrm{P}}\right) + b_{\mathrm{BJS}}\left(\boldsymbol{u}_r^{\mathrm{P}};\boldsymbol{v}_r^{\mathrm{P}}\right) + b_\Gamma\left(\boldsymbol{v}_f^{\mathrm{S}},\boldsymbol{v}_r^{\mathrm{P}},\boldsymbol{w}_s^{\mathrm{P}};\lambda\right) \\
& - b^{\mathrm{S}}(q^{\mathrm{S}},\boldsymbol{u}_f^{\mathrm{S}}) - b_f^{\mathrm{P}}(q^{\mathrm{P}},\boldsymbol{u}_r^{\mathrm{P}}) + b_\Gamma\left(\boldsymbol{u}_f^{\mathrm{S}},\boldsymbol{u}_r^{\mathrm{P}},0;\mu\right),
\end{aligned}
$$

whereas the right-hand side terms are denoted by the form F, given by:

$$
\mathrm{F}(\vec{\boldsymbol{y}}) := (\boldsymbol{f}_{\mathrm{S}},\boldsymbol{v}_f^{\mathrm{S}}) + (2\rho_f\phi\boldsymbol{f}_{\mathrm{P}},\boldsymbol{v}_r^{\mathrm{P}}) + (\rho_p\boldsymbol{f}_{\mathrm{P}},\boldsymbol{w}_s^{\mathrm{P}}) + (\rho_f\phi\boldsymbol{f}_{\mathrm{P}},\boldsymbol{w}_s^{\mathrm{P}}) + (\rho_f^{-1}\theta,q^{\mathrm{P}}).
$$

The weak formulation reads: for $t \in (0,T]$, find $\vec{\boldsymbol{x}}(t) \in \vec{\mathbf{X}}$ for given initial conditions such that

$$
\mathrm{E}(\partial_t\vec{\boldsymbol{x}},\vec{\boldsymbol{y}}) + \mathrm{H}(\vec{\boldsymbol{x}},\vec{\boldsymbol{y}}) = \mathrm{F}(\vec{\boldsymbol{y}}), \tag{3.1}
$$

for all $\vec{\boldsymbol{y}} \in \vec{\mathbf{X}}$, where the balance of normal stress, the BJS conditions, and conservation of momentum (2.5b)-(2.5e) have been utilized naturally in the derivation of the weak formulation, while the essential conservation of mass (2.5a) is imposed weakly (see also [7, 33]).

3.1. *Assumptions*

(H.1) ϕ is such that $\phi, 1/\phi, (1-\phi)$ and $1/(1-\phi)$ belong to $W^{s,r}(\Omega)$ with $s > d/r$, see [11, Lemma 13] and there exist constants $\underline{\phi}$ and $\overline{\phi}$ such that $0 < \underline{\phi} \leq \phi \leq \overline{\phi} < \frac{\rho_s}{\rho_s+\rho_f} < 1$ a.e. in Ω.

(H.2) The source term θ represents a fluid sink. These terms naturally arise in poromechanical mass conservation laws, and handling them is crucial for the well-posedness and stability analysis (see, e.g., [24]).

(H.3) κ is symmetric and positive-definite, i.e., $\exists C_k > 0 : \left(\phi^2\kappa^{-1}\boldsymbol{v}_r^{\mathrm{P}},\boldsymbol{v}_r^{\mathrm{P}}\right) \geq C_k\|\boldsymbol{v}_r^{\mathrm{P}}\|_{0,\Omega_{\mathrm{P}}}^2$, for all $\boldsymbol{v}_r^{\mathrm{P}} \in \mathbf{V}_r$.

From these assumptions, we obtain ellipticity properties to be used in the well-posedness analysis, the stability analysis and the energy estimates. We point out that (H.2) is used to simplify the proof of existence and stability of solutions. However, it can be relaxed by exploiting an exponential scaling of the velocity. Nevertheless, this can turn the analysis much more involved.

Remark 3.1. *We do not analyze the original formulation directly using standard Hilbert space theory or Galerkin-type methods, as commonly employed in Stokes–poroelasticity and Stokes–elasticity couplings. In that context, non-coercivity typically stems from coupling or boundary/interface conditions and can often be handled via an inf-sup condition, yielding bounds in $H(\mathrm{div})$. In contrast, our formulation involves a more severe form of non-coercivity: the energy terms involving $\boldsymbol{u}_r^{\mathrm{P}}$ and $\partial_t\boldsymbol{y}_s^{\mathrm{P}}$ cancel out due to symmetry, which prevents coercivity on $\boldsymbol{u}_r^{\mathrm{P}}$ in the $\mathbf{H}^1$ norm. This cancellation hinders uniform energy control, making it difficult to apply compactness arguments or derive a priori estimates needed for Lions–Magenes-type evolution theory. Although the system is linear, the lack of coercivity in the variational formulation makes classical Galerkin methods not straightforwardly applicable.*

Similar non-coercivity issues also arise in poromechanics models for incompressible or nearly incompressible media, as analyzed in [12], where well-posedness was established using a combination of semigroup and variational techniques, together with inf-sup and T-coercivity arguments. Consequently, we introduce a mixed formulation in Appendix A as an auxiliary tool to establish the well-posedness of the primal formulation at both continuous and discrete levels. Once existence is established, we revert to the original formulation for all subsequent analyses (stability, error estimates, and numerical experiments), as our primary focus is the study of the original formulation.

3.2. *Existence and uniqueness of solution of the original formulation*

In this section we discuss how the well-posedness of formulation (3.1) follows from the existence of a solution of (A.2). First we recall that $\boldsymbol{u}_s^{\mathrm{P}}$ is the structure velocity, so the displacement solution can be recovered from

$$\boldsymbol{y}_s^{\mathrm{P}}(t) = \boldsymbol{y}_{s,0} + \int_0^t \boldsymbol{u}_s^{\mathrm{P}}(s)\,\mathrm{d}s, \quad \forall t \in (0,T]. \tag{3.2}$$

Since $\boldsymbol{u}_s^{\mathrm{P}} \in L^\infty(0,T;\mathbf{V}_s)$ (in Appendix A), then $\boldsymbol{y}_s^{\mathrm{P}} \in W^{1,\infty}(0,T;\mathbf{V}_s)$ for any $\boldsymbol{y}_{s,0} \in \mathbf{V}_s$. By construction, $\boldsymbol{u}_s^{\mathrm{P}} = \partial_t \boldsymbol{y}_s^{\mathrm{P}}$ and $\boldsymbol{y}_s^{\mathrm{P}}(0) = \boldsymbol{y}_{s,0}$.

Theorem 3.2. *Assume (H.1)–(H.3). Then, for data $\boldsymbol{f}_{\mathrm{S}} \in W^{1,1}(0,T;\mathbf{L}^2(\Omega_{\mathrm{S}}))$, $\boldsymbol{f}_{\mathrm{P}} \in W^{1,1}(0,T;\mathbf{L}^2(\Omega_{\mathrm{P}}))$, $\theta \in W^{1,1}(0,T;\mathrm{W}_f')$, and $(\boldsymbol{u}_f^{\mathrm{S}}(0) = \boldsymbol{u}_{f,0}, \boldsymbol{u}_r^{\mathrm{P}}(0) = \boldsymbol{u}_{r,0}, \boldsymbol{y}_s^{\mathrm{P}}(0) = \boldsymbol{y}_{s,0}, \boldsymbol{u}_s^{\mathrm{P}}(0) = \boldsymbol{u}_{s,0}, p^{\mathrm{P}}(0) = p^{\mathrm{P},0}) \in \mathbf{V}_f \times \mathbf{V}_r \times \mathbf{V}_s \times \mathbf{W}_s \times \mathrm{W}_p$, where $(\boldsymbol{u}_{f,0}, \boldsymbol{u}_{r,0}, \boldsymbol{y}_{s,0}, \boldsymbol{u}_{s,0}, p^{\mathrm{P},0})$ are compatible initial data, there exists a unique solution $\vec{\boldsymbol{x}} \in W^{1,\infty}\left(0,T;\mathbf{V}_f\right) \times W^{1,\infty}(0,T;\mathbf{V}_r) \times W^{1,\infty}(0,T;\mathbf{V}_s) \times W^{1,\infty}(0,T;\mathbf{W}_s) \times L^\infty\left(0,T;\mathrm{W}_f\right) \times W^{1,\infty}(0,T;\mathrm{W}_p) \times L^\infty(0,T;\Lambda)$ solving (3.1).*

Proof. We use the solvabilty of (A.2) to establish that of (3.1). Let $(\vec{\boldsymbol{u}}, \vec{p})$ (see Appendix A) be a solution to (A.2), and define $\boldsymbol{y}_s^{\mathrm{P}}$ as in (3.2) so that $\boldsymbol{u}_s^{\mathrm{P}} = \partial_t \boldsymbol{y}_s^{\mathrm{P}}$. The only difference between (A.2) and (3.1) lies in the terms $a_s^{\mathrm{P}}(\boldsymbol{y}_s^{\mathrm{P}}, \boldsymbol{w}_s^{\mathrm{P}})$ versus $b_{\mathrm{sig}}(\boldsymbol{v}_s^{\mathrm{P}}, \boldsymbol{\sigma}^{\mathrm{P}})$. Selecting the terms in (A.2) with $\boldsymbol{\tau}^{\mathrm{P}} \in \mathbf{Z}$, where $\mathbf{Z} = \mathbb{L}^2_{\mathrm{sym}}(\Omega_{\mathrm{P}})$, we obtain $(\partial_t(A\boldsymbol{\sigma}^{\mathrm{P}} - \boldsymbol{\varepsilon}(\boldsymbol{y}_s^{\mathrm{P}})), \boldsymbol{\tau}^{\mathrm{P}})_{\Omega_{\mathrm{P}}} = 0$. Since $\boldsymbol{\varepsilon}(\mathbf{V}_s) \subseteq \mathbf{Z}$, it follows that $\partial_t(A\boldsymbol{\sigma}^{\mathrm{P}} - \boldsymbol{\varepsilon}(\boldsymbol{y}_s^{\mathrm{P}})) = \mathbf{0}$. Integrating from 0 to $t \in (0,T]$ and using $\boldsymbol{\sigma}^{\mathrm{P}}(0) = A^{-1}\boldsymbol{\varepsilon}(\boldsymbol{y}_s^{\mathrm{P}}(0))$, we conclude that $\boldsymbol{\sigma}^{\mathrm{P}}(t) = A^{-1}\boldsymbol{\varepsilon}(\boldsymbol{y}_s^{\mathrm{P}}(t))$. Consequently, $b_{\mathrm{sig}}(\boldsymbol{v}_s^{\mathrm{P}}, \boldsymbol{\sigma}^{\mathrm{P}}) = (\boldsymbol{\sigma}^{\mathrm{P}}, \boldsymbol{\varepsilon}(\boldsymbol{v}_s^{\mathrm{P}}))_{\Omega_{\mathrm{P}}} = (A^{-1}\boldsymbol{\varepsilon}(\boldsymbol{y}_s^{\mathrm{P}}), \boldsymbol{\varepsilon}(\boldsymbol{v}_s^{\mathrm{P}}))_{\Omega_{\mathrm{P}}} = a_s^{\mathrm{P}}(\boldsymbol{y}_s^{\mathrm{P}}, \boldsymbol{v}_s^{\mathrm{P}})$, so that the bilinear and linear terms in (3.1) follow as in (A.2). Thus, $(\boldsymbol{u}_f^{\mathrm{S}}, \boldsymbol{u}_r^{\mathrm{P}}, \boldsymbol{y}_{s,0} + \int_0^t \boldsymbol{u}_s^{\mathrm{P}}(s)\,\mathrm{d}s, \boldsymbol{u}_s^{\mathrm{P}}, p^{\mathrm{S}}, p^{\mathrm{P}}, \lambda)$ solves (3.1) and, in particular, $\boldsymbol{u}_s^{\mathrm{P}} \in \mathbf{W}_s$. ☐

Next, we provide a stability bound for the solution of (3.1).

Lemma 3.3. *Assuming sufficient regularity of the data as well as (H.1)–(H.3), there exists a positive constant $\hat{C}$ (possibly depending on $K, \kappa, \rho_f, \rho_s, \lambda_p, \mu_f, \mu_p, \phi, \alpha_{\mathrm{BJS}}, C_K$) such that*

$$\begin{aligned}
&\|\boldsymbol{u}_f^{\mathrm{S}}\|^2_{L^\infty\left(0,T;\mathbf{L}^2(\Omega_{\mathrm{S}})\right)} + \|\boldsymbol{u}_r^{\mathrm{P}}\|^2_{L^\infty\left(0,T;\mathbf{L}^2(\Omega_{\mathrm{P}})\right)} + \|\boldsymbol{y}_s^{\mathrm{P}}\|^2_{L^\infty\left(0,T;\mathbf{H}^1(\Omega_{\mathrm{P}})\right)} + \|p^{\mathrm{P}}\|^2_{L^\infty(0,T;L^2(\Omega_{\mathrm{P}}))} + \|\boldsymbol{u}_s^{\mathrm{P}}\|^2_{L^\infty\left(0,T;\mathbf{L}^2(\Omega_{\mathrm{P}})\right)}\\
&\quad + \left|\boldsymbol{u}_f^{\mathrm{S}} - \partial_t \boldsymbol{y}_s^{\mathrm{P}}\right|^2_{L^2(0,T;\mathrm{BJS})} + \left|\boldsymbol{u}_r^{\mathrm{P}}\right|^2_{L^2(0,T;\mathrm{BJS})} + \|\boldsymbol{u}_f^{\mathrm{S}}\|^2_{L^2\left(0,T;\mathbf{H}^1(\Omega_{\mathrm{S}})\right)} + \|\boldsymbol{u}_r^{\mathrm{P}}\|^2_{L^2\left(0,T;\mathbf{L}^2(\Omega_{\mathrm{P}})\right)} + \|p^{\mathrm{S}}\|^2_{L^2(0,T;L^2(\Omega_{\mathrm{S}}))}\\
&\quad + \|p^{\mathrm{P}}\|^2_{L^2\left(0,T;L^2(\Omega_{\mathrm{P}})\right)} + \|\lambda\|^2_{L^2(0,T;\Lambda)} \lesssim \|\boldsymbol{f}_{\mathrm{P}}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega_{\mathrm{P}}))} + \|\boldsymbol{f}_{\mathrm{S}}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} + \|\theta\|^2_{L^2(0,T;L^2(\Omega_{\mathrm{S}}))}\\
&\quad \|\boldsymbol{u}_f^{\mathrm{S}}(0)\|^2_{0,\Omega_{\mathrm{S}}} + \|\boldsymbol{u}_r^{\mathrm{P}}(0)\|^2_{0,\Omega_{\mathrm{P}}} + \|\boldsymbol{y}_s^{\mathrm{P}}(0)\|^2_{1,\Omega_{\mathrm{P}}} + \|p^{\mathrm{P}}(0)\|^2_{0,\Omega_{\mathrm{P}}} + \|\boldsymbol{u}_s^{\mathrm{P}}(0)\|^2_{0,\Omega_{\mathrm{P}}},
\end{aligned}$$

where we introduced the notation $|\boldsymbol{v}|^2_{L^2(0,T;\mathrm{BJS})} := \int_0^t |\boldsymbol{v}|^2_{\mathrm{BJS}}\,ds$.

Proof. We test the system against $\vec{\boldsymbol{y}} = (\boldsymbol{u}_f^{\mathrm{S}}, \boldsymbol{u}_r^{\mathrm{P}}, \partial_t \boldsymbol{y}_s^{\mathrm{P}}, \boldsymbol{u}_s^{\mathrm{P}}, p^{\mathrm{S}}, p^{\mathrm{P}}, \lambda)$ and add the first three equations. Using the inequality (A.14) and we can derive the following estimate

$$\begin{aligned}
&\frac{1}{2}\partial_t\Big(\left(\rho_s(1-\phi)\boldsymbol{u}_s^{\mathrm{P}}, \boldsymbol{u}_s^{\mathrm{P}}\right)_{\Omega_{\mathrm{P}}} + \left(\sqrt{\rho_f\phi}(\boldsymbol{u}_r^{\mathrm{P}} + \boldsymbol{u}_s^{\mathrm{P}}), \sqrt{\rho_f\phi}(\boldsymbol{u}_r^{\mathrm{P}} + \boldsymbol{u}_s^{\mathrm{P}})\right) + \left((1-\phi)^2 K^{-1} p^{\mathrm{P}}, p^{\mathrm{P}}\right)_{\Omega_{\mathrm{P}}} + (\rho_f \boldsymbol{u}_f^{\mathrm{S}}, \boldsymbol{u}_f^{\mathrm{S}})\\
&\quad + \left(2\mu_p \boldsymbol{\varepsilon}(\boldsymbol{y}_s^{\mathrm{P}}), \boldsymbol{\varepsilon}(\boldsymbol{y}_s^{\mathrm{P}})\right)_{\Omega_{\mathrm{P}}} + (\lambda_p \nabla\cdot\boldsymbol{y}_s^{\mathrm{P}}, \nabla\cdot\boldsymbol{y}_s^{\mathrm{P}})_{\Omega_{\mathrm{P}}}\Big) + \left|\boldsymbol{u}_f^{\mathrm{S}} - \partial_t \boldsymbol{y}_s^{\mathrm{P}}\right|^2_{\mathrm{BJS}} + \left|\boldsymbol{u}_r^{\mathrm{P}}\right|^2_{\mathrm{BJS}} + \left(\phi^2\kappa^{-1}\boldsymbol{u}_r^{\mathrm{P}}, \boldsymbol{u}_r^{\mathrm{P}}\right)_{\Omega_{\mathrm{P}}}\\
&\quad + (2\mu_f \boldsymbol{\varepsilon}(\boldsymbol{u}_f^{\mathrm{S}}), \boldsymbol{\varepsilon}(\boldsymbol{u}_f^{\mathrm{S}}))_{\Omega_{\mathrm{S}}} \le \langle \boldsymbol{f}_{\mathrm{S}}, \boldsymbol{u}_f^{\mathrm{S}}\rangle_{\Omega_{\mathrm{S}}} + \left(\rho_p \boldsymbol{f}_{\mathrm{P}}, \boldsymbol{u}_s^{\mathrm{P}}\right)_{\Omega_{\mathrm{P}}} + \left(\rho_f\phi \boldsymbol{f}_{\mathrm{P}}, \boldsymbol{u}_s^{\mathrm{P}}\right)_{\Omega_{\mathrm{P}}} + (2\rho_f\phi \boldsymbol{f}_{\mathrm{P}}, \boldsymbol{u}_r^{\mathrm{P}})_{\Omega_{\mathrm{P}}} + (\rho_f^{-1}\theta, p^{\mathrm{P}})_{\Omega_{\mathrm{P}}}.
\end{aligned} \tag{3.3}$$

Applying Young's inequality with $\varepsilon_1 > 0$ to $\mathtt{RHS}_{(3.3)}$, together with hypotheses (H.1)–(H.3), Korn's inequality, and the estimate $\rho_f\,\phi\,\|\boldsymbol{u}_r^{\mathrm{P}} + \boldsymbol{u}_s^{\mathrm{P}}\|^2_{0,\Omega_{\mathrm{P}}} \ge \rho_f\,\phi\left(\frac{1}{2}\|\boldsymbol{u}_r^{\mathrm{P}}\|^2_{0,\Omega_{\mathrm{P}}} - \|\boldsymbol{u}_s^{\mathrm{P}}\|^2_{0,\Omega_{\mathrm{P}}}\right)$, and integrating in time over $(0,t]$ for any

$t \in (0,T]$ on $\mathtt{LHS}_{(3.3)}$, with c_1 and c_2 data-dependent, we combine lower and upper bounds to obtain an estimate. Taking the supremum over $t \in (0,T]$ and using $\int_0^t \varphi(s)\,ds \le T\|\varphi\|_\infty$, the desired result follows:

$$\begin{aligned}
&(c_1 - c_2 T\varepsilon_1)\Big(\|\boldsymbol{u}_f^{\mathrm{S}}\|^2_{L^\infty(0,T;\mathbf{L}^2(\Omega_{\mathrm{S}}))} + \|\boldsymbol{u}_r^{\mathrm{P}}\|^2_{L^\infty(0,T;\mathbf{L}^2(\Omega_{\mathrm{P}}))} + \|\boldsymbol{y}_s^{\mathrm{P}}\|^2_{L^\infty(0,T;\mathbf{H}^1(\Omega_{\mathrm{P}}))} + \|p^{\mathrm{P}}\|^2_{L^\infty(0,T;L^2(\Omega_{\mathrm{P}}))}\\
&\quad + \|\boldsymbol{u}_s^{\mathrm{P}}\|^2_{L^\infty(0,T;\mathbf{L}^2(\Omega_{\mathrm{P}}))}\Big) + c_1\int_0^t |\boldsymbol{u}_f^{\mathrm{S}} - \partial_t \boldsymbol{y}_s^{\mathrm{P}}|^2_{\mathrm{BJS}} + c_1\int_0^t |\boldsymbol{u}_r^{\mathrm{P}}|^2_{\mathrm{BJS}}\,\mathrm{d}s + (c_1 - c_2\varepsilon_1)\|\boldsymbol{u}_f^{\mathrm{S}}\|^2_{L^2(0,T;\mathbf{H}^1(\Omega_{\mathrm{S}}))}\\
&\quad + c_1\|\boldsymbol{u}_r^{\mathrm{P}}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega_{\mathrm{P}}))} \le \frac{c_2}{\varepsilon_1}\Big(\|\boldsymbol{f}_{\mathrm{P}}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega_{\mathrm{P}}))} + \|\boldsymbol{f}_{\mathrm{S}}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega_{\mathrm{S}}))}\Big) + \frac{c_2}{\varepsilon_1}\|\theta\|^2_{L^2(0,T;L^2(\Omega_{\mathrm{P}}))}\\
&\quad + c_2\varepsilon_1(\|\boldsymbol{u}_r^{\mathrm{P}}\|_{L^2(0,T;\mathbf{L}^2(\Omega_{\mathrm{P}}))} + \|p^{\mathrm{S}}\|_{L^2(0,T;L^2(\Omega_{\mathrm{S}}))}) + c_2(\|\boldsymbol{u}_f^{\mathrm{S}}(0)\|^2_{0,\Omega_{\mathrm{S}}} + \|\boldsymbol{u}_r^{\mathrm{P}}(0)\|^2_{0,\Omega_{\mathrm{P}}} + \|\boldsymbol{y}_s^{\mathrm{P}}(0)\|^2_{1,\Omega_{\mathrm{P}}}\\
&\quad + \|p^{\mathrm{P}}(0)\|^2_{0,\Omega_{\mathrm{P}}} + \|\boldsymbol{u}_s^{\mathrm{P}}(0)\|^2_{0,\Omega_{\mathrm{P}}}).
\end{aligned} \tag{3.4}$$

Finally, we use the inf-sup condition (A.5b) for $\tilde{p}_{\mathrm{S}}, \tilde{p}_{\mathrm{P}}, \tilde{\lambda}$ together with (3.1) and then using the continuity bounds in Lemmas A.1 and A.2, and applying integration in time, we have

$$\begin{aligned}
\varepsilon_2\int_0^t \left(\|p^{\mathrm{S}}\|^2_{0,\Omega_{\mathrm{S}}} + \|p^{\mathrm{P}}\|^2_{0,\Omega_{\mathrm{P}}} + \|\lambda\|^2_\Lambda\right)\mathrm{d}s \le \tilde{C}\varepsilon_2\int_0^t \Big(&\|\boldsymbol{u}_f^{\mathrm{S}}\|^2_{1,\Omega_{\mathrm{S}}} + \|\boldsymbol{u}_r^{\mathrm{P}}\|^2_{0,\Omega_{\mathrm{P}}} + \|\boldsymbol{u}_f^{\mathrm{S}}\|^2_{0,\Omega_{\mathrm{S}}} + \|\boldsymbol{u}_f^{\mathrm{S}} - \partial_t\boldsymbol{y}_s^{\mathrm{P}}\|^2_{\mathrm{BJS}}\\
&+ \|\boldsymbol{u}_r^{\mathrm{P}}\|^2_{\mathrm{BJS}} + \|\boldsymbol{f}_{\mathrm{S}}\|^2_{0,\Omega_{\mathrm{S}}} + \|\boldsymbol{f}_{\mathrm{P}}\|^2_{0,\Omega_{\mathrm{P}}}\Big)\,\mathrm{d}s.
\end{aligned} \tag{3.5}$$

Adding (3.4) and (3.5), and choosing ε_2 and ε_1 small enough, readily yields the desired result. $\square$

Corollary 3.4. *Assume (H.1)–(H.3). Then, there exists a unique* $\vec{\boldsymbol{x}}$ *in* $W^{1,\infty}(0,T;\mathbf{V}_f) \times W^{1,\infty}(0,T;\mathbf{V}_r) \times W^{1,\infty}(0,T;\mathbf{V}_s) \times W^{1,\infty}(0,T;\mathbf{W}_s) \times L^\infty(0,T;\mathrm{W}_f) \times W^{1,\infty}(0,T;\mathrm{W}_p) \times L^\infty(0,T;\Lambda)$*, solving* (3.1).

Proof. Let $\vec{\boldsymbol{x}}^i$, $i = 1,2$, be two solutions of (3.1) with the same data. Subtracting both weak formulations yields homogeneous initial data and zero RHS. Consider $\tilde{\vec{\boldsymbol{x}}} = \vec{\boldsymbol{x}}^1 - \vec{\boldsymbol{x}}^2$ and substitute $\tilde{\vec{\boldsymbol{x}}}$ instead of $\vec{\boldsymbol{x}}$ with homogeneous initial data and forcing terms in Lemma 3.3. This gives that the solution to problem (3.1) is unique. $\square$

4. Semi-discrete formulation

Suppose that $\mathscr{T}_h^{\mathrm{S}}$ and $\mathscr{T}_h^{\mathrm{P}}$ are shape-regular quasi-uniform partitions of Ω_{S} and Ω_{P}, respectively, both consisting of affine elements with maximal element diameter h. The two partitions may not match at the interface Σ. For the discretization of the fluid velocity and pressure, we choose Inf-sup stable Taylor–Hood FE spaces $\mathbf{V}_{f,h} \subset \mathbf{V}_f$ and $\mathrm{W}_{f,h} \subset \mathrm{W}_f$. For the discretization of the generalized Biot unknowns, we define $\mathrm{X}_h^k = \{q \in C(\Omega) : q|_K \in \mathbb{P}_k(K) \quad \forall K \in \mathscr{T}_h\}$ where $\mathbb{P}_k(K)$ denotes the space of polynomials of degree $k \ge 1$ defined over K. With them, we define the following conforming discrete spaces:

$$\mathbf{V}_{r,h} = \mathbf{V}_r \cap \left[\mathrm{X}_h^{k+1}\right]^d, \quad \mathbf{V}_{s,h} = \mathbf{V}_s \cap \left[\mathrm{X}_h^{k+1}\right]^d, \quad \mathrm{W}_{p,h} = \mathrm{W}_p \cap \mathrm{X}_h^k, \quad \mathbf{W}_{s,h} = \mathbf{W}_s \cap \left[\mathrm{X}_h^k\right]^d.$$

On $\mathbf{V}_{f,h}, \mathbf{V}_{r,h}$ and $\mathbf{V}_{s,h}$ we prescribe homogeneous boundary conditions on Γ_{S} and Γ_{P}, and the discrete space for the Lagrange multiplier is $\Lambda_h = \mathbf{V}_{r,h} \cdot \boldsymbol{n}_{\mathrm{P}}|_\Sigma$, equipped with the norm $\|\cdot\|_{\Lambda_h} = \|\cdot\|_{H_{00}^{-1/2}(\Sigma)}$.

The first semi-discrete problem reads: find $\vec{\boldsymbol{x}}_h \in W^{1,\infty}(0,T;\mathbf{V}_{f,h}) \times W^{1,\infty}(0,T;\mathbf{V}_{r,h}) \times W^{1,\infty}(0,T;\mathbf{V}_{s,h}) \times W^{1,\infty}(0,T;\mathbf{W}_{s,h}) \times L^\infty(0,T;\mathrm{W}_{f,h}) \times W^{1,\infty}(0,T;\mathrm{W}_{p,h}) \times L^\infty(0,T;\Lambda_h)$, such that

$$\mathrm{E}(\partial_t \vec{\boldsymbol{x}}_h, \vec{\boldsymbol{y}}_h) + \mathrm{H}(\vec{\boldsymbol{x}}_h, \vec{\boldsymbol{y}}_h) = \mathrm{F}(\vec{\boldsymbol{y}}), \tag{4.1}$$

for all $\vec{\boldsymbol{y}}_h \in \vec{\mathbf{X}}_h$. The initial conditions $\boldsymbol{u}_{f,h}^{\mathrm{S}}(0), \boldsymbol{u}_{r,h}^{\mathrm{P}}(0), \boldsymbol{y}_{s,h}^{\mathrm{P}}(0), \boldsymbol{u}_{s,h}^{\mathrm{P}}(0)$, and $p_h^{\mathrm{P}}(0)$ are suitable approximations of $\boldsymbol{u}_{f,0}, \boldsymbol{u}_{r,0}, \boldsymbol{y}_{s,0}, \boldsymbol{u}_{s,0}$, and $p^{\mathrm{P},0}$, respectively. To prove that (4.1) is well-posed we follow the same strategy as in the continuous case.

As a corollary of Theorem A.18 and Remark A.19, we obtain the following well-posedness result for the original semidiscrete problem (4.1) The proof is identical to the proof of Theorem 3.2.

Theorem 4.1. *Under the same assumptions as Theorem A.18, and compatible initial data in* $\mathbf{V}_{f,h}\times\mathbf{V}_{r,h}\times\mathbf{V}_{s,h}\times\mathbf{W}_{s,h}\times\mathrm{W}_{p,h}$ *as above, there exists a unique solution* $\vec{\boldsymbol{x}}_h\in W^{1,\infty}\left(0,T;\mathbf{V}_{f,h}\right)\times W^{1,\infty}\left(0,T;\mathbf{V}_{r,h}\right)\times W^{1,\infty}\left(0,T;\mathbf{V}_{s,h}\right)\times W^{1,\infty}\left(0,T;W_{s,h}\right)\times L^{\infty}\left(0,T;\mathrm{W}_{f,h}\right)\times W^{1,\infty}\left(0,T;\mathrm{W}_{p,h}\right)\times L^{\infty}(0,T;\Lambda_h)$ *of* (4.1).

The proof of the following stability result is identical to the proof of Theorem 3.3.

Lemma 4.2. *For the solution of* (4.1)*, assuming (H.1)–(H.3) as well as sufficient regularity of the data, there exists* $\hat{C}(K,\kappa,\rho_f,\rho_s,\lambda_p,\mu_f,\mu_p,\phi,\alpha_{\mathrm{BJS}},C_K)>0$ *such that*

$$\begin{aligned}
&\|\boldsymbol{u}^{\mathrm{S}}_{f,h}\|_{L^\infty\left(0,T;\mathbf{L}^2(\Omega_{\mathrm{S}})\right)}+\|\boldsymbol{u}^{\mathrm{P}}_{r,h}\|_{L^\infty\left(0,T;\mathbf{L}^2(\Omega_{\mathrm{P}})\right)}+\|\boldsymbol{y}^{\mathrm{P}}_{s,h}\|_{L^\infty\left(0,T;\mathbf{H}^1(\Omega_{\mathrm{P}})\right)}+\|p^{\mathrm{P}}_h\|_{L^\infty(0,T;L^2(\Omega_{\mathrm{P}}))}+\|\boldsymbol{u}^{\mathrm{P}}_{s,h}\|_{L^\infty\left(0,T;\mathbf{L}^2(\Omega_{\mathrm{P}})\right)}\\
&\quad+\left|\boldsymbol{u}^{\mathrm{S}}_{f,h}-\partial_t\boldsymbol{y}^{\mathrm{P}}_{s,h}\right|_{L^2(0,T;\mathrm{BJS})}+\left|\boldsymbol{u}^{\mathrm{P}}_{r,h}\right|_{L^2(0,T;\mathrm{BJS})}+\|\boldsymbol{u}^{\mathrm{S}}_{f,h}\|_{L^2\left(0,T;\mathbf{H}^1(\Omega_{\mathrm{S}})\right)}++\|\boldsymbol{u}^{\mathrm{P}}_{r,h}\|_{L^2\left(0,T;\mathbf{L}^2(\Omega_{\mathrm{P}})\right)}+\|p^{\mathrm{S}}_h\|_{L^2(0,T;L^2(\Omega_{\mathrm{S}}))}\\
&\quad+\|p^{\mathrm{P}}_h\|_{L^2\left(0,T;L^2(\Omega_{\mathrm{P}})\right)}+\|\lambda_h\|_{L^2(0,T;\Lambda)}\lesssim\|\boldsymbol{f}_{\mathrm{P}}\|_{L^2(0,T;\mathbf{L}^2(\Omega_{\mathrm{P}}))}+\|\boldsymbol{f}_{\mathrm{S}}\|_{L^2(0,T;\mathbf{L}^2(\Omega))}+\|\theta\|_{L^2(0,T;L^2(\Omega_{\mathrm{S}}))}\\
&\quad+\|\boldsymbol{u}^{\mathrm{S}}_{f,h}(0)\|_{0,\Omega_{\mathrm{S}}}+\|\boldsymbol{u}^{\mathrm{P}}_{r,h}(0)\|_{0,\Omega_{\mathrm{P}}}+\|\boldsymbol{y}^{\mathrm{P}}_{s,h}(0)\|_{1,\Omega_{\mathrm{P}}}+\|p^{\mathrm{P}}_h(0)\|_{0,\Omega_{\mathrm{P}}}+\|\boldsymbol{u}^{\mathrm{P}}_{s,h}(0)\|_{0,\Omega_{\mathrm{P}}}.
\end{aligned}$$

4.1. *Error analysis for the semi discrete scheme*

In this section we analyze the spatial discretization error. Let k_f and s_f be the degrees of polynomials in $\mathbf{V}_{f,h}$ and $\mathrm{W}_{f,h}$, let k_p and s_p be the degrees of polynomials in $\mathbf{V}_{r,h}$ and $\mathrm{W}_{p,h}$ respectively, and let k_s and s_s be the polynomial degree in $\mathbf{V}_{s,h}$ and $\mathbf{W}_{s,h}$.

4.1.1. Approximation error

Let $Q_{f,h},Q_{p,h}$, and $\mathbf{Q}_{s,h}$ be the L^2-projection operators onto $\mathrm{W}_{f,h},\mathrm{W}_{p,h}$, and $\mathbf{W}_{s,h}$ respectively:

$$\left(p^{\mathrm{S}}-Q_{f,h}p^{\mathrm{S}},q^{\mathrm{S}}_h\right)_{\Omega_{\mathrm{S}}}=0\qquad\forall q^{\mathrm{S}}_h\in\mathrm{W}_{f,h},\tag{4.2a}$$

$$\left(p^{\mathrm{P}}-Q_{p,h}p^{\mathrm{P}},q^{\mathrm{P}}_h\right)_{\Omega_{\mathrm{P}}}=0\qquad\forall q^{\mathrm{P}}_h\in\mathrm{W}_{p,h},\tag{4.2b}$$

$$\left\langle\boldsymbol{u}^{\mathrm{P}}_s-\mathbf{Q}_{s,h}\boldsymbol{u}^{\mathrm{P}}_s,\boldsymbol{v}^{\mathrm{P}}_{s,h}\right\rangle_{\Omega_{\mathrm{P}}}=0\qquad\forall\boldsymbol{v}^{\mathrm{P}}_{s,h}\in\mathbf{W}_{s,h}.\tag{4.2c}$$

These operators satisfy the approximation properties [38]:

$$\|p^{\mathrm{S}}-Q_{f,h}p^{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}}\le C^\star_1h^{r_{sf}+1}\|p^{\mathrm{S}}\|_{r_{sf}+1,\Omega_{\mathrm{S}}}\qquad 0\le r_{s_f}\le s_f,\tag{4.3a}$$

$$\|p^{\mathrm{P}}-Q_{p,h}p^{\mathrm{P}}\|_{0,\Omega_{\mathrm{P}}}\le C^\star_1h^{r_{sp}+1}\|p^{\mathrm{P}}\|_{r_{sp}+1,\Omega_{\mathrm{P}}}\qquad 0\le r_{s_p}\le s_p,\tag{4.3b}$$

$$\|\boldsymbol{u}^{\mathrm{P}}_s-\mathbf{Q}_{s,h}\boldsymbol{u}^{\mathrm{P}}_s\|_{0,\Omega_{\mathrm{P}}}\le C^\star_1h^{r_{ss}+1}\|\boldsymbol{u}^{\mathrm{P}}_s\|_{r_{ss}+1,\Omega_{\mathrm{P}}}\qquad 0\le r_{s_s}\le s_s.\tag{4.3c}$$

The definition $\Lambda_h=\mathbf{V}_{r,h}\cdot\boldsymbol{n}_{\mathrm{P}}\big|_{\Sigma}$ implies the following approximation property [48, Appendix A]

$$\|\lambda-Q_{\lambda,h}\lambda\|_{\Lambda_h}\le C^\star_1h^{r_{\tilde{k}_p}+\frac{1}{2}}\|\lambda\|_{r_{\tilde{k}_p},\Sigma}\quad -1/2\le r_{\tilde{k}_p}\le\tilde{k}_p-1/2.\tag{4.4}$$

Next, for all $\boldsymbol{v}^{\mathrm{S}}_f\in\mathbf{V}_f$, we define a Stokes-like projector $\left(\mathbf{S}_{f,h},R_{f,h}\right):\mathbf{V}_f\to\mathbf{V}_{f,h}\times\mathrm{W}_{f,h}$, as

$$a^{\mathrm{S}}_f\left(\mathbf{S}_{f,h}\boldsymbol{v}^{\mathrm{S}}_f,\boldsymbol{v}^{\mathrm{S}}_{f,h}\right)-b^{\mathrm{S}}_f\left(\boldsymbol{v}^{\mathrm{S}}_{f,h},R_{f,h}\boldsymbol{v}^{\mathrm{S}}_f\right)=a^{\mathrm{S}}_f\left(\boldsymbol{v}^{\mathrm{S}}_f,\boldsymbol{v}^{\mathrm{S}}_{f,h}\right)\qquad\forall\boldsymbol{v}^{\mathrm{S}}_{f,h}\in\mathbf{V}_{f,h},\tag{4.5a}$$

$$b_f^{\mathrm{S}}\left(\mathbf{S}_{f,h}\boldsymbol{v}_f^{\mathrm{S}}, q_h^{\mathrm{S}}\right) = b_f\left(\boldsymbol{v}_f^{\mathrm{S}}, q_h^{\mathrm{S}}\right) \qquad \forall q_h^{\mathrm{S}} \in \mathrm{W}_{f,h}. \tag{4.5b}$$

The operator $\mathbf{S}_{f,h}$ satisfies the approximation property [4, 31]:

$$\|\boldsymbol{v}_f^{\mathrm{S}} - \mathbf{S}_{f,h}\boldsymbol{v}_f^{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} \leq C_1^\star h^{r_{k_f}} \|\boldsymbol{v}_f^{\mathrm{S}}\|_{r_{k_f}+1,\Omega_{\mathrm{S}}} \quad 0 \leq r_{k_f} \leq k_f. \tag{4.6}$$

Let $\boldsymbol{\Pi}_{r,h}$ be the Stokes projection onto $\mathbf{V}_{r,h}$ satisfying for all $\boldsymbol{v}_r^{\mathrm{P}} \in \mathbf{V}_r$,

$$\left(\nabla\cdot\boldsymbol{\Pi}_{r,h}\boldsymbol{v}_r^{\mathrm{P}}, q_h^{\mathrm{P}}\right) = \left(\nabla\cdot\boldsymbol{v}_r^{\mathrm{P}}, q_h^{\mathrm{P}}\right) \qquad \forall q_h^{\mathrm{P}} \in \mathrm{W}_{p,h}, \tag{4.7a}$$

$$\left\langle\boldsymbol{\Pi}_{r,h}\boldsymbol{v}_r^{\mathrm{P}}\cdot\boldsymbol{n}_{\mathrm{P}}, \boldsymbol{v}_{r,h}^{\mathrm{P}}\cdot\boldsymbol{n}_{\mathrm{P}}\right\rangle_\Sigma = \left\langle\boldsymbol{v}_r^{\mathrm{P}}\cdot\boldsymbol{n}_{\mathrm{P}}, \boldsymbol{v}_{r,h}^{\mathrm{P}}\cdot\boldsymbol{n}_{\mathrm{P}}\right\rangle_\Sigma \qquad \forall \boldsymbol{v}_{r,h}^{\mathrm{P}} \in \mathbf{V}_{r,h}. \tag{4.7b}$$

We will make use of the following estimates regarding $\boldsymbol{\Pi}_{r,h}$:

$$\|\boldsymbol{v}_r^{\mathrm{P}} - \boldsymbol{\Pi}_{r,h}\boldsymbol{v}_r^{\mathrm{P}}\|_{0,\Omega_{\mathrm{P}}} \leq C_1^\star h^{r_{k_p}+1}\|\boldsymbol{v}_r^{\mathrm{P}}\|_{H^{r_{k_p}+1}(\Omega_{\mathrm{P}})} \quad 0 \leq r_{k_p} \leq k_p, \tag{4.8a}$$

$$\left\|\boldsymbol{\Pi}_{r,h}\boldsymbol{v}_r^{\mathrm{P}}\right\|_{1,\Omega_p} \leq C_1^\star\left\|\boldsymbol{v}_r^{\mathrm{P}}\right\|_{1,\Omega_p}. \tag{4.8b}$$

Finally, let $\mathbf{S}_{s,h}$ be the Scott–Zhang interpolant from $\mathbf{V}_s$ onto $\mathbf{V}_{s,h}$, satisfying [41]:

$$\|\boldsymbol{y}_s^{\mathrm{P}} - \mathbf{S}_{s,h}\boldsymbol{y}_s^{\mathrm{P}}\|_{0,\Omega_{\mathrm{P}}} + h\left|\boldsymbol{y}_s^{\mathrm{P}} - \mathbf{S}_{s,h}\boldsymbol{y}_s^{\mathrm{P}}\right|_{1,\Omega_{\mathrm{P}}} \leq C_1^\star h^{r_{k_s}+1}\|\boldsymbol{y}_s^{\mathrm{P}}\|_{r_{ks}+1,\Omega_{\mathrm{P}}} \quad 0 \leq r_{k_s} \leq k_s. \tag{4.9}$$

4.1.2. Construction of a weakly-continuous interpolant

In this section we use the operators defined above to build an operator onto a space with weak continuity of normal velocities. Let us consider

$$\mathbf{U} = \left\{\left(\boldsymbol{v}_f^{\mathrm{S}}, \boldsymbol{v}_r^{\mathrm{P}}, \boldsymbol{w}_s^{\mathrm{P}}\right) \in \mathbf{V}_f\times\mathbf{V}_r\times\mathbf{V}_s : \boldsymbol{v}_f^{\mathrm{S}}\cdot\boldsymbol{n}_{\mathrm{S}} + \boldsymbol{v}_r^{\mathrm{P}}\cdot\boldsymbol{n}_{\mathrm{P}} + \boldsymbol{w}_s^{\mathrm{P}}\cdot\boldsymbol{n}_{\mathrm{P}} = 0 \quad \text{on } \Sigma\right\},$$

$$\mathbf{U}_h = \left\{\left(\boldsymbol{v}_{f,h}^{\mathrm{S}}, \boldsymbol{v}_{r,h}^{\mathrm{P}}, \boldsymbol{w}_{s,h}^{\mathrm{P}}\right) \in \mathbf{V}_{f,h}\times\mathbf{V}_{r,h}\times\mathbf{V}_{s,h} : b_\Gamma\left(\boldsymbol{v}_{f,h}^{\mathrm{S}}, \boldsymbol{v}_{r,h}^{\mathrm{P}}, \boldsymbol{w}_{s,h}^{\mathrm{P}}; \mu_h\right) = 0, \forall \mu_h \in \Lambda_h\right\}.$$

We will construct an interpolation operator $\mathbf{I}_h : \mathbf{U} \to \mathbf{U}_h$ as a triple $\mathbf{I}_h(\boldsymbol{v}_f^{\mathrm{S}}, \boldsymbol{v}_r^{\mathrm{P}}, \boldsymbol{w}_s^{\mathrm{P}}) = (\mathbf{I}_{f,h}\boldsymbol{v}_f^{\mathrm{S}}, \mathbf{I}_{r,h}\boldsymbol{v}_r^{\mathrm{P}}, \mathbf{I}_{s,h}\boldsymbol{w}_s^{\mathrm{P}})$, with the following properties:

$$b_\Gamma\left(\mathbf{I}_{f,h}\boldsymbol{v}_f^{\mathrm{S}}, \mathbf{I}_{r,h}\boldsymbol{v}_r^{\mathrm{P}}, \mathbf{I}_{s,h}\boldsymbol{w}_s^{\mathrm{P}}; \mu_h\right) = 0 \qquad \forall \mu_h \in \Lambda_h, \tag{4.10a}$$

$$b_f^{\mathrm{S}}\left(\mathbf{I}_{f,h}\boldsymbol{v}_f^{\mathrm{S}} - \boldsymbol{v}_f^{\mathrm{S}}, q_h^{\mathrm{S}}\right) = 0 \qquad \forall q_h^{\mathrm{S}} \in \mathrm{W}_{f,h}, \tag{4.10b}$$

$$b_f^{\mathrm{P}}\left(\mathbf{I}_{r,h}\boldsymbol{v}_r^{\mathrm{P}} - \boldsymbol{v}_r^{\mathrm{P}}, q_h^{\mathrm{P}}\right) = 0 \qquad \forall q_h^{\mathrm{P}} \in \mathrm{W}_{p,h}. \tag{4.10c}$$

We let $\mathbf{I}_{f,h} := \mathbf{S}_{f,h}$ and $\mathbf{I}_{s,h} := \mathbf{S}_{s,h}$. To construct $\mathbf{I}_{r,h}$, we first consider an auxiliary Stokes problem

$$-\boldsymbol{\Delta\zeta} + \nabla p^{\mathrm{S}} = \mathbf{0} \quad \text{and} \quad \nabla\cdot\boldsymbol{\zeta} = 0 \qquad \text{in } \Omega_{\mathrm{P}}, \tag{4.11}$$

$$\boldsymbol{\zeta} = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{P}}, \quad \text{and} \quad \boldsymbol{\zeta}\cdot\boldsymbol{n}_{\mathrm{P}} = \left(\boldsymbol{v}_f^{\mathrm{S}} - \mathbf{I}_{f,h}\boldsymbol{v}_f^{\mathrm{S}}\right)\cdot\boldsymbol{n}_{\mathrm{S}} + \left(\boldsymbol{w}_s^{\mathrm{P}} - \mathbf{I}_{s,h}\boldsymbol{w}_s^{\mathrm{P}}\right)\cdot\boldsymbol{n}_{\mathrm{P}} \qquad \text{on } \Sigma.$$

Define $\boldsymbol{w} = \boldsymbol{\zeta} + \boldsymbol{v}_r^{\mathrm{P}}$. From (4.11) we have

$$\nabla\cdot\boldsymbol{w} = \nabla\cdot\boldsymbol{\zeta} + \nabla\cdot\boldsymbol{v}_r^{\mathrm{P}} = \nabla\cdot\boldsymbol{v}_r^{\mathrm{P}} \qquad \text{in } \Omega_{\mathrm{P}}, \tag{4.12a}$$

$$\boldsymbol{w}\cdot\boldsymbol{n}_{\mathrm{P}} = \boldsymbol{\zeta}\cdot\boldsymbol{n}_{\mathrm{P}} + \boldsymbol{v}_r^{\mathrm{P}}\cdot\boldsymbol{n}_{\mathrm{P}} = -\mathbf{I}_{f,h}\boldsymbol{v}_f^{\mathrm{S}}\cdot\boldsymbol{n}_{\mathrm{S}} - \mathbf{I}_{s,h}\boldsymbol{w}_s^{\mathrm{P}}\cdot\boldsymbol{n}_{\mathrm{P}} \qquad \text{on } \Sigma. \tag{4.12b}$$

We now let $\mathbf{I}_{r,h}\boldsymbol{v}_r^{\mathrm{P}} = \boldsymbol{\Pi}_{r,h}\boldsymbol{w}$. Next, we verify that the operator $\mathbf{I}_h = \left(\mathbf{I}_{f,h}, \mathbf{I}_{r,h}, \mathbf{I}_{s,h}\right)$ satisfies (4.10a)-(4.10c). Property (4.10b) follows immediately from (4.5b), while, using (4.12a) and (4.7a), property (4.10c) follows from

$$\left(\nabla\cdot\mathbf{I}_{r,h}\boldsymbol{v}_r^{\mathrm{P}}, q_h^{\mathrm{P}}\right)_{\Omega_{\mathrm{P}}} = \left(\nabla\cdot\boldsymbol{\Pi}_{r,h}\boldsymbol{w}, q_h^{\mathrm{P}}\right)_{\Omega_{\mathrm{P}}} = \left(\nabla\cdot\boldsymbol{w}, q_h^{\mathrm{P}}\right)_{\Omega_{\mathrm{P}}} = \left(\nabla\cdot\boldsymbol{v}_r^{\mathrm{P}}, q_h^{\mathrm{P}}\right)_{\Omega_{\mathrm{P}}} \quad \forall q_h^{\mathrm{P}}\in \mathrm{W}_{p,h}.$$

Using (4.12b) and (4.7b), we have for all $\mu_h\in\Lambda_h$,

$$\left\langle \mathbf{I}_{r,h}\boldsymbol{v}_r^{\mathrm{P}}\cdot\boldsymbol{n}_{\mathrm{P}}, \mu_h\right\rangle_\Sigma = \left\langle \boldsymbol{\Pi}_{r,h}\boldsymbol{w}\cdot\boldsymbol{n}_{\mathrm{P}}, \mu_h\right\rangle_\Sigma = \langle \boldsymbol{w}\cdot\boldsymbol{n}_{\mathrm{P}}, \mu_h\rangle_\Sigma = \left\langle -\mathbf{I}_{f,h}\boldsymbol{v}_f^{\mathrm{S}}\cdot\boldsymbol{n}_{\mathrm{S}} - \mathbf{I}_{s,h}\boldsymbol{w}_s^{\mathrm{P}}\cdot\boldsymbol{n}_{\mathrm{P}}, \mu_h\right\rangle_\Sigma,$$

which implies (4.10a).

The approximation properties of the components of $\mathbf{I}_h$ are the following.

Lemma 4.3. *For smooth $\boldsymbol{v}_f^{\mathrm{S}}, \boldsymbol{v}_r^{\mathrm{P}}$, $\boldsymbol{w}_s^{\mathrm{P}}$, and for $0\le r_{k_p}\le k_p$, $0\le r_{k_f}\le k_f$, $0\le r_{k_s}\le k_s$, there holds*

$$\|\boldsymbol{v}_f^{\mathrm{S}} - \mathbf{I}_{f,h}\boldsymbol{v}_f^{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} \le C_1^\star h^{r_{k_f}}\|\boldsymbol{v}_f^{\mathrm{S}}\|_{r_{k_f}+1,\Omega_{\mathrm{S}}}, \tag{4.13a}$$

$$\|\boldsymbol{w}_s^{\mathrm{P}} - \mathbf{I}_{s,h}\boldsymbol{w}_s^{\mathrm{P}}\|_{0,\Omega_{\mathrm{P}}} + h\left|\boldsymbol{w}_s^{\mathrm{P}} - \mathbf{I}_{s,h}\boldsymbol{w}_s^{\mathrm{P}}\right|_{1,\Omega_{\mathrm{P}}} \le C_1^\star h^{r_{k_s}+1}\|\boldsymbol{w}_s^{\mathrm{P}}\|_{r_{k_s}+1,\Omega_{\mathrm{P}}}, \tag{4.13b}$$

$$\|\boldsymbol{v}_r^{\mathrm{P}} - \mathbf{I}_{r,h}\boldsymbol{v}_r^{\mathrm{P}}\|_{0,\Omega_{\mathrm{P}}} \le C_1^\star(h^{r_{k_p}+1}\|\boldsymbol{v}_r^{\mathrm{P}}\|_{r_{k_p}+1,\Omega_{\mathrm{P}}} + h^{r_k}\|\boldsymbol{v}_f^{\mathrm{S}}\|_{r_{k_f}+1,\Omega_{\mathrm{S}}} + h^{r_{k_s}}\|\boldsymbol{w}_s^{\mathrm{P}}\|_{r_{k_s}+1,\Omega_{\mathrm{P}}}). \tag{4.13c}$$

Proof. The proof follows the approach of [4, Lemma 5.1]. □

4.2. *Error estimates*

Theorem 4.4. *Assuming (H.1)-(H.3) and sufficient smoothness for the solution of (3.1), the solution of (4.1) with $\boldsymbol{u}_{f,h}^{\mathrm{S}}(0) = \mathbf{I}_{f,h}\boldsymbol{u}_{f,0}$, $\boldsymbol{u}_{r,h}^{\mathrm{P}}(0) = \mathbf{I}_{r,h}\boldsymbol{u}_{r,0}$, $\boldsymbol{y}_{s,h}^{\mathrm{P}}(0) = \mathbf{I}_{s,h}\boldsymbol{y}_{s,0}$, $\boldsymbol{u}_{s,h}^{\mathrm{P}}(0) = \mathbf{Q}_{s,h}\boldsymbol{u}_{s,0}$, and $p_h^{\mathrm{P}}(0) = Q_{r,h}p^{p,0}$ satisfies*

$$\begin{aligned}
&\|\boldsymbol{u}_s^{\mathrm{P}} - \boldsymbol{u}_{s,h}^{\mathrm{P}}\|_{L^\infty(0,T;\mathbf{L}^2(\Omega_{\mathrm{P}}))} + \|p^{\mathrm{P}} - p_h^{\mathrm{P}}\|_{L^\infty(0,T;L^2(\Omega))} + \|\boldsymbol{y}_s^{\mathrm{P}} - \boldsymbol{y}_{s,h}^{\mathrm{P}}\|_{L^\infty(0,T;\mathbf{H}^1(\Omega_{\mathrm{P}}))} + \|\boldsymbol{u}_r^{\mathrm{P}} - \boldsymbol{u}_{r,h}^{\mathrm{P}}\|_{L^\infty(0,T;\mathbf{L}^2(\Omega_{\mathrm{P}}))}\\
&\quad + \|\boldsymbol{u}_f^{\mathrm{S}} - \boldsymbol{u}_{f,h}^{\mathrm{S}}\|_{L^2(0,T;\mathbf{H}^1(\Omega_{\mathrm{S}}))} + \|\boldsymbol{u}_r^{\mathrm{P}} - \boldsymbol{u}_{r,h}^{\mathrm{P}}\|_{L^2(0,T;\mathbf{L}^2(\Omega_{\mathrm{P}}))} + \left|\left(\boldsymbol{u}_f^{\mathrm{S}} - \partial_t\boldsymbol{y}_s^{\mathrm{P}}\right) - \left(\boldsymbol{u}_{f,h}^{\mathrm{S}} - \partial_t\boldsymbol{y}_{s,h}^{\mathrm{P}}\right)\right|_{L^2(0,T;\mathrm{BJS})}\\
&\quad + \left|\boldsymbol{u}_r^{\mathrm{P}} - \boldsymbol{u}_{r,h}^{\mathrm{P}}\right|_{L^2(0,T;\mathrm{BJS})} + \|p^{\mathrm{S}} - p_h^{\mathrm{S}}\|_{L^2(0,T;L^2(\Omega_{\mathrm{S}}))} + \|p^{\mathrm{P}} - p_h^{\mathrm{P}}\|_{L^2(0,T;L^2(\Omega_{\mathrm{P}}))} + \|\lambda - \lambda_h\|_{L^2(0,T;\Lambda_h)}\\
&\le C\sqrt{\exp(T)}\Bigg[h^{r_{k_f}}\left(\|\boldsymbol{u}_f^{\mathrm{S}}\|_{L^2(0,T;\mathbf{H}^{r_{k_f}+1}(\Omega_{\mathrm{S}}))} + \|\partial_t\boldsymbol{u}_f^{\mathbf{S}}\|_{L^2(0,T;\mathbf{H}^{r_{k_f}+1}(\Omega_{\mathrm{S}}))}\right) + h^{r_{s_f}+1}\|p^{\mathrm{S}}\|_{L^2(0,T;H^{r_{s_f}+1}(\Omega_{\mathrm{S}}))}\\
&\quad + h^{r_{k_p}-1}\left(\|\boldsymbol{u}_r^{\mathrm{P}}\|_{L^2(0,T;\mathbf{H}^{r_{k_p}+1}(\Omega_{\mathrm{P}}))} + \|\boldsymbol{u}_r^{\mathrm{P}}\|_{L^\infty(0,T;\mathbf{H}^{r_{k_p}+1}(\Omega_{\mathrm{P}}))} + \|\partial_t\boldsymbol{u}_r^{\mathrm{P}}\|_{L^2(0,T;\mathbf{H}^{r_{k_p}+1}(\Omega_{\mathrm{P}}))}\right)\\
&\quad + h^{r_{s_p}+1}\left(\|p^{\mathrm{P}}\|_{L^\infty(0,T;H^{r_{s_p}+1}(\Omega_{\mathrm{P}}))} + \|p^{\mathrm{P}}\|_{L^2(0,T;H^{r_{s_p}+1}(\Omega_{\mathrm{P}}))} + \|\partial_t p^{\mathrm{P}}\|_{L^2(0,T;H^{r_{s_p}+1}(\Omega_{\mathrm{P}}))}\right)\\
&\quad + h^{r_{k_s}-1}\Big(\|\boldsymbol{y}_s^{\mathrm{P}}\|_{L^\infty(0,T;\mathbf{H}^{r_{k_s}+1}(\Omega_{\mathrm{P}}))} + \|\boldsymbol{y}_s^{\mathrm{P}}\|_{L^2(0,T;\mathbf{H}^{r_{k_s}+1}(\Omega_{\mathrm{P}}))} + \|\partial_t\boldsymbol{y}_s^{\mathrm{P}}\|_{L^2(0,T;\mathbf{H}^{r_{k_s}+1}(\Omega_{\mathrm{P}}))}\\
&\qquad\qquad + \|\partial_t\boldsymbol{y}_s^{\mathrm{P}}\|_{L^\infty(0,T;\mathbf{H}^{r_{k_s}+1}(\Omega_{\mathrm{P}}))} + \|\partial_{tt}\boldsymbol{y}_s^{\mathrm{P}}\|_{L^2(0,T;\mathbf{H}^{r_{k_s}+1}(\Omega_{\mathrm{P}}))}\Big)\\
&\quad + h^{r_{s_s}+1}\left(\|\boldsymbol{u}_s^{\mathrm{P}}\|_{L^2(0,T;\mathbf{H}^{r_{s_s}+1}(\Omega_{\mathrm{P}}))} + \|\partial_t\boldsymbol{u}_s^{\mathrm{P}}\|_{L^2(0,T;\mathbf{H}^{r_{s_s}+1}(\Omega_{\mathrm{P}}))}\right)\\
&\quad + h^{r_{\tilde{k}_p}+\frac{1}{2}}\left(\|\lambda\|_{L^2(0,T;H^{r_{\tilde{k}_p}}(\Sigma))} + \|\lambda\|_{L^\infty(0,T;H^{r_{\tilde{k}_p}}(\Sigma))} + \|\partial_t\lambda\|_{L^2(0,T;H^{r_{\tilde{k}_p}}(\Sigma))}\right)\Bigg],
\end{aligned}$$

where $0\le r_{k_f}\le k_f$, $0\le r_{s_f}\le s_f$, $1\le r_{k_p}\le k_p$, $0\le r_{s_p}\le s_p$, $1\le r_{k_s}\le k_s$, $0\le r_{s_s}\le s_s$, $-1/2\le r_{\tilde{k}_p}\le\tilde{k}_p-1/2$.

Proof. We recall that, due to (3.1), $(\boldsymbol{u}_f^{\mathrm{S}}, \boldsymbol{u}_r^{\mathrm{P}}, \partial_t \boldsymbol{y}_s^{\mathrm{P}}) \in \mathbf{U}$ and so $(\mathbf{I}_{f,h}\boldsymbol{u}_f^{\mathrm{S}}, \mathbf{I}_{r,h}\boldsymbol{u}_r^{\mathrm{P}}, \mathbf{I}_{s,h}\partial_t \boldsymbol{y}_s^{\mathrm{P}}) \in \mathbf{U}_h$ for any $t \in (0,T]$. We split individual errors into approximation and discretization contributions

$$
\begin{aligned}
\mathbf{e}_i &:= \boldsymbol{u}_i^{\mathrm{S}} - \boldsymbol{u}_{i,h}^{\mathrm{S}} = \left(\boldsymbol{u}_i^{\mathrm{S}} - \mathbf{I}_{i,h}\boldsymbol{u}_i^{\mathrm{S}}\right) + \left(\mathbf{I}_{i,h}\boldsymbol{u}_i^{\mathrm{S}} - \boldsymbol{u}_{i,h}^{\mathrm{S}}\right) := \boldsymbol{\chi}_i + \boldsymbol{\phi}_{i,h}, \quad i \in \{f,r\},\\
\mathbf{e}_s &:= \boldsymbol{y}_s^{\mathrm{P}} - \boldsymbol{y}_{s,h}^{\mathrm{P}} = \left(\boldsymbol{y}_s^{\mathrm{P}} - \mathbf{I}_{s,h}\boldsymbol{y}_s^{\mathrm{P}}\right) + \left(\mathbf{I}_{s,h}\boldsymbol{y}_s^{\mathrm{P}} - \boldsymbol{y}_{s,h}^{\mathrm{P}}\right) := \boldsymbol{\chi}_s + \boldsymbol{\phi}_{s,h},\\
\mathbf{e}_{ss} &:= \boldsymbol{u}_s^{\mathrm{P}} - \boldsymbol{u}_{s,h}^{\mathrm{P}} = \left(\boldsymbol{u}_s^{\mathrm{P}} - \mathbf{Q}_{s,h}\boldsymbol{u}_s^{\mathrm{P}}\right) + \left(\mathbf{Q}_{s,h}\boldsymbol{u}_s^{\mathrm{P}} - \boldsymbol{u}_{s,h}^{\mathrm{P}}\right) := \boldsymbol{\chi}_{ss} + \boldsymbol{\phi}_{ss,h},\\
e_{fp} &:= p^{\mathrm{S}} - p_h^{\mathrm{S}} = \left(p^{\mathrm{S}} - Q_{f,h}p^{\mathrm{S}}\right) + \left(Q_{f,h}p^{\mathrm{S}} - p_h^{\mathrm{S}}\right) := \chi_{fp} + \phi_{fp,h},\\
e_{pp} &:= p^{\mathrm{P}} - p_h^{\mathrm{P}} = \left(p^{\mathrm{P}} - Q_{p,h}p^{\mathrm{P}}\right) + \left(Q_{p,h}p^{\mathrm{P}} - p_h^{\mathrm{P}}\right) := \chi_{pp} + \phi_{pp,h},\\
e_\lambda &:= \lambda - \lambda_h = \left(\lambda - Q_{\lambda,h}\lambda\right) + \left(Q_{\lambda,h}\lambda - \lambda_h\right) := \chi_\lambda + \phi_{\lambda,h}.
\end{aligned}
$$

Subtracting (4.1) from (3.1) yields the following error equation

$$
\begin{aligned}
&a_f^{\mathrm{S}}(\mathbf{e}_f, \boldsymbol{v}_{f,h}^{\mathrm{S}}) + a_f^{\mathrm{P}}(\mathbf{e}_r, \boldsymbol{w}_{s,h}^{\mathrm{P}}) + a_s^{\mathrm{P}}(\mathbf{e}_s, \boldsymbol{w}_{s,h}^{\mathrm{P}}) + a_f^{\mathrm{P}}(\mathbf{e}_r, \boldsymbol{v}_{r,h}^{\mathrm{P}}) + a_f^{\mathrm{P}}(\partial_t\mathbf{e}_s, \boldsymbol{v}_{r,h}^{\mathrm{P}}) + a_{\mathrm{BJS}}(\mathbf{e}_f, \partial_t\mathbf{e}_s; \boldsymbol{v}_{f,h}^{\mathrm{S}}, \boldsymbol{w}_{s,h}^{\mathrm{P}}) + a_f^{\mathrm{P}}(\partial_t\mathbf{e}_s, \boldsymbol{w}_{s,h}^{\mathrm{P}})\\
&\quad + b^{\mathrm{S}}(\boldsymbol{v}_{f,h}^{\mathrm{S}}, e_{fp}) + b_s^{\mathrm{P}}(\boldsymbol{w}_{s,h}^{\mathrm{P}}, e_{pp}) + b_f^{\mathrm{P}}(\boldsymbol{v}_{r,h}^{\mathrm{P}}, e_{pp}) + b_\Gamma(\boldsymbol{v}_{f,h}^{\mathrm{S}}, \boldsymbol{v}_{r,h}^{\mathrm{P}}, \boldsymbol{w}_{s,h}^{\mathrm{P}}; e_\lambda) - b_s^{\mathrm{P}}(\partial_t\mathbf{e}_s, q_h^{\mathrm{P}}) - b_f^{\mathrm{P}}(\mathbf{e}_r, q_h^{\mathrm{P}}) - b^{\mathrm{S}}(\mathbf{e}_f, q_h^{\mathrm{S}})\\
&\quad - m_\theta(\mathbf{e}_r, \boldsymbol{w}_{s,h}^{\mathrm{P}}) - m_\theta(\partial_t\mathbf{e}_s, \boldsymbol{w}_{s,h}^{\mathrm{P}}) - m_\theta(\mathbf{e}_r, \boldsymbol{v}_{r,h}^{\mathrm{P}}) - m_\theta(\partial_t\mathbf{e}_s, \boldsymbol{v}_{r,h}^{\mathrm{P}}) + m_{\phi^2/\kappa}(\mathbf{e}_r, \boldsymbol{v}_{r,h}^{\mathrm{P}}) + b_{\mathrm{BJS}}(\mathbf{e}_r; \boldsymbol{v}_{r,h}^{\mathrm{P}}) + m_{\rho_f\phi}(\partial_t\mathbf{e}_r, \boldsymbol{w}_{s,h}^{\mathrm{P}})\\
&\quad + m_{\rho_p}(\partial_t\mathbf{e}_{ss}, \boldsymbol{w}_{s,h}^{\mathrm{P}}) + m_{\rho_f\phi}(\partial_t\mathbf{e}_r, \boldsymbol{v}_{r,h}^{\mathrm{P}}) + m_{\rho_f}(\partial_t\mathbf{e}_f, \boldsymbol{v}_{f,h}^{\mathrm{S}}) + m_{\rho_f\phi}(\partial_t\mathbf{e}_{ss}, \boldsymbol{v}_{r,h}^{\mathrm{P}}) + \left((1-\phi)^2K^{-1}\partial_t e_{pp}, q_h^{\mathrm{P}}\right)_{\Omega_{\mathrm{P}}} = 0.
\end{aligned}
$$

Next, setting $\boldsymbol{v}_{f,h}^{\mathrm{S}} = \boldsymbol{\phi}_{f,h}, \boldsymbol{v}_{r,h}^{\mathrm{P}} = \boldsymbol{\phi}_{r,h}, \boldsymbol{w}_{s,h}^{\mathrm{P}} = \partial_t\boldsymbol{\phi}_{s,h}, \boldsymbol{v}_{s,h}^{\mathrm{P}} = \boldsymbol{\phi}_{ss,h}, q_h^{\mathrm{S}} = \phi_{fp,h},$ and $q_h^{\mathrm{P}} = \phi_{pp,h}$, we get

$$
b^{\mathrm{S}}\left(\boldsymbol{\chi}_f, \phi_{fp,h}\right) = b_f^{\mathrm{P}}\left(\boldsymbol{\chi}_r, \phi_{pp,h}\right) = b_f^{\mathrm{P}}\left(\boldsymbol{\phi}_{r,h}, \chi_{pp}\right) = 0, \quad \left((1-\phi)^2K^{-1}\partial_t\chi_{pp}, \phi_{pp,h}\right) = 0,
$$

where we have used the properties of the projection operators (4.2b), (4.10b), and (4.10c). Moreover, from (4.10a) and (4.1) we obtain $b_\Gamma(\boldsymbol{\phi}_{f,h}, \boldsymbol{\phi}_{r,h}, \partial_t\boldsymbol{\phi}_{s,h}; \phi_{\lambda,h}) = 0$. On the other hand, by definition of $H_{00}^{-1/2}(\Sigma)$ [48], there exists $\boldsymbol{w} \in \mathbf{H}^1(\Omega_{\mathrm{P}})$ such that

$$
\langle \lambda, \boldsymbol{w}\cdot\boldsymbol{n}_{\mathrm{P}}\rangle_\Sigma = \|\lambda\|_{\Lambda_h}^2, \quad \text{and} \quad \|\boldsymbol{w}\|_{1,\Omega_{\mathrm{P}}} = \|\lambda\|_{\Lambda_h}. \tag{4.14}
$$

This implies that $\langle \chi_\lambda, \boldsymbol{\phi}_{f,h}\cdot\boldsymbol{n}_{\mathrm{P}}\rangle_\Sigma = \langle \chi_\lambda, \boldsymbol{\phi}_{r,h}\cdot\boldsymbol{n}_{\mathrm{P}}\rangle_\Sigma = \|\chi_\lambda\|_{\Lambda_h}^2$. Rearranging terms, the error equation becomes

$$
\begin{aligned}
&a_f^{\mathrm{S}}(\boldsymbol{\phi}_{f,h}, \boldsymbol{\phi}_{f,h}) + a_f^{\mathrm{P}}(\boldsymbol{\phi}_{r,h}, \partial_t\boldsymbol{\phi}_{s,h}) + a_f^{\mathrm{P}}(\boldsymbol{\phi}_{r,h}, \boldsymbol{\phi}_{r,h}) + a_f^{\mathrm{P}}(\partial_t\boldsymbol{\phi}_{s,h}, \boldsymbol{\phi}_{r,h}) + a_f^{\mathrm{P}}(\partial_t\boldsymbol{\phi}_{s,h}, \partial_t\boldsymbol{\phi}_{s,h}) + a_s^{\mathrm{P}}(\boldsymbol{\phi}_{s,h}, \partial_t\boldsymbol{\phi}_{s,h})\\
&\quad + a_{\mathrm{BJS}}(\boldsymbol{\phi}_{f,h}, \partial_t\boldsymbol{\phi}_{s,h}; \boldsymbol{\phi}_{f,h}, \partial_t\boldsymbol{\phi}_{s,h}) + b_{\mathrm{BJS}}(\boldsymbol{\phi}_{r,h}; \boldsymbol{\phi}_{r,h}) + m_{\rho_f\phi}(\partial_t\boldsymbol{\phi}_{r,h}, \partial_t\boldsymbol{\phi}_{s,h}) + m_{\rho_p}(\partial_t\boldsymbol{\phi}_{ss,h}, \partial_t\boldsymbol{\phi}_{s,h})\\
&\quad + m_{\rho_f\phi}(\partial_t\boldsymbol{\phi}_{r,h}, \boldsymbol{\phi}_{r,h}) + m_{\rho_f\phi}(\partial_t\boldsymbol{\phi}_{ss,h}, \boldsymbol{\phi}_{r,h}) + m_{\rho_f}(\partial_t\boldsymbol{\phi}_{f,h}, \boldsymbol{\phi}_{f,h}) + ((1-\phi)^2K^{-1}\partial_t\phi_{pp,h}, \phi_{pp,h})_{\Omega_{\mathrm{P}}}\\
&\quad - m_\theta(\boldsymbol{\phi}_{r,h}, \partial_t\boldsymbol{\phi}_{s,h}) - m_\theta(\partial_t\boldsymbol{\phi}_{s,h}, \partial_t\boldsymbol{\phi}_{s,h}) - m_\theta(\boldsymbol{\phi}_{r,h}, \boldsymbol{\phi}_{r,h}) - m_\theta(\partial_t\boldsymbol{\phi}_{s,h}, \boldsymbol{\phi}_{r,h}) + m_{\phi^2/\kappa}(\boldsymbol{\phi}_{r,h}, \boldsymbol{\phi}_{r,h})\\
&\quad = \mathscr{I}_1 + \mathscr{I}_2 + \mathscr{I}_3 + \mathscr{I}_4 + \mathscr{I}_5 + \mathscr{I}_6,
\end{aligned} \tag{4.15}
$$

where the RHS terms are defined as follows

$$
\begin{aligned}
\mathscr{I}_1 &:= -a_f^{\mathrm{S}}\left(\boldsymbol{\chi}_f, \boldsymbol{\phi}_{f,h}\right) + m_\theta(\boldsymbol{\chi}_r, \boldsymbol{\phi}_{ss,h}) + m_\theta\left(\boldsymbol{\chi}_r, \boldsymbol{\phi}_{r,h}\right) + m_\theta(\partial_t\boldsymbol{\chi}_s, \boldsymbol{\phi}_{r,h}) - m_{\phi^2/\kappa}(\boldsymbol{\chi}_r, \boldsymbol{\phi}_{r,h}) - m_{\rho_p}(\partial_t\boldsymbol{\chi}_{ss}, \boldsymbol{\phi}_{ss,h})\\
&\qquad - m_{\rho_f\phi}(\partial_t\boldsymbol{\chi}_r, \boldsymbol{\phi}_{r,h}) - m_{\rho_f\phi}\left(\partial_t\boldsymbol{\chi}_{ss}, \boldsymbol{\phi}_{r,h}\right) + m_\theta(\partial_t\boldsymbol{\chi}_s, \boldsymbol{\phi}_{ss,h}) - m_{\rho_f\phi}(\partial_t\boldsymbol{\chi}_r, \boldsymbol{\phi}_{ss,h}) - m_{\rho_f}(\partial_t\boldsymbol{\chi}_{f,h}, \boldsymbol{\phi}_{f,h}),\\
\mathscr{I}_2 &:= -a_f^{\mathrm{P}}\left(\boldsymbol{\chi}_r, \boldsymbol{\phi}_{r,h}\right) - a_f^{\mathrm{P}}\left(\partial_t\boldsymbol{\chi}_s, \boldsymbol{\phi}_{r,h}\right),
\end{aligned}
$$

$$\mathscr{J}_3 := -\sum_{j=1}^{d-1}\left\langle \mu_f \alpha_{\rm BJS}\sqrt{Z_j^{-1}}\left(\boldsymbol{\chi}_f - \partial_t \boldsymbol{\chi}_s\right)\cdot \boldsymbol{\tau}_{f,j}, \left(\boldsymbol{\phi}_{f,h} - \partial_t \boldsymbol{\phi}_{s,h}\right)\cdot \boldsymbol{\tau}_{f,j}\right\rangle_\Sigma - \sum_{j=1}^{d-1}\left\langle \mu_f \alpha_{\rm BJS}\sqrt{Z_j^{-1}}\boldsymbol{\chi}_r\cdot \boldsymbol{\tau}_{f,j}, \boldsymbol{\phi}_{r,h}\cdot\boldsymbol{\tau}_{f,j}\right\rangle_\Sigma,$$
$$\mathscr{J}_4 := -b^{\rm S}\left(\boldsymbol{\phi}_{f,h}, \chi_{fp}\right) + b_s^{\rm P}\left(\partial_t\boldsymbol{\chi}_s, \phi_{pp,h}\right) - b_\Gamma\left(\boldsymbol{\phi}_{f,h}, \boldsymbol{\phi}_{r,h}, 0; \chi_\lambda\right),$$
$$\mathscr{J}_5 := -a_f^{\rm P}\left(\boldsymbol{\chi}_r, \partial_t\boldsymbol{\phi}_{s,h}\right) - a_s^{\rm P}\left(\boldsymbol{\chi}_s, \partial_t\boldsymbol{\phi}_{s,h}\right) - a_f^{\rm P}\left(\partial_t\boldsymbol{\chi}_s, \partial_t\boldsymbol{\phi}_{s,h}\right),\ \mathscr{J}_6 := -\langle \partial_t\boldsymbol{\phi}_{s,h}\cdot \boldsymbol{n}_{\rm P}, \chi_\lambda\rangle_\Sigma - b_s^{\rm P}(\partial_t\boldsymbol{\phi}_{s,h}, \chi_{pp}).$$

It is important to remark that we have the equation

$$\begin{aligned} &m_{\rho_f\phi}(\partial_t\boldsymbol{\phi}_{r,h}, \partial_t\boldsymbol{\phi}_{s,h}) + m_{\rho_s(1-\phi)}\left(\partial_t\boldsymbol{\phi}_{ss,h}, \partial_t\boldsymbol{\phi}_{s,h}\right) + m_{\rho_f\phi}\left(\partial_t\boldsymbol{\phi}_{ss,h}, \partial_t\boldsymbol{\phi}_{s,h}\right) + m_{\rho_f\phi}(\partial_t\boldsymbol{\phi}_{r,h}, \boldsymbol{\phi}_{r,h}) \\ &\quad + m_{\rho_f\phi}(\partial_t\boldsymbol{\phi}_{ss,h}, \boldsymbol{\phi}_{r,h}) + m_{\rho_f}(\partial_t\boldsymbol{\phi}_{f,h}, \boldsymbol{\phi}_{f,h}) \\ &= \frac{1}{2}\partial_t\left(\|\sqrt{\rho_f\phi}(\boldsymbol{\phi}_{r,h} + \boldsymbol{\phi}_{ss,h})\|_{0,\Omega_{\rm P}}^2 + \|\sqrt{\rho_s(1-\phi)}\boldsymbol{\phi}_{ss,h}\|_{0,\Omega_{\rm P}}^2 + \|\sqrt{\rho_f}\boldsymbol{\phi}_{f,h}\|_{0,\Omega_{\rm S}}^2\right). \end{aligned}$$

Next, we combine the above equation with assumptions (H.1)–(H.3), inequality (A.14), and the estimate $\rho_f\,\phi\,\|\boldsymbol{\phi}_{r,h} + \boldsymbol{\phi}_{ss,h}\|_{0,\Omega_{\rm P}}^2 \geq \rho_f\,\phi\left(\frac{1}{2}\|\boldsymbol{\phi}_{r,h}\|_{0,\Omega_{\rm P}}^2 - \|\boldsymbol{\phi}_{ss,h}\|_{0,\Omega_{\rm P}}^2\right)$, and then use the respective coercivity to get

$$\begin{aligned} \texttt{LHS}_{(4.15)} \gtrsim \frac{1}{2}\partial_t\left(\|\boldsymbol{\phi}_{f,h}\|_{0,\Omega_{\rm S}}^2 + \|\boldsymbol{\phi}_{r,h}\|_{0,\Omega_{\rm P}}^2 + \|\boldsymbol{\phi}_{s,h}\|_{1,\Omega_{\rm P}}^2 + \|\phi_{pp,h}\|_{0,\Omega_{\rm P}}^2 + \|\boldsymbol{\phi}_{ss,h}\|_{0,\Omega_{\rm P}}^2\right) + \left|\boldsymbol{\phi}_{f,h} - \partial_t\boldsymbol{\phi}_{s,h}\right|_{\rm BJS}^2 \\ + \left|\boldsymbol{\phi}_{r,h}\right|_{\rm BJS}^2 + \|\boldsymbol{\phi}_{r,h}\|_{0,\Omega_{\rm P}}^2 + \|\boldsymbol{\phi}_{f,h}\|_{1,\Omega_{\rm S}}^2. \end{aligned} \tag{4.16}$$

We proceed to bound the terms on the RHS in (4.15). Using Lemma A.1, Cauchy–Schwarz and Young's inequalities, as well as (4.14), we have

$$\begin{aligned} \mathscr{J}_1 \leq C\varepsilon_1^{-1}\left(\|\boldsymbol{\chi}_f\|_{1,\Omega_{\rm S}}^2 + \|\boldsymbol{\chi}_r\|_{0,\Omega_{\rm P}}^2 + \|\partial_t\boldsymbol{\chi}_s\|_{0,\Omega_{\rm P}}^2 + \|\partial_t\boldsymbol{\chi}_{ss}\|_{0,\Omega_{\rm P}}^2 + \|\partial_t\boldsymbol{\chi}_r\|_{0,\Omega_{\rm P}}^2 + \|\partial_t\boldsymbol{\chi}_f\|_{0,\Omega_{\rm S}}^2\right) \\ + \varepsilon_1\left(\|\boldsymbol{\phi}_{f,h}\|_{1,\Omega_{\rm S}}^2 + \|\boldsymbol{\phi}_{ss,h}\|_{0,\Omega_{\rm P}}^2 + \|\boldsymbol{\phi}_{r,h}\|_{0,\Omega_{\rm P}}^2 + \|\boldsymbol{\phi}_{f,h}\|_{0,\Omega_{\rm S}}^2\right). \end{aligned}$$

Next we estimate the terms involving $\boldsymbol{\phi}_{r,h}$ in $\mathbf{H}^1(\Omega_{\rm P})$, since we do not have bounds on $\boldsymbol{\phi}_{r,h}$ in the energy norm on the LHS. We use inverse, Cauchy–Schwarz, and Young's inequalities. Taking $\varepsilon = \varepsilon_1 h^2$, we get

$$\mathscr{J}_2 \leq \varepsilon_1\|\boldsymbol{\phi}_{r,h}\|_{0,\Omega_{\rm P}}^2 + C\varepsilon_1^{-1}h^{-2}(\|\partial_t\boldsymbol{\chi}_s\|_{1,\Omega_{\rm P}}^2 + \|\boldsymbol{\chi}_r\|_{1,\Omega_{\rm P}}^2). \tag{4.17}$$

Similarly, using Cauchy–Schwarz, trace and Young's inequalities, we obtain

$$\mathscr{J}_3 \leq \varepsilon_1(\left|\boldsymbol{\phi}_{f,h} - \partial_t\boldsymbol{\phi}_{s,h}\right|_{\rm BJS}^2 + \left|\boldsymbol{\phi}_{r,h}\right|_{\rm BJS}^2) + C\varepsilon_1^{-1}\left(\|\boldsymbol{\chi}_f\|_{1,\Omega_{\rm S}}^2 + \|\partial_t\boldsymbol{\chi}_s\|_{1,\Omega_{\rm P}}^2 + \|\boldsymbol{\chi}_r\|_{1,\Omega_{\rm P}}^2\right).$$

Finally, using Cauchy–Schwarz and Young's inequalities as well as (4.14), we can derive the bound

$$\mathscr{J}_4 \leq C\varepsilon_1^{-1}\left(\|\chi_{fp}\|_{0,\Omega_{\rm S}}^2 + \|\partial_t\boldsymbol{\chi}_s\|_{1,\Omega_{\rm P}}^2\right) + \varepsilon_1\left(\|\boldsymbol{\phi}_{f,h}\|_{1,\Omega_{\rm S}}^2 + \|\phi_{pp,h}\|_{0,\Omega_{\rm P}}^2\right) + C\|\chi_\lambda\|_{\Lambda_h}. \tag{4.18}$$

Combining (4.15)-(4.18), integrating over $[0,t]$, where $0 < t \leq T$, and taking ε_1 small enough, gives

$$\begin{aligned} &\|\boldsymbol{\phi}_{f,h}(t)\|_{0,\Omega_{\rm S}}^2 + \|\boldsymbol{\phi}_{r,h}(t)\|_{0,\Omega_{\rm P}}^2 + \|\boldsymbol{\phi}_{s,h}(t)\|_{1,\Omega_{\rm P}}^2 + \|\phi_{pp,h}(t)\|_{0,\Omega_{\rm P}}^2 + \|\boldsymbol{\phi}_{ss,h}(t)\|_{0,\Omega_{\rm P}}^2 + \int_0^t \left|\boldsymbol{\phi}_{f,h} - \partial_t\boldsymbol{\phi}_{s,h}\right|_{\rm BJS}^2 \\ &+ \int_0^t \left|\boldsymbol{\phi}_{r,h}\right|_{\rm BJS}^2 + \|\boldsymbol{\phi}_{r,h}\|_{L^2(0,T;\mathbf{L}^2(\Omega_{\rm P}))}^2 + \|\boldsymbol{\phi}_{f,h}\|_{L^2(0,T;\mathbf{H}^1(\Omega_{\rm S}))}^2 \end{aligned}$$

$$
\begin{aligned}
&\le C\varepsilon_1^{-1}\Big(\|\partial_t\boldsymbol{\chi}_f\|^2_{L^2(0,T;\mathbf{L}^2(\Omega_{\rm P}))}+\|\boldsymbol{\chi}_f\|^2_{L^2(0,T;\mathbf{H}^1(\Omega_{\rm S}))}+\|\partial_t\boldsymbol{\chi}_s\|^2_{L^2(0,T;\mathbf{H}^1(\Omega_{\rm P}))}+\|\boldsymbol{\chi}_r\|^2_{L^2(0,T;\mathbf{L}^2(\Omega_{\rm P}))}\\
&\quad+\|\boldsymbol{\chi}_r\|^2_{L^2(0,T;\mathbf{H}^1(\Omega_{\rm P}))}+\|\partial_t\boldsymbol{\chi}_s\|^2_{L^2(0,T;\mathbf{L}^2(\Omega_{\rm P}))}+\|\partial_t\boldsymbol{\chi}_{ss}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega_{\rm P}))}+\|\partial_t\boldsymbol{\chi}_r\|^2_{L^2(0,T;\mathbf{L}^2(\Omega_{\rm P}))}\\
&\quad+\|\chi_{fp}\|^2_{L^2(0,T;L^2(\Omega_{\rm S}))}+h^{-2}\|\partial_t\boldsymbol{\chi}_s\|^2_{L^2(0,T;\mathbf{H}^1(\Omega_{\rm P}))}+h^{-2}\|\boldsymbol{\chi}_r\|^2_{L^2(0,T;\mathbf{L}^2(\Omega_{\rm P}))}\Big)\\
&\quad+C\|\chi_\lambda\|_{L^2(0,T;\Lambda_h})^2+\|\boldsymbol{\phi}_{ss,h}(0)\|^2_{0,\Omega_{\rm P}}+\varepsilon_1\Big(\|\boldsymbol{\phi}_{f,h}\|^2_{L^2(0,T;\mathbf{H}^1(\Omega_{\rm S}))}+\|\boldsymbol{\phi}_{ss,h}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega_{\rm P}))}\\
&\quad+\|\boldsymbol{\phi}_{r,h}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega_{\rm P}))}+\|\phi_{pp,h}\|^2_{L^2(0,T;L^2(\Omega_{\rm P}))}+\|\boldsymbol{\phi}_{f,h}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega_{\rm S}))}+\int_0^t\left|\boldsymbol{\phi}_{f,h}-\partial_t\boldsymbol{\phi}_{s,h}\right|^2_{\rm BJS}\\
&\quad+\int_0^t\left|\boldsymbol{\phi}_{r,h}\right|^2_{\rm BJS}\Big)+\|\boldsymbol{\phi}_{f,h}(0)\|^2_{0,\Omega_{\rm P}}+\|\boldsymbol{\phi}_{r,h}(0)\|^2_{0,\Omega_{\rm P}}+\|\boldsymbol{\phi}_{s,h}(0)\|^2_{1,\Omega_{\rm P}}+\|\phi_{pp,h}(0)\|^2_{L^2(\Omega)}\\
&\quad+\|\boldsymbol{\phi}_{ss,h}(0)\|^2_{1,\Omega_{\rm P}}+C\int_0^t(\mathscr{J}_5+\mathscr{J}_6)\,\mathrm{d}s. \qquad (4.19)
\end{aligned}
$$

Taking $\boldsymbol{u}^{\rm S}_{f,h}(0)=\mathbf{I}_{f,h}\boldsymbol{u}_{f,0}$, $\boldsymbol{u}^{\rm P}_{r,h}(0)=\mathbf{I}_{r,h}\boldsymbol{u}_{r,0}$, $\boldsymbol{y}^{\rm P}_{s,h}(0)=\mathbf{I}_{s,h}\boldsymbol{y}_{s,0}$, $p^{\rm P}_{p,h}(0)=Q_{p,h}p^{p,0}$, $\boldsymbol{u}^{\rm P}_{s,h}(0)=\mathbf{Q}_{s,h}\boldsymbol{u}_{s,0}$, gives $\boldsymbol{\phi}_{f,h}(0)=0$, $\boldsymbol{\phi}_{r,h}(0)=0$, $\boldsymbol{\phi}_{s,h}(0)=0$, $\phi_{pp,h}(0)=0$ $\boldsymbol{\phi}_{ss,h}(0)=0$.

Next, we bound the terms on the RHS involving $\partial_t\boldsymbol{\phi}_{s,h}$. By applying integration by parts over $[0,t]$, where $0<t\le T$, along with Cauchy–Schwarz and Young's inequalities, Lemma A.1, and (4.14), we obtain

$$
\begin{aligned}
\int_0^t\mathscr{J}_5\,\mathrm{d}s&=a^{\rm P}_f(\boldsymbol{\chi}_r,\boldsymbol{\phi}_{s,h})\big|_0^t-\int_0^t a^{\rm P}_f(\partial_t\boldsymbol{\chi}_r,\boldsymbol{\phi}_{s,h})\,\mathrm{d}s+a^{\rm P}_s(\boldsymbol{\chi}_s,\boldsymbol{\phi}_{s,h})\big|_0^t-\int_0^t a^{\rm P}_s(\partial_t\boldsymbol{\chi}_s,\boldsymbol{\phi}_{s,h})\,\mathrm{d}s\\
&\quad+a^{\rm P}_f(\partial_t\boldsymbol{\chi}_s,\boldsymbol{\phi}_{s,h})\,|_0^t-\int_0^t a^{\rm P}_f(\partial_{tt}\boldsymbol{\chi}_s,\boldsymbol{\phi}_{s,h})\,\mathrm{d}s\\
&\le C\Big(\|\partial_t\boldsymbol{\chi}_r\|^2_{L^2(0,T;\mathbf{H}^1(\Omega_{\rm P}))}+\|\partial_t\boldsymbol{\chi}_s\|^2_{L^2(0,T;\mathbf{H}^1(\Omega_{\rm P}))}+\|\partial_{tt}\boldsymbol{\chi}_s\|^2_{L^2(0,T;\mathbf{H}^1(\Omega_{\rm P}))}+\|\boldsymbol{\phi}_{s,h}\|^2_{L^2(0,T;\mathbf{H}^1(\Omega_{\rm P}))}\Big)\\
&\quad+C\varepsilon_1^{-1}\left(\|\boldsymbol{\chi}_r(t)\|^2_{1,\Omega_{\rm P}}+\|\boldsymbol{\chi}_s(t)\|^2_{1,\Omega_{\rm P}}+\|\partial_t\boldsymbol{\chi}_s(t)\|^2_{1,\Omega_{\rm P}}\right)+\varepsilon_1\|\boldsymbol{\phi}_{s,h}(t)\|^2_{1,\Omega_{\rm P}}.
\end{aligned}
$$

Employing again Cauchy–Schwarz and Young's inequalities, we have

$$
\begin{aligned}
\int_0^t\mathscr{J}_6\,\mathrm{d}s&=\left\langle\boldsymbol{\phi}_{s,h}\cdot\boldsymbol{n}_{\rm P},\chi_\lambda\right\rangle_\Sigma\Big|_0^t-\int_0^t\left\langle\boldsymbol{\phi}_{s,h}\cdot\boldsymbol{n}_{\rm P},\partial_t\chi_\lambda\right\rangle_\Sigma\mathrm{d}s+b^{\rm P}_s(\boldsymbol{\phi}_{s,h},\chi_{pp})\big|_0^t-\int_0^t b^{\rm P}_s(\boldsymbol{\phi}_{s,h},\partial_t\chi_{pp})\,\mathrm{d}s\\
&\le C\Big(\varepsilon_1^{-1}\|\chi_{pp}(t)\|^2_{0,\Omega_{\rm P}}+\|\partial_t\chi_{pp}\|^2_{L^2(0,T;L^2(\Omega_{\rm P}))}+\|\partial_t\chi_\lambda\|^2_{L^2(0,T;\Lambda_h)}\Big)\\
&\quad+\|\chi_\lambda(t)\|^2_{\Lambda_h}+\varepsilon_1\|\boldsymbol{\phi}_{s,h}(t)\|^2_{1,\Omega_{\rm P}}+\|\boldsymbol{\phi}_{s,h}\|^2_{L^2(0,T;\mathbf{H}^1(\Omega_{\rm P}))}. \qquad (4.20)
\end{aligned}
$$

Choosing a sufficiently small ε_1 in (4.19)-(4.20), we derive the following bound from (4.19)

$$
\begin{aligned}
&\|\boldsymbol{\phi}_{f,h}(t)\|^2_{0,\Omega_{\rm S}}+\|\boldsymbol{\phi}_{r,h}(t)\|^2_{0,\Omega_{\rm P}}+\|\boldsymbol{\phi}_{s,h}(t)\|^2_{1,\Omega_{\rm P}}+\|\phi_{pp,h}(t)\|^2_{0,\Omega_{\rm P}}+\|\boldsymbol{\phi}_{ss,h}(t)\|^2_{0,\Omega_{\rm P}}+|\boldsymbol{\phi}_{f,h}-\partial_t\boldsymbol{\phi}_{s,h}|^2_{L^2(0,T;{\rm BJS})}\\
&\quad+\left|\boldsymbol{\phi}_{r,h}\right|^2_{L^2(0,T;{\rm BJS})}+\|\boldsymbol{\phi}_{r,h}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega_{\rm P}))}+\|\boldsymbol{\phi}_{f,h}\|^2_{L^2(0,T;\mathbf{H}^1(\Omega_{\rm S}))}\le\frac{C}{\varepsilon_1}\left(\|\boldsymbol{\chi}_f\|^2_{L^2(0,T;\mathbf{H}^1(\Omega_{\rm S}))}+\|\boldsymbol{\chi}_r\|^2_{L^2(0,T;\mathbf{L}^2(\Omega_{\rm P}))}\right.\\
&\quad+h^{-2}\|\boldsymbol{\chi}_r\|^2_{L^2(0,T;\mathbf{H}^1(\Omega_{\rm P}))}+h^{-2}\|\partial_t\boldsymbol{\chi}_s\|^2_{L^2(0,T;\mathbf{H}^1(\Omega_{\rm P}))}+\|\partial_t\boldsymbol{\chi}_s\|^2_{L^2(0,T;\mathbf{L}^2(\Omega_{\rm P}))}+\|\partial_t\boldsymbol{\chi}_{ss}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega_{\rm P}))}\\
&\quad+\|\boldsymbol{\chi}_s(t)\|^2_{1,\Omega_{\rm P}}+\|\partial_t\boldsymbol{\chi}_r\|^2_{L^2(0,T;\mathbf{L}^2(\Omega_{\rm P}))}+\|\chi_{fp}\|^2_{L^2(0,T;L^2(\Omega_{\rm S}))}+\|\boldsymbol{\chi}_r(t)\|^2_{1,\Omega_{\rm P}}+\|\partial_t\boldsymbol{\chi}_s(t)\|^2_{1,\Omega_{\rm P}}+\|\chi_\lambda(t)\|^2_{\Lambda_h}\\
&\quad\left.+\|\chi_{pp}(t)\|^2_{0,\Omega_{\rm P}}\right)+C\Big(\|\partial_t\boldsymbol{\chi}_r\|^2_{\mathbf{L}^2(0,T;\mathbf{H}^1(\Omega_{\rm P}))}+\|\partial_t\chi_\lambda\|_{L^2(0,T;\Lambda_h})^2+\|\partial_t\chi_{pp}\|^2_{L^2(0,T;L^2(\Omega_{\rm P}))}\\
&\quad+\|\partial_{tt}\boldsymbol{\chi}_s\|^2_{L^2(0,T;\mathbf{H}^1(\Omega_{\rm P}))}+\|\chi_\lambda\|_{L^2(0,T;\Lambda_h})^2+\|\boldsymbol{\phi}_{s,h}\|^2_{\mathbf{L}^2(0,T;\mathbf{H}^1(\Omega_{\rm P}))}+\|\partial_t\boldsymbol{\chi}_f\|^2_{L^2(0,T;\mathbf{L}^2(\Omega_{\rm P}))}
\end{aligned}
$$

$$+\|\partial_t \boldsymbol{\chi}_s\|^2_{L^2\left(0,T;\mathbf{H}^1(\Omega_{\mathrm{P}})\right)}\Big). \tag{4.21}$$

Next, we employ the inf–sup condition (A.23b) with the choice $(q^{\mathrm{S}}, q^{\mathrm{P}}, \mu_h) = \left(\phi_{fp,h}, \phi_{pp,h}, \phi_{\lambda,h}\right)$ and utilize the error equation derived by subtracting (4.1) from (3.1). We treat the term $a_f^{\mathrm{P}}\left(\mathbf{e}_r, \boldsymbol{v}_{r,h}^{\mathrm{P}}\right)$ similarly as in (4.17). Integrating over the interval $(0,T]$ and applying Lemma A.1 along with the trace inequality, we obtain

$$\begin{aligned}
&\|\phi_{fp,h}\|^2_{L^2(0,T;L^2(\Omega_{\mathrm{S}}))} + \|\phi_{pp,h}\|^2_{L^2(0,T;L^2(\Omega_{\mathrm{P}}))} + \|\phi_{\lambda,h}\|_{L^2(0,T;\Lambda_h)})^2 \lesssim \|\boldsymbol{\phi}_{f,h}\|^2_{L^2(0,T;\mathbf{H}^1(\Omega_{\mathrm{S}}))} \\
&\quad + \|\boldsymbol{\phi}_{r,h}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega_{\mathrm{P}}))} + |\boldsymbol{\phi}_{f,h} - \partial_t \boldsymbol{\phi}_{s,h}|^2_{L^2(0,T;\mathrm{BJS})} + \|\boldsymbol{\phi}_{r,h}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega_{\mathrm{P}}))} + \|\boldsymbol{\chi}_f\|^2_{L^2(0,T;\mathbf{H}^1(\Omega_{\mathrm{S}}))} \\
&\quad + \|\boldsymbol{\chi}_r\|^2_{L^2(0,T;\mathbf{H}^1(\Omega_{\mathrm{P}}))} + \|\boldsymbol{\chi}_r\|^2_{L^2(0,T;\mathbf{L}^2(\Omega_{\mathrm{P}}))} + \|\chi_{fp}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega_{\mathrm{P}}))} + \|\chi_{pp}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega_{\mathrm{P}}))} + \|\chi_\lambda\|_{L^2(0,T;\Lambda_h})^2.
\end{aligned} \tag{4.22}$$

Adding (4.21) and (4.22), and taking ε_2 small enough, and then ε_1 small enough, gives

$$\begin{aligned}
&\|\boldsymbol{\phi}_{f,h}(t)\|^2_{0,\Omega_{\mathrm{S}}} + \|\boldsymbol{\phi}_{r,h}(t)\|^2_{0,\Omega_{\mathrm{P}}} + \|\boldsymbol{\phi}_{s,h}(t)\|^2_{1,\Omega_{\mathrm{P}}} + \|\phi_{pp,h}(t)\|^2_{L^2(\Omega)} + \|\boldsymbol{\phi}_{ss,h}(t)\|^2_{0,\Omega_{\mathrm{P}}} + \left|\boldsymbol{\phi}_{f,h} - \partial_t \boldsymbol{\phi}_{s,h}\right|^2_{L^2(0,T;\mathrm{BJS})} \\
&\quad + \left|\boldsymbol{\phi}_{r,h}\right|^2_{L^2(0,T;\mathrm{BJS})} + \|\boldsymbol{\phi}_{r,h}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega_{\mathrm{P}}))} + \|\boldsymbol{\phi}_{f,h}\|^2_{L^2(0,T;\mathbf{H}^1(\Omega_{\mathrm{S}}))} + \|\phi_{fp,h}\|^2_{L^2(0,T;L^2(\Omega_{\mathrm{S}}))} \\
&\quad + \|\phi_{pp,h}\|^2_{L^2(0,T;L^2(\Omega_{\mathrm{P}}))} + \|\phi_{\lambda,h}\|_{L^2(0,T;\Lambda_h})^2 \leq C\big(\|\boldsymbol{\phi}_{s,h}\|^2_{\mathbf{L}^2\left(0,T;\mathbf{H}^1(\Omega_{\mathrm{P}})\right)} \\
&\quad + \|\boldsymbol{\chi}_f\|^2_{L^2(0,T;\mathbf{H}^1(\Omega_{\mathrm{S}}))} + h^{-2}\|\boldsymbol{\chi}_r\|^2_{L^2(0,T;\mathbf{H}^1(\Omega_{\mathrm{P}}))} + h^{-2}\|\partial_t \boldsymbol{\chi}_s\|^2_{L^2(0,T;\mathbf{H}^1(\Omega_{\mathrm{P}}))} + \|\boldsymbol{\chi}_r\|^2_{L^2(0,T;\mathbf{L}^2(\Omega_{\mathrm{P}}))} \\
&\quad + \|\partial_t \boldsymbol{\chi}_s\|^2_{L^2(0,T;\mathbf{L}^2(\Omega_{\mathrm{P}}))} + \|\partial_t \boldsymbol{\chi}_{ss}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega_{\mathrm{P}}))} + \|\partial_t \boldsymbol{\chi}_r\|^2_{L^2(0,T;\mathbf{L}^2(\Omega_{\mathrm{P}}))} + \|\chi_{fp}\|^2_{L^2(0,T;L^2(\Omega_{\mathrm{S}}))} \\
&\quad + \|\boldsymbol{\chi}_r(t)\|^2_{1,\Omega_{\mathrm{P}}} + \|\boldsymbol{\chi}_s(t)\|^2_{1,\Omega_{\mathrm{P}}} + \|\chi_\lambda(t)\|_{\Lambda_h} + \|\chi_{pp}(t)\|^2_{0,\Omega_{\mathrm{P}}} + \|\partial_t \boldsymbol{\chi}_s(t)\|^2_{1,\Omega_{\mathrm{P}}} + \|\partial_t \boldsymbol{\chi}_r\|^2_{\mathbf{L}^2\left(0,T;\mathbf{H}^1(\Omega_{\mathrm{P}})\right)} \\
&\quad + \|\partial_t \chi_\lambda\|_{L^2(0,T;\Lambda_h)}) + \|\partial_t \chi_{pp}\|^2_{L^2\left(0,T;L^2(\Omega_{\mathrm{P}})\right)} + \|\partial_{tt} \boldsymbol{\chi}_s\|_{L^2(0,T;\mathbf{H}^1(\Omega_{\mathrm{P}}))} + \|\chi_{pp}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega_{\mathrm{P}}))} \\
&\quad + \|\chi_\lambda\|_{L^2(0,T;\Lambda_h})^2 + \|\partial_t \boldsymbol{\chi}_f\|^2_{L^2(0,T;\mathbf{L}^2(\Omega_{\mathrm{P}}))} + \|\partial_t \boldsymbol{\chi}_s\|^2_{\mathbf{L}^2\left(0,T;\mathbf{H}^1(\Omega_{\mathrm{P}})\right)}\big).
\end{aligned}$$

Grönwall's and triangle inequalities alongside the approximation properties (4.3a)-(4.3c), (4.4), and (4.13a)-(4.13c), imply the result. $\square$

Remark 4.5. *We observe that relative velocity and solid displacement exhibit sub-optimal convergence. Their corresponding blocks in the system matrix diagonal makes it difficult to derive an energy bound.*

5. Fully discrete formulation

5.1. *Definition and unique solvability*

For the time discretization we employ the backward Euler method with constant time-step τ, $T = N\tau$, and let $t_n = n\tau$, $0 \leq n \leq N$. Let $d_\tau u^n := \tau^{-1}(u^n - u^{n-1})$ be the first order (backward) discrete time derivative, where $u^n \approx u(t_n)$. The fully discrete problem reads: given $\boldsymbol{u}^0_{f,h} = \boldsymbol{u}^{\mathrm{S}}_{f,h}(0)$, $\boldsymbol{u}^0_{r,h} = \boldsymbol{u}^{\mathrm{P}}_{r,h}(0)$, $\boldsymbol{y}^0_{s,h} = \boldsymbol{y}^{\mathrm{P}}_{s,h}(0)$, $p_h^{\mathrm{P},0} = p_h^{\mathrm{P}}(0)$, and $\boldsymbol{u}^0_{s,h} = \boldsymbol{u}^{\mathrm{P}}_{s,h}(0)$, find $\vec{\boldsymbol{x}}_h^n \in \vec{\mathbf{X}}_h$, such that for $1 \leq n \leq N$, there holds

$$\mathrm{E}(\frac{1}{\tau}\vec{\boldsymbol{x}}_h^n, \vec{\boldsymbol{y}}_h) + \mathrm{H}(\vec{\boldsymbol{x}}_h^n, \vec{\boldsymbol{y}}_h) = \mathrm{F}^n(\vec{\boldsymbol{y}}_h) + \mathrm{E}(\frac{1}{\tau}\vec{\boldsymbol{x}}_h^{n-1}, \vec{\boldsymbol{y}}_h), \tag{5.1}$$

for all $\vec{\boldsymbol{y}}_h \in \vec{\mathbf{X}}_h$ and where F^n stands for the evaluation of F at time t_n.

Theorem 5.1. *The fully discrete method* (5.1) *has a unique solution under the assumptions* (H.1)-(H.3).

Proof. The aim is to show that $\mathrm{E}(\frac{1}{\tau}\vec{\boldsymbol{x}}_h^n, \vec{\boldsymbol{x}}_h) + \mathrm{H}(\vec{\boldsymbol{x}}_h^n, \vec{\boldsymbol{x}}_h)$ is coercive. Consider

$$
\begin{aligned}
\mathrm{E}(\frac{1}{\tau}\vec{\boldsymbol{x}}_h^n, \vec{\boldsymbol{x}}_h) + \mathrm{H}(\vec{\boldsymbol{x}}_h^n, \vec{\boldsymbol{x}}_h) &= a_f^{\mathrm{S}}(\boldsymbol{u}_{f,h}^{\mathrm{S},n}, \boldsymbol{u}_{f,h}^{\mathrm{S},n}) + \frac{1}{\tau} a_f^{\mathrm{P}}(\boldsymbol{u}_{r,h}^{\mathrm{P},n}, \boldsymbol{y}_{s,h}^{\mathrm{P},n}) + \frac{1}{\tau} a_s^{\mathrm{P}}(\boldsymbol{y}_{s,h}^{\mathrm{P},n}, \boldsymbol{y}_{s,h}^{\mathrm{P},n}) + a_f^{\mathrm{P}}(\boldsymbol{u}_{r,h}^{\mathrm{P},n}, \boldsymbol{u}_{r,h}^{\mathrm{P},n}) + \frac{1}{\tau} a_f^{\mathrm{P}}(\boldsymbol{y}_{s,h}^{\mathrm{P},n}, \boldsymbol{u}_{r,h}^{\mathrm{P},n}) \\
&+ \frac{1}{\tau^2} a_f^{\mathrm{P}}(\boldsymbol{y}_{s,h}^{\mathrm{P},n}, \boldsymbol{y}_{s,h}^{\mathrm{P},n}) + \frac{1}{\tau^2} a_{\mathrm{BJS}}(\boldsymbol{u}_{f,h}^{\mathrm{S},n}, \boldsymbol{y}_{s,h}^{\mathrm{P},n}; \boldsymbol{u}_{f,h}^{\mathrm{S},n}, \boldsymbol{y}_{s,h}^{\mathrm{P},n}) - \frac{1}{\tau} m_\theta(\boldsymbol{u}_{r,h}^{\mathrm{P},n}, \boldsymbol{y}_{s,h}^{\mathrm{P},n}) - \frac{1}{\tau^2} m_\theta(\boldsymbol{y}_{s,h}^{\mathrm{P},n}, \boldsymbol{y}_{s,h}^{\mathrm{P},n}) - m_\theta(\boldsymbol{u}_{r,h}^{\mathrm{P},n}, \boldsymbol{u}_{r,h}^{\mathrm{P},n}) \\
&- \frac{1}{\tau} m_\theta(\boldsymbol{y}_{s,h}^{\mathrm{P},n}, \boldsymbol{u}_{r,h}^{\mathrm{P},n}) + m_{\phi^2/\kappa}(\boldsymbol{u}_{r,h}^{\mathrm{P},n}, \boldsymbol{u}_{r,h}^{\mathrm{P},n}) + \frac{1}{\tau^2} m_{\rho_f\phi}(\boldsymbol{u}_{r,h}^{\mathrm{P},n}, \boldsymbol{y}_{s,h}^{\mathrm{P},n}) + \frac{1}{\tau^2} m_{\rho_p}(\boldsymbol{u}_{s,h}^{\mathrm{P},n}, \boldsymbol{y}_{s,h}^{\mathrm{P},n}) + \frac{1}{\tau} m_{\rho_f\phi}(\boldsymbol{u}_{r,h}^{\mathrm{P},n}, \boldsymbol{u}_{r,h}^{\mathrm{P},n}) \\
&+ \frac{1}{\tau} m_{\rho_f}(\boldsymbol{u}_{f,h}^{\mathrm{S},n}, \boldsymbol{u}_{f,h}^{\mathrm{S},n}) + \frac{1}{\tau} m_{\rho_f\phi}(\boldsymbol{u}_{s,h}^{\mathrm{P},n}, \boldsymbol{u}_{r,h}^{\mathrm{P},n}) + \frac{1}{\tau}((1-\phi)^2 K^{-1} p_h^{\mathrm{P},n}, p_h^{\mathrm{P}})_{\Omega_{\mathrm{P}}} + \frac{1}{\tau^2} b_{\mathrm{BJS}}(\boldsymbol{u}_{r,h}^{\mathrm{P},n}; \boldsymbol{u}_{r,h}^{\mathrm{P},n}).
\end{aligned}
$$

We bound some of the terms above using the inequality (A.14). This gives

$$
\begin{aligned}
&\frac{1}{\tau} a_f^{\mathrm{P}}(\boldsymbol{u}_{r,h}^{\mathrm{P},n}, \boldsymbol{y}_{s,h}^{\mathrm{P},n}) + \frac{1}{\tau} a_f^{\mathrm{P}}(\boldsymbol{y}_{s,h}^{\mathrm{P},n}, \boldsymbol{u}_{r,h}^{\mathrm{P},n}) - \frac{1}{\tau} m_\theta(\boldsymbol{u}_{r,h}^{\mathrm{P},n}, \boldsymbol{y}_{s,h}^{\mathrm{P},n}) - \frac{1}{\tau} m_\theta(\boldsymbol{y}_{s,h}^{\mathrm{P},n}, \boldsymbol{u}_{r,h}^{\mathrm{P},n}) + \frac{1}{\tau^2} m_{\rho_f\phi}(\boldsymbol{u}_{r,h}^{\mathrm{P},n}, \boldsymbol{y}_{s,h}^{\mathrm{P},n}) \\
&+ \frac{1}{\tau} m_{\rho_f\phi}(\boldsymbol{u}_{s,h}^{\mathrm{P},n}, \boldsymbol{u}_{r,h}^{\mathrm{P},n}) + \frac{1}{\tau^2} m_{\rho_p}(\boldsymbol{u}_{s,h}^{\mathrm{P},n}, \boldsymbol{y}_{s,h}^{\mathrm{P},n}) \geq -\frac{1}{\tau} a_f^{\mathrm{P}}\left(\boldsymbol{u}_{r,h}^{\mathrm{P},n}, \boldsymbol{u}_{r,h}^{\mathrm{P},n}\right) - \frac{1}{\tau} a_f^{\mathrm{P}}(\boldsymbol{y}_{s,h}^{\mathrm{P},n}, \boldsymbol{y}_{s,h}^{\mathrm{P},n}) + \frac{1}{\tau} m_\theta(\boldsymbol{u}_{r,h}^{\mathrm{P},n}, \boldsymbol{u}_{r,h}^{\mathrm{P},n}) \\
&+ \frac{1}{\tau} m_\theta(\boldsymbol{y}_{s,h}^{\mathrm{P},n}, \boldsymbol{y}_{s,h}^{\mathrm{P},n}) - \frac{1}{\tau} m_{\rho_f\phi}(\boldsymbol{u}_{r,h}^{\mathrm{P},n}, \boldsymbol{u}_{r,h}^{\mathrm{P},n}) - \frac{1}{\tau} m_{\rho_f\phi}(\boldsymbol{u}_{s,h}^{\mathrm{P},n}, \boldsymbol{u}_{s,h}^{\mathrm{P},n}) + \frac{1}{\tau} m_{\rho_p}(\boldsymbol{u}_{s,h}^{\mathrm{P},n}, \boldsymbol{u}_{s,h}^{\mathrm{P},n}).
\end{aligned}
$$

By combining both estimates above and using Lemma A.1, we arrive at

$$
\begin{aligned}
\mathrm{E}(\frac{1}{\tau}\vec{\boldsymbol{x}}_h^n, \vec{\boldsymbol{x}}_h) + \mathrm{H}(\vec{\boldsymbol{x}}_h^n, \vec{\boldsymbol{x}}_h) &\gtrsim \|\boldsymbol{u}_{f,h}^{\mathrm{S},n}\|_{1,\Omega_{\mathrm{P}}}^2 + \|\boldsymbol{u}_{r,h}^{\mathrm{P},n}\|_{1,\Omega_{\mathrm{P}}}^2 + \|\boldsymbol{y}_{s,h}^{\mathrm{P},n}\|_{1,\Omega_{\mathrm{P}}}^2 + |\boldsymbol{u}_{f,h}^{\mathrm{S},n} - \boldsymbol{y}_{s,h}^{\mathrm{P},n}|_{\mathrm{BJS}}^2 + \|\boldsymbol{y}_{s,h}^{\mathrm{P},n}\|_{0,\Omega_{\mathrm{P}}}^2 + \|\boldsymbol{u}_{r,h}^{\mathrm{P},n}\|_{0,\Omega_{\mathrm{P}}}^2 \\
&+ \|\boldsymbol{u}_{s,h}^{\mathrm{P},n}\|_{0,\Omega_{\mathrm{P}}}^2 + \|p_h^{\mathrm{P},n}\|_{0,\Omega_{\mathrm{P}}}^2 + \|\boldsymbol{u}_{f,h}^{\mathrm{S},n}\|_{0,\Omega_{\mathrm{S}}} + |\boldsymbol{u}_{r,h}^{\mathrm{P},n}|_{\mathrm{BJS}}^2.
\end{aligned}
$$

It is clear that all RHS are positive. Hence, the bilinear form on the LHS is positive-definite and consequently the matrix obtained from the system (5.1) is non-singular. The uniqueness follows from the fact that a linear system with a non-singular matrix admits a unique solution. □

5.2. *Stability analysis of the fully discrete scheme*

In this section we will make use of the following discrete space–time norms

$$
\|\phi\|_{l^2(0,T;X)}^2 := \tau \sum_{n=1}^{N} \|\phi^n\|_X^2, \qquad \|\phi\|_{l^\infty(0,T;X)}^2 := \max_{0\leq n\leq N} \|\phi^n\|_X^2, \qquad |\boldsymbol{\phi}|_{l^2(0,T;\mathrm{BJS})} := \tau \sum_{n=1}^{N} |\boldsymbol{\phi}|_{\mathrm{BJS}}^2.
$$

Lemma 5.2. *Under assumptions* (H.1)-(H.3)*, the fully discrete solution to* (5.1) *satisfies*

$$
\begin{aligned}
&\|\boldsymbol{u}_{f,h}^{\mathrm{S}}\|_{l^\infty(0,T;\mathbf{L}^2(\Omega_{\mathrm{S}}))}^2 + \|\boldsymbol{u}_{r,h}^{\mathrm{P}}\|_{l^\infty(0,T;\mathbf{L}^2(\Omega_{\mathrm{P}}))} + \|p_h^{\mathrm{P}}\|_{l^\infty(0,T;L^2(\Omega_{\mathrm{P}}))} + \|\boldsymbol{y}_{s,h}^{\mathrm{P}}\|_{l^\infty(0,T;\mathbf{H}^1(\Omega_{\mathrm{P}}))} + \|\boldsymbol{u}_{s,h}^{\mathrm{P}}\|_{l^\infty(0,T;\mathbf{L}^2(\Omega_{\mathrm{P}}))} \\
&+ \left|\boldsymbol{u}_{f,h}^{\mathrm{S}} - d_\tau \boldsymbol{y}_{s,h}^{\mathrm{P}}\right|_{l^2(0,T;\mathrm{BJS})} + \left|\boldsymbol{u}_{r,h}^{\mathrm{P}}\right|_{l^2(0,T;\mathrm{BJS})} + \|\boldsymbol{u}_{f,h}^{\mathrm{S}}\|_{l^2(0,T;\mathbf{H}^1(\Omega_{\mathrm{S}}))} + \|\boldsymbol{u}_{r,h}^{\mathrm{P}}\|_{l^2(0,T;L^2(\Omega_{\mathrm{P}}))} + \|\boldsymbol{u}_{s,h}^{\mathrm{P}}\|_{l^2(0,T;\mathbf{L}^2(\Omega_{\mathrm{P}}))} \\
&+ \|p_h^{\mathrm{S}}\|_{l^2(0,T;L^2(\Omega_{\mathrm{S}}))} + \|p_h^{\mathrm{P}}\|_{l^2(0,T;L^2(\Omega_{\mathrm{P}}))} + \|\lambda_h\|_{l^2(0,T;\Lambda_h)} \leq \hat{C}\Big(\|\boldsymbol{f}_{\mathrm{P}}\|_{l^2(0,T;\mathbf{L}^2(\Omega_{\mathrm{P}}))} + \|\boldsymbol{f}_{\mathrm{S}}\|_{l^2(0,T;\mathbf{L}^2(\Omega_{\mathrm{S}}))} \\
&+ \|\theta\|_{l^2(0,T;L^2(\Omega_{\mathrm{P}}))} + \|\boldsymbol{u}_{f,h}^0\|_{0,\Omega_{\mathrm{S}}} + \|\boldsymbol{u}_{r,h}^0\|_{0,\Omega_{\mathrm{P}}} + \|\boldsymbol{y}_{s,h}^0\|_{1,\Omega_{\mathrm{P}}} + \|p_h^{\mathrm{P},0}\|_{0,\Omega_{\mathrm{P}}} + \|\boldsymbol{u}_{s,h}^0\|_{0,\Omega_{\mathrm{P}}}\Big),
\end{aligned}
$$

where $\hat{C}(K, \kappa, \rho_f, \rho_s, \lambda_p, \mu_f, \mu_p, \phi, \alpha_{\mathrm{BJS}}, C_T, C_I, C_K)$ *is a positive constant.*

Proof. We choose $\vec{\boldsymbol{y}}_h = \vec{\boldsymbol{x}}_h^n$ in (5.1) and using (A.14), we have

$$
\begin{aligned}
&a_f^{\mathrm{S}}(\boldsymbol{u}_{f,h}^{\mathrm{S},n},\boldsymbol{u}_{f,h}^{\mathrm{S},n}) + a_s^{\mathrm{P}}(\boldsymbol{y}_{s,h}^{\mathrm{P},n}, d_\tau \boldsymbol{y}_{s,h}^{\mathrm{P},n}) + ((1-\phi)^2 K^{-1} d_\tau p_h^{\mathrm{P},n}, p_h^{\mathrm{P},n})_{\Omega_{\mathrm{P}}} + m_{\rho_s(1-\phi)}(d_\tau \boldsymbol{u}_{s,h}^{\mathrm{P},n}, d_\tau \boldsymbol{y}_{s,h}^{\mathrm{P},n}) + m_{\rho_f}(d_\tau \boldsymbol{u}_{f,h}^{\mathrm{S},n}, \boldsymbol{u}_{f,h}^{\mathrm{S},n})\\
&\quad + m_{\rho_f\phi}(d_\tau(\boldsymbol{u}_{r,h}^{\mathrm{P},n} + \boldsymbol{u}_{s,h}^{\mathrm{P},n}), (\boldsymbol{u}_{r,h}^{\mathrm{P},n} + \boldsymbol{u}_{s,h}^{\mathrm{P},n})) + a_{\mathrm{BJS}}(\boldsymbol{u}_{f,h}^{\mathrm{S},n}, d_\tau \boldsymbol{y}_{s,h}^{\mathrm{P},n}; \boldsymbol{u}_{f,h}^{\mathrm{S},n}, d_\tau \boldsymbol{y}_{s,h}^{\mathrm{P},n}) + b_{\mathrm{BJS}}(\boldsymbol{u}_{r,h}^{\mathrm{P},n}; \boldsymbol{u}_{r,h}^{\mathrm{P},n}) + m_{\phi^2/\kappa}(\boldsymbol{u}_{r,h}^{\mathrm{P},n}, \boldsymbol{u}_{r,h}^{\mathrm{P},n})\\
&\quad = (\boldsymbol{f}_{\mathrm{S}}^n, \boldsymbol{u}_{f,h}^{\mathrm{S},n})_{\Omega_{\mathrm{P}}} + (\rho_p \boldsymbol{f}_{\mathrm{P}}^n, d_\tau \boldsymbol{y}_{s,h}^{\mathrm{P},n})_{\Omega_{\mathrm{P}}} + (\rho_f \phi \boldsymbol{f}_{\mathrm{P}}^n, d_\tau \boldsymbol{y}_{s,h}^{\mathrm{P},n})_{\Omega_{\mathrm{P}}} + (2\rho_f \phi \boldsymbol{f}_{\mathrm{P}}^n, \boldsymbol{u}_{r,h}^{\mathrm{P},n})_{\Omega_{\mathrm{P}}} + (\rho_f^{-1}\theta^n, p_h^{\mathrm{P},n})_{\Omega_{\mathrm{P}}}.
\end{aligned}
$$

Now, as a consequence of the following identity

$$
\int_{\Omega_{\mathrm{P}}} \Upsilon^n d_\tau \Upsilon^n = \frac{1}{2} d_\tau \|\Upsilon^n\|_{0,\Omega_{\mathrm{P}}}^2 + \frac{1}{2}\tau \|d_\tau \Upsilon^n\|_{0,\Omega_{\mathrm{P}}}^2, \tag{5.2}
$$

we readily obtain an energy inequality. By using Cauchy–Schwarz and Young's inequalities, together with Lemma A.1 under assumptions (H.1)–(H.3), and using the estimate $\rho_f \phi \left\| \boldsymbol{u}_{r,h}^{\mathrm{P}} + \boldsymbol{u}_{s,h}^{\mathrm{P}} \right\|_{0,\Omega_{\mathrm{P}}}^2 \geq \rho_f \phi \left(\frac{1}{2} \|\boldsymbol{u}_{r,h}^{\mathrm{P}}\|_{0,\Omega_{\mathrm{P}}}^2 - \|\boldsymbol{u}_{s,h}^{\mathrm{P}}\|_{0,\Omega_{\mathrm{P}}}^2 \right)$, we then sum over $n = 1, \ldots, N$ and multiply by τ to obtain

$$
\begin{aligned}
&\|\boldsymbol{u}_{f,h}^{\mathrm{S},N}\|_{0,\Omega_{\mathrm{S}}}^2 + \|\boldsymbol{u}_{r,h}^{\mathrm{P},N}\|_{0,\Omega_{\mathrm{P}}}^2 + \|\boldsymbol{y}_{s,h}^{\mathrm{P},N}\|_{1,\Omega_{\mathrm{P}}}^2 + \|p_h^{\mathrm{P},N}\|_{0,\Omega_{\mathrm{P}}}^2 + \|\boldsymbol{u}_{s,h}^{\mathrm{P},N}\|_{0,\Omega_{\mathrm{P}}}^2 + \tau \sum_{n=1}^N (|\boldsymbol{u}_{f,h}^{\mathrm{S},n} - d_\tau \boldsymbol{y}_{s,h}^{\mathrm{P},n}|_{\mathrm{BJS}}^2 + |\boldsymbol{u}_{r,h}^{\mathrm{P},n}|_{\mathrm{BJS}}^2\\
&+ \|\boldsymbol{u}_{r,h}^{\mathrm{P},n}\|_{0,\Omega_{\mathrm{P}}}^2 + \|\boldsymbol{u}_{f,h}^{\mathrm{S},n}\|_{1,\Omega_{\mathrm{S}}}^2) + \tau^2 \sum_{n=1}^N (\|d_\tau \boldsymbol{u}_{f,h}^{\mathrm{S},n}\|_{0,\Omega_{\mathrm{S}}}^2 + \|d_\tau \boldsymbol{u}_{r,h}^{\mathrm{P},n}\|_{0,\Omega_{\mathrm{P}}}^2 + \|d_\tau \boldsymbol{y}_{s,h}^{\mathrm{P},n}\|_{1,\Omega_{\mathrm{P}}}^2 + \|d_\tau p_h^{\mathrm{P},n}\|_{0,\Omega_{\mathrm{P}}}^2 + \|d_\tau \boldsymbol{u}_{s,h}^{\mathrm{P},n}\|_{0,\Omega_{\mathrm{P}}}^2)\\
&\quad \lesssim (\|\boldsymbol{u}_{f,h}^0\|_{0,\Omega_{\mathrm{S}}}^2 + \|\boldsymbol{u}_{r,h}^0\|_{0,\Omega_{\mathrm{P}}}^2 + \|\boldsymbol{y}_{s,h}^0\|_{1,\Omega_{\mathrm{P}}}^2 + \|p_h^{\mathrm{P},0}\|_{0,\Omega_{\mathrm{P}}}^2 + \|\boldsymbol{u}_{s,h}^0\|_{0,\Omega_{\mathrm{P}}}^2 + \varepsilon_1{}^{-1}\tau \sum_{n=1}^N (\|\boldsymbol{f}_{\mathrm{P}}(t_n)\|_{0,\Omega_{\mathrm{P}}}^2 + \|\theta(t_n)\|_{0,\Omega_{\mathrm{P}}}^2\\
&\quad + \|\boldsymbol{f}_{\mathrm{S}}(t_n)\|_{0,\Omega_{\mathrm{S}}}^2) + \varepsilon_1 \tau \sum_{n=1}^N (\|\boldsymbol{u}_{f,h}^{\mathrm{S},n}\|_{1,\Omega_{\mathrm{S}}}^2 + \|p_h^{\mathrm{P},n}\|_{0,\Omega_{\mathrm{P}}}^2 + \|p_h^{\mathrm{S},n}\|_{L^2(\Omega_{\mathrm{S}})}^2 + \|\boldsymbol{u}_{s,h}^{\mathrm{P},n}\|_{0,\Omega_{\mathrm{P}}}^2 + \|\boldsymbol{u}_{r,h}^{\mathrm{P},n}\|_{0,\Omega_{\mathrm{P}}}^2)).
\end{aligned} \tag{5.3}
$$

Next, employing the inf–sup condition for $(p_h^{\mathrm{S},n}, p_h^{\mathrm{P},n}, \lambda_h^n)$, in a similar way, we readily obtain

$$
\begin{aligned}
&\varepsilon_2 \tau \sum_{n=1}^N (\|p_h^{\mathrm{S},n}\|_{\mathrm{W}_{f,h}}^2 + \|p_h^{\mathrm{P},n}\|_{\mathrm{W}_{p,h}}^2 + \|\lambda_h^n\|_{\Lambda_h}^2)\\
&\quad \leq \tilde{C}\varepsilon_2 \tau \sum_{n=1}^N (\|\boldsymbol{u}_{f,h}^{\mathrm{S},n}\|_{1,\Omega_{\mathrm{S}}}^2 + \|\boldsymbol{u}_{r,h}^{\mathrm{P},n}\|_{0,\Omega_{\mathrm{P}}}^2 + |\boldsymbol{u}_{f,h}^{\mathrm{S},n} - d_\tau \boldsymbol{y}_{s,h}^{\mathrm{P},n}|_{\mathrm{BJS}}^2 + \|\boldsymbol{f}_{\mathrm{S}}(t_n)\|_{0,\Omega_{\mathrm{S}}}^2 + \|\boldsymbol{f}_{\mathrm{P}}(t_n)\|_{0,\Omega_{\mathrm{P}}}^2).
\end{aligned} \tag{5.4}
$$

Combining (5.3) and (5.4), and taking ε_2 and in turn ε_1 small enough, we obtain the desired result. $\square$

5.3. *Error estimates for the fully discrete scheme*

Theorem 5.3. *Assuming (H.1)-(H.3) and sufficient smoothness for the solution of (3.1), then the solution of the fully discrete problem (5.1) satisfies*

$$
\begin{aligned}
&\|\boldsymbol{u}_f^{\mathrm{S}} - \boldsymbol{u}_{f,h}^{\mathrm{S}}\|_{l^2(0,T;\mathbf{H}^1(\Omega_{\mathrm{S}}))} + \|\boldsymbol{u}_r^{\mathrm{P}} - \boldsymbol{u}_{r,h}^{\mathrm{P}}\|_{l^\infty(0,T;\mathbf{L}^2(\Omega_{\mathrm{P}}))} + \|\boldsymbol{y}_s^{\mathrm{P}} - \boldsymbol{y}_{s,h}^{\mathrm{P}}\|_{l^\infty(0,T;\mathbf{H}^1(\Omega_{\mathrm{P}}))} + \|p^{\mathrm{P}} - p_h^{\mathrm{P}}\|_{l^\infty(0,T;L^2(\Omega_{\mathrm{P}}))}\\
&\quad + \|\boldsymbol{u}_s^{\mathrm{P}} - \boldsymbol{u}_{s,h}^{\mathrm{P}}\|_{l^\infty(0,T;\mathbf{L}^2(\Omega_{\mathrm{P}}))} + \|\boldsymbol{u}_r^{\mathrm{P}} - \boldsymbol{u}_{r,h}^{\mathrm{P}}\|_{l^2(0,T;\mathbf{L}^2(\Omega_{\mathrm{P}}))} + |(\boldsymbol{u}_f^{\mathrm{S}} - d_\tau \boldsymbol{y}_s^{\mathrm{P}}) - (\boldsymbol{u}_{f,h}^{\mathrm{S}} - d_\tau \boldsymbol{y}_{s,h}^{\mathrm{P}})|_{l^2(0,T;\mathrm{BJS})}\\
&\quad + |\boldsymbol{u}_r^{\mathrm{P}} - \boldsymbol{u}_{r,h}^{\mathrm{P}}|_{l^2(0,T;\mathrm{BJS})} + \|p^{\mathrm{S}} - p_h^{\mathrm{S}}\|_{l^2(0,T;L^2(\Omega_{\mathrm{S}}))} + \|p^{\mathrm{P}} - p_h^{\mathrm{P}}\|_{l^2(0,T;L^2(\Omega_{\mathrm{P}}))} + \|\lambda - \lambda_h\|_{l^2(0,T;\Lambda_h)})\\
&\leq C\sqrt{\exp(T)} \Bigg[h^{r_{k_f}} \left(\|\boldsymbol{u}_f^{\mathrm{S}}\|_{l^2(0,T;\mathbf{H}^{r_{k_f}+1}(\Omega_{\mathrm{S}}))} + \|\partial_t \boldsymbol{u}_f^{\mathrm{S}}\|_{L^2(0,T;\mathbf{H}^{r_{k_f}+1}(\Omega_{\mathrm{S}}))} \right)
\end{aligned}
$$

$$+h^{r_{s_s}+1}(\|\boldsymbol{u}_s^{\mathrm{P}}\|_{l^2(0,T;\mathbf{H}^{r_{s_s}+1}(\Omega_{\mathrm{P}}))}+\|\partial_t\boldsymbol{u}_s^{\mathrm{P}}\|_{L^2(0,T;\mathbf{H}^{r_{s_s}+1}(\Omega_{\mathrm{P}}))})$$
$$+h^{r_{k_p}-1}(\|\boldsymbol{u}_r^{\mathrm{P}}\|_{l^2(0,T;\mathbf{H}^{r_{k_p}+1}(\Omega_{\mathrm{P}}))}+\|\boldsymbol{u}_r^{\mathrm{P}}\|_{l^\infty(0,T;\mathbf{H}^{r_{k_p}+1}(\Omega_{\mathrm{P}}))}+\|\partial_t\boldsymbol{u}_r^{\mathrm{P}}\|_{L^2(0,T;\mathbf{H}^{r_{k_p}+1}(\Omega_{\mathrm{P}}))})$$
$$+h^{r_{s_p}+1}(\|p^{\mathrm{P}}\|_{l^\infty(0,T;H^{r_{s_p}+1}(\Omega_{\mathrm{P}}))}+\|p^{\mathrm{P}}\|_{l^2(0,T;H^{r_{s_p}+1}(\Omega_{\mathrm{P}}))}+\|\partial_t p^{\mathrm{P}}\|_{L^2\left(0,T;H^{r_{s_p}+1}(\Omega_{\mathrm{P}})\right)})$$
$$+h^{r_{k_s}-1}\big(\|\boldsymbol{y}_s^{\mathrm{P}}\|_{l^\infty(0,T;\mathbf{H}^{r_{k_s}+1}(\Omega_{\mathrm{P}}))}+\|\boldsymbol{y}_s^{\mathrm{P}}\|_{l^2(0,T;\mathbf{H}^{r_{k_s}+1}(\Omega_{\mathrm{P}}))}+\|\partial_t\boldsymbol{y}_s^{\mathrm{P}}\|_{L^2(0,T;\mathbf{H}^{r_{k_s}+1}(\Omega_{\mathrm{P}}))}$$
$$+\|\partial_t\boldsymbol{y}_s^{\mathrm{P}}\|_{L^\infty(0,T;\mathbf{H}^{r_{k_s}+1}(\Omega_{\mathrm{P}}))}+\|\partial_{tt}\boldsymbol{y}_s^{\mathrm{P}}\|_{L^2(0,T;\mathbf{H}^{r_{k_s}+1}(\Omega_{\mathrm{P}}))}\big)$$
$$+h^{r_{s_f}+1}\|p^{\mathrm{S}}\|_{l^2(0,T;H^{r_{s_f}+1}(\Omega_{\mathrm{S}}))}+h^{r_{\tilde{k}_p}+\frac{1}{2}}(\|\lambda\|_{l^2(0,T;H^{r_{\tilde{k}_p}}(\Sigma))}+\|\lambda\|_{l^\infty(0,T;H^{r_{\tilde{k}_p}}(\Sigma))}+\|\partial_t\lambda\|_{L^2(0,T;H^{r_{\tilde{k}_p}}(\Sigma))})\Big]$$
$$+\tau\Big[\|\partial_{tt}\boldsymbol{y}_s^{\mathrm{P}}\|_{L^2(0,T;\mathbf{H}^1(\Omega_{\mathrm{P}}))}+\|\partial_{tt}\boldsymbol{y}_s^{\mathrm{P}}\|_{L^2(0,T;\mathbf{L}^2(\Omega_{\mathrm{P}}))}+\|\partial_{tt}\boldsymbol{y}_s^{\mathrm{P}}\|_{L^\infty(0,T;\mathbf{H}^1(\Omega_{\mathrm{P}}))}+\|\partial_{ttt}\boldsymbol{y}_s^{\mathrm{P}}\|_{L^2(0,T;\mathbf{H}^1(\Omega_{\mathrm{P}}))}$$
$$+\|\partial_{tt}\boldsymbol{u}_r^{\mathrm{P}}\|_{L^2(0,T;\mathbf{L}^2(\Omega_{\mathrm{P}}))}+\|\partial_{tt}\boldsymbol{u}_s^{\mathrm{P}}\|_{L^2(0,T;\mathbf{L}^2(\Omega_{\mathrm{P}}))}+\|\partial_{tt}p^{\mathrm{P}}\|_{L^2(0,T;L^2(\Omega_{\mathrm{P}}))}+\|\partial_{tt}\boldsymbol{u}_f^{\mathrm{S}}\|_{L^2(0,T;\mathbf{L}^2(\Omega_{\mathrm{P}}))}\Big],$$

where $0\le r_{k_f}\le k_f$, $0\le r_{s_f}\le s_f$, $1\le r_{k_p}\le k_p$, $0\le r_{s_p}\le s_p$, $1\le r_{k_s}\le k_s$, $0\le r_{s_s}\le s_s$, $-1/2\le r_{\tilde{k}_p}\le\tilde{k}_p-1/2$.

Proof. We split errors into approximation and discretization errors similarly as before. Subtracting (5.1) from (3.1) yields the error equations. Let r_n denote the difference between the time derivative and its discrete analogue $r_n(\theta)=\partial_t\theta(t_n)-d_\tau\theta^n$. The proof follows that of Theorem 4.4: First, substitute $(\boldsymbol{v}_{f,h}^{\mathrm{S}},\boldsymbol{v}_{r,h}^{\mathrm{P}},\boldsymbol{w}_{s,h}^{\mathrm{P}},\boldsymbol{v}_{s,h}^{\mathrm{P}},q_h^{\mathrm{S}},q_h^{\mathrm{P}})=(\boldsymbol{\phi}_{f,h}^n,\boldsymbol{\phi}_{r,h}^n,d_\tau\boldsymbol{\phi}_{s,h}^n,\boldsymbol{\phi}_{ss,h}^n,\phi_{fp,h}^n,\phi_{pp,h}^n)$ in the error equation, split the individual errors, and apply the properties of the projection operators. This procedure gives

$$\begin{aligned}
&a_f^{\mathrm{S}}\left(\boldsymbol{\phi}_{f,h}^n,\boldsymbol{\phi}_{f,h}^n\right)+a_f^{\mathrm{P}}\left(\boldsymbol{\phi}_{r,h}^n,d_\tau\boldsymbol{\phi}_{s,h}^n\right)+a_f^{\mathrm{P}}\left(\boldsymbol{\phi}_{r,h}^n,\boldsymbol{\phi}_{r,h}^n\right)+a_f^{\mathrm{P}}\left(d_\tau\boldsymbol{\phi}_{s,h}^n,\boldsymbol{\phi}_{r,h}^n\right)+a_f^{\mathrm{P}}\left(d_\tau\boldsymbol{\phi}_{s,h}^n,d_\tau\boldsymbol{\phi}_{s,h}^n\right)\\
&\quad+a_s^{\mathrm{P}}\left(\boldsymbol{\phi}_{s,h}^n,d_\tau\boldsymbol{\phi}_{s,h}^n\right)+m_{\rho_f\phi}\left(d_\tau\boldsymbol{\phi}_{r,h}^n,d_\tau\boldsymbol{\phi}_{s,h}^n\right)+m_{\rho_p}\left(d_\tau\boldsymbol{\phi}_{ss,h}^n,d_\tau\boldsymbol{\phi}_{s,h}^n\right)+m_{\rho_f\phi}\left(d_\tau\boldsymbol{\phi}_{r,h}^n,\boldsymbol{\phi}_{r,h}^n\right)\\
&\quad+m_{\rho_f}(d_\tau\boldsymbol{\phi}_{f,h}^{\mathrm{S}},\boldsymbol{\phi}_{f,h}^{\mathrm{S}})+m_{\rho_f\phi}\left(d_\tau\boldsymbol{\phi}_{ss,h}^n,\boldsymbol{\phi}_{r,h}^n\right)+\left((1-\phi)^2K^{-1}d_\tau\phi_{pp,h}^n,\phi_{pp,h}^n\right)_{\Omega_{\mathrm{P}}}-m_\theta\left(\boldsymbol{\phi}_{r,h}^n,d_\tau\boldsymbol{\phi}_{s,h}^n\right)\\
&\quad-m_\theta\left(d_\tau\boldsymbol{\phi}_{s,h}^n,d_\tau\boldsymbol{\phi}_{s,h}^n\right)-m_\theta\left(\boldsymbol{\phi}_{r,h}^n,\boldsymbol{\phi}_{r,h}^n\right)-m_\theta\left(d_\tau\boldsymbol{\phi}_{s,h}^n,\boldsymbol{\phi}_{r,h}^n\right)+m_{\phi^2/\kappa}\left(\boldsymbol{\phi}_{r,h}^n,\boldsymbol{\phi}_{r,h}^n\right)\\
&\quad+a_{\mathrm{BJS}}\left(\boldsymbol{\phi}_{f,h}^n,d_\tau\boldsymbol{\phi}_{s,h}^n;\boldsymbol{\phi}_{f,h}^n,d_\tau\boldsymbol{\phi}_{s,h}^n\right)+b_{\mathrm{BJS}}\left(\boldsymbol{\phi}_{r,h}^n;\boldsymbol{\phi}_{r,h}^n\right)=\mathscr{E}+\mathscr{H},
\end{aligned}\tag{5.5}$$

where

$$\begin{aligned}
\mathscr{E}:=&-a_f^{\mathrm{S}}\left(\boldsymbol{\chi}_f^n,\boldsymbol{\phi}_{f,h}^n\right)-a_f^{\mathrm{P}}\left(\boldsymbol{\chi}_r^n,\boldsymbol{\phi}_{r,h}^n\right)-a_f^{\mathrm{P}}\left(d_\tau\boldsymbol{\chi}_s^n,\boldsymbol{\phi}_{r,h}^n\right)-a_{\mathrm{BJS}}\left(\boldsymbol{\chi}_f^n,d_\tau\boldsymbol{\chi}_s^n;\boldsymbol{\phi}_{f,h}^n,d_\tau\boldsymbol{\phi}_{s,h}^n\right)-b_{\mathrm{BJS}}\left(\boldsymbol{\chi}_r^n;\boldsymbol{\phi}_r^n\right)\\
&-b^{\mathrm{S}}\left(\boldsymbol{\phi}_{f,h}^n,\chi_{fp}^n\right)-b_\Gamma\left(\boldsymbol{\phi}_{f,h}^n,\boldsymbol{\phi}_{r,h}^n,\mathbf{0};\chi_\lambda^n\right)+b_s^{\mathrm{P}}\left(d_\tau\boldsymbol{\chi}_s^n,\phi_{pp,h}^n\right)+m_\theta(\boldsymbol{\chi}_r^n,d_\tau\boldsymbol{\phi}_{s,h}^n)+m_\theta(d_\tau\boldsymbol{\chi}_s^n,d_\tau\boldsymbol{\phi}_{s,h}^n)\\
&+m_\theta(\boldsymbol{\chi}_r^n,\boldsymbol{\phi}_{r,h}^n)+m_\theta(d_\tau\boldsymbol{\chi}_s^n,\boldsymbol{\phi}_{r,h}^n)-m_{\phi^2/\kappa}(\boldsymbol{\chi}_r^n,\boldsymbol{\phi}_{r,h}^n)-m_{\rho_f\phi}(d_\tau\boldsymbol{\chi}_r^n,d_\tau\boldsymbol{\phi}_{s,h}^n)-m_{\rho_p}(d_\tau\boldsymbol{\chi}_{ss}^n,d_\tau\boldsymbol{\phi}_{s,h}^n)\\
&-m_{\rho_f\phi}(d_\tau\boldsymbol{\chi}_r^n,\boldsymbol{\phi}_{r,h}^n)-m_{\rho_f\phi}(d_\tau\boldsymbol{\chi}_{ss}^n,\boldsymbol{\phi}_{r,h}^n)-m_{\rho_f}(d_\tau\boldsymbol{\chi}_f^n,\boldsymbol{\phi}_{f,h}^n)-a_f^{\mathrm{P}}\left(r_n(\boldsymbol{y}_s^{\mathrm{P}}),\boldsymbol{\phi}_{r,h}^n\right)+m_\theta(r_n(\boldsymbol{y}_s^{\mathrm{P}}),d_\tau\boldsymbol{\phi}_{s,h}^n)\\
&+m_\theta(r_n(\boldsymbol{y}_s^{\mathrm{P}}),\boldsymbol{\phi}_{r,h}^n)-m_{\rho_f\phi}(r_n(\boldsymbol{u}_r^{\mathrm{P}}),d_\tau\boldsymbol{\phi}_{s,h}^n)-m_{\rho_p}(r_n(\boldsymbol{y}_s^{\mathrm{P}}),d_\tau\boldsymbol{\phi}_{s,h}^n)-m_{\rho_f\phi}(r_n(\boldsymbol{u}_r^{\mathrm{P}}),\boldsymbol{\phi}_{r,h}^n)\\
&-m_{\rho_f\phi}(r_n(\boldsymbol{u}_s^{\mathrm{P}}),\boldsymbol{\phi}_{r,h}^n)+b_s^{\mathrm{P}}\left(r_n(\boldsymbol{y}_s^{\mathrm{P}}),\phi_{pp,h}^n\right)-\left((1-\phi)^2K^{-1}r_n(p),\phi_{pp,h}^n\right)_{\Omega_{\mathrm{P}}}-m_{\rho_f}(r_n(\boldsymbol{u}_f^{\mathrm{S}}),\boldsymbol{\phi}_{f,h}^n),\\
\mathscr{H}:=&-a_s^{\mathrm{P}}\left(\boldsymbol{\chi}_s^n,d_\tau\boldsymbol{\phi}_{s,h}^n\right)-a_f^{\mathrm{P}}\left(\boldsymbol{\chi}_r^n,d_\tau\boldsymbol{\phi}_{s,h}^n\right)-a_f^{\mathrm{P}}\left(d_\tau\boldsymbol{\chi}_s^n,d_\tau\boldsymbol{\phi}_{s,h}^n\right)-b_s^{\mathrm{P}}\left(d_\tau\boldsymbol{\phi}_{s,h}^n,\chi_{pp}^n\right)\\
&-a_f^{\mathrm{P}}\left(r_n(\boldsymbol{y}_s^{\mathrm{P}}),d_\tau\boldsymbol{\phi}_{s,h}^n\right)-a_{\mathrm{BJS}}\left(\mathbf{0},r_n(\boldsymbol{y}_s^{\mathrm{P}});\boldsymbol{\phi}_{f,h}^n,d_\tau\boldsymbol{\phi}_{s,h}^n\right)-b_\Gamma(\mathbf{0},\mathbf{0},d_\tau\boldsymbol{\phi}_{s,h}^n;\chi_\lambda^n).
\end{aligned}$$

Using inequality (A.14) and identity (5.2), together with assumptions (H.1)–(H.3) and the estimate $\rho_f\,\phi\,\|\boldsymbol{\phi}^n_{r,h}+\boldsymbol{\phi}^n_{ss,h}\|^2_{0,\Omega_{\rm P}} \geq \rho_f\,\phi\left(\frac{1}{2}\|\boldsymbol{\phi}^n_{r,h}\|^2_{0,\Omega_{\rm P}} - \|\boldsymbol{\phi}^n_{ss,h}\|^2_{0,\Omega_{\rm P}}\right)$, the left-hand side of (5.5) becomes

$$\begin{aligned}\mathtt{LHS}_{(5.5)} \geq&\frac{1}{2}d_\tau\left(\|\boldsymbol{\phi}^n_{f,h}\|^2_{0,\Omega_{\rm S}}+\|\boldsymbol{\phi}^n_{r,h}\|^2_{0,\Omega_{\rm P}}+\|\boldsymbol{\phi}^n_{s,h}\|^2_{1,\Omega_{\rm P}}+\|\phi^n_{pp,h}\|^2_{0,\Omega_{\rm P}}+\|\boldsymbol{\phi}^n_{ss,h}\|^2_{0,\Omega_{\rm P}}\right)+\frac{\tau}{2}\Big(\|d_\tau\boldsymbol{\phi}^n_{f,h}\|^2_{0,\Omega_{\rm S}}\\&+\|d_\tau\boldsymbol{\phi}^n_{r,h}\|^2_{0,\Omega_{\rm P}}+\|d_\tau\boldsymbol{\phi}^n_{s,h}\|^2_{1,\Omega_{\rm P}}+\|d_\tau\phi^n_{pp,h}\|^2_{0,\Omega_{\rm P}}+\|d_\tau\boldsymbol{\phi}^n_{ss,h}\|^2_{0,\Omega_{\rm P}}\Big)+\left|\boldsymbol{\phi}^n_{f,h}-d_\tau\boldsymbol{\phi}^n_{s,h}\right|^2_{\rm BJS}\\&+\left|\boldsymbol{\phi}^n_{r,h}\right|^2_{\rm BJS}+\|\boldsymbol{\phi}^n_{r,h}\|_{0,\Omega_{\rm P}}+\|\boldsymbol{\phi}^n_{f,h}\|^2_{1,\Omega_{\rm S}}.\end{aligned}$$

Now, we bound the terms on the RHS similarly as in Theorem 4.4. This yields

$$\begin{aligned}\mathscr{E}\leq&\varepsilon_1^{-1}\big(\|\boldsymbol{\chi}^n_f\|^2_{1,\Omega_{\rm S}}+\|\boldsymbol{\chi}^n_r\|^2_{1,\Omega_{\rm P}}+\|d_\tau\boldsymbol{\chi}^n_s\|^2_{1,\Omega_{\rm P}}+\|\boldsymbol{\chi}^n_r\|^2_{0,\Omega_{\rm P}}+\|d_\tau\boldsymbol{\chi}^n_s\|^2_{0,\Omega_{\rm P}}+\|d_\tau\boldsymbol{\chi}^n_{ss}\|^2_{0,\Omega_{\rm P}}+\|d_\tau\boldsymbol{\chi}^n_r\|^2_{0,\Omega_{\rm P}}\\&+\|\chi^n_{fp}\|^2_{0,\Omega_{\rm S}}+\|\chi^n_\lambda\|^2_{L^2(\Sigma)}+\|d_\tau\boldsymbol{\chi}^n_f\|^2_{0,\Omega_{\rm S}}+\|r_n(\boldsymbol{y}^{\rm P}_s)\|^2_{1,\Omega_{\rm P}}+\|r_n(\boldsymbol{y}^{\rm P}_s)\|^2_{0,\Omega_{\rm P}}+\|r_n(\boldsymbol{u}^{\rm P}_r)\|^2_{0,\Omega_{\rm P}}+\|r_n(\boldsymbol{u}^{\rm P}_s)\|^2_{0,\Omega_{\rm P}}\\&+\|r_n(p^{\rm P})\|^2_{0,\Omega_{\rm P}}+\|r_n(\boldsymbol{u}^{\rm S}_f)\|^2_{0,\Omega_{\rm S}}\big)+C\|\chi^n_\lambda\|^2_{\Lambda_h}+\varepsilon_1\big(\|\boldsymbol{\phi}^n_{f,h}\|^2_{1,\Omega_{\rm S}}+\|\boldsymbol{\phi}^n_{r,h}\|^2_{0,\Omega_{\rm P}}+\|\boldsymbol{\phi}^n_{ss,h}\|^2_{0,\Omega_{\rm P}}\\&+\|\boldsymbol{\phi}^n_{r,h}\|^2_{0,\Omega_{\rm P}}+\left|\boldsymbol{\phi}^n_{f,h}-\partial_t\boldsymbol{\phi}^n_{s,h}\right|^2_{\rm BJS}+\left|\boldsymbol{\phi}^n_{r,h}\right|^2_{\rm BJS}+\|\phi^n_{pp,h}\|^2_{0,\Omega_{\rm P}}\big).\end{aligned}\tag{5.6}$$

Combining (5.5) and (5.6), summing over $n=1,\ldots,N$, and multiplying by τ, from Lemma A.1 we get

$$\begin{aligned}&\|\boldsymbol{\phi}^N_{f,h}\|^2_{0,\Omega_{\rm S}}+\|\boldsymbol{\phi}^N_{r,h}\|^2_{0,\Omega_{\rm P}}+\|\boldsymbol{\phi}^N_{s,h}\|^2_{1,\Omega_{\rm P}}+\|\phi^N_{pp,h}\|^2_{0,\Omega_{\rm P}}+\|\boldsymbol{\phi}^N_{ss,h}\|^2_{0,\Omega_{\rm P}}+\tau^2\sum_{n=1}^N\Big(\|d_\tau\boldsymbol{\phi}^n_{f,h}\|^2_{0,\Omega_{\rm S}}+\|d_\tau\boldsymbol{\phi}^n_{r,h}\|^2_{0,\Omega_{\rm P}}\\&+\|d_\tau\boldsymbol{\phi}^n_{s,h}\|^2_{1,\Omega_{\rm P}}+\|d_\tau\phi^n_{pp,h}\|^2_{0,\Omega_{\rm P}}+\|d_\tau\boldsymbol{\phi}^n_{ss,h}\|^2_{0,\Omega_{\rm P}}\Big)+\tau\sum_{n=1}^N\bigg(\left|\boldsymbol{\phi}^n_{f,h}-d_\tau\boldsymbol{\phi}^n_{s,h}\right|^2_{\rm BJS}+\left|\boldsymbol{\phi}^n_{r,h}\right|^2_{\rm BJS}+\|\boldsymbol{\phi}^n_{r,h}\|^2_{0,\Omega_{\rm P}}\\&+\|\boldsymbol{\phi}^n_{r,h}\|^2_{0,\Omega_{\rm P}}+\|\boldsymbol{\phi}^n_{f,h}\|^2_{1,\Omega_{\rm S}}\bigg)\leq C\Big[\|\boldsymbol{\phi}^0_{f,h}\|^2_{0,\Omega_{\rm S}}+\|\boldsymbol{\phi}^0_{r,h}\|^2_{0,\Omega_{\rm P}}+\|\boldsymbol{\phi}^0_{s,h}\|^2_{1,\Omega_{\rm P}}+\|\phi^0_{pp,h}\|^2_{0,\Omega_{\rm P}}+\|\boldsymbol{\phi}^0_{ss,h}\|^2_{0,\Omega_{\rm P}}\\&+\varepsilon_1^{-1}\tau\sum_{n=1}^N\Big(\|\boldsymbol{\chi}^n_f\|^2_{1,\Omega_{\rm S}}+\|\boldsymbol{\chi}^n_r\|^2_{1,\Omega_{\rm P}}+\|d_\tau\boldsymbol{\chi}^n_s\|^2_{1,\Omega_{\rm P}}+\|\boldsymbol{\chi}^n_r\|^2_{0,\Omega_{\rm P}}+\|d_\tau\boldsymbol{\chi}^n_s\|^2_{0,\Omega_{\rm P}}+\|d_\tau\boldsymbol{\chi}^n_{ss}\|^2_{0,\Omega_{\rm P}}+\|d_\tau\boldsymbol{\chi}^n_f\|^2_{0,\Omega_{\rm S}}\\&+\|d_\tau\boldsymbol{\chi}^n_r\|^2_{0,\Omega_{\rm P}}+\|\chi^n_{fp}\|^2_{0,\Omega_{\rm S}}+\|r_n(\boldsymbol{y}^{\rm P}_s)\|^2_{1,\Omega_{\rm P}}+\|r_n(\boldsymbol{y}^{\rm P}_s)\|^2_{0,\Omega_{\rm P}}+\|r_n(\boldsymbol{u}^{\rm P}_r)\|^2_{0,\Omega_{\rm P}}+\|r_n(\boldsymbol{u}^{\rm P}_s)\|^2_{0,\Omega_{\rm P}}\\&+\|r_n(p^{\rm P})\|^2_{0,\Omega_{\rm P}}+\|r_n(\boldsymbol{u}^{\rm S}_f)\|^2_{0,\Omega_{\rm S}}\Big)+\varepsilon_1\tau\sum_{n=1}^N\big(\|\boldsymbol{\phi}^n_{f,h}\|^2_{1,\Omega_{\rm S}}+\|\boldsymbol{\phi}^n_{r,h}\|^2_{0,\Omega_{\rm P}}+\|\boldsymbol{\phi}^n_{ss,h}\|^2_{0,\Omega_{\rm P}}+\|\boldsymbol{\phi}^n_{r,h}\|^2_{0,\Omega_{\rm P}}\\&+\left|\boldsymbol{\phi}^n_{f,h}-d_\tau\boldsymbol{\phi}^n_{s,h}\right|^2_{\rm BJS}+\left|\boldsymbol{\phi}^n_{r,h}\right|^2_{\rm BJS}+\|\boldsymbol{\phi}^n_{r,h}\|_{0,\Omega_{\rm P}}+\|\phi^n_{pp,h}\|^2_{0,\Omega_{\rm P}}\big)+C\tau\sum_{n=1}^N\big(\|\chi^n_\lambda\|^2_{\Lambda_h}+\mathscr{H}\big)\Big].\end{aligned}\tag{5.7}$$

Next, for each term in $\mathscr{H}$, we use the summation by parts $\tau\sum_{n=1}^N\left(\upsilon(t_n),d_\tau\boldsymbol{\phi}^n_{s,h}\right)=\left(\upsilon(t_N),\boldsymbol{\phi}^N_{s,h}\right)-\left(\upsilon(0),\boldsymbol{\phi}^0_{s,h}\right)-\tau\sum_{n=1}^{N-1}\left(d_\tau\upsilon^n,\boldsymbol{\phi}^n_{s,h}\right)$, where υ is any of the functions in $\mathscr{H}$. Then, we apply Cauchy–Schwarz and Young's inequalities to obtain

$$\begin{aligned}\tau\sum_{n=1}^N\left(\upsilon(t_n),d_\tau\boldsymbol{\phi}^n_{s,h}\right)\leq&\frac{\varepsilon_1}{2}\|\boldsymbol{\phi}^N_{s,h}\|^2_{0,\Omega_{\rm P}}+\frac{1}{2\varepsilon_1}\|\upsilon(t_N)\|^2_{0,\Omega_{\rm P}}+\frac{\tau}{2}\sum_{n=1}^{N-1}\|\boldsymbol{\phi}^n_{s,h}\|^2_{0,\Omega_{\rm P}}\\&+\frac{1}{2}\big(\|\boldsymbol{\phi}^0_{s,h}\|^2_{0,\Omega_{\rm P}}+\|\upsilon(0)\|^2_{0,\Omega_{\rm P}}+\tau\sum_{n=1}^{N-1}\|d_\tau\upsilon^n\|^2_{0,\Omega_{\rm P}}\big).\end{aligned}$$

Next we proceed to bound each term of $\mathscr{H}$. It follows that

$$\tau\sum_{n=1}^{N}\mathscr{H} \le \varepsilon_1\|\boldsymbol{\phi}^N_{s,h}\|^2_{1,\Omega_P} + \tau\sum_{n=1}^{N-1}\|\boldsymbol{\phi}^n_{s,h}\|^2_{1,\Omega_P} + \|\boldsymbol{\phi}^0_{s,h}\|^2_{1,\Omega_P} + \varepsilon_1^{-1}\left(\|\boldsymbol{\chi}^N_s\|^2_{1,\Omega_P} + \|\boldsymbol{\chi}^N_r\|^2_{1,\Omega_P} + \|d_\tau\boldsymbol{\chi}^N_s\|^2_{1,\Omega_P}\right.$$
$$\left. + \|\chi^N_{pp}\|^2_{0,\Omega_P} + \|r_N(\boldsymbol{y}^P_s)\|^2_{1,\Omega_P} + \|\chi^N_\lambda\|^2_{H^{-1/2}(\Omega_P)}\right) + C\left(\|\boldsymbol{\chi}^0_s\|^2_{1,\Omega_P} + \|\boldsymbol{\chi}^0_r\|^2_{1,\Omega_P} + \|d_\tau\boldsymbol{\chi}^0_s\|^2_{1,\Omega_P}\right.$$
$$\left. + \|\chi^0_{pp}\|^2_{0,\Omega_P} + \|r_0(\boldsymbol{y}^P_s)\|^2_{1,\Omega_P} + \|\chi^0_\lambda\|^2_{H^{-1/2}(\Omega_P)}\right) + \tau\sum_{n=1}^{N-1}\left(\|d_\tau\boldsymbol{\chi}^n_s\|^2_{1,\Omega_P} + \|d_\tau\boldsymbol{\chi}^n_r\|^2_{1,\Omega_P}\right.$$
$$\left. + \|d_\tau d_\tau\boldsymbol{\chi}^n_s\|^2_{1,\Omega_P} + \|d_\tau\chi^n_{pp}\|^2_{0,\Omega_P} + \|d_\tau r_n(\boldsymbol{y}^P_s)\|^2_{1,\Omega_P} + \|d_\tau\chi^n_\lambda\|^2_{-1/2,\Omega_P}\right).$$

For the initial conditions, we set $\boldsymbol{u}^S_{f,h} = \mathbf{I}_{f,h}\boldsymbol{u}_{f,0}$, $\boldsymbol{u}^P_{r,h} = \mathbf{I}_{r,h}\boldsymbol{u}_{r,0}$, $\boldsymbol{y}^P_{s,h}(0) = \mathbf{I}_{s,h}\boldsymbol{y}_{s,0}$, $p^P_h(0) = Q_{p,h}p^{p,0}$, and $\boldsymbol{u}^P_{s,h} = \mathbf{Q}_{s,h}\boldsymbol{u}_{s,0}$, implying $\boldsymbol{\phi}^0_{f,h} = \mathbf{0}$, $\boldsymbol{\phi}^0_{r,h} = \mathbf{0}$, $\boldsymbol{\phi}^0_{ss,h} = \mathbf{0}$, $\boldsymbol{\phi}^0_{s,h} = \mathbf{0}$, $\phi^0_{pp,h} = 0$. And analogously to (4.22), we have

$$\varepsilon_2\tau\sum_{n=1}^{N}\left(\|\phi^n_{fp,h}\|^2_{L^2(\Omega_S)} + \|\phi^n_{pp,h}\|^2_{0,\Omega_P} + \|\phi^n_{\lambda,h}\|^2_{-1/2,\Sigma}\right) \lesssim \varepsilon_2\tau\sum_{n=1}^{N}\left(\|\boldsymbol{\phi}^n_{f,h}\|^2_{1,\Omega_S} + h^{-2}\|\boldsymbol{\phi}^n_{r,h}\|^2_{0,\Omega_P} + \|\boldsymbol{\phi}^n_{r,h}\|^2_{0,\Omega_P}\right.$$
$$\left. + |\boldsymbol{\phi}^n_{f,h} - d_\tau\boldsymbol{\phi}^n_{s,h}|^2_{\mathrm{BJS}} + \|\boldsymbol{\chi}^n_f\|^2_{1,\Omega_S} + \|\boldsymbol{\chi}^n_r\|^2_{1,\Omega_P} + \|\boldsymbol{\chi}^n_r\|^2_{0,\Omega_P} + \|\chi^n_{fp}\|^2_{0,\Omega_P} + \|\chi^n_{pp}\|^2_{0,\Omega_P} + \|\chi^n_\lambda\|^2_{-1/2,\Sigma}\right).$$

Note that the terms involving d_τ on the RHS require a special treatment as in [40]. Consequently

$$\tau\sum_{n=1}^{N-1}\left(\|d_\tau\boldsymbol{\chi}^n_s\|^2_{1,\Omega_P} + \|d_\tau\boldsymbol{\chi}^n_r\|^2_{1,\Omega_P} + \|d_\tau\chi^n_{pp}\|^2_{0,\Omega_P} + \|d_\tau\chi^n_\lambda\|_{-1/2,\Omega_P} + \|d_\tau\boldsymbol{\chi}^n_s\|^2_{0,\Omega_P} + \|d_\tau\boldsymbol{\chi}^n_r\|^2_{0,\Omega_P} + \|d_\tau\boldsymbol{\chi}^n_{ss}\|^2_{0,\Omega_P}\right.$$
$$\left. + \|d_\tau\boldsymbol{\chi}^n_f\|^2_{0,\Omega_S}\right) \le \int_0^{t_N}\left(\|\partial_t\boldsymbol{\chi}_s\|^2_{1,\Omega_P} + \|\partial_t\boldsymbol{\chi}_r\|^2_{1,\Omega_P} + \|\partial_t\chi_{pp}\|^2_{0,\Omega_P} + \|\partial_t\chi_\lambda\|_{-1/2,\Omega_P} + \|\partial_t\boldsymbol{\chi}^n_s\|^2_{0,\Omega_P} + \|\partial_t\boldsymbol{\chi}^n_r\|^2_{0,\Omega_P}\right.$$
$$\left. + \|\partial_t\boldsymbol{\chi}^n_{ss}\|^2_{0,\Omega_P} + \|\partial_t\boldsymbol{\chi}^n_f\|^2_{0,\Omega_S}\right).$$

To bound $\|d_\tau d_\tau\boldsymbol{\chi}^n_s\|^2_{1,\Omega_P}$, we use the Integral Mean Value and Mean Value Theorems. Therefore

$$\tau\sum_{n=1}^{N-1}\|d_\tau d_\tau\boldsymbol{\chi}^n_s\|^2_{1,\Omega_P} \le C\operatorname*{esssup}_{t\in(0,t_N)}\|\partial_{tt}\boldsymbol{\chi}_s\|^2_{1,\Omega_P}.$$

On the other hand, regarding the time discretization error, Taylor's expansion gives

$$\tau\sum_{n=1}^{N}\|r_n(\phi)\|^2_{H^k(S)} \le C\tau^2\|\partial_{tt}\phi\|^2_{L^2(0,T;H^k(S))}, \quad \tau\sum_{n=1}^{N}\|d_\tau r_n(\phi)\|^2_{H^k(S)} \le C\tau^2\|\partial_{ttt}\phi\|^2_{L^2(0,T;H^k(S))}. \tag{5.8}$$

The assertion of the theorem follows from combining (5.7)-(5.8), the discrete Grönwall inequality [38] for $a_n = |\boldsymbol{\phi}^N_{s,h}|^2_{1,\Omega_P}$, triangle inequality, and the approximation properties (4.3a)-(4.3c), (4.4), (4.13a)-(4.13c). □

6. Numerical tests

All routines have been implemented using the open-source FE library FEniCS [1], along with the module Multiphenics [10] (to handle specific terms related to subdomains and boundaries). The solvers used in this work are monolithic. We utilized the MUMPS [6] distributed direct solver for the linear systems in the first three examples and UMFPACK [29] for the fourth example. While the model, the continuous and discrete analyses are valid also in the 3D case, our numerical tests were only performed in 2D.

6.1. *Convergence tests against manufactured solutions*

The accuracy of the discretization is verified using the following closed-form solutions defined on the domains $\Omega_\mathrm{P} = (0,1)\times(0,1), \Omega_\mathrm{S} = (0,1)\times(1,2)$, separated by the interface $\Sigma = (0,1)\times\{1\}$

$$\begin{aligned} \boldsymbol{u}_f^\mathrm{S} = \sin(t)\begin{pmatrix} -\cos(\pi x)\sin(\pi y) \\ \sin(\pi x)\cos(\pi y)\end{pmatrix}, \quad p^\mathrm{S} = \sin(t)\cos(\pi x)\cos(\pi y), \\ \boldsymbol{u}_r^\mathrm{P} = \begin{pmatrix} t^2\sin^2(4\pi y) - tx^3\cos(4\pi y) \\ t^2\sin^2(4\pi y) + 2tx^3\sin(4\pi y)\end{pmatrix}, \\ \boldsymbol{u}_s^\mathrm{P} = \begin{pmatrix} tx^3\cos(4\pi y) \\ -2tx^3\sin(4\pi y)\end{pmatrix}, \quad \boldsymbol{y}_s^\mathrm{P} = \begin{pmatrix} 0.5t^2x^3\cos(4\pi y) \\ -t^2x^3\sin(4\pi y)\end{pmatrix}, \quad p^\mathrm{P} = t^2(1-\sin(4\pi x)\sin(4\pi y)). \end{aligned} \tag{6.1}$$

The synthetic model parameters are taken as $\lambda_p = \mu_p = \mu_f = 10$, $\alpha_\mathrm{BJS} = 1$, $\phi = 0.1$, $\kappa = \rho_p = \rho_f = K = 1$, $\theta = -0.01$, all regarded non-dimensional, as we will be simply testing the convergence of the FE approximations. These functions do not necessarily fulfill the interface conditions, so additional terms are required giving modified relations on Σ:

$$\begin{gathered} \boldsymbol{u}_f^\mathrm{S}\cdot\boldsymbol{n}_\mathrm{S} + \left(\partial_t\boldsymbol{y}_s^\mathrm{P} + \boldsymbol{u}_r^\mathrm{P}\right)\cdot\boldsymbol{n}_\mathrm{P} = m^1_{\Sigma,\mathrm{ex}}, \quad -\left(\boldsymbol{\sigma}^\mathrm{S}\boldsymbol{n}_\mathrm{S}\right)\cdot\boldsymbol{n}_\mathrm{S} = -\left(\boldsymbol{\sigma}_f^\mathrm{P}\boldsymbol{n}_\mathrm{P}\right)\cdot\boldsymbol{n}_\mathrm{P} + m^2_{\Sigma,\mathrm{ex}} = \lambda, \\ \boldsymbol{\sigma}_f^\mathrm{S}\boldsymbol{n}_\mathrm{S} + \boldsymbol{\sigma}_f^\mathrm{P}\boldsymbol{n}_\mathrm{P} + \boldsymbol{\sigma}_s^\mathrm{P}\boldsymbol{n}_\mathrm{P} = m^3_{\Sigma,\mathrm{ex}}, \quad -\left(\boldsymbol{\sigma}_f^\mathrm{S}\boldsymbol{n}_\mathrm{S}\right)\cdot\boldsymbol{\tau}_{f,j} = \mu_f\alpha_\mathrm{BJS}\sqrt{Z_j^{-1}}\left(\boldsymbol{u}_f^\mathrm{S} - \partial_t\boldsymbol{y}_s^\mathrm{P}\right)\cdot\boldsymbol{\tau}_{f,j} + m^4_{\Sigma,\mathrm{ex}}, \\ -(\boldsymbol{\sigma}_f^\mathrm{P}\boldsymbol{n}_\mathrm{P})\cdot\tau_{f,j} = \mu_f\alpha_\mathrm{BJS}\sqrt{Z_j^{-1}}\boldsymbol{u}_r^\mathrm{P}\cdot\boldsymbol{\tau}_{f,j} + m^5_{\Sigma,\mathrm{ex}}, \end{gathered}$$

and the additional scalar and vector terms $m^i_{\Sigma,\mathrm{ex}}$ (computed with the exact solutions (6.1) entail the following changes in the linear functionals

$$\begin{gathered} F(\boldsymbol{v}_f^\mathrm{S}) = \int_{\Omega_\mathrm{S}} \boldsymbol{f}_\mathrm{S}\boldsymbol{v}_f^\mathrm{S} - \langle m^4_{\Sigma,\mathrm{ex}}, \boldsymbol{v}_f^\mathrm{S}\cdot\tau_{f,j}\rangle_\Sigma, \quad F(\boldsymbol{v}_r^\mathrm{P}) := \int_{\Omega_\mathrm{P}} \rho_f\phi\boldsymbol{f}_\mathrm{P}\boldsymbol{v}_f^\mathrm{S} + \langle m^2_{\Sigma,\mathrm{ex}}, \boldsymbol{v}_r^\mathrm{P}\cdot\boldsymbol{n}_\mathrm{P}\rangle_\Sigma - \langle m^5_{\Sigma,\mathrm{ex}}, \boldsymbol{v}_r^\mathrm{P}\cdot\tau_{f,j}\rangle_\Sigma, \\ F(\boldsymbol{w}_s^\mathrm{P}) := \int_{\Omega_\mathrm{P}} \rho_p\boldsymbol{f}_\mathrm{P}\boldsymbol{w}_s^\mathrm{P} + \langle m^3_{\Sigma,\mathrm{ex}}, \boldsymbol{w}_s^\mathrm{P} + \langle m^4_{\Sigma,\mathrm{ex}}, \boldsymbol{w}_s^\mathrm{P}\cdot\tau_{f,j}\rangle_\Sigma\rangle_\Sigma, \quad F(\mu) := -\langle m^1_{\Sigma,\mathrm{ex}}, \mu\rangle_\Sigma. \end{gathered}$$

We generate successively refined simplicial grids and use a sufficiently small (non dimensional) time step $\tau = h^2$ and final time $T = 1$, to guarantee that the error produced by the time discretization does not dominate. Errors between the approximate and exact solutions are shown in Table 6.1. Theoretically, we observe that both the relative velocity and solid displacement exhibit sub-optimal convergence, but in practice only the relative velocity exhibits this behavior in our tests. Additionally, the fluid pressure shows super-convergence.

The backward Euler method is assessed for time convergence and verified by partitioning the time interval $(0,1)$ into successively refined uniform discretizations and computing cumulative errors $\hat{e}_s = \left(\sum_{n=1}^N \tau\|s(t_{n+1}) - s_h^{n+1}\|^2_\star\right)^{1/2}$, where $\|\cdot\|_\star$ is the appropriate space norm for the generic vector or scalar field s. For this test we use a fixed mesh involving 923915 DoFs. The results are shown in Table 6.2, confirming the expected first-order convergence.

6.2. *Simulation of subsurface fracture flow*

This test illustrates the applicability of the formulation in hydraulic fracturing and problem setup is similar to [3, 40]. Consider a rectangular domain Ω with dimensions $(0,3.048)\times(0,6.096)$, consisting of a macro void or open channel Ω_S filled with an incompressible fluid, and the poro-hyperelastic domain is defined as $\Omega_\mathrm{P} = \Omega\setminus\Omega_\mathrm{S}$. The permeability κ and porosity ϕ (heterogeneous but isotropic in the xy-plane) are derived

DoFs	h	$\|e_{\boldsymbol{u}_f^{\mathrm{S}}}\|_{l^2(\mathbf{H}^1)}$	rate	$\|e_{p^{\mathrm{S}}}\|_{l^2(L^2)}$	rate	$\|e_{\boldsymbol{u}_r}\|_{l^2(\mathbf{L}^2)}$	rate	$\|e_{p^{\mathrm{P}}}\|_{l^2(L^2)}$	rate
1107	0.2795	0.214600	–	28.57000	–	4.998000	–	0.375800	–
3995	0.1398	0.035700	2.586	0.900800	4.987	0.136400	5.196	0.060180	2.643
15147	0.0699	0.009108	1.972	0.335000	1.427	0.045020	1.599	0.016930	1.830
58955	0.0349	0.002272	2.003	0.047570	2.816	0.007665	2.554	0.004460	1.924
232587	0.0175	0.000567	2.003	0.006113	2.960	0.001311	2.547	0.001058	2.075

DoFs	h	$\|e_{\boldsymbol{y}_s^{\mathrm{P}}}\|_{l^2(\mathbf{H}^1)}$	rate	$\|e_{\boldsymbol{u}_s^{\mathrm{P}}}\|_{l^2(\mathbf{L}^2)}$	rate	$\|e_{\lambda}\|_{l^2(H^{-1/2})}$	rate
1107	0.2795	0.454600	–	0.241100	–	13.81000	–
3995	0.1398	0.181500	1.325	0.048640	2.309	0.297700	5.536
15147	0.0699	0.048740	1.897	0.012250	1.989	0.080220	1.892
58955	0.0349	0.012440	1.970	0.003070	1.996	0.007744	3.373
232587	0.0175	0.003128	1.991	0.000763	2.009	0.000654	3.566

TABLE 6.1 *Experimental errors related to spatial discretization and convergence rates are computed for the approximate solutions* $\boldsymbol{u}_f^{\mathrm{S}}, p^{\mathrm{S}}, \boldsymbol{u}_r^{\mathrm{P}}, p_h, \boldsymbol{y}_s^{\mathrm{P}}, \boldsymbol{u}_s^{\mathrm{P}}$ *and* λ_h, *using* $\mathbb{P}_2^2 - \mathbb{P}_1 - \mathbb{P}_2^2 - \mathbb{P}^1 - \mathbb{P}_2^2 - \mathbb{P}_1^2 - \mathbb{P}_1$. *The computations are performed at the last time step.*

τ	$\hat{e}_{\boldsymbol{u}_f^{\mathrm{S}}}$	rate	$\hat{e}_{p^{\mathrm{S}}}$	rate	$\hat{e}_{\boldsymbol{u}_r^{\mathrm{P}}}$	rate	$\hat{e}_{p^{\mathrm{P}}}$	rate	$\hat{e}_{\boldsymbol{y}_s^{\mathrm{P}}}$	rate
0.5	0.0293	–	0.5374	–	0.0514	–	0.1013	–	1.3238	–
0.225	0.0146	1.005	0.2688	1.000	0.0231	1.150	0.0504	1.008	0.5736	1.207
0.125	0.0073	1.000	0.1345	0.990	0.0109	1.078	0.0251	1.003	0.2645	1.117
0.0625	0.0036	0.999	0.0673	0.998	0.0053	1.033	0.0126	1.001	0.1266	1.063
0.03125	0.0018	0.998	0.0337	0.998	0.0027	1.001	0.0063	0.999	0.0619	1.033
0.015625	0.0009	0.997	0.0169	0.997	0.0014	0.967	0.0031	0.996	0.0306	1.018

τ	$\hat{e}_{\boldsymbol{u}_s^{\mathrm{P}}}$	rate	$\|\hat{e}_{\lambda}\|_{l^2(H^{-1/2})}$	rate
0.5	0.1458	–	0.1505	–
0.225	0.0729	1.000	0.0749	1.007
0.125	0.0365	1.000	0.0374	1.003
0.0625	0.0182	1.000	0.0187	1.002
0.03125	0.0091	1.000	0.0093	1.000
0.015625	0.0046	0.999	0.0047	1.000

TABLE 6.2 *Experimental cumulative errors associated with the temporal discretization and convergence rates for the approximate solutions* $\boldsymbol{u}_f^{\mathrm{S}}, p^{\mathrm{S}}, \boldsymbol{u}_r^{\mathrm{P}}, p_h, \boldsymbol{y}_s^{\mathrm{P}}$, *and* $\boldsymbol{u}_s^{\mathrm{P}}$, *using a backward Euler scheme.*

from the non-smooth pattern found in the SPE – 10 benchmark data/model 2[1]. We rescale this pattern as in

[1] www.spe.org/web/csp

[4] and map it onto a piecewise constant field using an unstructured triangular mesh for the poro-hyperelastic region. Note that the present formulation requires smooth porosity. Therefore, we project both the porosity and permeability data onto a $\mathbb{P}_1$ field (Figure 6.1). There are 85 distinct layers within two general categories and we choose layer 80 from the dataset (Upper Ness region exhibiting a fluvial fan pattern – flux channels of higher permeability and porosity). No gravity and no external loads are considered and the unstructured triangular mesh has 1629 elements for the Stokes region and 18897 elements for the poro-hyperelastic domain. In the poroelastic region we incorporate $\mu_f \phi^2 \kappa^{-1} \boldsymbol{u}_r$ instead of $\phi^2 \kappa^{-1} \boldsymbol{u}_r$. The boundary conditions are

$$\begin{aligned}
\text{Injection: } & \boldsymbol{u}_f^{\mathrm{S}} \cdot \boldsymbol{n}_{\mathrm{S}} = 10, \quad \boldsymbol{u}_f^{\mathrm{S}} \cdot \boldsymbol{\tau}_f = 0 && \text{on } \Sigma_{\text{inflow}},\\
\text{stress free: } & \left(\boldsymbol{\sigma}_s^{\mathrm{P}} \boldsymbol{n}_{\mathrm{P}}\right) = \mathbf{0}, \left(\boldsymbol{\sigma}_f^{\mathrm{P}} \boldsymbol{n}_{\mathrm{P}}\right) = \mathbf{0} && \text{on } \Sigma_{\text{left}},\\
\text{Normal relative velocity: } & \boldsymbol{u}_r^{\mathrm{P}} \cdot \boldsymbol{n}_{\mathrm{P}} = 0 && \text{on } \Sigma_{\text{bottom}} \cup \Sigma_{\text{right}} \cup \Sigma_{\text{top}},\\
\text{Normal displacement: } & \boldsymbol{y}_s^{\mathrm{P}} \cdot \boldsymbol{n}_{\mathrm{P}} = 0 && \text{on } \Sigma_{\text{bottom}} \cup \Sigma_{\text{right}} \cup \Sigma_{\text{top}},\\
\text{Shear traction: } & \left(\boldsymbol{\sigma}_s^{\mathrm{P}} \boldsymbol{n}_{\mathrm{P}}\right) \cdot \boldsymbol{\tau}_{f,j} = 0, \left(\boldsymbol{\sigma}_f^{\mathrm{P}} \boldsymbol{n}_{\mathrm{P}}\right) \cdot \boldsymbol{\tau}_{f,j} = 0 && \text{on } \Sigma_{\text{bottom}} \cup \Sigma_{\text{right}} \cup \Sigma_{\text{top}}.
\end{aligned}$$

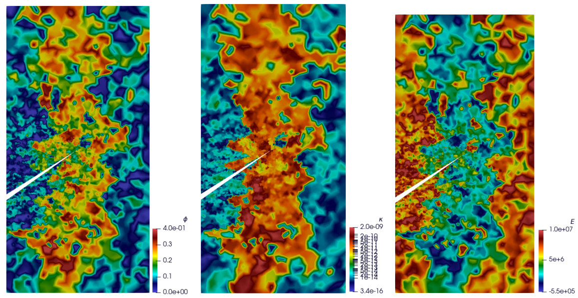
The initial conditions are set accordingly to $\boldsymbol{u}_f^{\mathrm{S}}(\mathbf{0}) = \mathbf{0}, \quad \boldsymbol{u}_r^{\mathrm{P}}(\mathbf{0}) = \mathbf{0}, \quad \boldsymbol{y}_s^{\mathrm{P}}(\mathbf{0}) = \mathbf{0}, \quad \boldsymbol{u}_s^{\mathrm{P}}(\mathbf{0}) = \mathbf{0}$ and $p^{\mathrm{P}}(0) = 0$. The total simulation time is $T = 10$ hours and the time step is $\tau = 30$ s. The Lamé coefficients and bulk modulus are determined from the Young's modulus E and the Poisson's ratio ν via the relationships $\lambda_p = E\nu/[(1+\nu)(1-2\nu)]$ and $\mu_p = E/[2(1+\nu)]$. Given the porosity ϕ, the Young's modulus is determined from the law $E(\boldsymbol{x}) = 10^7(1-2\phi(\boldsymbol{x}))^{2.1}$. The model parameters are taken as $\nu = 0.2$, $\mu_f = 10^{-6}$, $\alpha_{\mathrm{BJS}} = 1$, $\rho_f = 1000$, $\rho_p = 1016$, $\theta = 0$, $c_0 = 6.89 \times 10^{-2}$, $K = (1-\phi)^2/c_0$. These parameters are typical for hydraulic fracturing [22, 36].

For this test, we use Taylor–Hood $\mathbb{P}_2^2$–$\mathbb{P}_1$ elements for the fluid velocity and pressure in the fracture region, and $\mathbb{P}_2^2$–$\mathbb{P}_1$–$\mathbb{P}_2^2$–$\mathbb{P}_1^2$ elements for the relative velocity, pressure, solid displacement, and solid velocity in the porous medium, with continuous $\mathbb{P}_1$ elements for the Lagrange multiplier. Snapshots of the approximate solutions (relative velocity, solid velocity, solid displacement, poro-hyperelastic pressure, and fluid pressure and velocity magnitude in the Stokes region) for fluid injection into a fractured porous medium using the SPE10-based benchmark are shown in Figure 6.2. Most leak-off occurs at the fracture tip, where the relative velocity in the porous medium is largest in the adjacent high-permeability channel. The injected fluid increases the interfacial pressure, producing the expected channel-like filtration from the Stokes to the poro-hyperelastic domain, visible in the porous pressure plot on the left of Figure 6.2 with higher values near the tip. The model also accommodates spatially varying porosity, which explains leakage near the base of the injection (downward) and an additional upward flow about two-thirds along the tip.

6.3. *Channel filtration and stress build-up on interface deformation*

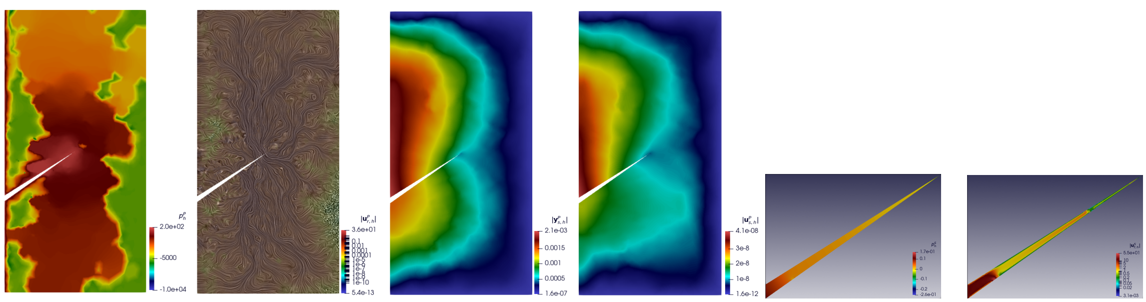
We continue our set of numerical simulations addressing an important practical problem. We adopt a similar setup as in [40]. Consider the domains $\Omega_{\mathrm{S}} = (-1,1)\times(0,2), \Omega_{\mathrm{P}} = (-1,1)\times(-2,0)$, separated by the interface $\Sigma = (-1,1)\times\{0\}$. Even if the present model is stated in the limit of small strains, it is possible to have large displacements, likely located near the interface (and without violating the model assumptions). We smoothly move the fluid domain and the mesh to avoid distortions generated near the interface. For this we use a standard harmonic extension that is solved at each time step: Find $\mathbf{d}_h^* = \mathbf{d}_h + \widehat{\mathbf{d}}_h$ such that $-\Delta\widehat{\mathbf{d}}_h = \mathbf{0}$ in Ω_S, $\widehat{\mathbf{d}}_h = \mathbf{d}_h$ on Σ, $\nabla\widehat{\mathbf{d}}_h \cdot \boldsymbol{n}_{\mathrm{S}} = \mathbf{0}$ on the walls of Ω_S, and $\widehat{\mathbf{d}}_h = \mathbf{0}$ on the inlet. Note that, in contrast to [40], we are using a homogeneous Neumann boundary condition at the sides. Then, we perform an $\mathbf{L}^2$-projection of both $\mathbf{d}_h$ and $\widehat{\mathbf{d}}_h$ into $\mathbf{V}_{s,h} + \mathbf{V}_{f,h}$ and add them to obtain the global displacement $\mathbf{d}_h^*$. We assume that there are no body forces or gravity acting on the system.

The boundary conditions is as follows, assuming that the flow is driven by pressure differences only. On the top segment we impose the fluid pressure $p_{\text{in}}^{\mathrm{S}} = 2\sin^2(\pi t)$, and on the outlet (the bottom segment) the fluid



(a) Porosity (b) Permeability (c) Young's modulus

FIG. 6.1. Material properties (porosity $\phi(\boldsymbol{x})$, permeability $\kappa(\boldsymbol{x})$, and Young's modulus $E(\boldsymbol{x})$) from layer 80 of the SPE10 benchmark dataset for reservoir simulations, projected onto a $\mathbb{P}_1$ field for the poro-hyperelastic sub-domain.



(a) porous press. (b) rel. vel. (c) solid displ. (d) solid vel. (e) Stokes press. (f) Stokes vel.

FIG. 6.2. Approximate solutions for fluid injection into a fractured porous medium using the SPE10-based benchmark test.

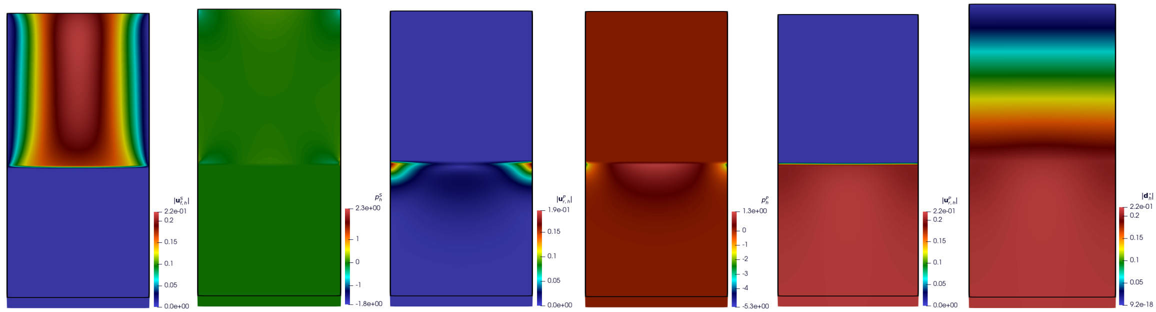
pressure $p^{\mathrm{P}}_{\mathrm{out}} = 0$. On the vertical walls of Ω_{S} we set $\boldsymbol{u}^{\mathrm{S}}_f = \mathbf{0}$ while on the vertical walls of Ω_{P} we set the slip conditions $\boldsymbol{y}^{\mathrm{P}}_s \cdot \boldsymbol{n}_{\mathrm{P}} = 0$ and $\boldsymbol{u}^{\mathrm{P}}_r \cdot \boldsymbol{n}_{\mathrm{P}} = 0$. The model parameters (all adimensional) are taken as $\kappa = 0.005$, $\lambda_p = 10$, $\mu_p = 5$, $\rho_p = 1.07$, $\rho_f = 1$, $\alpha_{\mathrm{BJS}} = 0.1$, $\mu_f = 0.8$, $\theta = 0$, $c_0 = 0.02$, $\phi = 0.3$, $K = (1-\phi)^2/c_0$.

This example uses the same FEs as before and the numerical results are presented in Figure 6.3. The effect of the interface is clearly seen in the poroelastic domain. Close to the interface, the relative velocity, the solid displacement, and the fluid pressure are heterogeneous in the horizontal direction before recovering the expected constant value (constant in the horizontal direction) expected in the far field. Also, we plot the overall displacement in the domain. From Figure 6.4 one can see that for large enough interfacial displacements, the elements close to it exhibit a large distortion.

6.4. *Interfacial flow in the brain*

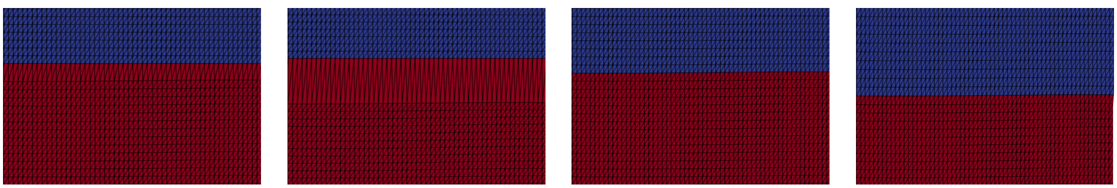
To conclude this section, we present a 2D simulation related to brain biomechanics. More specifically, we investigate how the incoming cerebrospinal fluid (CSF) flow from the spinal canal effects the brain tissues. In other words, we can say that the chosen problem is motivated by real-world applications like modeling the glymphatic system, where the brain porous tissue interacts with the surrounding CSF. For example, a heartbeat creates pressure waves in the CSF around the brain, which then spread through the brain. A number of mechanical processes constantly affect the brain function: blood entering and leaving, fluid movements such as CSF and interstitial fluid in and around the brain and spine, pressures inside the skull, brain tissue shifts, and fluid flow between cells.

Following [18], we represent the brain parenchyma as a 2D poro-hyperelastic sub-domain Ω_P, and the surrounding CSF-filled spaces as a free fluid (Stokes) sub-domain denoted by Ω_S. The sub-domains share a



(a) Stokes vel. (b) Stokes press. (c) rel. vel. (d) porous press. (e) solid displ. (f) global displ.

FIG. 6.3. Filtration into a deformable porous medium at time $t = 2$ with $\tau = 0.1$. The solid line indicates the undeformed domain.



(a) (b) (c) (d)

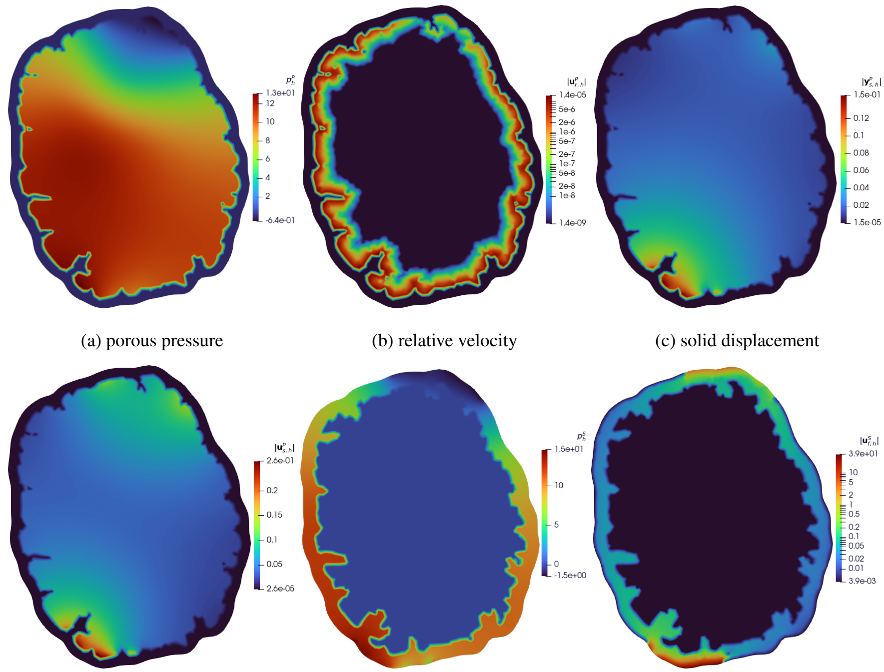
FIG. 6.4. Zoom of the meshes on the interface at times $t = 1$ (a, c) and $t = 2$ (b,d). We show the results without (a,b) and with (c,d) the harmonic extension.

common boundary $\Sigma = \Omega_S \cap \Omega_P$ with normal vector $\boldsymbol{n}_S$, pointing from Ω_S to Ω_P on Σ and outwards on the boundary $\partial\Omega_S$. The following parameter values are adopted from [18, 25, 46]: $\mu_f = 7 \times 10^{-7}$, $\alpha_{\mathrm{BJS}} = 1$, $c_0 = 2 \times 10^{-5}$, $\phi = 0.2$, $K = (1-\phi)^2/c_0$, $\kappa = 1 \times 10^{-8}$, $\mu_p = 267 \times 10^{-3}$, $\lambda_p = 26488 \times 10^{-3}$, $\theta = 0$, $\rho_f = 1.005$, $\rho_p = 1.03$. The traction boundary conditions are applied as follows: at the top right $(\boldsymbol{\sigma}_f^{\mathrm{S}} \cdot \boldsymbol{n}_{\mathrm{S}} = (10, 10))$ and bottom right $(\boldsymbol{\sigma}_f^{\mathrm{S}} \cdot \boldsymbol{n}_{\mathrm{S}} = (1, 1))$ regions in the axial slices of the brain, and at the top $(\boldsymbol{\sigma}_f^{\mathrm{S}} \cdot \boldsymbol{n}_{\mathrm{S}} = (10, 10))$ region in the coronal slices of the brain. A homogeneous Dirichlet boundary condition is imposed on the remaining parts of the boundary. The snapshots of the approximate solutions (interstitial fluid pressure, interstitial fluid velocity, brain tissue displacement, brain tissue velocity, CSF pressure, CSF velocity) for the axial and coronal slices, which illustrate the interfacial flow in the brain, are shown in Figures 6.5 and 6.6, respectively.

For the axial slices, excess pore pressure in the parenchyma drains across the interface. Under the considered flow and loading rates, the boundary conditions at the bottom left localize the Stokes pressure and velocity, producing steep pore pressure gradients and higher velocities in that region. The Biot pressure then dissipates through the porous domain, with permeation patterns following brain displacement. Near the top-right interface, displacement is noticeably smaller than in the parenchyma center. In the coronal slices, fluid pressures in the subarachnoid space (CSF) and parenchyma show a stronger gradient near the top interface, leading to smooth displacements with slightly higher magnitudes in the top-center region. On average, CSF pressure remains higher than parenchyma pressure. In both views, interstitial fluid (ISF) velocity within the parenchyma is consistently low compared to CSF flow.

7. Conclusion

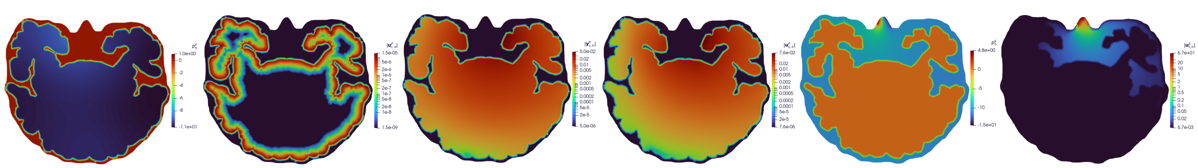
In this paper, we propose a model for the coupling between free fluid and a generalized poroelastic body. This model is a novel contribution to the field of theoretical and numerical partial differential equations in interface coupled problems. We employed the Brinkman equation for fluid flow in the porous medium, incorporating inertial effects into the fluid dynamics. A Lagrange multiplier–based formulation is proposed, and an alternative formulation is used in which the primary variables are the elastic stress and structural



(a) porous pressure (b) relative velocity (c) solid displacement

(d) solid velocity (e) Stokes pressure (f) Stokes velocity

FIG. 6.5. Snapshots of the approximate solutions for the interfacial flow in an idealized geometry at $T = 1$ with $dt = 0.005$. The traction boundary conditions in the top right and bottom left corners (axial slices), respectively.



(a) porous pressure (b) relative velocity (c) solid displacement (d) solid velocity (e) Stokes pressure (f) Stokes velocity

FIG. 6.6. Snapshots of the approximate solutions for the interfacial flow in an idealized geometry at $T = 1$ with $dt = 0.005$. The traction boundary conditions at the top (coronal slices), respectively.

velocity. This formulation serves as a mathematical tool to establish the unique solvability of the governing equations, and corresponding a priori error estimates for both the semi- and fully discrete schemes are derived.

Theoretically, we observe that both relative velocity and solid displacement exhibit sub-optimal convergence, but only the relative velocity exhibits such a behavior in practice in our tests. The solid and relative velocity blocks in the diagonal of the system matrix makes it difficult to derive a bound in the energy norm. We also conducted numerical validation of spatio-temporal accuracy, and additionally observe superconvergence for Stokes pressure. We performed tests of realistic applications using this model, studying the behavior of poromechanical filtration in subsurface hydraulic fracture with challenging heterogeneous material parameters and channel filtration when stress builds up on interface deformation. The set of tests also includes a typical application in biomechanical modeling of the brain to study how the incoming CSF flow from the spinal canal effects the brain tissues. Further perspectives of this work include the extension to the fully nonlinear regime, as well as other types of transmission conditions that would allow greater generality in the poromechanical problems we can tackle, together with the development of robust iterative solvers that would enable the use of this model in 3D scenarios.

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A. Mixed formulation

Following [43], we consider a mixed elasticity formulation in terms of structure velocity and elastic stress. Let us recall the inverse stress-strain relation

$$A\boldsymbol{\sigma}^{\mathrm{P}} = \boldsymbol{\varepsilon}(\boldsymbol{y}_s^{\mathrm{P}}), \tag{A.1}$$

where A is a symmetric and positive definite compliance tensor. Similarly to $\boldsymbol{y}_{s,0}$ in the original formulation, the initial stress $\boldsymbol{\sigma}_0^{\mathrm{P}}$ is determined from $p^{\mathrm{P},0}$ using (2.3c). In particular, we will show that $\boldsymbol{\sigma}_0^{\mathrm{P}} = A^{-1}\boldsymbol{\varepsilon}(\boldsymbol{y}_{s,0})$. In the isotropic case $A\boldsymbol{\sigma}^{\mathrm{P}} = \frac{1}{2\mu_p}(\boldsymbol{\sigma}^{\mathrm{P}} - \frac{\lambda_p}{2\mu_p+d\lambda_p}\operatorname{tr}\left(\boldsymbol{\sigma}^{\mathrm{P}}\right)\mathbf{I})$, with $A^{-1}\varepsilon = 2\mu_p\varepsilon + \lambda_p\operatorname{tr}(\varepsilon)\mathbf{I}$. The regularity of the displacement implies that the functional space for the elastic stress is $\mathbf{Z} = \mathbb{L}^2_{\mathrm{sym}}(\Omega_{\mathrm{P}})$ with the norm $\|\boldsymbol{\sigma}^{\mathrm{P}}\|^2_{\mathbf{Z}} := \sum_{i,j=1}^d \|(\boldsymbol{\sigma}^{\mathrm{P}})_{i,j}\|^2_{0,\Omega_{\mathrm{P}}}$. The mixed formulation handles (2.4c) in a different manner. We still test this equation, as before, against $\boldsymbol{v}_s^{\mathrm{P}} \in \mathbf{V}_s$ and integrate by parts, but we use the constitutive relation (2.3c). This yields

$$-\int_{\Omega_{\mathrm{P}}} \boldsymbol{\sigma}_s^{\mathrm{P}} : \nabla\boldsymbol{v}_s^{\mathrm{P}}\,\mathrm{d}\boldsymbol{x} = \int_{\Omega_{\mathrm{P}}} \left(\boldsymbol{\sigma}^{\mathrm{P}} : \boldsymbol{\varepsilon}(\boldsymbol{v}_s^{\mathrm{P}}) - (1-\phi)p^{\mathrm{P}}\nabla\cdot\boldsymbol{v}_s^{\mathrm{P}}\right)\mathrm{d}\boldsymbol{x} - \int_{\Sigma} \boldsymbol{\sigma}_s^{\mathrm{P}}\boldsymbol{n}_{\mathrm{P}}\cdot\boldsymbol{v}_s^{\mathrm{P}}\,\mathrm{d}s.$$

We remove $\boldsymbol{y}_s^{\mathrm{P}}$ by differentiating (A.1) in time and writing $\boldsymbol{u}_s^{\mathrm{P}}$ instead of $\partial_t\boldsymbol{y}_s^{\mathrm{P}} \in \mathbf{V}_s$. Testing against $\boldsymbol{\tau}^{\mathrm{P}} \in \mathbf{Z}$ gives

$$\int_{\Omega_{\mathrm{P}}} \left(A\partial_t\boldsymbol{\sigma}^{\mathrm{P}} : \boldsymbol{\tau}^{\mathrm{P}} - \boldsymbol{\varepsilon}\left(\boldsymbol{u}_s^{\mathrm{P}}\right) : \boldsymbol{\tau}^{\mathrm{P}}\right)\mathrm{d}\boldsymbol{x} = 0.$$

The rest of the equations are handled in the same way as in the original formulation, resulting in the same functionals and interfacial terms. Next, we define $b^{\mathrm{P}}_{\mathrm{sig}}(\cdot,\cdot) : \mathbf{V}_s\times\mathbf{Z}\to\mathbb{R}$ and $a^{\mathrm{P}}_p(\cdot,\cdot) : \mathbf{Z}\times\mathbf{Z}\to\mathbb{R}$ by

$$b^{\mathrm{P}}_{\mathrm{sig}}\left(\boldsymbol{u}_s^{\mathrm{P}},\boldsymbol{\tau}^{\mathrm{P}}\right) := (\boldsymbol{\varepsilon}(\boldsymbol{u}_s^{\mathrm{P}}),\boldsymbol{\tau}^{\mathrm{P}})_{\Omega_{\mathrm{P}}}, \quad a^{\mathrm{P}}_p(\boldsymbol{\sigma}^{\mathrm{P}},\boldsymbol{\tau}^{\mathrm{P}}) := \left(A\boldsymbol{\sigma}^{\mathrm{P}},\boldsymbol{\tau}^{\mathrm{P}}\right)_{\Omega_{\mathrm{P}}},$$

and then proceed to group the trial and test spaces and functions in the following manner

$$\begin{aligned}
\vec{\boldsymbol{u}} := \left(\boldsymbol{u}_r^{\mathrm{P}},\boldsymbol{u}_s^{\mathrm{P}},\boldsymbol{u}_f^{\mathrm{S}}\right), \quad \vec{\boldsymbol{v}} := \left(\boldsymbol{v}_r^{\mathrm{P}},\boldsymbol{v}_s^{\mathrm{P}},\boldsymbol{v}_f^{\mathrm{P}}\right) \quad &\in \vec{\mathbf{V}} := \mathbf{V}_r\times\mathbf{V}_s\times\mathbf{V}_f,\\
\vec{p} := \left(p^{\mathrm{P}},\boldsymbol{\sigma}^{\mathrm{P}},p^{\mathrm{S}},\lambda\right), \quad \vec{q} := \left(q^{\mathrm{P}},\boldsymbol{\tau}^{\mathrm{P}},q^{\mathrm{S}},\mu\right) \quad &\in \vec{Q} := \mathrm{W}_p\times\mathbf{Z}\times\mathrm{W}_f\times\Lambda,
\end{aligned}$$

and use the norms $\|\vec{\boldsymbol{u}}\|_{\vec{\mathbf{V}}} := \|\boldsymbol{u}_r^{\mathrm{P}}\|_{\mathbf{V}_r} + \|\boldsymbol{u}_s^{\mathrm{P}}\|_{\mathbf{V}_s} + \|\boldsymbol{u}_f^{\mathrm{S}}\|_{\mathbf{V}_f}$, $\|\vec{p}\|_{\vec{Q}} := \|p^{\mathrm{P}}\|_{\mathrm{W}_p} + \|\boldsymbol{\sigma}^{\mathrm{P}}\|_{\mathbf{Z}} + \|p^{\mathrm{S}}\|_{\mathrm{W}_f} + \|\lambda\|_{\Lambda}$. With this, the weak formulation is written as a degenerate mixed evolution problem

$$\partial_t\mathscr{E}_1\vec{\boldsymbol{u}}(t) + \mathscr{A}\vec{\boldsymbol{u}}(t) + \mathscr{B}^T\vec{p}(t) = \boldsymbol{f}(t) \qquad \text{in } \vec{\mathbf{V}}', \tag{A.2a}$$

$$\partial_t\mathscr{E}_2\vec{p}(t) - \mathscr{B}\vec{\boldsymbol{u}}(t) + \mathscr{C}\vec{p}(t) = g(t) \qquad \text{in } \vec{Q}', \tag{A.2b}$$

where the operators $\mathscr{A}:\vec{\mathbf{V}}\to\vec{\mathbf{V}}',\mathscr{B}:\vec{\mathbf{V}}\to\vec{Q}',\mathscr{C}:\vec{Q}\to\vec{Q}',\mathscr{E}_1:\vec{\mathbf{V}}\to\vec{\mathbf{V}}',\mathscr{E}_2:\vec{Q}\to\vec{Q}'$, and the functionals $\boldsymbol{f}\in\vec{\mathbf{V}}', g\in\vec{Q}'$ are defined as follows (where we are also using the notation $\langle\mathscr{B}^{\mathrm{P}}_{\mathrm{sig}}\cdot_1,\cdot_2\rangle = b^{\mathrm{P}}_{\mathrm{sig}}(\cdot_1,\cdot_2)$):

$$\mathscr{A}=\left[\begin{array}{ccc} \mathscr{A}_f^{\mathrm{P}}-\mathscr{M}_\theta+\mathscr{K}+\mathscr{A}_{rr}^{\mathrm{BJS}} & -M_\theta+\mathscr{A}_f^{\mathrm{P}} & 0\\ -M_\theta+\mathscr{A}_f^{\mathrm{P}} & -M_\theta+\mathscr{A}_f^{\mathrm{P}}+A_{ss}^{\mathrm{BJS}} & (\mathscr{A}_{fs}^{\mathrm{BJS}})^*\\ 0 & \mathscr{A}_{fs}^{\mathrm{BJS}} & \mathscr{A}_f^{\mathrm{S}}+\mathscr{A}_{ff}^{\mathrm{BJS}} \end{array}\right],$$

$$\mathscr{B}^T=\left[\begin{array}{cccc} \mathscr{B}_f^{\mathrm{P}} & 0 & 0 & \mathscr{B}_{f,\Gamma}\\ \mathscr{B}_s^{\mathrm{P}} & \mathscr{B}_{\mathrm{sig}}^{\mathrm{P}} & 0 & \mathscr{B}_{p,\Gamma}\\ 0 & 0 & \mathscr{B}^{\mathrm{S}} & \mathscr{B}_{e,\Gamma} \end{array}\right],\quad \mathscr{E}_1=\begin{pmatrix} \mathscr{M}_{\rho_f\phi} & \mathscr{M}_{\rho_f\phi} & 0\\ \mathscr{M}_{\rho_f\phi} & \mathscr{M}_{\rho_p} & 0\\ 0 & 0 & \mathscr{M}_{\rho_f}\end{pmatrix},$$

$$\mathscr{E}_2=\begin{pmatrix} \mathscr{M}_{(1-\phi)^2K^{-1}} & 0 & 0 & 0\\ 0 & \mathscr{M}_A & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},\quad \mathscr{C}=[\mathbf{0}]_{4\times 4},\quad \boldsymbol{f}=\begin{pmatrix} 2\rho_f\phi\boldsymbol{f}_{\mathrm{P}}\\ \rho_p\boldsymbol{f}_{\mathrm{P}}+\rho_f\phi\boldsymbol{f}_{\mathrm{P}}\\ \boldsymbol{f}_{\mathrm{S}}\end{pmatrix},\quad g=\begin{pmatrix}\theta\\0\\0\\0\end{pmatrix}.$$

Lemma A.1. *Assuming (H.1)-(H.3), the linear operators $\mathscr{A}$, $\mathscr{E}_1$ and $\mathscr{E}_2$ are continuous and monotone.*

Proof. From Cauchy–Schwarz and Young inequalities, there exist $C_f, C_s, C_r, C_{\mathrm{BJS}}>0$ such that

$$\begin{aligned} a_f^{\mathrm{S}}(\boldsymbol{u}_f^{\mathrm{S}},\boldsymbol{v}_f^{\mathrm{S}}) &\le C_f\|\boldsymbol{u}_f^{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}\|\boldsymbol{v}_f^{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}},\\ a_f^{\mathrm{P}}(\boldsymbol{u}_s^{\mathrm{P}},\boldsymbol{v}_s^{\mathrm{P}})-m_\theta(\boldsymbol{u}_s^{\mathrm{P}},\boldsymbol{v}_s^{\mathrm{P}}) &\le C_s\|\boldsymbol{u}_s^{\mathrm{P}}\|_{1,\Omega_{\mathrm{P}}}\|\boldsymbol{v}_s^{\mathrm{P}}\|_{1,\Omega_{\mathrm{P}}},\\ a_f^{\mathrm{P}}(\boldsymbol{u}_r^{\mathrm{P}},\boldsymbol{v}_r^{\mathrm{P}})-m_\theta(\boldsymbol{u}_r^{\mathrm{P}},\boldsymbol{v}_r^{\mathrm{P}})+m_{\phi^2/\kappa}(\boldsymbol{u}_r^{\mathrm{P}},\boldsymbol{v}_r^{\mathrm{P}}) &\le C_r\|\boldsymbol{u}_r^{\mathrm{P}}\|_{1,\Omega_{\mathrm{P}}}\|\boldsymbol{v}_r^{\mathrm{P}}\|_{1,\Omega_{\mathrm{P}}},\\ a_{\mathrm{BJS}}(\boldsymbol{u}_f^{\mathrm{S}},\boldsymbol{u}_s^{\mathrm{P}};\boldsymbol{v}_f^{\mathrm{S}},\boldsymbol{v}_s^{\mathrm{P}})+b_{\mathrm{BJS}}(\boldsymbol{u}_r^{\mathrm{P}};\boldsymbol{v}_r^{\mathrm{P}}) &\le C_{\mathrm{BJS}}\left(\|\boldsymbol{u}_f^{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}+\|\boldsymbol{u}_s^{\mathrm{P}}\|_{1,\Omega_{\mathrm{P}}}+\|\boldsymbol{u}_r^{\mathrm{P}}\|_{1,\Omega_{\mathrm{P}}}\right)\\ &\qquad\left(\|\boldsymbol{v}_f^{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}+\|\boldsymbol{v}_s^{\mathrm{P}}\|_{1,\Omega_{\mathrm{P}}}+\|\boldsymbol{v}_r^{\mathrm{P}}\|_{1,\Omega_{\mathrm{P}}}\right), \end{aligned}$$

where we have also used the trace inequality. Thus, $\mathscr{A}$, $\mathscr{E}_1$ and $\mathscr{E}_2$ are continuous.

On the other hand, there exist positive constants $\alpha_f,\alpha_s,\alpha_r$ such that

$$\begin{aligned} a_f^{\mathrm{S}}\left(\boldsymbol{v}_f^{\mathrm{S}},\boldsymbol{v}_f^{\mathrm{S}}\right) \ge \alpha_f\|\boldsymbol{v}_f^{\mathrm{S}}\|^2_{1,\Omega_{\mathrm{S}}},\quad a_f^{\mathrm{P}}(\boldsymbol{v}_s^{\mathrm{P}},\boldsymbol{v}_s^{\mathrm{P}})-m_\theta(\boldsymbol{v}_s^{\mathrm{P}},\boldsymbol{v}_s^{\mathrm{P}}) &\ge \alpha_s\|\boldsymbol{v}_s^{\mathrm{P}}\|^2_{1,\Omega_{\mathrm{P}}},\\ a_f^{\mathrm{P}}(\boldsymbol{v}_r^{\mathrm{P}},\boldsymbol{v}_r^{\mathrm{P}})-m_\theta(\boldsymbol{v}_r^{\mathrm{P}},\boldsymbol{v}_r^{\mathrm{P}})+m_{\phi^2/\kappa}(\boldsymbol{v}_r^{\mathrm{P}},\boldsymbol{v}_r^{\mathrm{P}}) &\ge \alpha_r\|\boldsymbol{v}_r^{\mathrm{P}}\|^2_{1,\Omega_{\mathrm{P}}},\\ a_{\mathrm{BJS}}\left(\boldsymbol{v}_f^{\mathrm{S}},\boldsymbol{w}_s^{\mathrm{P}};\boldsymbol{v}_f^{\mathrm{S}},\boldsymbol{w}_s^{\mathrm{P}}\right)+b_{\mathrm{BJS}}\left(\boldsymbol{v}_r^{\mathrm{P}};\boldsymbol{v}_r^{\mathrm{P}}\right) &\ge \mu_f\alpha_{\mathrm{BJS}}K_{\max}^{-1/2}\left(|\boldsymbol{v}_f^{\mathrm{S}}-\boldsymbol{w}_s^{\mathrm{P}}|^2_{\mathrm{BJS}}+|\boldsymbol{v}_r^{\mathrm{P}}|^2_{\mathrm{BJS}}\right), \end{aligned}$$

where we have used Korn's inequality and Section 3.1. Therefore $\mathscr{A}$ is monotone. In addition, from

$$\langle\mathscr{E}_1\vec{\boldsymbol{v}},\vec{\boldsymbol{v}}\rangle=\rho_f\phi\|\boldsymbol{v}_r^{\mathrm{P}}+\boldsymbol{v}_s^{\mathrm{P}}\|^2_{0,\Omega_{\mathrm{P}}}+(1-\phi)\rho_s\|\boldsymbol{v}_s^{\mathrm{P}}\|^2_{0,\Omega_{\mathrm{P}}},\quad (\mathscr{E}_2\vec{q},\vec{q})=\|(1-\phi)K^{-1/2}q^{\mathrm{P}}\|^2_{0,\Omega_{\mathrm{P}}}+\|A^{1/2}\boldsymbol{\tau}^{\mathrm{P}}\|^2_{0,\Omega_{\mathrm{P}}},$$

we get the monotonicity of $\mathscr{E}_1,\mathscr{E}_2$. □

Lemma A.2. *The operator $\mathscr{B}$ and its adjoint $\mathscr{B}^*$ are bounded and continuous.*

Proof. For all $\vec{\boldsymbol{v}}=(\boldsymbol{v}_r^{\mathrm{P}},\boldsymbol{v}_s^{\mathrm{P}},\boldsymbol{v}_f^{\mathrm{S}})\in\vec{\mathbf{V}}$ and $\vec{q}=\left(q^{\mathrm{P}},\boldsymbol{\tau}^{\mathrm{P}},q^{\mathrm{S}},\mu\right)\in\vec{Q}$ we have

$$\begin{aligned}\langle\mathscr{B}(\vec{\boldsymbol{v}}),\vec{q}\rangle &\lesssim \|\boldsymbol{v}_f^{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}\|q^{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}}+\|\boldsymbol{v}_s^{\mathrm{P}}\|_{1,\Omega_{\mathrm{P}}}\|q^{\mathrm{P}}\|_{0,\Omega_{\mathrm{P}}}+\|\boldsymbol{v}_r^{\mathrm{P}}\|_{1,\Omega_{\mathrm{P}}}\|q^{\mathrm{P}}\|_{0,\Omega_{\mathrm{P}}}+\|\boldsymbol{v}_s^{\mathrm{P}}\|_{1,\Omega_{\mathrm{P}}}\|\boldsymbol{\tau}^{\mathrm{P}}\|_{0,\Omega_{\mathrm{P}}}\\ &\quad+\|\boldsymbol{v}_f^{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}\|\mu\|_\Lambda+\|\boldsymbol{v}_r^{\mathrm{P}}\|_{1,\Omega_{\mathrm{P}}}\|\mu\|_\Lambda+\|\boldsymbol{v}_s^{\mathrm{P}}\|_{1,\Omega_{\mathrm{P}}}\|\mu\|_\Lambda\lesssim\|\vec{\boldsymbol{v}}\|_{\vec{\mathbf{V}}}\|\vec{q}\|_{\vec{Q}}.\end{aligned}$$

□

Lemma A.3. *There exists a constant* $\xi_1(\Omega) > 0$ *such that*

$$\inf_{(q^{\mathrm{S}},\mathbf{0},q^{\mathrm{P}},0)\in\vec{Q}} \sup_{\left(\boldsymbol{v}_r^{\mathrm{P}},\mathbf{0},\boldsymbol{v}_f^{\mathrm{S}}\right)\in\vec{\mathbf{V}}} \frac{b^{\mathrm{S}}(\boldsymbol{v}_f^{\mathrm{S}},q^{\mathrm{S}}) + b_f^{\mathrm{P}}\left(\phi\boldsymbol{v}_r^{\mathrm{P}},q^{\mathrm{P}}\right)}{\|(\boldsymbol{v}_r^{\mathrm{P}},\mathbf{0},\boldsymbol{v}_f^{\mathrm{S}})\|_{\vec{\mathbf{V}}}\|(q^{\mathrm{P}},\mathbf{0},q^{\mathrm{S}},0)\|_{\vec{Q}}} \geq \xi_1 > 0.$$

Proof. It follows from Stokes inf-sup condition [17] and its weighted form in [11, Lemma 14]. □

Lemma A.4. *There is a constant* $\xi_2(\Omega) > 0$*, such that*

$$\inf_{(0,\mathbf{0},0,\mu)\in\vec{Q}} \sup_{\left(\boldsymbol{v}_r^{\mathrm{P}},\mathbf{0},\boldsymbol{v}_f^{\mathrm{S}}\right)\in\vec{\mathbf{V}}} \frac{b_\Gamma(\boldsymbol{v}_r^{\mathrm{P}},\mathbf{0},\boldsymbol{v}_f^{\mathrm{S}};\mu)}{\|(\boldsymbol{v}_r^{\mathrm{P}},\mathbf{0},\boldsymbol{v}_f^{\mathrm{S}})\|_{\vec{\mathbf{V}}}\|(0,\mathbf{0},0,\mu)\|_{\vec{Q}}} \geq \xi_2 > 0.$$

Proof. Owing to the Riesz representation theorem, for $\mu \in H_{00}^{-1/2}(\Sigma)$ there exists $\tilde{\xi} \in H_{00}^{1/2}(\Sigma)$ and assuming $\boldsymbol{n}_{\mathrm{S}}$ sufficiently smooth, we have $\|\tilde{\xi}\|_{H_{00}^{1/2}(\Sigma)} = \|\mu\|_\Lambda$. Let us consider the following problem

$$\begin{aligned} -\boldsymbol{\Delta}\hat{\boldsymbol{v}}_f^{\mathrm{S}} + \nabla\zeta = \mathbf{0}, \quad \nabla\cdot\hat{\boldsymbol{v}}_f^{\mathrm{S}} = 0 \quad \text{in } \Omega_{\mathrm{S}}, \\ \hat{\boldsymbol{v}}_f^{\mathrm{S}} = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{S}}, \quad \hat{\boldsymbol{v}}_f^{\mathrm{S}} = \tilde{\xi}\boldsymbol{n}_{\mathrm{S}} \quad \text{on } \Sigma. \end{aligned} \tag{A.3}$$

Thanks to [34], we can assert that there exists a unique velocity solution to (A.3), for which there holds

$$\|\hat{\boldsymbol{v}}_f^{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} \lesssim \|\hat{\xi}\boldsymbol{n}_{\mathrm{S}}\|_{H_{00}^{1/2}(\Sigma)} \lesssim \|\tilde{\xi}\|_\Lambda \quad \text{and} \quad \|\hat{\boldsymbol{v}}_f^{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} \lesssim \|\mu\|_\Lambda, \tag{A.4}$$

and so $\hat{\boldsymbol{v}}_f^{\mathrm{S}} \in \mathbf{H}_\star^1(\Omega_{\mathrm{S}}) = \{\boldsymbol{v}_f^{\mathrm{S}} \in \mathbf{H}^1(\Omega_{\mathrm{S}}) : \boldsymbol{v}_f|_\Sigma^{\mathrm{S}} = \tilde{\xi}\boldsymbol{n}_{\mathrm{S}}\}$. In this way, we can choose $\boldsymbol{v}_r^{\mathrm{P}} = \mathbf{0}$ and write it as

$$\sup_{\left(\boldsymbol{v}_r^{\mathrm{P}},\mathbf{0},\boldsymbol{v}_f^{\mathrm{S}}\right)\in\vec{\mathbf{V}}} \frac{b_\Gamma(\boldsymbol{v}_r^{\mathrm{P}},\mathbf{0},\boldsymbol{v}_f^{\mathrm{S}};\mu)}{\|(\boldsymbol{v}_r^{\mathrm{P}},\mathbf{0},\boldsymbol{v}_f^{\mathrm{S}})\|_{\vec{\mathbf{V}}}} \geq \frac{\langle\hat{\boldsymbol{v}}_f^{\mathrm{S}}\cdot\boldsymbol{n}_{\mathrm{S}},\mu\rangle_\Sigma}{\|(\mathbf{0},\mathbf{0},\boldsymbol{v}_f^{\mathrm{S}})\|_{\vec{\mathbf{V}}}} = \frac{\langle\tilde{\xi}\boldsymbol{n}_{\mathrm{S}}\cdot\boldsymbol{n}_{\mathrm{S}},\mu\rangle_\Sigma}{\|\hat{\boldsymbol{v}}_f^{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}} = \frac{\langle\tilde{\xi},\mu\rangle_\Sigma}{\|\hat{\boldsymbol{v}}_f^{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}} = \frac{\|\mu\|_\Lambda^2}{\|\hat{\boldsymbol{v}}_f^{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}} \gtrsim \xi_2\|\mu\|_\Lambda,$$

where we have used (A.4) and hence this concludes the result. □

Lemma A.5. *There exist constants* $\xi_3(\Omega),\xi_4(\Omega),\xi_5(\Omega) > 0$*, such that*

$$\inf_{(\mathbf{0},\boldsymbol{v}_s^{\mathrm{P}},\mathbf{0})\in\vec{\mathbf{V}}} \sup_{(0,\boldsymbol{\tau}^{\mathrm{P}},0,0)\in\vec{Q}} \frac{b_{sig}^{\mathrm{P}}\left(\boldsymbol{v}_s^{\mathrm{P}},\boldsymbol{\tau}^{\mathrm{P}}\right)}{\|(\mathbf{0},\boldsymbol{v}_s^{\mathrm{P}},\mathbf{0})\|_{\vec{\mathbf{V}}}\|(0,\boldsymbol{\tau}^{\mathrm{P}},0,0)\|_{\vec{Q}}} \geq \xi_3, \tag{A.5a}$$

$$\inf_{(q^{\mathrm{P}},\mathbf{0},q^{\mathrm{S}},\mu)\in\vec{Q}} \sup_{(\boldsymbol{v}_r^{\mathrm{P}},\mathbf{0},\boldsymbol{v}_f^{\mathrm{S}})\in\vec{\mathbf{V}}} \frac{b^{\mathrm{S}}(\boldsymbol{v}_f^{\mathrm{S}},q^{\mathrm{S}}) + b_f^{\mathrm{P}}\left(\boldsymbol{v}_r^{\mathrm{P}},q^{\mathrm{P}}\right) + b_\Gamma\left(\boldsymbol{v}_r^{\mathrm{P}},\boldsymbol{v}_f^{\mathrm{S}},\mathbf{0};\mu\right)}{\|(\boldsymbol{v}_r^{\mathrm{P}},\mathbf{0},\boldsymbol{v}_f^{\mathrm{S}})\|_{\vec{\mathbf{V}}}\|(q^{\mathrm{P}},\mathbf{0},q^{\mathrm{S}},\lambda)\|_{\vec{Q}}} \geq \xi_4, \tag{A.5b}$$

$$\inf_{\vec{q}\in\vec{Q}} \sup_{\vec{\mathbf{v}}\in\vec{\mathbf{V}}} \frac{\langle\mathscr{B}(\vec{\mathbf{v}}),\vec{q}\rangle}{\|\vec{\mathbf{v}}\|_{\vec{\mathbf{V}}}\|\vec{q}\|_{\vec{Q}}} \geq \xi_5. \tag{A.5c}$$

Proof. By choosing $\boldsymbol{\tau}^{\mathrm{P}} = \boldsymbol{\varepsilon}(\boldsymbol{v}_s^{\mathrm{P}})$ and applying Korn's inequality gives (A.5a). Combining Lemmas A.3 and A.4 implies the result stated in (A.5b). Therefore, (A.5c) follows from (A.5a) and (A.5b). □

The next result (cf. [42, Theorem 6.1(b)]) is used to establish existence of solution to (A.2).

Theorem A.6. *Let the linear, symmetric and monotone operator $\mathscr{N}$ be given for the real vector space E to its algebraic dual E^*, and let E_b' be the Hilbert space which is the dual of E with the seminorm*

$$|x|_b = \langle \mathscr{N} x, x\rangle^{1/2}, \quad x \in E.$$

Let $\mathscr{M}^\star \subset E \times E_b'$ be a relation with domain $D = \{x \in E : \mathscr{M}^\star(x) \neq \emptyset\}$. Assume that $\mathscr{M}^\star$ is monotone and $\mathrm{Rg}(\mathscr{N} + \mathscr{M}^\star) = E_b'$. Then, for each $u_0 \in \mathbb{D}$ and for each $f \in W^{1,1}\left(0,T;E_b'\right)$, there is a solution of

$$\frac{d}{dt}(\mathscr{N} u(t)) + \mathscr{M}^\star(u(t)) \ni f(t), \quad 0 < t < T,$$

with $\mathscr{N} u \in W^{1,\infty}\left(0,T;E_b'\right)$, $u(t) \in \mathbb{D}$ for all $0 \le t \le T$, and $\mathscr{N} u(0) = \mathscr{N} u_0$.

Whenever possible we will use the shorthand notation as $\vec{\alpha} = (\boldsymbol{u}_r^{\mathrm{P}}, \boldsymbol{u}_s^{\mathrm{P}}, \boldsymbol{u}_f^{\mathrm{S}}, p^{\mathrm{P}}, \boldsymbol{\sigma}^{\mathrm{P}})$, $\vec{\beta} = (\boldsymbol{v}_r^{\mathrm{P}}, \boldsymbol{v}_s^{\mathrm{P}}, \boldsymbol{v}_f^{\mathrm{S}}, q^{\mathrm{P}}, \boldsymbol{\tau}^{\mathrm{P}})$. Note first that the seminorm induced by $\mathscr{E}_1$ is $|\vec{\boldsymbol{v}}|^2_{\mathscr{E}_1} = \rho_f \phi \|\boldsymbol{v}_r^{\mathrm{P}} + \boldsymbol{v}_s^{\mathrm{P}}\|^2_{0,\Omega_{\mathrm{P}}} + \rho_s(1-\phi)\|\boldsymbol{v}_s^{\mathrm{P}}\|^2_{0,\Omega_{\mathrm{P}}} + \rho_f\|\boldsymbol{v}_f^{\mathrm{S}}\|^2_{0,\Omega_{\mathrm{S}}}$ is equivalent to $\|\boldsymbol{v}_r^{\mathrm{P}}\|^2_{0,\Omega_{\mathrm{P}}} + \|\boldsymbol{v}_s^{\mathrm{P}}\|^2_{0,\Omega_{\mathrm{P}}} + \|\boldsymbol{v}_f^{\mathrm{S}}\|^2_{0,\Omega_{\mathrm{S}}}$. We denote by $\mathbf{W}_{r,2}$, $\mathbf{W}_{s,2}$,$\mathbf{W}_{f,2}$, $W_{p,2}$ and $\mathbf{Z}_2$ the closure of $\mathbf{V}_r$, $\mathbf{V}_s$,$\mathbf{V}_f$, W_p and $\mathbf{Z}$ with respect to the norms

$$\begin{aligned}
&\|\boldsymbol{u}_r^{\mathrm{P}}\|^2_{\mathbf{W}_{r,2}} := (\rho_f\phi\boldsymbol{u}_r^{\mathrm{P}}, \boldsymbol{u}_r^{\mathrm{P}})_{\Omega_{\mathrm{P}}}, \quad \|\boldsymbol{u}_s^{\mathrm{P}}\|^2_{\mathbf{W}_{s,2}} := (\rho_p\boldsymbol{u}_s^{\mathrm{P}}, \boldsymbol{u}_s^{\mathrm{P}})_{\Omega_{\mathrm{P}}}, \quad \|\boldsymbol{u}_f^{\mathrm{S}}\|^2_{\mathbf{W}_{f,2}} := (\rho_f\boldsymbol{u}_f^{\mathrm{S}}, \boldsymbol{u}_f^{\mathrm{S}})_{\Omega_{\mathrm{S}}},\\
&\|p^{\mathrm{P}}\|^2_{W_{p,2}} := ((1-\phi)^2K^{-1}p^{\mathrm{P}}, p^{\mathrm{P}})_{\Omega_{\mathrm{P}}}, \quad \|\boldsymbol{\tau}^{\mathrm{P}}\|^2_{\mathbf{Z}_2} := (A\boldsymbol{\tau}^{\mathrm{P}}, \boldsymbol{\tau}^{\mathrm{P}})_{\Omega_{\mathrm{P}}}.
\end{aligned} \tag{A.6}$$

Let $\mathbb{S}_2 := \mathbf{W}_{r,2} \times \mathbf{W}_{s,2} \times \mathbf{W}_{f,2} \times W_{p,2} \times \mathbf{Z}_2$. We introduce the inner product $(\cdot,\cdot)_{\mathbb{S}_2}$ by

$$\begin{aligned}
(\vec{\alpha}, \vec{\beta})_{\mathbb{S}_2} := &\left(\rho_f\phi\boldsymbol{u}_r^{\mathrm{P}}, \boldsymbol{v}_r^{\mathrm{P}}\right)_{\Omega_{\mathrm{P}}} + \left(\rho_f\phi\boldsymbol{u}_r^{\mathrm{P}}, \boldsymbol{v}_s^{\mathrm{P}}\right)_{\Omega_{\mathrm{P}}} + \left(\rho_f\phi\boldsymbol{u}_s^{\mathrm{P}}, \boldsymbol{v}_r^{\mathrm{P}}\right)_{\Omega_{\mathrm{P}}} + \left(\rho_f\boldsymbol{u}_f^{\mathrm{S}}, \boldsymbol{v}_f^{\mathrm{S}}\right)\\
&+ \left(\rho_p\boldsymbol{u}_s^{\mathrm{P}}, \boldsymbol{v}_s^{\mathrm{P}}\right)_{\Omega_{\mathrm{P}}} + \left((1-\phi)^2K^{-1}p^{\mathrm{P}}, q^{\mathrm{P}}\right)_{\Omega_{\mathrm{P}}} + \left(A\boldsymbol{\sigma}^{\mathrm{P}}, \boldsymbol{\tau}^{\mathrm{P}}\right)_{\Omega_{\mathrm{P}}}.
\end{aligned}$$

Next we define the domain $\mathbb{D} \subseteq \mathbb{S}_2$ as

$$\mathbb{D} := \{\vec{\alpha} \in \mathbf{V}_r \times \mathbf{V}_s \times \mathbf{V}_f \times \mathrm{W}_p \times \mathbf{Z} : \exists (p^{\mathrm{S}}, \lambda) \in \mathrm{W}_f \times \Lambda \text{ s.t. } \forall (\vec{\boldsymbol{v}}, \vec{q}) \in \vec{\mathbf{V}} \times \vec{Q} : \text{ (A.8) holds for } \vec{\boldsymbol{f}} \in \mathbb{S}_2'\}. \tag{A.7}$$

Setting $\vec{\boldsymbol{f}} = (\vec{f}, \vec{g}) = ((\bar{f}_r, \bar{f}_s, \bar{f}_f), (\bar{g}_p, \bar{g}_e, 0, \mathbf{0}) \in \mathbb{S}_2'$, the equations defined on the domain are

$$(\mathscr{E}_1 + \mathscr{A})\vec{\boldsymbol{u}} + \mathscr{B}'\vec{p} = \vec{f}, \tag{A.8a}$$

$$-\mathscr{B}\vec{\boldsymbol{u}} + \mathscr{E}_2\vec{p} = \vec{g}, \tag{A.8b}$$

and the corresponding set of equations are

$$\begin{aligned}
&a_f^{\mathrm{S}}\left(\boldsymbol{u}_f^{\mathrm{S}}, \boldsymbol{v}_f^{\mathrm{S}}\right) + a_f^{\mathrm{P}}(\boldsymbol{u}_r^{\mathrm{P}}, \boldsymbol{v}_s^{\mathrm{P}}) + a_f^{\mathrm{P}}\left(\boldsymbol{u}_r^{\mathrm{P}}, \boldsymbol{v}_r^{\mathrm{P}}\right) + a_f^{\mathrm{P}}(\boldsymbol{u}_s^{\mathrm{P}}, \boldsymbol{v}_r^{\mathrm{P}}) + a_f^{\mathrm{P}}(\boldsymbol{u}_s^{\mathrm{P}}, \boldsymbol{v}_s^{\mathrm{P}}) + b^{\mathrm{S}}(\boldsymbol{v}_f^{\mathrm{S}}, p^{\mathrm{S}}) + b_s^{\mathrm{P}}(\boldsymbol{v}_s^{\mathrm{P}}, p^{\mathrm{P}}) + b_f^{\mathrm{P}}(\phi\boldsymbol{v}_r^{\mathrm{P}}, p^{\mathrm{P}})\\
&+ b_\Gamma(\boldsymbol{v}_f^{\mathrm{S}}, \boldsymbol{v}_r^{\mathrm{P}}, \boldsymbol{v}_s^{\mathrm{P}}; \lambda) - m_\theta(\boldsymbol{u}_r^{\mathrm{P}}, \boldsymbol{v}_s^{\mathrm{P}}) - m_\theta(\boldsymbol{u}_s^{\mathrm{P}}, \boldsymbol{v}_s^{\mathrm{P}}) - m_\theta(\boldsymbol{u}_s^{\mathrm{P}}, \boldsymbol{v}_r^{\mathrm{P}}) - m_\theta(\boldsymbol{u}_r^{\mathrm{P}}, \boldsymbol{v}_r^{\mathrm{P}}) + m_{\phi^2/\kappa}(\boldsymbol{u}_r^{\mathrm{P}}, \boldsymbol{v}_r^{\mathrm{P}}) + m_{\rho_f\phi}(\boldsymbol{u}_r^{\mathrm{P}}, \boldsymbol{v}_s^{\mathrm{P}})\\
&+ m_{\rho_p}(\boldsymbol{u}_s^{\mathrm{P}}, \boldsymbol{v}_s^{\mathrm{P}}) + m_{\rho_f\phi}(\boldsymbol{u}_r^{\mathrm{P}}, \boldsymbol{v}_r^{\mathrm{P}}) + m_{\rho_f\phi}(\boldsymbol{u}_s^{\mathrm{P}}, \boldsymbol{v}_r^{\mathrm{P}}) + m_{\rho_f}(\boldsymbol{u}_f^{\mathrm{S}}, \boldsymbol{v}_f^{\mathrm{S}}) + b_{\mathrm{sig}}^{\mathrm{P}}(\boldsymbol{v}_s^{\mathrm{P}}, \boldsymbol{\sigma}^{\mathrm{P}}) + b_{\mathrm{BJS}}(\boldsymbol{u}_r^{\mathrm{P}}; \boldsymbol{v}_r^{\mathrm{P}}) + a_{\mathrm{BJS}}(\boldsymbol{u}_f^{\mathrm{S}}, \boldsymbol{u}_s^{\mathrm{P}}; \boldsymbol{v}_f^{\mathrm{S}}, \boldsymbol{v}_s^{\mathrm{P}})\\
&= \left(\rho_p\bar{f}_s, \boldsymbol{v}_s^{\mathrm{P}}\right)_{\Omega_{\mathrm{P}}} + \left(\rho_f\phi\bar{f}_r, \boldsymbol{v}_r^{\mathrm{P}}\right)_{\Omega_{\mathrm{P}}} + \left(\rho_f\phi\bar{f}_r, \boldsymbol{v}_s^{\mathrm{P}}\right)_{\Omega_{\mathrm{P}}} + (\rho_f\phi\bar{f}_s, \boldsymbol{v}_r^{\mathrm{P}})_{\Omega_{\mathrm{P}}} + (\rho_f\bar{f}_f, \boldsymbol{v}_f^{\mathrm{S}})_{\Omega_{\mathrm{S}}},
\end{aligned} \tag{A.9a}$$

$$\begin{aligned}
&\left((1-\phi)^2K^{-1}p^{\mathrm{P}}, q^{\mathrm{P}}\right)_{\Omega_{\mathrm{P}}} - b_s^{\mathrm{P}}(\boldsymbol{u}_s^{\mathrm{P}}, q^{\mathrm{P}}) - b_f^{\mathrm{P}}(\phi\boldsymbol{u}_r^{\mathrm{P}}, q^{\mathrm{P}}) - b^{\mathrm{S}}(\boldsymbol{u}_f^{\mathrm{S}}, q^{\mathrm{S}}) + a_p^{\mathrm{P}}(\boldsymbol{\sigma}^{\mathrm{P}}, \boldsymbol{\tau}^{\mathrm{P}}) - b_{\mathrm{sig}}^{\mathrm{P}}(\boldsymbol{u}_s^{\mathrm{P}}, \boldsymbol{\tau}^{\mathrm{P}})\\
&= \left((1-\phi)^2K^{-1}\bar{g}_p, q^{\mathrm{P}}\right)_{\Omega_{\mathrm{P}}} + (A\bar{g}_e, \boldsymbol{\tau}^{\mathrm{P}})_{\Omega_{\mathrm{P}}},
\end{aligned} \tag{A.9b}$$

$$b_\Gamma(\boldsymbol{u}_f^{\mathrm{S}}, \boldsymbol{u}_r^{\mathrm{P}}, \boldsymbol{u}_s^{\mathrm{P}}; \mu) = 0. \tag{A.9c}$$

As there may be more than one $\vec{\boldsymbol{f}} \in \mathbb{S}_2'$ generating the same $\vec{\alpha} \in \mathbb{D}$, we introduce (denoting $\vec{\boldsymbol{f}} - \vec{\alpha} = (\bar{f}_r - \boldsymbol{u}_r^{\mathrm{P}}, \bar{f}_s - \boldsymbol{u}_s^{\mathrm{P}}, \bar{f}_f - \boldsymbol{u}_f^{\mathrm{S}}, \bar{g}_p - p^{\mathrm{P}}, \bar{g}_e - \boldsymbol{\sigma}^{\mathrm{P}})$) the multivalued operator $\mathscr{M}(\cdot)$ with domain $\mathbb{D}$ as

$$\mathscr{M}(\vec{\alpha}) := \{\vec{\boldsymbol{f}} - \vec{\alpha} \in \mathbb{S}_2' : \vec{\alpha} \text{ solves (A.8) for } \vec{\boldsymbol{f}} \in \mathbb{S}_2'\}. \tag{A.10}$$

Next, we consider the problem: given $\boldsymbol{h}_f \in W^{1,1}(0,T;\mathbf{W}_{f,2}'), \boldsymbol{h}_r \in W^{1,1}(0,T;\mathbf{W}_{r,2}')$, $\boldsymbol{h}_s \in W^{1,1}(0,T;\mathbf{W}_{s,2}')$, $h_p \in W^{1,1}(0,T;W_{p,2}')$ and $h_e \in W^{1,1}(0,T;\mathbf{Z}_2')$, find $\vec{\alpha} \in \mathbb{D}$ such that

$$\frac{d}{dt}\vec{\alpha}(t) + \mathscr{M}\left(\vec{\alpha}(t)\right) \ni \vec{\mathbf{h}}(t). \tag{A.11}$$

where $\vec{\mathbf{h}}(t) = \left(\boldsymbol{h}_f(t), \boldsymbol{h}_r(t), \boldsymbol{h}_s(t), h_p(t), h_e(t)\right)$. Using Theorem A.6, we can show that (A.2) is well-posed.

Theorem A.7. *Suppose (H.1)–(H.3). For each* $\boldsymbol{f}_{\mathrm{S}} \in W^{1,1}(0,T;\mathbf{L}^2(\Omega_{\mathrm{S}})), \boldsymbol{f}_{\mathrm{P}} \in W^{1,1}(0,T;\mathbf{L}^2(\Omega_{\mathrm{P}}))$, $\theta \in W^{1,1}(0,T;\mathrm{W}_p')$, *and* $(\boldsymbol{u}_r^{\mathrm{P}}(0) = \boldsymbol{u}_{r,0}, \boldsymbol{u}_s^{\mathrm{P}}(0) = \boldsymbol{u}_{s,0}, \boldsymbol{u}_f^{\mathrm{S}}(0) = \boldsymbol{u}_{f,0}, p^{\mathrm{P}}(0) = p^{\mathrm{P},0}, \boldsymbol{\sigma}^{\mathrm{P}}(0) = \boldsymbol{\sigma}_0^{\mathrm{P}}) \in \mathbf{V}_r \times \mathbf{V}_s \times \mathbf{V}_f \times \mathrm{W}_p \times \mathbf{Z}$, *where* $(\boldsymbol{u}_{r,0}, \boldsymbol{u}_{s,0}, \boldsymbol{u}_{f,0}, p^{\mathrm{P},0}, \boldsymbol{\sigma}_0^{\mathrm{P}}) \in \mathbb{D}$ *are compatible initial data, there exists a unique solution of* (A.2) *with* $\vec{\boldsymbol{u}} \times \vec{p} \in W^{1,\infty}(0,T;\mathbf{V}_r) \times W^{1,\infty}(0,T;\mathbf{V}_s) \times W^{1,\infty}\left(0,T;\mathbf{V}_f\right) \times W^{1,\infty}(0,T;\mathrm{W}_p) \times W^{1,\infty}(0,T;\mathbf{Z}) \times L^\infty\left(0,T;\mathrm{W}_f\right) \times L^\infty(0,T;\Lambda)$.

To prove Theorem A.7 we proceed in the following manner. **Step 1**: Establish that the domain $\mathbb{D}$ defined above is nonempty; **Step 2**: Show solvability of the parabolic problem (A.11); and **Step 3**: Show that the original problem (A.2) is a special case of (A.11). We address each step in what follows.

Step 1: The domain $\mathbb{D}$ is nonempty. We first introduce operators that will be used to regularize the problem. Let $R_s : \mathbf{V}_s \to \mathbf{V}_s', R_r : \mathbf{V}_r \to \mathbf{L}^2, L_f : \mathrm{W}_f \to \mathrm{W}_f'$, $L_p : \mathrm{W}_p \to \mathrm{W}_p'$ be defined by

$$\langle R_s\left(\boldsymbol{u}_s^{\mathrm{P}}\right), \boldsymbol{v}_s^{\mathrm{P}}\rangle := r_s\left(\boldsymbol{u}_s^{\mathrm{P}}, \boldsymbol{v}_s^{\mathrm{P}}\right) = \left(\boldsymbol{\varepsilon}\left(\boldsymbol{u}_s^{\mathrm{P}}\right), \boldsymbol{\varepsilon}(\boldsymbol{v}_s^{\mathrm{P}})\right)_{\Omega_{\mathrm{P}}}, \quad \langle R_r(\boldsymbol{u}_r^{\mathrm{P}}), \boldsymbol{v}_r^{\mathrm{P}}\rangle := r_r\left(\boldsymbol{u}_r^{\mathrm{P}}, \boldsymbol{v}_r^{\mathrm{P}}\right) = \left(\boldsymbol{\varepsilon}\left(\boldsymbol{u}_r^{\mathrm{P}}\right), \boldsymbol{\varepsilon}\left(\boldsymbol{v}_r^{\mathrm{P}}\right)\right)_{\Omega_{\mathrm{P}}},$$
$$\langle L_f\left(p^{\mathrm{S}}\right), q^{\mathrm{S}}\rangle := l_f\left(p^{\mathrm{S}}, q^{\mathrm{S}}\right) = \left(p^{\mathrm{S}}, q^{\mathrm{S}}\right)_{\Omega_{\mathrm{S}}}, \quad \langle L_p\left(p^{\mathrm{P}}\right), q^{\mathrm{P}}\rangle := l_p\left(p^{\mathrm{P}}, q^{\mathrm{P}}\right) = \left(p^{\mathrm{P}}, q^{\mathrm{P}}\right)_{\Omega_{\mathrm{P}}}.$$

Lemma A.8. *The operators R_s, R_r, L_f, and L_p are bounded, continuous, and coercive.*

Proof. The coercivity bounds follow directly from the definitions, using Korn's inequality [19] for R_s and R_r, whereas the continuity bounds follow from Cauchy–Schwarz and Young's inequalities. □

For the regularization of the Lagrange multiplier, let $\psi(\lambda) \in H^1(\Omega_{\mathrm{P}})$ solve the auxiliary problem

$$-\nabla \cdot \nabla \psi(\lambda) = 0 \quad \text{in} \quad \Omega_{\mathrm{P}},$$
$$\nabla \psi(\lambda) \cdot \boldsymbol{n} = \lambda \quad \text{on } \Sigma, \quad \psi(\lambda) = 0 \quad \text{on } \Gamma_{\mathrm{P}}.$$

From the continuous dependence on data and trace continuity, there exist constants $c^\star, C^\star > 0$, such that

$$c^\star \|\psi(\lambda)\|_{1,\Omega_{\mathrm{P}}} \leq \|\lambda\|_\Lambda \leq C^\star \|\psi(\lambda)\|_{1,\Omega_{\mathrm{P}}}. \tag{A.12}$$

Lemma A.9. *The operator $L_\Sigma : \Lambda \to \Lambda'$, defined as*

$$\langle L_\Sigma \lambda, \xi\rangle = l_\Sigma(\lambda, \xi) := (\nabla\psi(\lambda), \nabla\psi(\xi))_{\Omega_{\mathrm{P}}},$$

is continuous and coercive.

Proof. It follows from (A.12) that there exist positive constants $c^\star$ and $C^\star$ such that

$$\langle L_\Sigma \lambda, \xi\rangle \le C^\star \|\lambda\|_\Lambda \|\xi\|_\Lambda, \quad \langle L_\Sigma \lambda, \lambda\rangle \ge c^\star \|\lambda\|_\Lambda^2, \quad \forall \lambda, \xi \in \Lambda,$$

which completes the proof. $\square$

In order to establish that $\mathbb{D}$ is nonempty we first show that there exists a solution to a regularization of (A.9a)-(A.9c) and then send the regularization parameter to zero.

Lemma A.10. *The domain $\mathbb{D}$ specified by* (A.7) *is nonempty.*

Proof. We follow four steps.
1. Regularization of (A.8): For $\vec{\boldsymbol{v}}^i = (\boldsymbol{v}_r^{\mathrm{P},i}, \boldsymbol{v}_s^{\mathrm{P},i}, \boldsymbol{v}_f^{\mathrm{S},i}) \in \vec{\mathbf{V}}, \vec{q}^i = \left(q^{\mathrm{P},i}, \boldsymbol{\tau}^{\mathrm{P},i}, q^{\mathrm{S},i}, \mu^i\right) \in \vec{Q}, i = 1,2,$ define the operators $\mathscr{R} : \vec{\mathbf{V}} \to \vec{\mathbf{V}}'$ and $\mathscr{L} : \vec{Q} \to \vec{Q}'$ as

$$\begin{aligned}
\langle \mathscr{R}\vec{\boldsymbol{v}}^1, \vec{\boldsymbol{v}}^2\rangle &:= \langle R_s(\boldsymbol{v}_s^{\mathrm{P},1}), \boldsymbol{v}_s^{\mathrm{P},2}\rangle + \langle R_r\left(\boldsymbol{v}_r^{\mathrm{P},1}\right), \boldsymbol{v}_r^{\mathrm{P},2}\rangle = r_s\left(\boldsymbol{v}_s^{\mathrm{P},1}, \boldsymbol{v}_s^{\mathrm{P},2}\right) + r_r\left(\boldsymbol{v}_r^{\mathrm{P},1}, \boldsymbol{v}_r^{\mathrm{P},2}\right), \\
\langle \mathscr{L}\vec{q}^1, \vec{q}^2\rangle &:= \langle L_f(q^{\mathrm{S},1}), q^{\mathrm{S},2}\rangle + \langle L_p(q^{\mathrm{P},1}), q^{\mathrm{P},2}\rangle + \langle L_\Sigma\left(\mu^1\right), \mu^2\rangle.
\end{aligned}$$

For $\varepsilon > 0$, consider the regularized problem: given $(\vec{f}, \vec{g}) \in \mathbb{S}_2'$, determine $\vec{\boldsymbol{v}}_\varepsilon \in \vec{\mathbf{V}}, \vec{q}_\varepsilon \in \vec{Q}$ satisfying

$$(\varepsilon\mathscr{R} + \mathscr{E}_1 + \mathscr{A})\vec{\boldsymbol{v}}_\varepsilon + \mathscr{B}'\vec{q}_\varepsilon = \vec{f} \tag{A.13a}$$

$$-\mathscr{B}\vec{\boldsymbol{v}}_\varepsilon + (\varepsilon\mathscr{L} + \mathscr{E}_2)\vec{q}_\varepsilon = \vec{g}. \tag{A.13b}$$

2. Existence of unique solution of (A.13a)-(A.13b): We define the map $\mathscr{O} : \vec{\mathbf{V}} \times \vec{Q} \to (\vec{\mathbf{V}} \times \vec{Q})'$ as

$$\mathscr{O}\begin{pmatrix} \vec{\boldsymbol{v}} \\ \vec{q} \end{pmatrix} = \begin{pmatrix} \varepsilon\mathscr{R} + \mathscr{E}_1 + \mathscr{A} & \mathscr{B}' \\ -\mathscr{B} & \varepsilon\mathscr{L} + \mathscr{E}_2 \end{pmatrix} \begin{bmatrix} \vec{\boldsymbol{v}} \\ \vec{q} \end{bmatrix}.$$

From Lemmas A.1-A.2 it can be shown that $\mathscr{O}$ is bounded and continuous. Additionally, leveraging Lemmas A.1 and A.9, Assumptions in Section 3.1, and the inequality

$$-(\xi a, a) - (\xi b, b) \le 2(\xi a, b) \le (\xi a, a) + (\xi b, b), \tag{A.14}$$

we can conclude that

$$\begin{aligned}
\left\langle \mathscr{O}\begin{pmatrix} \vec{\boldsymbol{v}} \\ \vec{q} \end{pmatrix}, \begin{pmatrix} \vec{\boldsymbol{v}} \\ \vec{q} \end{pmatrix} \right\rangle &\ge C\Big(\varepsilon\|\boldsymbol{\varepsilon}(\boldsymbol{v}_r^{\mathrm{P}})\|_{0,\Omega_{\mathrm{P}}}^2 + \varepsilon\|\boldsymbol{\varepsilon}(\boldsymbol{v}_s^{\mathrm{P}})\|_{0,\Omega_{\mathrm{P}}}^2 + \|\boldsymbol{v}_r^{\mathrm{P}}\|_{0,\Omega_{\mathrm{P}}}^2 + \left|\boldsymbol{v}_f^{\mathrm{S}} - \boldsymbol{v}_s^{\mathrm{P}}\right|_{\mathrm{BJS}}^2 + \left|\boldsymbol{v}_r^{\mathrm{P}}\right|_{\mathrm{BJS}}^2 + \|\boldsymbol{\varepsilon}(\boldsymbol{v}_f^{\mathrm{S}})\|_{0,\Omega_{\mathrm{S}}}^2 \\
&\quad + \|\boldsymbol{\tau}^{\mathrm{P}}\|_{0,\Omega_{\mathrm{P}}}^2 + \varepsilon\|q^{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}}^2 + (1-\phi)^2 K^{-1}\|q^{\mathrm{P}}\|_{0,\Omega_{\mathrm{P}}}^2 + \varepsilon\|\mu\|_\Lambda^2 + \|\boldsymbol{u}_s^{\mathrm{P}}\|_{0,\Omega_{\mathrm{P}}}^2 + \|\boldsymbol{u}_f^{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}}^2\Big).
\end{aligned} \tag{A.15}$$

It follows that $\mathscr{O}$ is coercive. Thus, the Lax–Miligram Lemma establishes the existence of a solution $(\vec{\boldsymbol{u}}_\varepsilon, \vec{p}_\varepsilon) \in \vec{\mathbf{V}} \times \vec{Q}$ of (A.13a)-(A.13b), where $\vec{\boldsymbol{u}}_\varepsilon = \left(\boldsymbol{u}_{r,\varepsilon}^{\mathrm{P}}, \boldsymbol{u}_{s,\varepsilon}^{\mathrm{P}}, \boldsymbol{u}_{f,\varepsilon}^{\mathrm{S}}\right)$ and $\vec{p}_\varepsilon = \left(p_\varepsilon^{\mathrm{P}}, \boldsymbol{\sigma}_\varepsilon^{\mathrm{P}}, p_\varepsilon^{\mathrm{S}}, \lambda_\varepsilon\right)$.

3. Uniform boundedness: From inequality (A.15) and (A.13a)-(A.13b), we have that

$$\begin{aligned}
&\varepsilon\|\boldsymbol{u}_{r,\varepsilon}^{\mathrm{P}}\|_{1,\Omega_{\mathrm{P}}}^2 + \varepsilon\|\boldsymbol{u}_{s,\varepsilon}^{\mathrm{P}}\|_{1,\Omega_{\mathrm{P}}}^2 + \|\boldsymbol{u}_{f,\varepsilon}^{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}^2 + \left|\boldsymbol{u}_{f,\varepsilon}^{\mathrm{S}} - \boldsymbol{u}_{s,\varepsilon}^{\mathrm{P}}\right|_{\mathrm{BJS}}^2 + \left|\boldsymbol{u}_{r,\varepsilon}^{\mathrm{P}}\right|_{\mathrm{BJS}}^2 + \|\boldsymbol{u}_{r,\varepsilon}^{\mathrm{P}}\|_{0,\Omega_{\mathrm{P}}}^2 \\
&\quad + \|\boldsymbol{\sigma}_\varepsilon^{\mathrm{P}}\|_{0,\Omega_{\mathrm{P}}}^2 + \varepsilon\|p_\varepsilon^{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}}^2 + (1-\phi)^2 K^{-1}\|p_\varepsilon^{\mathrm{P}}\|_{0,\Omega_{\mathrm{P}}}^2 + \|\boldsymbol{u}_{s,\varepsilon}^{\mathrm{P}}\|_{0,\Omega_{\mathrm{P}}}^2 + \varepsilon\|\lambda_\varepsilon\|_\Lambda^2 + \|\boldsymbol{u}_{f,\varepsilon}^{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}}^2 \\
&\le C\big(\|\bar{f}_s\|_{0,\Omega_{\mathrm{P}}}\|\boldsymbol{u}_{s,\varepsilon}^{\mathrm{P}}\|_{0,\Omega_{\mathrm{P}}} + \|\bar{f}_r\|_{0,\Omega_{\mathrm{P}}}\|\boldsymbol{u}_{r,\varepsilon}^{\mathrm{P}}\|_{0,\Omega_{\mathrm{P}}} + \|\bar{f}_r\|_{0,\Omega_{\mathrm{P}}}\|\boldsymbol{u}_{s,\varepsilon}^{\mathrm{P}}\|_{0,\Omega_{\mathrm{P}}} + \|\bar{f}_s\|_{0,\Omega_{\mathrm{P}}}\|\boldsymbol{u}_{r,\varepsilon}^{\mathrm{P}}\|_{0,\Omega_{\mathrm{P}}}
\end{aligned}$$

$$+\|\bar{g}_p\|_{0,\Omega_\mathrm{P}}\|p^\mathrm{P}_\varepsilon\|_{0,\Omega_\mathrm{P}} + \|\bar{g}_e\|_{0,\Omega_\mathrm{P}}\|\boldsymbol{\sigma}^\mathrm{P}_\varepsilon\|_{0,\Omega_\mathrm{P}} + \|\bar{f}_f\|_{0,\Omega_\mathrm{P}}\|\boldsymbol{u}^\mathrm{S}_{f,\varepsilon}\|_{0,\Omega_\mathrm{S}}\Big). \tag{A.16}$$

On the other hand, as a consequence of (A.9b), it follows that $\boldsymbol{\sigma}^\mathrm{P}_\varepsilon$ and $\boldsymbol{v}_{s,\varepsilon}$ satisfy

$$a^\mathrm{P}_p\left(\boldsymbol{\sigma}^\mathrm{P}_\varepsilon, \boldsymbol{\tau}^\mathrm{P}\right) - b^\mathrm{P}_\mathrm{sig}\left(\boldsymbol{u}^\mathrm{P}_{s,\varepsilon}, \boldsymbol{\tau}^\mathrm{P}\right) = \left(A\bar{g}_e, \boldsymbol{\tau}^\mathrm{P}\right)_{\Omega_\mathrm{P}} \quad \forall \boldsymbol{\tau}^\mathrm{P} \in \mathbf{Z}.$$

Therefore, applying the inf-sup condition (A.5a), we obtain:

$$\|\boldsymbol{u}^\mathrm{P}_{s,\varepsilon}\|_{1,\Omega_\mathrm{P}} \lesssim \sup_{(0,\boldsymbol{\tau}^\mathrm{P},0,0)\in\vec{Q}} \frac{b^\mathrm{P}_\mathrm{sig}\left(\boldsymbol{u}^\mathrm{P}_{s,\varepsilon}, \boldsymbol{\tau}^\mathrm{P}\right)}{\|(0,\boldsymbol{\tau}^\mathrm{P},0,0)\|_{\vec{Q}}} = \sup_{(0,\boldsymbol{\tau}^\mathrm{P},0,0)\in\vec{Q}} \frac{a^\mathrm{P}_p\left(\boldsymbol{\sigma}^\mathrm{P}_\varepsilon, \boldsymbol{\tau}^\mathrm{P}\right) - \left(A\bar{g}_e, \boldsymbol{\tau}^\mathrm{P}\right)_{\Omega_\mathrm{P}}}{\|(0,\boldsymbol{\tau}^\mathrm{P},0,0)\|_{\vec{Q}}} \lesssim \|\boldsymbol{\sigma}^\mathrm{P}_\varepsilon\|_{0,\Omega_\mathrm{P}} + \|\bar{g}_e\|_{0,\Omega_\mathrm{P}}. \tag{A.17}$$

Combining (A.16) and (A.17), and using Young's inequality, we eventually obtain

$$\begin{aligned}\|\boldsymbol{u}^\mathrm{P}_{s,\varepsilon}\|^2_{1,\Omega_\mathrm{P}} &+ \varepsilon\|\boldsymbol{u}^\mathrm{P}_{r,\varepsilon}\|^2_{1,\Omega_\mathrm{P}} + \|\boldsymbol{u}^\mathrm{S}_{f,\varepsilon}\|^2_{1,\Omega_\mathrm{S}} + \left|\boldsymbol{u}^\mathrm{S}_{f,\varepsilon} - \boldsymbol{u}^\mathrm{P}_{s,\varepsilon}\right|^2_\mathrm{BJS} + \left|\boldsymbol{u}^\mathrm{P}_{r,\varepsilon}\right|^2_\mathrm{BJS} + \|\boldsymbol{u}^\mathrm{P}_{r,\varepsilon}\|^2_{0,\Omega_\mathrm{P}} + \|\boldsymbol{u}^\mathrm{P}_{s,\varepsilon}\|^2_{0,\Omega_\mathrm{P}} \\ &+ \|\boldsymbol{\sigma}^\mathrm{P}_\varepsilon\|^2_{0,\Omega_\mathrm{P}} + \|\boldsymbol{u}^\mathrm{S}_{f,\varepsilon}\|^2_{0,\Omega_\mathrm{S}} \lesssim \|\bar{g}_p\|_{0,\Omega_\mathrm{P}}\|p^\mathrm{P}_\varepsilon\|_{0,\Omega_\mathrm{P}} + \|\bar{f}_r\|^2_{0,\Omega_\mathrm{P}} + \|\bar{f}_s\|^2_{0,\Omega_\mathrm{P}} + \|\bar{g}_e\|^2_{0,\Omega_\mathrm{P}} + \|\bar{f}_f\|^2_{0,\Omega_\mathrm{P}}.\end{aligned} \tag{A.18}$$

To obtain bounds for $p^\mathrm{P}_\varepsilon, p^\mathrm{S}_\varepsilon$, and λ_ε we use (A.5b). With $\vec{p} = \left(p^\mathrm{P}_\varepsilon, 0, p^\mathrm{S}_\varepsilon, \lambda_\varepsilon\right) \in \vec{Q}$, we have

$$\begin{aligned}\|p^\mathrm{S}_\varepsilon\|_{0,\Omega_\mathrm{S}} + \|p^\mathrm{P}_\varepsilon\|_{0,\Omega_\mathrm{P}} + \|\lambda_\varepsilon\|_\Lambda &\le C \sup_{\left(\boldsymbol{v}^\mathrm{P}_r,\mathbf{0},\boldsymbol{v}^\mathrm{S}_f\right)\in\vec{\mathbf{V}}} \frac{b^\mathrm{S}(\boldsymbol{v}^\mathrm{S}_f, p^\mathrm{S}_\varepsilon) + b^\mathrm{P}_f(\boldsymbol{v}^\mathrm{P}_r, p^\mathrm{P}_\varepsilon) + b_\Gamma(\boldsymbol{v}^\mathrm{P}_r, \boldsymbol{v}^\mathrm{S}_f, \mathbf{0}; \lambda_\varepsilon)}{\|(\boldsymbol{v}^\mathrm{P}_r, \mathbf{0}, \boldsymbol{v}^\mathrm{S}_f)\|_{\vec{\mathbf{V}}}} \\ &\lesssim \varepsilon\|\boldsymbol{u}^\mathrm{P}_{r,\varepsilon}\|_{1,\Omega_\mathrm{P}} + \|\boldsymbol{u}^\mathrm{S}_{f,\varepsilon}\|_{1,\Omega_\mathrm{S}} + \|\boldsymbol{u}^\mathrm{P}_{s,\varepsilon}\|_{1,\Omega_\mathrm{P}} + \left|\boldsymbol{u}^\mathrm{S}_{f,\varepsilon} - \boldsymbol{u}^\mathrm{P}_{s,\varepsilon}\right|_\mathrm{BJS} \\ &\quad + \left|\boldsymbol{u}^\mathrm{P}_{r,\varepsilon}\right|^2_\mathrm{BJS} + \|\boldsymbol{u}^\mathrm{P}_{s,\varepsilon}\|_{0,\Omega_\mathrm{P}} + \|\boldsymbol{u}^\mathrm{P}_{r,\varepsilon}\|_{0,\Omega_\mathrm{P}} + \|\bar{f}_r\|_{0,\Omega_\mathrm{P}} + \|\bar{f}_f\|_{0,\Omega_\mathrm{S}}.\end{aligned} \tag{A.19}$$

Employing again Young's inequality, (A.18), and (A.19), we arrive at

$$\begin{aligned}&\|\boldsymbol{u}^\mathrm{P}_{s,\varepsilon}\|^2_{1,\Omega_\mathrm{P}} + \varepsilon\|\boldsymbol{u}^\mathrm{P}_{r,\varepsilon}\|^2_{1,\Omega_\mathrm{P}} + \|\boldsymbol{u}^\mathrm{S}_{f,\varepsilon}\|^2_{1,\Omega_\mathrm{S}} + \left|\boldsymbol{u}^\mathrm{S}_{f,\varepsilon} - \boldsymbol{u}^\mathrm{P}_{s,\varepsilon}\right|^2_\mathrm{BJS} + \left|\boldsymbol{u}^\mathrm{P}_{r,\varepsilon}\right|^2_\mathrm{BJS} + \|\boldsymbol{u}^\mathrm{P}_{r,\varepsilon}\|^2_{0,\Omega_\mathrm{P}} + \|\boldsymbol{u}^\mathrm{P}_{s,\varepsilon}\|^2_{0,\Omega_\mathrm{P}} + \|\boldsymbol{\sigma}^\mathrm{P}_\varepsilon\|^2_{0,\Omega_\mathrm{P}} \\ &+ \|p^\mathrm{S}_\varepsilon\|^2_{0,\Omega_\mathrm{S}} + \|p^\mathrm{P}_\varepsilon\|^2_{0,\Omega_\mathrm{P}} + \|\lambda_\varepsilon\|^2_\Lambda + \|\boldsymbol{u}^\mathrm{S}_{f,\varepsilon}\|^2_{0,\Omega_\mathrm{S}} \lesssim \|\bar{g}_p\|^2_{0,\Omega_\mathrm{P}} + \|\bar{f}_r\|^2_{0,\Omega_\mathrm{P}} + \|\bar{f}_s\|^2_{0,\Omega_\mathrm{P}} + \|\bar{g}_e\|^2_{0,\Omega_\mathrm{P}} + \|\bar{f}_f\|^2_{0,\Omega_\mathrm{S}},\end{aligned}$$

which implies that all the quantities $\|\boldsymbol{u}^\mathrm{P}_{s,\varepsilon}\|_{1,\Omega_\mathrm{P}}$, $\|\boldsymbol{u}^\mathrm{S}_{f,\varepsilon}\|_{1,\Omega_\mathrm{S}}$, $\|\boldsymbol{u}^\mathrm{P}_{r,\varepsilon}\|_{0,\Omega_\mathrm{P}}$, $\|\boldsymbol{u}^\mathrm{P}_{s,\varepsilon}\|_{0,\Omega_\mathrm{P}}$, $\|\boldsymbol{\sigma}^\mathrm{P}_\varepsilon\|_{0,\Omega_\mathrm{P}}$, $\|p^\mathrm{S}_\varepsilon\|_{0,\Omega_\mathrm{S}}$, $\|p^\mathrm{P}_\varepsilon\|_{0,\Omega_\mathrm{P}}$, and $\|\lambda_\varepsilon\|_\Lambda$ are bounded independently of ε. Using (A.13a) and the continuity of R_r (cf. Lemma A.8), we can readily see that

$$\|\boldsymbol{u}^\mathrm{P}_{r,\varepsilon}\|_{1,\Omega_\mathrm{P}} \lesssim \|\boldsymbol{u}^\mathrm{P}_{r,\varepsilon}\|_{0,\Omega_\mathrm{P}} + \|\boldsymbol{u}^\mathrm{P}_{s,\varepsilon}\|_{1,\Omega_\mathrm{P}} + \|p^\mathrm{P}_\varepsilon\|_{0,\Omega_\mathrm{P}} + \|\lambda_\varepsilon\|_\Lambda + \|\bar{f}_r\|_{0,\Omega_\mathrm{P}} + \|\bar{f}_s\|_{0,\Omega_\mathrm{P}} + \|\bar{f}_f\|_{0,\Omega_\mathrm{S}}.$$

Therefore $\|\boldsymbol{u}^\mathrm{P}_{r,\varepsilon}\|_{1,\Omega_\mathrm{P}}$ is also bounded independently of ε.

4. Passing to the limit: Since $\vec{\mathbf{V}}$ and $\vec{Q}$ are reflexive Banach spaces, as $\varepsilon \to 0$ we can apply the Banach–Alaoglu–Bourbaki Theorem [20] to extract weakly convergent subsequences $\{\vec{\boldsymbol{v}}_{\varepsilon,n}\}_{n=1}^\infty$, $\{\vec{q}_{\varepsilon,n}\}_{n=1}^\infty$ and $\{\mathscr{A}\vec{\boldsymbol{v}}_{\varepsilon,n}\}_{n=1}^\infty$, such that $\vec{\boldsymbol{v}}_{\varepsilon,n} \rightharpoonup \vec{\boldsymbol{v}}$ in $\vec{\mathbf{V}}$, $\vec{q}_{\varepsilon,n} \rightharpoonup \vec{q}$ in $\vec{Q}$, $\mathscr{A}\vec{\boldsymbol{v}}_{\varepsilon,n} \rightharpoonup \zeta$ in $\vec{\mathbf{V}}'$. This implies

$$\zeta + \mathscr{E}_1\vec{\boldsymbol{v}} + \mathscr{B}'\vec{q} = \vec{f}, \quad \mathscr{E}_2\vec{q} - \mathscr{B}\vec{\boldsymbol{v}} = \vec{g}.$$

Moreover, from (A.13a)–(A.13b) we can infer that

$$\limsup_{\varepsilon\to 0}\left(\langle\mathscr{A}\vec{\boldsymbol{v}}_\varepsilon, \vec{\boldsymbol{v}}_\varepsilon\rangle + \langle\mathscr{E}_1\vec{\boldsymbol{v}}_\varepsilon, \vec{\boldsymbol{v}}_\varepsilon\rangle + \langle\mathscr{E}_2\vec{q}_\varepsilon, \vec{q}_\varepsilon\rangle\right) \le \vec{f}(\vec{\boldsymbol{v}}) + \vec{g}(\vec{q}) = \zeta(\vec{\boldsymbol{v}}) + \langle\mathscr{E}_1\vec{\boldsymbol{v}}, \vec{\boldsymbol{v}}\rangle + \langle\mathscr{E}_2\vec{q}, \vec{q}\rangle.$$

Since $\mathscr{A}+\mathscr{E}_1+\mathscr{E}_2$ is monotone and continuous, it follows (see [42, Def. on p. 38]) that $\mathscr{A}\vec{\boldsymbol{v}}=\zeta$. Hence, $\vec{\boldsymbol{v}}$ and $\vec{q}$ solve (A.9a)-(A.9c), which establishes that $\mathbb{D}$ is nonempty. $\square$

Corollary A.11. *Under the assumptions (H.1)-(H.3), we have that* $\mathrm{Rg}(I+\mathscr{M})=\mathbb{S}_2'$.

Proof. We need to show that for $\boldsymbol{f}\in\mathbb{S}_2'$ there exists $\boldsymbol{v}\in\mathbb{D}$ such that $\boldsymbol{f}\in(I+\mathscr{M})(\boldsymbol{v})$. Let $\vec{\boldsymbol{f}}\in\mathbb{S}_2'$. From Lemma A.10, there exists $\vec{\alpha}\in\mathbb{D}$ solving (A.9a)–(A.9c). Hence, $\vec{\boldsymbol{f}}-\vec{\alpha}\in\mathscr{M}(\vec{\alpha})$, and so $\vec{\boldsymbol{f}}\in(I+\mathscr{M})(\vec{\alpha})$. $\square$

Step 2: Solvability of the parabolic problem. We begin by showing that $\mathscr{M}$ (cf. (A.10)) is monotone.

Lemma A.12. *Under the assumptions (H.1)-(H.3), the operator $\mathscr{M}$ defined by (A.11) is monotone.*

Proof. For $\vec{\alpha}\in\mathbb{D}$, it holds that $\vec{\boldsymbol{f}}-\vec{\alpha}\in\mathscr{M}(\vec{\alpha})$, and $\vec{\beta}\in\mathbb{S}_2$. Using the definition of the inner product $(\cdot,\cdot)_{\mathbb{S}_2}$, (A.9a) and (A.9b) we have

$$
\begin{aligned}
(\vec{\boldsymbol{f}}-\vec{\alpha},\vec{\beta})_{\mathbb{S}_2} &= a_f^{\mathrm{S}}\left(\boldsymbol{u}_f^{\mathrm{S}},\boldsymbol{v}_f^{\mathrm{S}}\right)+a_f^{\mathrm{P}}(\boldsymbol{u}_r^{\mathrm{P}},\boldsymbol{v}_s^{\mathrm{P}})+a_f^{\mathrm{P}}\left(\boldsymbol{u}_r^{\mathrm{P}},\boldsymbol{v}_r^{\mathrm{P}}\right)+a_f^{\mathrm{P}}\left(\boldsymbol{u}_s^{\mathrm{P}},\boldsymbol{v}_r^{\mathrm{P}}\right)+a_f^{\mathrm{P}}\left(\boldsymbol{u}_s^{\mathrm{P}},\boldsymbol{v}_s^{\mathrm{P}}\right)\\
&\quad+a_{\mathrm{BJS}}\left(\boldsymbol{u}_f^{\mathrm{S}},\boldsymbol{u}_s^{\mathrm{P}};\boldsymbol{v}_f^{\mathrm{S}},\boldsymbol{v}_s^{\mathrm{P}}\right)+b_{\mathrm{BJS}}\left(\boldsymbol{u}_r^{\mathrm{P}};\boldsymbol{v}_r^{\mathrm{P}}\right)+b^{\mathrm{S}}\left(\boldsymbol{v}_f^{\mathrm{S}},p^{\mathrm{S}}\right)+b_s^{\mathrm{P}}\left(\boldsymbol{v}_s^{\mathrm{P}},p^{\mathrm{P}}\right)+b_f^{\mathrm{P}}\left(\phi\boldsymbol{v}_r^{\mathrm{P}},p^{\mathrm{P}}\right)\\
&\quad+b_\Gamma\left(\boldsymbol{v}_f^{\mathrm{S}},\boldsymbol{v}_r^{\mathrm{P}},\boldsymbol{v}_s^{\mathrm{P}};\lambda\right)-m_\theta(\boldsymbol{u}_r^{\mathrm{P}},\boldsymbol{v}_s^{\mathrm{P}})-m_\theta(\boldsymbol{u}_s^{\mathrm{P}},\boldsymbol{v}_s^{\mathrm{P}})-m_\theta(\boldsymbol{u}_s^{\mathrm{P}},\boldsymbol{v}_r^{\mathrm{P}})-m_\theta(\boldsymbol{u}_r^{\mathrm{P}},\boldsymbol{v}_r^{\mathrm{P}})\\
&\quad+m_{\phi^2/\kappa}(\boldsymbol{u}_r^{\mathrm{P}},\boldsymbol{v}_r^{\mathrm{P}})+b_{\mathrm{sig}}^{\mathrm{P}}\left(\boldsymbol{v}_s^{\mathrm{P}},\boldsymbol{\sigma}^{\mathrm{P}}\right)-b_s^{\mathrm{P}}\left(\boldsymbol{u}_s^{\mathrm{P}},q^{\mathrm{P}}\right)-b_f^{\mathrm{P}}\left(\phi\boldsymbol{u}_r^{\mathrm{P}},q^{\mathrm{P}}\right)-b^{\mathrm{S}}\left(\boldsymbol{u}_f^{\mathrm{S}},q^{\mathrm{S}}\right)-b_{\mathrm{sig}}^{\mathrm{P}}\left(\boldsymbol{u}_s^{\mathrm{P}},\boldsymbol{\tau}^{\mathrm{P}}\right).
\end{aligned}
\tag{A.20}
$$

As the model problem is linear, the coercivity of the bilinear forms in (A.20) (cf. Lemmas A.1 and A.5) suffices to assert the monotonicity of $\mathscr{M}$. $\square$

Lemma A.13. *Assume (H.1)-(H.3). Then, for each $\boldsymbol{h}_f\in W^{1,1}(0,T;\mathbf{W}_{f,2}')$, $\boldsymbol{h}_r\in W^{1,1}(0,T;\mathbf{W}_{r,2}')$, $\boldsymbol{h}_s\in W^{1,1}(0,T;\mathbf{W}_{s,2}')$, $h_p\in W^{1,1}(0,T;W_{p,2}')$ and $h_e\in W^{1,1}(0,T;\mathbf{Z}_2')$, and compatible initial data $\vec{\alpha}(0)\in\mathbf{V}_r\times\mathbf{V}_s\times\mathbf{V}_f\times\mathrm{W}_p\times\mathbf{Z}$, there exists a solution to (A.11) with $\vec{\alpha}\in W^{1,\infty}(0,T;\mathbf{V}_r)\times W^{1,\infty}(0,T;\mathbf{V}_s)\times W^{1,\infty}(0,T;\mathbf{V}_f)\times W^{1,\infty}(0,T;\mathrm{W}_p)\times W^{1,\infty}(0,T;\mathbf{Z})$.*

Proof. Applying Theorem A.6 with $\mathscr{N}=I,\mathscr{M}^\star=\mathscr{M},E=\mathbb{S}_2,E_b'=\mathbb{S}_2'$, and using Lemma A.12 and Corollary A.11, we obtain existence of a solution to (A.11). $\square$

Step 3: The mixed problem (A.2) is a special case of (A.11). Finally, we establish the existence of a solution to (A.2) as a corollary of Lemma A.13.

Lemma A.14. *If $\vec{\alpha}(t)\in\mathbb{D}$ solves (A.11) for $\vec{\boldsymbol{h}}=(\boldsymbol{0},\boldsymbol{f}_{\mathrm{P}},\boldsymbol{f}_{\mathrm{S}},\rho_f^{-1}(1-\phi)^{-2}K\theta,\boldsymbol{0})$, then it also solves (A.2).*

Proof. Let $\vec{\alpha}(t)\in\mathbb{D}$ solve (A.11). For $\vec{\boldsymbol{h}}=(\boldsymbol{f}_{\mathrm{P}},\boldsymbol{f}_{\mathrm{P}},\boldsymbol{f}_{\mathrm{S}},\rho_f^{-1}(1-\phi)^{-2}K\theta,\boldsymbol{0})$, there exists $\vec{\boldsymbol{f}}\in\mathbb{S}_2'$ such that $\vec{\boldsymbol{f}}-\vec{\alpha}\in\mathscr{M}(\vec{\alpha})$ which satisfies $\frac{d}{dt}\vec{\alpha}+(\vec{\boldsymbol{f}}-\vec{\alpha})=\vec{\boldsymbol{h}}$. Then $\left(\frac{d}{dt}\vec{\alpha},\vec{\beta}\right)_{\mathbb{S}_2}+(\vec{\boldsymbol{f}}-\vec{\alpha},\vec{\beta})_{\mathbb{S}_2}=(\vec{\boldsymbol{h}},\vec{\beta})_{\mathbb{S}_2}$. From the definition of $(\cdot,\cdot)_{\mathbb{S}_2}$, along with (A.9a) and (A.9b), we can deduce part of (A.2). Equation (A.9c), stemming directly from the definition of $\mathbb{D}$, implies the remainder of (A.2). $\square$

Proof of Theorem A.7. The solvability of (A.2) follows from Lemma A.13 and Lemma A.14. From Lemma A.13 we have that $\vec{\alpha}\in W^{1,\infty}(0,T;\mathbf{V}_r)\times W^{1,\infty}(0,T;\mathbf{V}_s)\times W^{1,\infty}\left(0,T;\mathbf{V}_f\right)\times W^{1,\infty}(0,T;\mathrm{W}_p)\times W^{1,\infty}(0,T;\mathbf{Z})$. Finally, the inf-sup condition (A.5b) and equation (A.2a) imply that $p^{\mathrm{S}}\in L^\infty(0,T;\mathrm{W}_f)$ and $\lambda\in L^\infty(0,T;\Lambda)$. $\square$

Theorem A.7 assumes that $(\boldsymbol{u}_{f,0},\boldsymbol{u}_{r,0},\boldsymbol{u}_{s,0},p^{\mathrm{P},0},\boldsymbol{\sigma}_0^{\mathrm{P}})\in\mathbb{D}$. We provide a procedure for obtaining such initial data.

Lemma A.15. *Assume (H.1)–(H.3) and* $(p^{\mathrm{P},0},\boldsymbol{u}_{r,0},\boldsymbol{u}_{s,0})\in \mathrm{W}_p\times\mathbf{V}_r\times\mathbf{V}_s$. *Then, there exist* $(\boldsymbol{u}_{f,0},\boldsymbol{\sigma}_0^{\mathrm{P}})\in\mathbf{V}_f\times\mathbf{Z}$ *and* $(p^{\mathrm{S},0},\lambda_0)\in\mathrm{W}_f\times\Lambda$ *such that* (A.9) *holds for a suitable* $\vec{\boldsymbol{f}}\in\mathbb{S}_2'$.

Proof. We solve a sequence of subproblems, using previously obtained solutions as data:

1.Define
$$\lambda_0=\boldsymbol{\sigma}_{f,0}^{\mathrm{P}}\boldsymbol{n}_{\mathrm{P}}\cdot\boldsymbol{n}_{\mathrm{P}}\big|_{\Sigma}=\left(2\mu_f\phi\boldsymbol{\varepsilon}\left(\boldsymbol{u}_{r,0}+\boldsymbol{u}_{s,0}\right)\boldsymbol{n}_{\mathrm{P}}\cdot\boldsymbol{n}_{\mathrm{P}}-\phi p^{\mathrm{P}}\right)\Big|_{\Sigma}\in\Lambda$$
(well-defined since $\boldsymbol{\sigma}_{f,0}^{\mathrm{P}}$ is meaningful for given $(p^{\mathrm{P},0},\boldsymbol{u}_{r,0},\boldsymbol{u}_{s,0})\in\mathrm{W}_p\times\mathbf{V}_r\times\mathbf{V}_s$). From (A.9a) and (A.9b), choose the bilinear forms with $\boldsymbol{v}_r^{\mathrm{P}}$ and q^{P}. For given data $\lambda_0\in\Lambda$, choose $\theta=0$, boundary conditions on Γ_{P}, and $-\boldsymbol{\sigma}_{f,0}^{\mathrm{P}}\boldsymbol{n}_{\mathrm{P}}\cdot\boldsymbol{n}_{\mathrm{P}}=\lambda_0$, $-\boldsymbol{\sigma}_{f,0}^{\mathrm{P}}\boldsymbol{n}_{\mathrm{P}}\cdot\tau_{f,j}=\mu_f\alpha_{\mathrm{BJS}}Z_J^{-1/2}\boldsymbol{u}_{r,0}\cdot\tau_{f,j}$ on Σ. This defines a well-posed generalized Brinkman problem (by Lemma A.1), which is satisfied by the given data $(\boldsymbol{u}_{r,0},\boldsymbol{u}_{s,0},p^{\mathrm{P},0})\in\mathbf{V}_r\times\mathbf{V}_s\times\mathrm{W}_p$.

2.Define $(\boldsymbol{u}_{f,0},p^{\mathrm{S},0})\in\mathbf{V}_f\times\mathrm{W}_f$. From (A.9a) and (A.9b), choose the bilinear forms with $\boldsymbol{v}_f^{\mathrm{S}}$ and q^{S}, and set $\boldsymbol{u}_{s,0}\cdot\tau_{f,j}=0$ in a_{BJS}. For given data $\lambda_0\in\Lambda$ and boundary conditions on Γ_{S} and $-\boldsymbol{\sigma}_{f,0}^{\mathrm{S}}\boldsymbol{n}_{\mathrm{S}}\cdot\boldsymbol{n}_{\mathrm{S}}=\lambda_0$, $-\boldsymbol{\sigma}_{f,0}^{\mathrm{S}}\boldsymbol{n}_{\mathrm{S}}\cdot\tau_{f,j}=\mu_f\alpha_{\mathrm{BJS}}Z_J^{-1/2}\boldsymbol{u}_{f,0}\cdot\tau_{f,j}$ on Σ, this is a well-posed Stokes problem (thanks to Lemma A.1).

3.Define $(\boldsymbol{\sigma}_0^{\mathrm{P}},\boldsymbol{u}_{r,0},\boldsymbol{y}_{s,0})\in\mathbf{Z}\times\mathbf{V}_r\times\mathbf{V}_s$. We choose $\theta=0$ from (A.9a) with test function $\boldsymbol{v}_s^{\mathrm{P}}$, adopt boundary conditions on Γ_P and $\boldsymbol{\sigma}_{f,0}^{\mathrm{S}}\boldsymbol{n}_{\mathrm{S}}+\boldsymbol{\sigma}_{f,0}^{\mathrm{P}}\boldsymbol{n}_{\mathrm{P}}+\boldsymbol{\sigma}_{s,0}^{\mathrm{P}}\boldsymbol{n}_{\mathrm{S}}=0$, $-\boldsymbol{\sigma}_{f,0}^{\mathrm{S}}\boldsymbol{n}_{\mathrm{S}}\cdot\boldsymbol{n}_{\mathrm{S}}=\lambda_0$, $-\boldsymbol{\sigma}_{f,0}^{\mathrm{S}}\boldsymbol{n}_{\mathrm{S}}\cdot\tau_{f,j}=\mu_f\alpha_{\mathrm{BJS}}Z_J^{-1/2}\boldsymbol{u}_{f,0}\cdot\tau_{f,j}$ on Σ. Then, owing to Lemma A.1, the elasticity problem
$$a_p^{\mathrm{P}}(\boldsymbol{\sigma}_0^{\mathrm{P}},\boldsymbol{\tau}^{\mathrm{P}})-b_{\mathrm{sig}}^{\mathrm{P}}(\boldsymbol{y}_{s,0},\boldsymbol{\tau}^{\mathrm{P}})=0\quad\forall\boldsymbol{\tau}^{\mathrm{P}}\in\mathbf{Z},\tag{A.21}$$
is well-posed. Note that $(p^{\mathrm{S},0},p^{\mathrm{P},0},\boldsymbol{u}_{f,0},\lambda_0)\in\mathrm{W}_f\times\mathrm{W}_p\times\mathbf{V}_f\times\Lambda$ are data, and that $\boldsymbol{y}_{s,0}$ is not part of the initial condition for the alternative formulation, but it will be used to recover $\boldsymbol{y}_{s,0}$ in the original formulation.

4.Define a suitable extension for $\boldsymbol{u}_{s,0}\in\mathbf{V}_s$. Choose $\boldsymbol{u}_{s,0}\in\mathbf{V}_s$ to satisfy (A.9c) and $\boldsymbol{u}_{s,0}\cdot\tau_{f,j}=0$ on Σ. Note that $(\boldsymbol{u}_{r,0},\boldsymbol{u}_{f,0})\in\mathbf{V}_r\times\mathbf{V}_f$ are given as data for this problem.

Then $(\boldsymbol{u}_{f,0},\boldsymbol{u}_{r,0},\boldsymbol{u}_{s,0},p^{\mathrm{P},0},\boldsymbol{\sigma}_0^{\mathrm{P}})\in\mathbf{V}_f\times\mathbf{V}_r\times\mathbf{V}_s\times\mathrm{W}_p\times\mathbf{Z}$ and $(p^{\mathrm{S},0},\lambda_0)\in\mathrm{W}_f\times\Lambda$ satisfy (A.9a)-(A.9c), with
$$\left((1-\phi)^2K^{-1}p^{\mathrm{P}},q^{\mathrm{P}}\right)_{\Omega_{\mathrm{P}}}-b_s^{\mathrm{P}}\left(\boldsymbol{u}_s^{\mathrm{P}},q^{\mathrm{P}}\right)-b_f^{\mathrm{P}}\left(\phi\boldsymbol{u}_r^{\mathrm{P}},q^{\mathrm{P}}\right)=\left((1-\phi)^2K^{-1}\bar{g}_p,q^{\mathrm{P}}\right)_{\Omega_{\mathrm{P}}},$$
$$a_p^{\mathrm{P}}\left(\boldsymbol{\sigma}^{\mathrm{P}},\boldsymbol{\tau}^{\mathrm{P}}\right)-b_{\mathrm{sig}}^{\mathrm{P}}\left(\boldsymbol{u}_s^{\mathrm{P}},\boldsymbol{\tau}^{\mathrm{P}}\right)=\left(A\bar{g}_e,\boldsymbol{\tau}^{\mathrm{P}}\right)_{\Omega_{\mathrm{P}}},$$
therefore obtaining the desired result. □

We will refer to $(\boldsymbol{u}_{f,0},\boldsymbol{u}_{r,0},\boldsymbol{u}_{s,0},p^{\mathrm{P},0},\boldsymbol{\sigma}_0^{\mathrm{P}})\in\mathbf{V}_f\times\mathbf{V}_r\times\mathbf{V}_s\times\mathrm{W}_p\times\mathbf{Z}$ and $(\boldsymbol{u}_{f,0},\boldsymbol{u}_{r,0},\boldsymbol{u}_{s,0},p^{\mathrm{P},0},\boldsymbol{y}_{s,0})\in\mathbf{V}_f\times\mathbf{V}_r\times\mathbf{W}_s\times\mathrm{W}_p\times\mathbf{V}_s$ constructed in Lemma A.15 as *compatible initial data* for the alternative and the original formulations, respectively. It follows from (A.21) that $\boldsymbol{\sigma}_0^{\mathrm{P}}=A^{-1}\boldsymbol{\varepsilon}(\boldsymbol{y}_{s,0})$.

A.1. *Mixed formulation for discrete analysis*

Let $\mathbf{V}_{s,h}$ consist of polynomials of degree at most k_s. The stress FE space $\mathbf{Z}_h\subset\mathbf{Z}$ is
$$\mathbf{Z}_h:=\left\{\boldsymbol{\sigma}^{\mathrm{P}}\in\mathbf{Z}:\,\boldsymbol{\sigma}^{\mathrm{P}}\big|_K\in\mathbb{P}_{k_s-1}^{\mathrm{sym}}(K)^{d\times d},\ \forall K\in\mathscr{T}_h^{\mathrm{P}}\right\}.$$

We group the spaces (endowed with the continuous norms) as well as trial and test functions as follows
$$\vec{\mathbf{V}}_h:=\mathbf{V}_{r,h}\times\mathbf{V}_{s,h}\times\mathbf{V}_{f,h},\quad\vec{Q}_h:=\mathrm{W}_{p,h}\times\mathbf{Z}_h\times\mathrm{W}_{f,h}\times\Lambda_h,\quad\vec{\boldsymbol{u}}_h:=(\boldsymbol{u}_{r,h}^{\mathrm{P}},\boldsymbol{u}_{s,h}^{\mathrm{P}},\boldsymbol{u}_{f,h}^{\mathrm{S}})\in\vec{\mathbf{V}}_h,$$

$$\vec{p}_h := (p_h^{\mathrm{P}}, \boldsymbol{\sigma}_h^{\mathrm{P}}, p_h^{\mathrm{S}}, \lambda_h) \in \vec{Q}_h, \quad \vec{\boldsymbol{v}}_h := (\boldsymbol{v}_{r,h}^{\mathrm{P}}, \boldsymbol{v}_{s,h}^{\mathrm{P}}, \boldsymbol{v}_{f,h}^{\mathrm{S}}) \in \vec{\mathbf{V}}_h, \quad \vec{q}_h := (q_h^{\mathrm{P}}, \boldsymbol{\tau}_h^{\mathrm{P}}, q_h^{\mathrm{S}}, \mu_h) \in \vec{Q}_h,$$

The semi-discrete formulation for the mixed method remains as before, except that instead of seeking $\boldsymbol{y}_{s,h}^{\mathrm{P}}$, we now seek $\boldsymbol{\sigma}_h^{\mathrm{P}} \in W^{1,\infty}(0,T;\mathbf{Z}_h)$ from the governing equations. The discrete mixed evolution problem is

$$\partial_t \mathscr{E}_{1,h} \vec{\boldsymbol{u}}_h(t) + \mathscr{A}_h \vec{\boldsymbol{u}}_h(t) + \mathscr{B}_h' \vec{p}_h(t) = \boldsymbol{f}(t) \qquad \text{in } \vec{\mathbf{V}}_h', \tag{A.22a}$$

$$\partial_t \mathscr{E}_{2,h} \vec{p}_h(t) - \mathscr{B}_h \vec{\boldsymbol{u}}_h(t) + \mathscr{C}_h \vec{p}_h(t) = g(t) \qquad \text{in } \vec{Q}_h', \tag{A.22b}$$

where the operators $\mathscr{A}_h : \vec{\mathbf{V}}_h \to \vec{\mathbf{V}}_h', \mathscr{B}_h : \vec{\mathbf{V}}_h \to \vec{Q}_h', \mathscr{C}_h : \vec{Q}_h \to \vec{Q}_h', \mathscr{E}_{1,h} : \vec{\mathbf{V}}_h \to \vec{\mathbf{V}}_h', \mathscr{E}_{2,h} : \vec{Q}_h \to \vec{Q}_h'$, and the functionals $\boldsymbol{f} \in \vec{\mathbf{V}}_h', g \in \vec{Q}_h'$ are the discrete counterparts of the operators introduced before.

We demonstrate the well-posedness of the mixed formulation and subsequently deduce the well-posedness of the primal formulation, akin to the continuous case. The initial conditions $\boldsymbol{v}_{r,h}^{\mathrm{P}}(0), \boldsymbol{v}_{s,h}^{\mathrm{P}}(0), \boldsymbol{v}_{f,h}^{\mathrm{S}}(0), p_h^{\mathrm{P}}(0)$ and $\boldsymbol{\sigma}_h^{\mathrm{P}}(0) = A^{-1}\boldsymbol{\varepsilon}(\boldsymbol{y}_{s,h}^{\mathrm{P}}(0))$ are suitable approximations of $\boldsymbol{v}_{r,0}, \boldsymbol{v}_{s,0}, \boldsymbol{v}_{f,0}, p^{\mathrm{P},0}$ and $\boldsymbol{\sigma}_0^{\mathrm{P}} = A^{-1}\boldsymbol{\varepsilon}\left(\boldsymbol{y}_{s,0}\right)$, respectively.

A.2. *Discrete inf-sup conditions*

Lemma A.16. *There exist constants $\beta_1, \beta_2, \beta_3 > 0$, independent of h such that*

$$\inf_{\left(\mathbf{0}, \boldsymbol{v}_{s,h}^{\mathrm{P}}, \mathbf{0}\right) \in \vec{\mathbf{V}}_h} \sup_{\left(0, \boldsymbol{\tau}_h^{\mathrm{P}}, 0, 0\right) \in \vec{Q}_h} \frac{b_{sig}^{\mathrm{P}}\left(\boldsymbol{v}_{s,h}^{\mathrm{P}}, \boldsymbol{\tau}_h^{\mathrm{P}}\right)}{\|(\mathbf{0}, \boldsymbol{v}_{s,h}^{\mathrm{P}}, \mathbf{0})\|_{\vec{\mathbf{V}}} \|(0, \boldsymbol{\tau}_h^{\mathrm{P}}, 0, 0)\|_{\vec{Q}}} \geq \beta_1, \tag{A.23a}$$

$$\inf_{\left(q_h^{\mathrm{P}}, \mathbf{0}, q_h^{\mathrm{S}}, \mu_h\right) \in \vec{Q}_h} \sup_{\left(\boldsymbol{v}_{r,h}^{\mathrm{P}}, \mathbf{0}, \boldsymbol{v}_{f,h}^{\mathrm{S}}\right) \in \vec{\mathbf{V}}_h} \frac{b^{\mathrm{S}}\left(\boldsymbol{v}_{f,h}^{\mathrm{S}}, q_h^{\mathrm{S}}\right) + b_f^{\mathrm{P}}\left(\boldsymbol{v}_{r,h}^{\mathrm{P}}, q_h^{\mathrm{P}}\right) + b_\Gamma\left(\boldsymbol{v}_{r,h}^{\mathrm{P}}, \boldsymbol{v}_{f,h}^{\mathrm{S}}, \mathbf{0}; \mu_h\right)}{\|(\boldsymbol{v}_{r,h}^{\mathrm{P}}, \mathbf{0}, \boldsymbol{v}_{f,h}^{\mathrm{S}})\|_{\vec{\mathbf{V}}} \|(q_h^{\mathrm{P}}, \mathbf{0}, q_h^{\mathrm{S}}, \lambda_h)\|_{\vec{Q}}} \geq \beta_2, \tag{A.23b}$$

$$\inf_{\vec{q}_h \in \vec{Q}} \sup_{\vec{\boldsymbol{v}}_h \in \vec{\mathbf{V}}} \frac{\langle \mathscr{B}(\vec{\boldsymbol{v}}_h), \vec{q}_h \rangle}{\|\vec{\boldsymbol{v}}_h\|_{\vec{\mathbf{V}}} \|\vec{q}_h\|_{\vec{Q}}} \geq \beta_3. \tag{A.23c}$$

Proof. Equation (A.23a) follows by choosing $\boldsymbol{\tau}_h^{\mathrm{P}} = \boldsymbol{\varepsilon}(\boldsymbol{v}_{s,h}^{\mathrm{P}})$ and and then applying Korn's inequality [19]. Equation (A.23b) follows from the Stokes discrete inf-sup condition [17] and its weighted variant in [11, Theorem 7]. The one for $b_\Gamma(\cdot,\cdot;\cdot)$ can be done as in [37, Corollary 3.5]. Hence, (A.23c) follows from (A.23a) and (A.23b). ☐

A.3. *Existence and uniqueness of discrete solution*

In order to show well-posedness of (A.22), we proceed as in the continuous problem. We introduce $\mathbf{W}_{r,h}^2$, $\mathbf{W}_{s,h}^2$, $\mathbf{W}_{f,h}^2$, $\mathrm{W}_{p,h}^2$ and $\mathbf{Z}_h^2$ as the closure of the spaces $\mathbf{V}_{r,h}$, $\mathbf{V}_{s,h}$, $\mathbf{V}_{f,h}$,$\mathrm{W}_{p,h}$ and $\mathbf{Z}_h$ with respect to the norms (A.6), and denote $\mathbb{S}_{2,h} = \mathbf{W}_{r,h}^2 \times \mathbf{W}_{s,h}^2 \times \mathbf{W}_{f,h}^2 \times \mathrm{W}_{p,h}^2 \times \mathbf{Z}_h^2$. Then, define

$$\begin{aligned}\mathbb{D}_h := \Big\{ \vec{\alpha}_h \in \mathbf{V}_{r,h} \times \mathbf{V}_{s,h} \times \mathbf{V}_{f,h} \times \mathrm{W}_{p,h} \times \mathbf{Z}_h : \exists \left(p_h^{\mathrm{S}}, \lambda_h\right) \in \mathrm{W}_{f,h} \times \Lambda_h \text{ such that} \\ \forall (\vec{\boldsymbol{v}}_h, \vec{q}_h) \in \vec{\mathbf{V}}_h \times \vec{Q}_h \text{ satisfy (A.25a)} - \text{(A.25b) for some } \vec{\boldsymbol{f}} \in \mathbb{S}_{2,h}' \Big\} \subset \mathbb{S}_{2,h}.\end{aligned} \tag{A.24}$$

Defining $\vec{\boldsymbol{f}} = (\vec{f}, \vec{g}) = ((\bar{f}_r, \bar{f}_s, \bar{f}_f), \bar{g}_p, \bar{g}_e, 0, \mathbf{0})) \in \mathbb{S}_2'$, the equations defined on the domain are

$$\left(\mathscr{E}_{1,h} + \mathscr{A}_h\right) \vec{\boldsymbol{v}}_h + \mathscr{B}_h' \vec{p}_h = \vec{f}, \tag{A.25a}$$

$$-\mathscr{B}_h\vec{\boldsymbol{v}}_h + \mathscr{E}_{2,h}\vec{p}_h = \vec{g}. \tag{A.25b}$$

As in the continuous case, we define $\mathscr{M}_h$ with domain $\mathbb{D}_h$ as

$$\mathscr{M}_h(\vec{\alpha}_h) := \{\vec{\boldsymbol{f}} - \vec{\alpha}_h \in \mathbb{S}'_{2,h} : \vec{\alpha}_h \text{ solves (A.25a)} - \text{(A.25b) for } \vec{\boldsymbol{f}} \in \mathbb{S}'_{2,h}\},$$

and for $\vec{\boldsymbol{h}} = \big(\boldsymbol{f}_{\mathrm{P}}, \boldsymbol{f}_{\mathrm{P}}, \boldsymbol{f}_{\mathrm{S}}, \rho_f^{-1}(1-\phi)^{-2}K\theta, \boldsymbol{0}\big)$, consider the problem

$$\frac{d}{dt}\vec{\alpha}_h(t) + \mathscr{M}_h\big(\vec{\alpha}_h(t)\big) \ni \vec{\boldsymbol{h}}. \tag{A.26}$$

Remark A.17 (Compatibility of discrete stress and displacement spaces)**.** *We establish well-posedness using the stress-based formulation, and derive stability for the displacement-based formulation. In the continuous setting, these formulations are equivalent due to the invertibility of the stress-strain relation and the corresponding regularity.*

At the semi-discrete level this is more subtle. In the steady case, compatibility between discrete stress and displacement is not guaranteed [8] (for example, enforcing stress symmetry and ensuring the divergence of the stress lies in the dual of the discrete displacement space is not possible without specifically designed FE spaces). Here the stress is reconstructed dynamically at each time step from $\boldsymbol{\sigma}_h^{\mathrm{P}} = A^{-1}\boldsymbol{\varepsilon}(\boldsymbol{y}_{s,h}^{\mathrm{P}})$*, thus stress-displacement compatibility is maintained throughout the time evolution. Key are the discrete spaces, which allow us to impose* $\boldsymbol{\sigma}_h^{\mathrm{P}} = A^{-1}\boldsymbol{\varepsilon}(\boldsymbol{y}_{s,h}^{\mathrm{P}})$ *strongly. The discrete displacement belongs to* $\mathbf{V}_{s,h} = \mathbf{V}_s \cap [X_h^{k+1}]^d$*, where each component lies in* $\mathbb{P}_{k+1}(K)$ *for all* $K \in \mathscr{T}_h^{\mathrm{P}}$*. The discrete strain* $\boldsymbol{\varepsilon}(\boldsymbol{y}_{s,h}^{\mathrm{P}})$ *then belongs to* $\mathbb{P}_k^{\mathrm{sym}}(K)^{d\times d}$*. In turn, the stress space is*

$$\mathbf{Z}_h := \{\boldsymbol{\sigma}_h^{\mathrm{P}} \in \mathbf{Z} : \boldsymbol{\sigma}_h^{\mathrm{P}}|_K \in \mathbb{P}_k^{\mathrm{sym}}(K)^{d\times d},\ \forall K \in \mathscr{T}_h^{\mathrm{P}}\},$$

so that $\boldsymbol{\varepsilon}(\boldsymbol{y}_{s,h}^{\mathrm{P}}) \subset \mathbf{Z}_h$ *by construction. Thus, the discrete stress* $\boldsymbol{\sigma}_h^{\mathrm{P}} = A^{-1}\boldsymbol{\varepsilon}(\boldsymbol{y}_{s,h}^{\mathrm{P}})$ *is symmetric, displacement-compatible, and preserves the structure needed for the equivalence of the mixed and primal formulations.*

Theorem A.18. *For each* $\boldsymbol{f}_{\mathrm{S}} \in W^{1,1}(0,T;\mathbf{L}^2(\Omega_{\mathrm{S}})), \boldsymbol{f}_{\mathrm{P}} \in W^{1,1}(0,T;\mathbf{L}^2(\Omega_{\mathrm{P}})), \mathrm{W}'_f)$*,* $\theta \in W^{1,1}\left(0,T;\mathrm{W}'_p\right)$*, and compatible initial data* $(\boldsymbol{u}_{r,h}^{\mathrm{P}}(0), \boldsymbol{u}_{s,h}^{\mathrm{P}}(0), \boldsymbol{u}_{f,h}^{\mathrm{S}}(0), p_h^{\mathrm{P}}(0), \boldsymbol{\sigma}_h^{\mathrm{P}}(0)) \in \mathbf{V}_{r,h} \times \mathbf{V}_{s,h} \times \mathbf{V}_{f,h} \times \mathrm{W}_{p,h} \times \mathbf{Z}_h$ *under assumptions (H.1)-(H.3), there exists a solution of (A.22) with* $(\vec{\boldsymbol{u}}_h, \vec{p}_h) \in W^{1,\infty}\left(0,T;\mathbf{V}_{r,h}\right) \times W^{1,\infty}\left(0,T;\mathbf{V}_{s,h}\right) \times W^{1,\infty}\left(0,T;\mathbf{V}_{f,h}\right) \times W^{1,\infty}\left(0,T;\mathrm{W}_{p,h}\right) \times W^{1,\infty}(0,T;\mathbf{Z}_h) \times L^\infty\left(0,T;\mathrm{W}_{f,h}\right) \times L^\infty(0,T;\Lambda_h)$*.*

To prove Theorem A.18 we proceed as in the continuous problem: first we show that $\mathbb{D}_h$ is nonempty, then we show solvability of (A.26), and finally show that the solution to (A.24) satisfies (A.22). With (A.23a) and (A.23b), the proof follows closely that of Theorem A.7. The only difference is that the operator L_Σ from Lemma A.9 is now defined as $L_\Sigma : \Lambda_h \to \Lambda'_h, \langle L_\Sigma \mu_{h,1}, \mu_{h,2}\rangle := \langle \mu_{h,1}, \mu_{h,2}\rangle_\Sigma$. Furthermore, L_Σ is bounded, continuous, coercive and monotone, following immediately from its definition, since $\langle L_\Sigma \mu_h, \mu_h\rangle^{1/2} = \|\mu_h\|_{\Lambda_h}$.

Remark A.19. *To satisfy the compatible initial data assumption for* $\vec{\alpha}_h(0) \in \mathbf{V}_{r,h} \times \mathbf{V}_{s,h} \times \mathbf{V}_{f,h} \times \mathrm{W}_{p,h} \times \mathbf{Z}_h$ *and* $(\boldsymbol{u}_{r,h}^{\mathrm{P}}(0), \boldsymbol{u}_{s,h}^{\mathrm{P}}(0), \boldsymbol{u}_{f,h}^{\mathrm{S}}(0), p_h^{\mathrm{P}}(0), \boldsymbol{y}_{s,h}^{\mathrm{P}}(0)) \in \mathbf{V}_{r,h} \times \mathbf{W}_{s,h} \times \mathbf{V}_{f,h} \times \mathrm{W}_{p,h} \times \mathbf{V}_{s,h}$*, we take* $(\vec{\boldsymbol{u}}_h(0), \vec{p}_h(0)) \in \vec{\mathbf{V}}_h \times \vec{Q}_h$ *to be the* $\mathbb{D}_h$*-elliptic projection of* $(\vec{\boldsymbol{u}}_0, \vec{p}_0)$ *constructed in Lemma A.15:*

$$\begin{aligned}
\left(\mathscr{E}_{1,h} + \mathscr{A}_h\right)\vec{\boldsymbol{u}}_h(0) + \mathscr{B}'_h\vec{p}_h(0) &= \left(\mathscr{E}_{1,h} + \mathscr{A}_h\right)\vec{\boldsymbol{u}}_0 + \mathscr{B}'_h\vec{p}_0 && \text{in } \vec{\mathbf{V}}'_h,\\
-\mathscr{B}_h\vec{\boldsymbol{u}}_h(0) + \mathscr{E}_{2,h}\vec{p}_h(0) &= -\mathscr{B}_h\vec{\boldsymbol{u}}_0 + \mathscr{E}_{2,h}\vec{p}_0 && \text{in } \vec{Q}'_h.
\end{aligned}$$