A perturbed threefold saddle-point formulation yielding new mixed finite element methods for poroelasticity with reduced symmetry

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Abstract

We propose and analyze new mixed finite element methods for the linear poroelasticity problem, which models the coupled phenomena of fluid diffusion and solid deformation. The formulation is based on the introduction of the vorticity and the strain tensor as auxiliary unknowns, which makes it possible to eliminate the fluid content from the system. The latter can then be recovered from the strain tensor and the pressure. Then, by incorporating a multiple of the pressure gradient as an additional unknown, we arrive at an operator equation showing a threefold saddle-point structure, which, in turn, is perturbed by a term depending on the pressure variable. The well-posedness of the continuous formulation is established through a suitable extension of the usual Babuška–Brezzi theory, which yields a new abstract result, along with a recently developed approach to analyze perturbed saddle-point problems. The discrete analysis follows a similar strategy, employing arbitrary finite element spaces satisfying suitable assumptions. In particular, we provide concrete examples based on PEERS elements and derive the corresponding convergence rates. Finally, several numerical experiments are presented, which confirm the theoretical results and illustrate the good performance of the methods.

Keywords: Poroelasticity, fully mixed finite element methods, threefold saddle-point problems, error analysis.

Mathematics subject classifications (2000): 74F10, 65N30, 65N15, 74S05.

1 Introduction

Scope. The equations of Biot poroelasticity describe the mechanical behavior of a fluid-saturated porous medium, where the coupling between fluid diffusion and solid deformation is taken into account. This model finds wide use in various fields that span from geomechanics and petroleum engineering to biomechanics and material science; see, e.g., [37, 38]. Numerical solutions of poroelasticity can be challenging, especially when dealing with heterogeneous materials, complex geometries, and multiphysics, and partly due to the presence of multiple scales and physical parameters (e.g., Lamé coefficients, storativity, permeability) as well as discretization parameters.

While the primary form of the governing equations is based on the solid displacement and the fluid pressure as main unknowns, it is well known that reformulations of the equations in fully mixed form can provide significant advantages in terms of robustness with respect to material parameters and of local mass

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conservation, for example. In this context, several mixed finite element and polytopal methods have been proposed in the literature; see, e.g., [7, 10, 13, 21, 24, 28, 32, 33, 39], which also include multiphysics couplings with diffusion, temperature, or interfacial effects. Depending on the specific structure and choice of the additional unknowns, many different methods can be derived, usually resulting in saddle-point formulations (symmetric, non-symmetric, single, twofold, threefold, etc.). Their analysis – solution existence and uniqueness, energy estimates, and error bounds – typically relies on the classical Babuška–Brezzi theory for mixed problems [11], or on its extensions to more complex saddle-point structures [9, 27].

The present work aims to introduce and analyze new mixed finite element methods for linear poroelasticity with mixed traction-loading boundary conditions, and based on a threefold saddle-point formulation. The proposed approach is inspired by traditional mixed formulations with weakly imposed symmetry for linear elasticity [3, 4, 16, 34]. Apart from solid displacement and fluid pressure, several other variables of mathematical and physical interest form part of the resulting system. Indeed, we introduce the full Biot stress, which, besides allowing the mixed boundary conditions to be incorporated more naturally, ensures that the balance of linear momentum is satisfied exactly. In turn, the balance of angular momentum is imposed weakly through the incorporation of the infinitesimal rotation tensor. In addition, we consider the infinitesimal strain, the discharge flux, and an additional unknown defined on a function space associated with the Neumann part of the boundary, which is related to the imposition of non-homogeneous boundary conditions on the discharge velocity (Darcy flux). While these extra unknowns often lead to larger algebraic systems, they also provide more accurate approximations of the corresponding physical quantities of interest, especially when low-regularity solutions are considered and in the presence of multiphysics couplings.

We prove that the resulting threefold saddle-point formulation is well-posed by extending the classical Babuška–Brezzi theory to this more complex setting, and by employing a recently developed framework for the analysis of perturbed saddle-point problems [18]. The developments in this paper can be therefore seen as a non-trivial extension of the abstract theory presented in [18] to the case of threefold saddle-point problems, which can be of use for other general applications in continuum mechanics. The arguments of the proofs involve kernel splittings and their identification/characterization when applied to the Biot equations.

At the discrete level it is possible to follow conforming or non-conforming schemes. In this work, we restrict ourselves to the conforming case and develop the discrete analysis under a set of explicit and verifiable assumptions on the finite element spaces involved. These hypotheses are formulated in a general manner, so that the resulting Galerkin scheme is not tied to a specific discretization. Within this setting, the discrete analysis follows a strategy that closely mirrors the proposed abstract continuous approach. Nevertheless, it departs from the ideas cited above when addressing approximation properties and the verification of suitable discrete inf-sup conditions.

We then particularize the abstract framework to PEERS elements combined with Raviart–Thomas and discontinuous polynomial spaces, showing that they satisfy all the required conditions, and we remark that other well-established mixed finite element families available in the literature also fit within the proposed setting. The discrete analysis requires additional assumptions on the regularity of the discretization of the Neumann sub-boundary, as well as on the compatibility between the finite element spaces for the Darcy flux, their normal traces, and the discrete Lagrange multiplier space. Another technical issue is that the discrete inf-sup conditions rely on auxiliary Poisson problems with mixed boundary conditions, which in turn impose mild restrictions on the class of admissible domains. The practical impact of these restrictions is investigated through numerical experiments, including a test on a domain that does not satisfy the required regularity assumptions and nevertheless exhibits the expected convergence behavior.

Outline. In the remainder of this section we include notational convention and preliminary definitions of spaces and operators needed for the functional setting. In Section 2 we recall the equations of poroelasticity in steady form, introducing also the auxiliary unknowns. Section 3 is devoted to deriving the weak formulation, stating and proving a new abstract result for threefold saddle-point problems, and using this theory for establishing the well-posedness of the fully mixed Biot equations. In Section 4 we define the Galerkin method, state regularity assumptions, and apply the new abstract theory to show existence and uniqueness of discrete solution for generic finite element spaces that satisfy inf-sup stability, space compatibility, suitable trace inequalities, and existence of adequate interpolation operators. There we also derive quasi-optimality results, which we combine with specific properties of PEERS and Raviart–Thomas finite elements to derive optimal convergence rates. We conclude in Section 6 with simple numerical tests in 2D and 3D that illustrate the convergence properties of the proposed finite element schemes.

Preliminaries. Let Ω be a bounded domain in R^n with polyhedral boundary $\partial\Omega$, and let Σ and Γ be two disjoint open subsets of $\partial\Omega$, such that $\partial\Omega = \overline{\Sigma} \cup \overline{\Gamma}$. We denote by $\boldsymbol{\nu}$ the outward unit normal vector on $\partial\Omega$. In what follows, standard notation is adopted for Lebesgue spaces $\mathrm{L}^p(\Omega)$ and Sobolev spaces $\mathrm{W}^{s,p}(\Omega)$, with $s \in \mathrm{R}$ and $p \geqslant 1$, whose corresponding norms, either for the scalar, vectorial, or tensorial case, are denoted by $\|\cdot\|_{0,p;\Omega}$ and $\|\cdot\|_{s,p;\Omega}$, respectively. In particular, given a non-negative integer m, $\mathrm{W}^{m,2}(\Omega)$ is also denoted by $\mathrm{H}^m(\Omega)$, and the notations of its norm and seminorm are simplified to $\|\cdot\|_{m,\Omega}$ and $|\cdot|_{m,\Omega}$, respectively. On the other hand, given any generic scalar functional space S, we let $\mathbf{S}$ and $\mathbb{S}$ be the corresponding vector and tensor counterparts, whereas $\|\cdot\|$, with no subscripts, will be employed for the norm of any element or operator whenever there is no confusion about the space to which they belong. Also, $|\cdot|$ denotes the Euclidean norm in both R^n and $\mathrm{R}^{n\times n}$, and as usual, $\mathbb{I}$ stands for the identity tensor in $\mathrm{R}^{n\times n}$. In addition, for normed vector spaces X and Y, with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively, we endow the product space $X \times Y$ with the natural norm

$$\|(x,y)\|_{X\times Y} := \|x\|_X + \|y\|_Y \qquad \forall\,(x,y) \in X \times Y\,.$$

Unless otherwise stated, the duality pairing between X and its topological dual X' is denoted by $[\cdot,\cdot]$. Furthermore, given a linear operator $A : X \to Y'$, its transpose is the operator $A^{\mathtt{t}} : Y \to X'$ characterized by the relation $[A(x), y] = [A^{\mathtt{t}}(y), x]$, for every $(x, y) \in X \times Y$. In this context, an operator $A : X \to X'$ is said to be symmetric if $A = A^{\mathtt{t}}$ and positive semi-definite if $[A(x), x] \geqslant 0$, for all $x \in X$. Given a closed subspace S of X, we define the operator $\Pi_S : \mathcal{L}(X, X') \to \mathcal{L}(S, S')$ through the relation

$$[\Pi_S A(u), v] = [A(u), v] \qquad \forall\, u, v \in S, \quad \forall\, A \in \mathcal{L}(X, X')\,. \tag{1.1}$$

Also, given any vector fields $\mathbf{v} = (v_i)_{i=1,n}$ and $\mathbf{w} = (w_i)_{i=1,n}$, we set the gradient and divergence as

$$\nabla\mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,n} \quad \text{and} \quad \operatorname{div}(\mathbf{v}) := \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j}\,,$$

whereas for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,n}$, we let $\mathbf{div}(\boldsymbol{\tau})$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, and the tensor inner product, respectively, as

$$\boldsymbol{\tau}^{\mathtt{t}} := (\tau_{ji})_{i,j=1,n}, \quad \operatorname{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{n} \tau_{ii}, \quad \text{and} \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{n} \tau_{ij}\,\zeta_{ij}\,.$$

In addition, $\mathrm{H}^{1/2}(\partial\Omega)$ is the space of traces of functions of $\mathrm{H}^1(\Omega)$, $\mathrm{H}^{-1/2}(\partial\Omega)$ denotes its dual, and $\langle\cdot,\cdot\rangle_{\partial\Omega}$ stands for the duality pairing between $\mathrm{H}^{-1/2}(\partial\Omega)$ (resp $\mathbf{H}^{-1/2}(\partial\Omega)$) and $\mathrm{H}^{1/2}(\partial\Omega)$ (resp. $\mathbf{H}^{1/2}(\partial\Omega)$). Furthermore, $\mathrm{H}^{1/2}(\Gamma)$ is the space of functions in $\mathrm{H}^{1/2}(\partial\Omega)$ when restricted to Γ. In turn, $E_{\Gamma,0} : \mathrm{H}^{1/2}(\Gamma) \to \mathrm{L}^2(\partial\Omega)$ denotes the extension by zero on $\partial\Omega\backslash\overline{\Gamma}$, and we define the Hilbert space

$$\mathrm{H}^{1/2}_{00}(\Gamma) := \left\{\psi \in \mathrm{H}^{1/2}(\Gamma) : \quad E_{\Gamma,0}(\psi) \in \mathrm{H}^{1/2}(\partial\Omega)\right\},$$

which is endowed with the inner product

$$\langle\psi,\varphi\rangle_{1/2,00;\Gamma} := \langle E_{\Gamma,0}(\psi), E_{\Gamma,0}(\varphi)\rangle_{1/2,\partial\Omega} \qquad \forall\,\psi,\varphi \in \mathrm{H}^{1/2}_{00}(\Gamma)\,. \tag{1.2}$$

Additionally, we denote by $\mathrm{H}_{00}^{-1/2}(\Gamma)$ the dual of $\mathrm{H}_{00}^{1/2}(\Gamma)$, and employ $\langle\cdot,\cdot\rangle_\Gamma$ to denote the duality pairing between them. Throughout the paper $\|\cdot\|_{1/2,00;\Gamma}$ is the norm induced by (1.2) and $\|\cdot\|_{-1/2,00;\Gamma}$ denotes the norm of $\mathrm{H}_{00}^{-1/2}(\Gamma)$. For further details, we refer to [22]. Next, we introduce the standard Hilbert spaces

$$\begin{aligned}\mathbf{H}(\mathrm{div};\Omega) &:= \Big\{\boldsymbol{\chi}\in\mathbf{L}^2(\Omega): \quad \mathrm{div}(\boldsymbol{\chi})\in\mathrm{L}^2(\Omega)\Big\} \quad\text{and}\\ \mathbb{H}(\mathbf{div};\Omega) &:= \Big\{\boldsymbol{\tau}\in\mathbb{L}^2(\Omega): \quad \mathbf{div}(\boldsymbol{\tau})\in\mathbf{L}^2(\Omega)\Big\},\end{aligned}$$

which are endowed with their usual norms

$$\begin{aligned}\|\boldsymbol{\chi}\|_{\mathrm{div};\Omega} &:= \Big(\|\boldsymbol{\chi}\|_{0,\Omega}^2 + \|\mathrm{div}(\boldsymbol{\chi})\|_{0,\Omega}^2\Big)^{1/2} \qquad \forall\,\boldsymbol{\chi}\in\mathbf{H}(\mathrm{div};\Omega)\,, \quad\text{and}\\ \|\boldsymbol{\tau}\|_{\mathbf{div};\Omega} &:= \Big(\|\boldsymbol{\tau}\|_{0,\Omega}^2 + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,\Omega}^2\Big)^{1/2} \qquad \forall\,\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div};\Omega)\,.\end{aligned}$$

Additionally, we recall the integration by parts formulas (cf. [22, Lemma 1.4])

$$\begin{aligned}\langle\boldsymbol{\chi}\cdot\boldsymbol{\nu},w\rangle_{\partial\Omega} &= \int_\Omega\boldsymbol{\chi}\cdot\nabla w + \int_\Omega w\,\mathrm{div}(\boldsymbol{\chi}) \qquad \forall\,w\in\mathrm{H}^1(\Omega)\,, \quad \forall\,\boldsymbol{\chi}\in\mathbf{H}(\mathrm{div};\Omega)\,, \quad\text{and}\\ \langle\boldsymbol{\tau}\,\boldsymbol{\nu},\mathbf{w}\rangle_{\partial\Omega} &= \int_\Omega\boldsymbol{\tau}:\nabla\mathbf{w} + \int_\Omega\mathbf{w}\cdot\mathbf{div}(\boldsymbol{\tau}) \qquad \forall\,\mathbf{w}\in\mathbf{H}^1(\Omega)\,, \quad \forall\,\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div};\Omega)\,.\end{aligned} \tag{1.3}$$

Finally, in what follows, we denote by 0 or $\mathbf{0}$ the null element of any vector space, and we use C to represent a generic constant independent of the discretization parameters, which may take different values in different contexts.

2 The model problem

Consider a fully-saturated poroelastic medium composed of isotropic and homogeneous fluid and solid phases, represented by Ω. Under suitable physical conditions, the medium is primarily influenced by a body force $\mathbf{f}:\Omega\to\mathrm{R}^n$, and the linear momentum conservation is expressed as

$$\mathbf{div}(\boldsymbol{\sigma}) = -\mathbf{f} \quad\text{in}\quad \Omega\,, \tag{2.1}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor, which, in turn, is symmetric due to the angular momentum conservation. Furthermore, the effective stress principle relates $\boldsymbol{\sigma}$ to the fluid pressure $p:\Omega\to\mathrm{R}$ and the solid displacement $\mathbf{u}:\Omega\to\mathrm{R}^n$ via

$$\boldsymbol{\sigma} = \mathcal{C}\big(\boldsymbol{e}(\mathbf{u})\big) - \alpha\,p\,\mathbb{I} \quad\text{in}\quad \Omega\,, \tag{2.2}$$

where $\mathcal{C}$ denotes the elasticity operator acting on the strain tensor $\boldsymbol{e}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u}+(\nabla\mathbf{u})^{\mathtt{t}})$, and $\alpha\in[0,1]$ is the Biot–Willis coefficient. Under the assumption of a linearized regime, the generalized Hooke's law provides a simplified relationship between stress and strain. In fact, denoting by λ and μ the Lamé coefficients, the elasticity operator is given by

$$\mathcal{C}(\boldsymbol{\tau}) \;=\; 2\mu\,\boldsymbol{\tau} + \lambda\,\mathrm{tr}(\boldsymbol{\tau})\,\mathbb{I} \qquad \forall\,\boldsymbol{\tau}\in\mathbb{L}^2(\Omega).$$

On the other hand, it is known that the fluid content $\vartheta:\Omega\to\mathrm{R}$, resulting from saturation and local volume dilation, is given by

$$\vartheta = c_0\,p + \alpha\,\mathrm{div}\,\mathbf{u} = c_0\,p + \alpha\,\mathrm{tr}\,\boldsymbol{e}(\mathbf{u})\,, \tag{2.3}$$

where c_0 is the constrained specific storage coefficient (storativity). Under Darcy flow, given the resultant flow $g:\Omega\to\mathrm{R}$ and the intrinsic permeability relative to fluid viscosity of the flow in the medium κ, there holds $\partial_t \vartheta - \operatorname{div}(\kappa \nabla p) = g$. Here, we note that under an appropriate semi-discrete transformation it is sufficient to consider the stationary form

$$\vartheta - \operatorname{div}\big(\kappa \nabla p\big) = g\,. \tag{2.4}$$

We assume that there exist positive constants κ_0 and κ_1 such that

$$\kappa_0 \leqslant \kappa^{-1}(\mathbf{x}) \leqslant \kappa_1 \qquad \text{for a.e. } \mathbf{x} \in \Omega\,.$$

In order to reformulate the system given by (2.1), (2.2), (2.3), (2.4) and the symmetry of $\boldsymbol{\sigma}$ in a mixed form, we begin by denoting the vorticity tensor and the strain tensor by

$$\boldsymbol{\gamma} = \frac{1}{2}\big(\nabla \mathbf{u} - (\nabla \mathbf{u})^{\mathtt{t}}\big) \quad \text{and} \quad \boldsymbol{\xi} = \boldsymbol{e}(\mathbf{u}) \quad \text{in} \quad \Omega\,, \tag{2.5}$$

respectively. It is worth noting that (2.5), together with (2.2), are equivalent to requiring that $\boldsymbol{\gamma}$ is skew-symmetric and

$$\boldsymbol{\sigma} = \mathcal{C}(\boldsymbol{\xi}) - \alpha\, p\, \mathbb{I} \quad \text{and} \quad \nabla \mathbf{u} = \boldsymbol{\gamma} + \boldsymbol{\xi} \quad \text{in} \quad \Omega\,,$$

provided that the symmetry of $\boldsymbol{\sigma}$ is already enforced. In addition, from (2.3), we realize that the fluid content ϑ is completely determined by the pressure p and the strain $\boldsymbol{\xi}$. So, by replacing this relation into the flow equation (2.4), we get

$$c_0\, p + \alpha \operatorname{tr}(\boldsymbol{\xi}) - \operatorname{div}(\boldsymbol{\eta}) = g\,,$$

where we introduced $\boldsymbol{\eta} := \kappa \nabla p$ as a further unknown. This enables us to eliminate ϑ from the system and recover it afterwards from $\boldsymbol{\xi}$ and p, using (2.3). Therefore, as a result of the previous discussion, we are able to rewrite the initial system as: find $\mathbf{u}$, $\boldsymbol{\sigma}$, $\boldsymbol{\eta}$, p, $\boldsymbol{\xi}$, $\boldsymbol{\gamma}$ in suitable spaces to be specified below such that $\boldsymbol{\gamma}$ is skew-symmetric, $\boldsymbol{\xi}$ is symmetric, and

$$\begin{aligned}
\boldsymbol{\sigma} \;=\; \mathcal{C}(\boldsymbol{\xi}) - \alpha\, p\, \mathbb{I}, \quad \nabla \mathbf{u} \;=\; \boldsymbol{\gamma} + \boldsymbol{\xi} \quad \text{in} \quad \Omega \\
-\mathbf{div}(\boldsymbol{\sigma}) \;=\; \mathbf{f}, \quad \boldsymbol{\sigma} \;=\; \boldsymbol{\sigma}^{\mathtt{t}}, \quad \boldsymbol{\eta} \;=\; \kappa \nabla p \quad \text{in} \quad \Omega \\
c_0\, p + \alpha \operatorname{tr}(\boldsymbol{\xi}) - \operatorname{div}(\boldsymbol{\eta}) = g \quad \text{in} \quad \Omega\,.
\end{aligned} \tag{2.6}$$

Furthermore, the system is complemented by mixed boundary conditions, incorporating the given boundary data $\mathbf{u}_{\mathrm{D}}$ and g_{N},

$$\begin{aligned}
\mathbf{u} \;=\; \mathbf{u}_{\mathrm{D}} \quad \text{on} \quad \Gamma\,, \quad \boldsymbol{\sigma}\boldsymbol{\nu} \;=\; 0 \quad \text{on} \quad \Sigma\,, \\
p \;=\; 0 \quad \text{on} \quad \Sigma\,, \quad \boldsymbol{\eta} \cdot \boldsymbol{\nu} \;=\; g_{\mathrm{N}} \quad \text{on} \quad \Gamma\,.
\end{aligned} \tag{2.7}$$

3 Weak formulation and its solvability analysis

3.1 Variational formulation

In this section, we derive the weak formulation of problem (2.6), together with the boundary conditions (2.7). To this end, we first assume that $\mathbf{u} \in \mathbf{H}^1(\Omega)$, and test the second equation in (2.6) against a tensor field $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\Omega)$, thus obtaining

$$\int_\Omega \nabla \mathbf{u} : \boldsymbol{\tau} = \int_\Omega \boldsymbol{\gamma} : \boldsymbol{\tau} + \int_\Omega \boldsymbol{\xi} : \boldsymbol{\tau} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\Omega)\,, \tag{3.1}$$

from which we deduce that all terms are well defined, provided that $\boldsymbol{\gamma}$ and $\boldsymbol{\xi}$ are sought in $\mathbb{L}^2(\Omega)$. Moreover, motivated by the boundary conditions (2.7), we introduce the space

$$\mathbb{H}_\Sigma(\mathbf{div};\Omega) := \Big\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\Omega) \;:\; \boldsymbol{\tau}\boldsymbol{\nu} = 0 \quad \text{on} \quad \Sigma \Big\}\,,$$

and we require that $\mathbf{u}_{\mathrm{D}} \in \mathbf{H}^{1/2}(\Gamma)$, so that the following relation holds:

$$\langle \boldsymbol{\tau}\,\boldsymbol{\nu}, \mathbf{u}\rangle_{\partial\Omega} = \langle (\boldsymbol{\tau}\,\boldsymbol{\nu})|_{\Gamma}, \mathbf{u}|_{\Gamma}\rangle_{\partial\Omega} = \langle \boldsymbol{\tau}\,\boldsymbol{\nu}, \mathbf{u}_{\mathrm{D}}\rangle_{\Gamma} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}_{\Sigma}(\mathbf{div};\Omega)\,, \tag{3.2}$$

where the last term $\boldsymbol{\tau}\,\boldsymbol{\nu}$ is understood as a functional acting on $\mathbf{H}^{1/2}(\Gamma)$. In fact, while usually $\boldsymbol{\tau}\,\boldsymbol{\nu}|_{\Gamma}$ belongs to $\mathbf{H}_{00}^{-1/2}(\Gamma)$, the fact that it vanishes on Σ guarantees that actually $\boldsymbol{\tau}\,\boldsymbol{\nu}|_{\Gamma} \in \mathbf{H}^{-1/2}(\Gamma)$, and hence here we also use $\langle \cdot, \cdot \rangle_{\Gamma}$ to denote the duality pairing between $\mathbf{H}^{-1/2}(\Gamma)$ and $\mathbf{H}^{1/2}(\Gamma)$. Then, by applying integration by parts (cf. (1.3)) in (3.1) and then using (3.2), we arrive at

$$\int_{\Omega} \boldsymbol{\xi} : \boldsymbol{\tau} + \int_{\Omega} \mathbf{u} \cdot \mathbf{div}(\boldsymbol{\tau}) + \int_{\Omega} \boldsymbol{\gamma} : \boldsymbol{\tau} = \langle \boldsymbol{\tau}\,\boldsymbol{\nu}, \mathbf{u}_{\mathrm{D}}\rangle_{\Gamma} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}_{\Sigma}(\mathbf{div};\Omega)\,. \tag{3.3}$$

Notice that this equation makes sense even if $\mathbf{u} \in \mathbf{L}^2(\Omega)$ instead of $\mathbf{H}^1(\Omega)$. Moreover, we claim that seeking $\mathbf{u}$ in $\mathbf{L}^2(\Omega)$ is equivalent to doing so in $\mathbf{H}^1(\Omega)$. Indeed, if (3.3) holds for $\mathbf{u} \in \mathbf{L}^2(\Omega)$, then, by restricting the test functions to compactly supported smooth tensor fields, we recover that $\nabla \mathbf{u} = \boldsymbol{\xi} + \boldsymbol{\gamma}$ in $\mathbf{L}^2(\Omega)$, so $\mathbf{u} \in \mathbf{H}^1(\Omega)$. In addition, testing the latter with $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\Omega)$, integrating by parts, and invoking (3.3), we obtain $\mathbf{u}\big|_{\Gamma} = \mathbf{u}_{\mathrm{D}}$ and (3.1). According to this equivalence, from now on we shall seek $\mathbf{u} \in \mathbf{L}^2(\Omega)$. Arguing similarly for the fifth equation in (2.6), this time using that $p = 0$ on Σ (cf. (2.7)), we obtain

$$\int_{\Omega} \kappa^{-1}\,\boldsymbol{\eta} \cdot \boldsymbol{\chi} + \int_{\Omega} p\,\mathrm{div}(\boldsymbol{\chi}) = \langle \boldsymbol{\chi} \cdot \boldsymbol{\nu}, p\rangle_{\Gamma} \qquad \forall\, \boldsymbol{\chi} \in \mathbf{H}(\mathrm{div};\Omega)\,,$$

which, after introducing the further unknown $\varphi := p\big|_{\Gamma} \in \mathrm{H}_{00}^{1/2}(\Gamma)$, reads

$$\int_{\Omega} \kappa^{-1}\,\boldsymbol{\eta} \cdot \boldsymbol{\chi} + \int_{\Omega} p\,\mathrm{div}(\boldsymbol{\chi}) - \langle \boldsymbol{\chi} \cdot \boldsymbol{\nu}, \varphi\rangle_{\Gamma} = 0 \qquad \forall\, \boldsymbol{\chi} \in \mathbf{H}(\mathrm{div};\Omega)\,. \tag{3.4}$$

Here, we notice that, by a reasoning similar to the one made for (3.3), we may require that $p \in \mathrm{L}^2(\Omega)$ and $\boldsymbol{\eta} \in \mathbf{L}^2(\Omega)$. In turn, the boundary condition for $\boldsymbol{\eta}$ (cf. (2.7)) is enforced as a weak constraint via

$$\langle \boldsymbol{\eta} \cdot \boldsymbol{\nu}, \psi\rangle_{\Gamma} = \langle g_{\mathrm{N}}, \psi\rangle_{\Gamma} \qquad \forall\, \psi \in \mathrm{H}_{00}^{1/2}(\Gamma)\,. \tag{3.5}$$

Next, we test the first equation in (2.6) against a tensor field $\boldsymbol{\rho} \in \mathbb{L}^2(\Omega)$, thereby obtaining

$$-\int_{\Omega} \boldsymbol{\sigma} : \boldsymbol{\rho} + \int_{\Omega} \mathcal{C}(\boldsymbol{\xi}) : \boldsymbol{\rho} - \alpha \int_{\Omega} p\,\mathrm{tr}(\boldsymbol{\rho}) = 0 \qquad \forall\, \boldsymbol{\rho} \in \mathbb{L}^2(\Omega)\,, \tag{3.6}$$

from which it follows that $\boldsymbol{\sigma}$ must be sought in $\mathbb{L}^2(\Omega)$. Certainly, the symmetry of the Cauchy stress tensor (cf. (2.6)) could be enforced by restricting the trial space to symmetric tensors. Nevertheless, this choice is not optimal from an implementation standpoint, so we instead enforce the symmetry as a weak constraint, which is accomplished by imposing

$$\int_{\Omega} \boldsymbol{\sigma} : \boldsymbol{\delta} = 0 \qquad \forall\, \boldsymbol{\delta} \in \mathbb{L}^2_{\mathrm{skew}}(\Omega)\,, \tag{3.7}$$

where

$$\mathbb{L}^2_{\mathrm{skew}}(\Omega) := \Big\{ \boldsymbol{\delta} \in \mathbb{L}^2(\Omega) \;:\; \boldsymbol{\delta} = -\boldsymbol{\delta}^{\mathrm{t}} \Big\}\,.$$

Now, observe that if $\boldsymbol{\sigma}$ satisfies (3.7), then, by testing with $\mathbb{L}^2_{\mathrm{skew}}(\Omega)$ in (3.6), we deduce that $\boldsymbol{\xi}$ is symmetric. In turn, the skew-symmetry of $\boldsymbol{\gamma}$ is enforced by simply requiring that $\boldsymbol{\gamma} \in \mathbb{L}^2_{\mathrm{skew}}(\Omega)$. Next, we test the momentum balance (cf. third equation of (2.6)) against a vector field $\mathbf{v} \in \mathbf{L}^2(\Omega)$, formally obtaining

$$-\int_{\Omega} \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\sigma}) = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \qquad \forall\, \mathbf{v} \in \mathbf{L}^2(\Omega)\,. \tag{3.8}$$

We observe that the previous equation is well defined when $\mathbf{f}$ and $\mathbf{div}(\boldsymbol{\sigma})$ belong to $\mathbf{L}^2(\Omega)$, the latter implying that $\boldsymbol{\sigma} \in \mathbb{H}(\mathbf{div};\Omega)$. Furthermore, we note that integration by parts is not necessary, since the symmetry required for a saddle-point formulation is already satisfied (cf. (3.3)). Finally, we test the last equation of (2.6) with a scalar field $q \in \mathrm{L}^2(\Omega)$, obtaining

$$c_0 \int_\Omega p\,q + \alpha \int_\Omega q\,\mathrm{tr}(\boldsymbol{\xi}) - \int_\Omega q\,\mathrm{div}(\boldsymbol{\eta}) = \int_\Omega g\,q \qquad \forall\, q \in \mathrm{L}^2(\Omega)\,,$$

from which we observe that each term is well defined provided $g \in \mathrm{L}^2(\Omega)$ and $\mathrm{div}(\boldsymbol{\eta}) \in \mathrm{L}^2(\Omega)$, which in turn implies that $\boldsymbol{\eta} \in \mathbf{H}(\mathrm{div};\Omega)$. In this way, adding (3.4) with (3.6), and suitably combining (3.3) with (3.5) and (3.7) with (3.8), the system describing our problem is: find $(\boldsymbol{\eta},\boldsymbol{\xi}) \in \mathbf{H}(\mathrm{div};\Omega) \times \mathbb{L}^2(\Omega)$, $p \in \mathrm{L}^2(\Omega)$, $(\varphi,\boldsymbol{\sigma}) \in \mathrm{H}^{1/2}_{00}(\Gamma) \times \mathbb{H}_\Sigma(\mathbf{div};\Omega)$, and $(\mathbf{u},\boldsymbol{\gamma}) \in \mathbf{L}^2(\Omega) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega)$, all of them such that

$$\begin{aligned}
\int_\Omega \kappa^{-1}\boldsymbol{\eta}\cdot\boldsymbol{\chi} + \int_\Omega p\,\mathrm{div}(\boldsymbol{\chi}) - \langle \boldsymbol{\chi}\cdot\boldsymbol{\nu},\varphi\rangle_\Gamma - \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\rho} + \int_\Omega \mathcal{C}(\boldsymbol{\xi}) : \boldsymbol{\rho} - \alpha \int_\Omega p\,\mathrm{tr}(\boldsymbol{\rho}) &= 0 \\
-\alpha \int_\Omega q\,\mathrm{tr}(\boldsymbol{\xi}) + \int_\Omega q\,\mathrm{div}(\boldsymbol{\eta}) - c_0 \int_\Omega p\,q &= -\int_\Omega g\,q \\
-\int_\Omega \boldsymbol{\xi} : \boldsymbol{\tau} - \int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) - \int_\Omega \boldsymbol{\gamma} : \boldsymbol{\tau} - \langle \boldsymbol{\eta}\cdot\boldsymbol{\nu},\psi\rangle_\Gamma &= -\langle \boldsymbol{\tau}\,\boldsymbol{\nu},\mathbf{u}_{\mathrm{D}}\rangle_\Gamma - \langle g_{\mathrm{N}},\psi\rangle_\Gamma \\
-\int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\sigma}) - \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\delta} &= \int_\Omega \mathbf{f}\cdot\mathbf{v}
\end{aligned}$$

for all $(\boldsymbol{\chi},\boldsymbol{\rho}) \in \mathbf{H}(\mathrm{div};\Omega) \times \mathbb{L}^2(\Omega)$, $q \in \mathrm{L}^2(\Omega)$, $(\psi,\boldsymbol{\tau}) \in \mathrm{H}^{1/2}_{00}(\Gamma) \times \mathbb{H}_\Sigma(\mathbf{div};\Omega)$, and $(\mathbf{v},\boldsymbol{\delta}) \in \mathbf{L}^2(\Omega) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega)$. This system can be rewritten in terms of linear operators, yielding the previously announced threefold saddle-point structure. To this end, we define the spaces

$$\mathbf{H} := \mathbf{H}_1 \times \mathbb{H}_2\,, \quad \mathrm{Q} := \mathrm{L}^2(\Omega)\,, \quad \mathbf{X} := \mathrm{X}_1 \times \mathbb{X}_2\,, \quad \text{and} \quad \mathbf{Y} := \mathbf{Y}_1 \times \mathbb{Y}_2\,,$$

where

$$\mathbf{H}_1 := \mathbf{H}(\mathrm{div};\Omega)\,, \quad \mathbb{H}_2 := \mathbb{L}^2(\Omega)\,, \quad \mathrm{X}_1 := \mathrm{H}^{1/2}_{00}(\Gamma)\,,$$
$$\mathbb{X}_2 := \mathbb{H}_\Sigma(\mathbf{div};\Omega)\,, \quad \mathbf{Y}_1 := \mathbf{L}^2(\Omega)\,, \quad \mathbb{Y}_2 := \mathbb{L}^2_{\mathrm{skew}}(\Omega)\,,$$

and set the following notation for trial and test functions, respectively

$$\begin{aligned}
\vec{\boldsymbol{\eta}} &:= (\boldsymbol{\eta},\boldsymbol{\xi}) \in \mathbf{H}\,, & \vec{\varphi} &:= (\varphi,\boldsymbol{\sigma}) \in \mathbf{X}\,, & \vec{\mathbf{u}} &:= (\mathbf{u},\boldsymbol{\gamma}) \in \mathbf{Y}\,, \\
\vec{\boldsymbol{\chi}} &:= (\boldsymbol{\chi},\boldsymbol{\rho}) \in \mathbf{H}\,, & \vec{\psi} &:= (\psi,\boldsymbol{\tau}) \in \mathbf{X}\,, & \vec{\mathbf{v}} &:= (\mathbf{v},\boldsymbol{\delta}) \in \mathbf{Y}\,,
\end{aligned}$$

which allows us to rewrite our system as: find $(\,\vec{\boldsymbol{\eta}}\,,\,p\,,\,\vec{\varphi}\,,\,\vec{\mathbf{u}}\,) \in \mathbf{H} \times \mathrm{Q} \times \mathbf{X} \times \mathbf{Y}$ such that

$$\begin{pmatrix} \mathcal{A}_1 & \mathcal{B}_1^{\mathrm{t}} & \mathcal{B}_2^{\mathrm{t}} & \\ \mathcal{B}_1 & -\mathcal{D}_1 & & \\ \mathcal{B}_2 & & & \mathcal{B}_3^{\mathrm{t}} \\ & & \mathcal{B}_3 & \end{pmatrix} \begin{pmatrix} \vec{\boldsymbol{\eta}} \\ p \\ \vec{\varphi} \\ \vec{\mathbf{u}} \end{pmatrix} = \begin{pmatrix} \mathcal{F} \\ \mathcal{G} \\ \mathcal{H} \\ \mathcal{I} \end{pmatrix}, \tag{3.9}$$

where the linear operators $\mathcal{A}_1 : \mathbf{H} \to \mathbf{H}'$, $\mathcal{B}_1 : \mathbf{H} \to \mathrm{Q}'$, $\mathcal{B}_2 : \mathbf{H} \to \mathbf{X}'$, $\mathcal{B}_3 : \mathbf{X} \to \mathbf{Y}'$ and $\mathcal{D}_1 : \mathrm{Q} \to \mathrm{Q}'$ are given by

$$[\mathcal{A}_1(\vec{\boldsymbol{\eta}}),\vec{\boldsymbol{\chi}}] := \int_\Omega \kappa^{-1}\boldsymbol{\eta}\cdot\boldsymbol{\chi} + \int_\Omega \mathcal{C}(\boldsymbol{\xi}) : \boldsymbol{\rho}\,, \quad [\mathcal{B}_1(\vec{\boldsymbol{\chi}}),q] := \int_\Omega q\,\mathrm{div}(\boldsymbol{\chi}) - \alpha \int_\Omega q\,\mathrm{tr}(\boldsymbol{\rho})\,,$$
$$[\mathcal{B}_2(\vec{\boldsymbol{\chi}}),\vec{\psi}] := -\int_\Omega \boldsymbol{\rho} : \boldsymbol{\tau} - \langle \boldsymbol{\chi}\cdot\boldsymbol{\nu},\psi\rangle_\Gamma\,, \quad [\mathcal{B}_3(\vec{\psi}),\vec{\mathbf{v}}] := -\int_\Omega \mathbf{v}\cdot\mathrm{div}(\boldsymbol{\tau}) - \int_\Omega \boldsymbol{\tau} : \boldsymbol{\delta}\,,$$
$$[\mathcal{D}_1(r),q] := c_0 \int_\Omega r\,q\,,$$

whereas the linear functionals $\mathcal{F} \in \mathbf{H}'$, $\mathcal{G} \in \mathrm{Q}'$, $\mathcal{H} \in \mathbf{X}'$ and $\mathcal{I} \in \mathbf{Y}'$ are given by

$$[\mathcal{F}, \vec{\chi}] := 0\,, \quad [\mathcal{G}, q] := -\int_\Omega g\, q\,, \quad [\mathcal{H}, \vec{\psi}\,] := -\langle \boldsymbol{\tau}\,\boldsymbol{\nu}, \mathbf{u}_{\mathrm{D}}\rangle_\Gamma - \langle g_{\mathrm{N}}, \psi\rangle_\Gamma\,, \quad \text{and} \quad [\mathcal{I}, \vec{\mathbf{v}}] := \int_\Omega \mathbf{f}\cdot\mathbf{v}\,,$$

respectively, for arbitrary inputs in their respective spaces. Furthermore, by applying the Cauchy–Schwarz inequality and bearing in mind the continuity of the normal trace operators $\gamma_{\boldsymbol{\nu}} : \mathbf{H}(\mathrm{div};\Omega) \to \mathrm{H}^{-1/2}(\partial\Omega)$ and $\boldsymbol{\gamma}_{\boldsymbol{\nu}} : \mathbb{H}(\mathbf{div};\Omega) \to \mathbf{H}^{-1/2}(\partial\Omega)$, we note that the foregoing operators and functionals satisfy the following stability properties:

$$\begin{gathered}
\|\mathcal{A}_1\| \leqslant \kappa_1^{-1} + 2\mu + \lambda\,, \quad \|\mathcal{B}_1\| \leqslant 1 + \alpha\,, \quad \|\mathcal{B}_2\| \leqslant 1 + \|\gamma_{\boldsymbol{\nu}}\|\,, \quad \|\mathcal{B}_3\| \leqslant \sqrt{2}\,, \quad \|\mathcal{D}_1\| \leqslant c_0\,, \\
\|\mathcal{F}\| = 0\,, \quad \|\mathcal{G}\| \leqslant \|g\|_{0,\Omega}\,, \quad \|\mathcal{H}\| \leqslant \|g_{\mathrm{N}}\|_{-1/2,00;\Gamma} + \|\boldsymbol{\gamma}_{\boldsymbol{\nu}}\|\,\|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} \quad \text{and} \quad \|\mathcal{I}\| \leqslant \|\mathbf{f}\|_{0,\Omega}\,.
\end{gathered} \tag{3.10}$$

3.2 A new abstract result

In this section, we establish an abstract result on the well-posedness of threefold saddle-point problems, which slightly extends the structure of (3.9). We emphasize that, although our problem is posed in Hilbert spaces, it is possible—without introducing unnecessary complications—to state the result in the broader setting of reflexive Banach spaces, which ensures greater generality and reusability. Our approach to proving well-posedness relies heavily on Babuška–Brezzi theory in combination with a recent result concerning perturbed saddle-point problems in reflexive Banach spaces (cf. [18]). This strategy has also been frequently applied in related contexts, namely, simple and twofold saddle-point problems, with or without perturbations (see, for instance, [14, 8, 31, 24, 15]).

We let H, Q, X and Y be reflexive real Banach spaces, and let $\mathbf{a} : H \to H'$, $\mathbf{b} : H \to Q'$, $\mathbf{c} : Q \to Q'$, $\mathbf{B} : H \times Q \to X'$, and $\mathbf{D} : H \times Q \times X \to Y'$ be linear and bounded operators. We aim to establish sufficient conditions to the well-posedness of the following problem: Given $(F, G, I, J) \in H' \times Q' \times X' \times Y'$, find $(u, \sigma, p, \psi) \in H \times Q \times X \times Y$ such that

$$\left(\begin{array}{cc|c|c}
\mathbf{a} & \mathbf{b}^{\mathrm{t}} & \mathbf{B}^{\mathrm{t}} & \mathbf{D}^{\mathrm{t}} \\
\mathbf{b} & -\mathbf{c} & & \\ \hline
\multicolumn{2}{c|}{\mathbf{B}} & & \\ \hline
\multicolumn{3}{c|}{\mathbf{D}} &
\end{array}\right)
\begin{pmatrix} u \\ \sigma \\ p \\ \psi \end{pmatrix} = \begin{pmatrix} F \\ G \\ I \\ J \end{pmatrix}. \tag{3.11}$$

To that end, we begin by defining the linear and bounded operators $\mathbf{A}_1 : H \times Q \to H' \times Q'$, $\mathbf{A}_2 : H \times Q \times X \to H' \times Q' \times X'$, and $\mathbf{A}_3 : H \times Q \times X \times Y \to H' \times Q' \times X' \times Y'$, as

$$\mathbf{A}_1 := \left(\begin{array}{c|c} \mathbf{a} & \mathbf{b}^{\mathrm{t}} \\ \hline \mathbf{b} & -\mathbf{c} \end{array}\right), \quad \mathbf{A}_2 := \left(\begin{array}{c|c} \mathbf{A}_1 & \mathbf{B}^{\mathrm{t}} \\ \hline \mathbf{B} & \end{array}\right), \quad \text{and} \quad \mathbf{A}_3 := \left(\begin{array}{c|c} \mathbf{A}_2 & \mathbf{D}^{\mathrm{t}} \\ \hline \mathbf{D} & \end{array}\right).$$

We then observe that studying the well-posedness of (3.11) reduces to analyzing the bijectivity of the operator $\mathbf{A}_3$. In this context, we recall from Babuška–Brezzi theory (cf. [20, Theorem 49.12]) that $\mathbf{A}_3$ is an isomorphism if and only if $\mathbf{D}$ is surjective and $\mathbf{A}_2$ defines an isomorphism from the kernel of $\mathbf{D}$ to its dual. More precisely, defining

$$\mathcal{K} := \Big\{(v, \tau, q) \in H \times Q \times X : \quad \mathbf{D}(v, \tau, q) = 0\Big\}\,, \tag{3.12}$$

we have that $\mathbf{A}_3$ is an isomorphism if and only if (cf. (1.1)) $\Pi_{\mathcal{K}}\mathbf{A}_2 : \mathcal{K} \to \mathcal{K}'$ is an isomorphism and $\mathbf{D}$ satisfies the inf-sup condition

$$\sup_{0 \neq (v,\tau,q) \in H \times Q \times X} \frac{[\mathbf{D}(v, \tau, q), \varphi]}{\|(v, \tau, q)\|} \geqslant \beta_3\, \|\varphi\| \qquad \forall\, \varphi \in Y\,, \tag{3.13}$$

for some positive constant β_3. Frequent use of the definition given by (1.1) is made in this section. Now, looking at the definition of $\mathbf{A}_2$, one would like to apply the same result to establish equivalent conditions to the bijectivity of the operator. This is possible only if we have that $\mathcal{K}$ has a product structure. In fact, assuming that $\mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_2 \times \mathcal{K}_3$, where $\mathcal{K}_1$, $\mathcal{K}_2$ and $\mathcal{K}_3$ are subspaces of H, Q and X, respectively, we have that $\Pi_{\mathcal{K}}\mathbf{A}_2$ is an isomorphism if and only if $\Pi_{\mathcal{V}}\mathbf{A}_1 : \mathcal{V} \to \mathcal{V}'$ is an isomorphism, where

$$\mathcal{V} := \Big\{ (v,\tau) \in \mathcal{K}_1 \times \mathcal{K}_2 : \quad [\mathbf{B}(v,\tau), q] = 0 \quad \forall\, q \in \mathcal{K}_3 \Big\}, \tag{3.14}$$

and there exists a positive constant β_2 such that

$$\sup_{0 \neq (v,\tau) \in \mathcal{K}_1 \times \mathcal{K}_2} \frac{[\mathbf{B}(v,\tau), q]}{\|(v,\tau)\|} \geqslant \beta_2 \, \|q\| \qquad \forall\, q \in \mathcal{K}_3 \,. \tag{3.15}$$

It only remains to translate the bijectivity condition of $\Pi_{\mathcal{V}}\mathbf{A}_1$. Noting that the structure of $\mathbf{A}_1$ differs from that of $\mathbf{A}_2$ and $\mathbf{A}_3$, due to the presence of a perturbation term, we cannot directly apply the previous result. Instead, we rely on [18, Theorem 3.4], which provides sufficient conditions for the bijectivity of an operator with this structure. In particular, we first observe that $\Pi_{\mathcal{V}}\mathbf{A}_1 : \mathcal{V} \to \mathcal{V}'$ is an isomorphism if and only if, for each $(\mathcal{F}, \mathcal{G}) \in \mathcal{V}'$, there exists a unique solution to the following problem: find $(u,\sigma) \in \mathcal{V}$ such that

$$\begin{aligned} \mathbf{a}(u) + \mathbf{b}^{\mathbf{t}}(\sigma) &= \mathcal{F}\,, \\ \mathbf{b}(u) - \mathbf{c}(\sigma) &= \mathcal{G}\,. \end{aligned}$$

Now, notice that, in order to apply [18, Theorem 3.4], $\mathcal{V}$ must possess a product structure. We then suppose that $\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2$, where $\mathcal{V}_1$ and $\mathcal{V}_2$ are subspaces of $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively, and define

$$\mathcal{W} := \Big\{ v \in \mathcal{V}_1 : \quad [\mathbf{b}(v), \tau] = 0 \quad \forall\, \tau \in \mathcal{V}_2 \Big\}. \tag{3.16}$$

Thus, by collecting the conditions mentioned earlier (cf. (3.13) and (3.15)), and by employing [18, Theorem 3.4], we arrive at the following result.

Theorem 3.1. *Let H, Q, X and Y be reflexive real Banach spaces, and let $\mathbf{a} : H \to H'$, $\mathbf{b} : H \to Q'$, $\mathbf{c} : Q \to Q'$, $\mathbf{B} : H \times Q \to X'$, and $\mathbf{D} : H \times Q \times X \to Y'$ be given linear and bounded operators. In addition, let us suppose that the kernel $\mathcal{K}$ of $\mathbf{D}$ (cf. (3.12)) can be written as a product space $\mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_2 \times \mathcal{K}_3$, where $\mathcal{K}_1$, $\mathcal{K}_2$ and $\mathcal{K}_3$ are subspaces of H, Q and X, respectively. Define $\mathcal{V}$ as in (3.14) and suppose that $\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2$, where $\mathcal{V}_1$ and $\mathcal{V}_2$ are subspaces of $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively. Furthermore, define $\mathcal{W}$ as in (3.16), and assume that the following conditions hold:*

(i) There exists a positive constant β_3 such that

$$\sup_{0 \neq (v,\tau,q) \in H \times Q \times X} \frac{[\mathbf{D}(v,\tau,q), \varphi]}{\|(v,\tau,q)\|} \geqslant \beta_3 \, \|\varphi\| \qquad \forall\, \varphi \in Y \,.$$

(ii) There exists a positive constant β_2 such that

$$\sup_{0 \neq (v,\tau) \in \mathcal{K}_1 \times \mathcal{K}_2} \frac{[\mathbf{B}(v,\tau), q]}{\|(v,\tau)\|} \geqslant \beta_2 \, \|q\| \qquad \forall\, q \in \mathcal{K}_3 \,.$$

(iii) There exists a positive constant β_1 such that

$$\sup_{0 \neq v \in \mathcal{V}_1} \frac{[\mathbf{b}(v), q]}{\|v\|} \geqslant \beta_1 \, \|q\| \qquad \forall\, q \in \mathcal{V}_2 \,.$$

(iv) There exists a positive constant α such that

$$\sup_{0 \neq \tau \in \mathcal{W}} \frac{[\mathbf{a}(v), \tau]}{\|\tau\|} \geqslant \alpha \, \|v\| \qquad \forall\, v \in \mathcal{W} \,.$$

(v) $\mathbf{a}$ *and* $\mathbf{c}$ *are symmetric and positive semi-definite in* $\mathcal{V}_1$*.*

Then, for each $(F,G,I,J) \in H' \times Q' \times X' \times Y'$*, there exists a unique* $(u,\sigma,p,\psi) \in H \times Q \times X \times Y$ *solution to* (3.11)*. Moreover, there exists a positive constant* C*, depending only on* α*,* β_1*,* β_2*,* β_3*,* $\|\mathbf{a}\|$*,* $\|\mathbf{b}\|$*,* $\|\mathbf{c}\|$*,* $\|\mathbf{B}\|$*, and* $\|\mathbf{D}\|$*, such that*

$$\|(u,\sigma,p,\psi)\|_{H\times Q\times X\times Y} \leqslant C\,\|(F,G,I,J)\|_{H'\times Q'\times X'\times Y'}\,. \tag{3.17}$$

Proof. The existence and uniqueness of solution follows from the previous discussion. In turn, (3.17) can be derived by using the *a priori* estimates given by [18, Theorem 3.4]. Alternatively, one can use the inf-sup conditions and perform standard techniques to arrive at the same estimate. □

We remark that in the previous discussion we could instead have considered [18, Theorem 3.1], which covers a different spectrum of problems, namely, when the transpose map $\mathbf{b}^{\mathtt{t}}$ has nontrivial kernel inside $\mathcal{V}_2$. This is not under the scope of this work, as our problem does not fall in this category.

3.3 Well-posedness of the continuous problem

In this section, we prove that (3.9) is well-posed by employing Theorem 3.1. For this purpose, we first define $\mathcal{K}_3$ as the kernel space of $\mathcal{B}_3$, which is characterized by

$$\mathcal{K}_3 = \mathrm{H}^{1/2}_{00}(\Gamma) \times \Big\{\boldsymbol{\tau} \in \mathbb{H}_\Sigma(\mathbf{div};\Omega):\ \boldsymbol{\tau} \in \mathbb{L}^2_{\mathrm{sym}}(\Omega) \quad\text{and}\quad \mathbf{div}(\boldsymbol{\tau}) = 0\Big\}, \tag{3.18}$$

where

$$\mathbb{L}^2_{\mathrm{sym}}(\Omega) := \Big\{\boldsymbol{\tau} \in \mathbb{L}^2(\Omega)\ :\ \ \boldsymbol{\tau}^{\mathtt{t}} = \boldsymbol{\tau}\Big\}.$$

In addition, we define the spaces

$$\mathcal{V} := \Big\{\vec{\boldsymbol{\chi}} \in \mathbf{H}\ :\ \ \big[\mathcal{B}_2(\vec{\boldsymbol{\chi}}),\vec{\psi}\,\big] = 0 \quad \forall\,\vec{\psi} \in \mathcal{K}_3\Big\} \quad\text{and}\quad \mathcal{W} := \Big\{\vec{\boldsymbol{\chi}} \in \mathcal{V}\ :\ \ [\mathcal{B}_1(\vec{\boldsymbol{\chi}}),q] = 0 \quad \forall\,q \in \mathrm{Q}\Big\},$$

which, denoting by $\mathcal{K}_3^{(2)}$ the second-component space of $\mathcal{K}_3$ (cf. (3.18)), can be characterized by

$$\begin{gathered}\mathcal{V} = \Big\{\vec{\boldsymbol{\chi}} := (\boldsymbol{\chi},\boldsymbol{\rho}) \in \mathbf{H}\ :\ \ \boldsymbol{\chi}\cdot\boldsymbol{\nu} = 0 \quad\text{on}\quad \Gamma \quad\text{and}\quad \int_\Omega \boldsymbol{\rho} : \boldsymbol{\tau} = 0 \quad \forall\,\boldsymbol{\tau} \in \mathcal{K}_3^{(2)}\Big\},\\ \text{and}\quad \mathcal{W} = \Big\{\vec{\boldsymbol{\chi}} := (\boldsymbol{\chi},\boldsymbol{\rho}) \in \mathcal{V}:\ \ \mathrm{div}(\boldsymbol{\chi}) = \alpha\,\mathrm{tr}(\boldsymbol{\rho})\Big\}.\end{gathered} \tag{3.19}$$

With these definitions in place, we begin our analysis by verifying the hypotheses of Theorem 3.1. First, observe that the splitting condition on $\mathcal{K}$ established in the previous section is trivially satisfied, as $\mathcal{B}_3$ acts only in $\mathbf{X}$. The following results establish the inf-sup conditions for $\mathcal{B}_3$, $\mathcal{B}_2$, and $\mathcal{B}_1$, respectively.

Lemma 3.2. *There exists a positive constant* β_3*, depending only on* Ω*, such that*

$$\sup_{0\neq\vec{\psi}\in\mathbf{X}} \frac{\big[\mathcal{B}_3(\vec{\psi}),\vec{\mathbf{v}}\,\big]}{\|\vec{\psi}\|_{\mathbf{X}}} \geqslant \beta_3\,\|\vec{\mathbf{v}}\|_{\mathbf{Y}} \qquad \forall\,\vec{\mathbf{v}} \in \mathbf{Y}.$$

Proof. It follows from a slightly modification of [22, Section 2.4.3.1]. □

Lemma 3.3. *There exists a positive constant* β_2*, depending only on* Ω*, such that*

$$\sup_{0\neq\vec{\boldsymbol{\chi}}\in\mathbf{H}} \frac{\big[\mathcal{B}_2(\vec{\boldsymbol{\chi}}),\vec{\psi}\,\big]}{\|\vec{\boldsymbol{\chi}}\|_{\mathbf{H}}} \geqslant \beta_2\,\|\vec{\psi}\|_{\mathbf{X}} \qquad \forall\,\vec{\psi} \in \mathcal{K}_3\,.$$

Proof. Given $\vec{\psi} := (\psi, \boldsymbol{\tau}) \in \mathcal{K}_3$, by the characterization given in (3.18), we have that $\|\boldsymbol{\tau}\|_{0,\Omega} = \|\boldsymbol{\tau}\|_{\mathbf{div};\Omega}$. Then, by choosing $\vec{\chi} = (0, \boldsymbol{\tau}) \in \mathbf{H}(\mathrm{div};\Omega) \times \mathbb{L}^2(\Omega)$ in the supremum, we obtain

$$\sup_{0 \neq \vec{\chi} \in \mathbf{H}} \frac{\big[\mathcal{B}_2(\vec{\chi}), \vec{\psi}\,\big]}{\|\vec{\chi}\|_{\mathbf{H}}} \geqslant \frac{\|\boldsymbol{\tau}\|_{0,\Omega}^2}{\|\boldsymbol{\tau}\|_{\mathbf{div};\Omega}} = \|\boldsymbol{\tau}\|_{\mathbf{div};\Omega}\,. \tag{3.20}$$

Similarly, taking $\vec{\chi} = (\boldsymbol{\chi}, 0)$ in the supremum, with arbitrary $\boldsymbol{\chi} \in \mathbf{H}(\mathrm{div};\Omega)$, leads to

$$\sup_{0 \neq \vec{\chi} \in \mathbf{H}} \frac{\big[\mathcal{B}_2(\vec{\chi}), \vec{\psi}\,\big]}{\|\vec{\chi}\|_{\mathbf{H}}} \geqslant \sup_{0 \neq \boldsymbol{\chi} \in \mathbf{H}(\mathrm{div};\Omega)} \frac{\langle \boldsymbol{\chi} \cdot \boldsymbol{\nu}, \psi \rangle_\Gamma}{\|\boldsymbol{\chi}\|_{\mathrm{div};\Omega}}. \tag{3.21}$$

We focus on bounding the right-hand side. To do so, we consider the following variational problem: Find $z \in \mathrm{H}^1_\Sigma(\Omega) := \big\{ w \in \mathrm{H}^1(\Omega) : \ w = 0 \ \text{ on } \ \Sigma \big\}$ such that

$$\int_\Omega \nabla z \cdot \nabla w = \big\langle \mathcal{R}_{00}^{-1}(\psi), w \big\rangle_\Gamma \qquad \forall\, w \in \mathrm{H}^1_\Sigma(\Omega)\,, \tag{3.22}$$

where $\mathcal{R}_{00} : \mathrm{H}_{00}^{-1/2}(\Gamma) \to \mathrm{H}_{00}^{1/2}(\Gamma)$ is the corresponding Riesz isomorphism. Since $\mathrm{H}^1_\Sigma(\Omega)$ is a Hilbert space, and owing to the well-known Poincaré inequality and the Lax–Milgram lemma, there exist a unique $z \in \mathrm{H}^1_\Sigma(\Omega)$ solution to (3.22), and a positive constant C_2 depending only on Ω, such that

$$|z|_{1,\Omega} \leqslant C_2\, \|\psi\|_{1/2,00;\Gamma}\,.$$

We then define $\widehat{\boldsymbol{\chi}} := \nabla z \in \mathbf{L}^2(\Omega)$, and observe from (3.22) that $\mathrm{div}(\widehat{\boldsymbol{\chi}}) = 0$, which yields $\widehat{\boldsymbol{\chi}} \in \mathbf{H}(\mathrm{div};\Omega)$, and

$$\|\widehat{\boldsymbol{\chi}}\|_{\mathrm{div};\Omega} = \|\widehat{\boldsymbol{\chi}}\|_{0,\Omega} = |z|_{1,\Omega} \leqslant C_2\, \|\psi\|_{1/2,00;\Gamma}\,. \tag{3.23}$$

Moreover, it is also clear from (3.22) that $\widehat{\boldsymbol{\chi}} \cdot \boldsymbol{\nu} = \mathcal{R}_{00}^{-1}(\psi)$ on Γ, and hence, using (3.23) we can assert that

$$\sup_{0 \neq \boldsymbol{\chi} \in \mathbf{H}(\mathrm{div};\Omega)} \frac{\langle \boldsymbol{\chi} \cdot \boldsymbol{\nu}, \psi \rangle_\Gamma}{\|\boldsymbol{\chi}\|_{\mathrm{div};\Omega}} \geqslant \frac{\langle \widehat{\boldsymbol{\chi}} \cdot \boldsymbol{\nu}, \psi \rangle_\Gamma}{\|\widehat{\boldsymbol{\chi}}\|_{\mathrm{div};\Omega}} = \frac{\big\langle \mathcal{R}_{00}^{-1}(\psi), \psi \big\rangle_\Gamma}{\|\widehat{\boldsymbol{\chi}}\|_{\mathrm{div};\Omega}} = \frac{\|\psi\|_{1/2,00;\Gamma}^2}{\|\widehat{\boldsymbol{\chi}}\|_{\mathrm{div};\Omega}} \geqslant C_2^{-1}\, \|\psi\|_{1/2,00;\Gamma}\,, \tag{3.24}$$

so that, replacing (3.24) in (3.21), and then adding it to (3.20), we arrive at the desired inf-sup condition with $\beta_2 = \frac{1}{2}\,(1 + C_2^{-1})$. □

Lemma 3.4. *There exists a positive constant β_1, depending only on Ω, such that*

$$\sup_{0 \neq \vec{\chi} \in \mathcal{V}} \frac{\big[\mathcal{B}_1(\vec{\chi}), q\big]}{\|\vec{\chi}\|_{\mathbf{H}}} \geqslant \beta_1\, \|q\|_{\mathrm{Q}} \qquad \forall\, q \in \mathrm{Q}\,.$$

Proof. Given $q \in \mathrm{Q}$, we consider the following boundary value problem:

$$\Delta z = q \quad \text{in} \quad \Omega\,, \quad z = 0 \quad \text{on} \quad \Sigma\,, \quad \nabla z \cdot \boldsymbol{\nu} = 0 \quad \text{on} \quad \Gamma\,,$$

whose unique weak solution $z \in \mathrm{H}^1(\Omega)$ satisfies the estimate $\|z\|_{1,\Omega} \leqslant C_1\, \|q\|_{0,\Omega}$ (cf. [22, Chapter 2.4.2]), for some positive constant C_1 depending only on the domain Ω. Then, define $\widetilde{\boldsymbol{\chi}} := \nabla z$, and note that $\mathrm{div}(\widetilde{\boldsymbol{\chi}}) = q \in \mathrm{L}^2(\Omega)$, which implies $\widetilde{\boldsymbol{\chi}} \in \mathbf{H}(\mathrm{div};\Omega)$. Moreover, since $\widetilde{\boldsymbol{\chi}} \cdot \boldsymbol{\nu} = 0$ on Γ, we deduce that $(\widetilde{\boldsymbol{\chi}}, 0) \in \mathcal{V}$ (cf. (3.19)), and from the *a priori* estimate of our auxiliary problem, there holds

$$\|\widetilde{\boldsymbol{\chi}}\|_{\mathrm{div};\Omega}^2 = |z|_{1,\Omega}^2 + \|q\|_{0,\Omega}^2 \leqslant (C_1^2 + 1)\, \|q\|_{0,\Omega}^2\,.$$

Hence, by taking $\vec{\chi} = (\widetilde{\boldsymbol{\chi}}, 0) \in \mathcal{V}$ in the supremum, we obtain that

$$\sup_{0 \neq \vec{\chi} \in \mathcal{V}} \frac{\big[\mathcal{B}_1(\vec{\chi}), q\big]}{\|\vec{\chi}\|_{\mathbf{H}}} \geqslant \frac{\displaystyle\int_\Omega q\, \mathrm{div}(\widetilde{\boldsymbol{\chi}})}{\|\widetilde{\boldsymbol{\chi}}\|_{\mathrm{div};\Omega}} = \frac{\|q\|_{0,\Omega}^2}{\|\widetilde{\boldsymbol{\chi}}\|_{\mathrm{div};\Omega}} \geqslant \frac{1}{(C_1^2 + 1)^{1/2}}\, \|q\|_{0,\Omega}\,,$$

which completes the proof with $\beta_1 = (1 + C_1^2)^{-1/2}$. □

Lemma 3.5. *$\mathcal{A}_1$ and $\mathcal{D}_1$ are symmetric and positive semi-definite. Moreover, there exists a positive constant α_1, depending only on κ_0, λ, α and μ, such that*

$$[\mathcal{A}_1(\vec{\boldsymbol{\chi}}), \vec{\boldsymbol{\chi}}] \geqslant \alpha_1 \|\vec{\boldsymbol{\chi}}\|^2_{\mathbf{H}} \qquad \forall\, \vec{\boldsymbol{\chi}} \in \mathcal{W}\,.$$

Proof. We first observe that $\mathcal{A}_1$ and $\mathcal{D}_1$ are symmetric and the latter is positive semi-definite. Thereby, we focus on proving that $\mathcal{A}_1$ is coercive in $\mathcal{W}$. In fact, given $\vec{\boldsymbol{\chi}} \in \mathbf{H}$, we have

$$[\mathcal{A}_1(\vec{\boldsymbol{\chi}}), \vec{\boldsymbol{\chi}}] = \int_\Omega \kappa^{-1} |\boldsymbol{\chi}|^2 + \int_\Omega \big(2\mu\,\boldsymbol{\rho} + \lambda\,\mathrm{tr}(\boldsymbol{\rho})\,\mathbb{I}\big) : \boldsymbol{\rho} \geqslant \kappa_0 \|\boldsymbol{\chi}\|^2_{0,\Omega} + 2\mu\|\boldsymbol{\rho}\|^2_{0,\Omega} + \lambda\,\|\mathrm{tr}(\boldsymbol{\rho})\|^2_{0,\Omega}\,, \tag{3.25}$$

which means that $\mathcal{A}_1$ is also positive semi-definite. Moreover, if $\vec{\boldsymbol{\chi}} \in \mathcal{W}$, we have, in particular, that $\mathrm{div}(\boldsymbol{\chi}) = \alpha\,\mathrm{tr}(\boldsymbol{\rho})$. Putting this into (3.25) yields

$$[\mathcal{A}_1(\vec{\boldsymbol{\chi}}), \vec{\boldsymbol{\chi}}] \geqslant \kappa_0\,\|\boldsymbol{\chi}\|^2_{0,\Omega} + 2\mu\|\boldsymbol{\rho}\|^2_{0,\Omega} + \frac{\lambda}{\alpha^2}\,\|\mathrm{div}(\boldsymbol{\chi})\|^2_{0,\Omega} \geqslant \min\{\kappa_0, \lambda\,\alpha^{-2}, 2\mu\}\,\|\vec{\boldsymbol{\chi}}\|^2_{\mathbf{H}}\,,$$

thereby proving the coercivity of $\mathcal{A}$ in $\mathcal{W}$ with $\alpha_1 = \min\{\kappa_0, \lambda\,\alpha^{-2}, 2\mu\}$. □

Theorem 3.6. *The problem (3.9) has a unique solution $(\vec{\boldsymbol{\eta}}, p, \vec{\varphi}, \vec{\mathbf{u}}) \in \mathbf{H} \times \mathrm{Q} \times \mathbf{X} \times \mathbf{Y}$. Moreover, there exists a positive constant C depending only on κ_0, κ_1, μ, λ, α, c_0 and Ω, such that*

$$\|(\vec{\boldsymbol{\eta}}, p, \vec{\varphi}, \vec{\mathbf{u}})\| \leqslant C \Big\{ \|g\|_{0,\Omega} + \|g_{\mathrm{N}}\|_{-1/2,00;\Gamma} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,\Omega} \Big\}\,.$$

Proof. Owing to Lemmas 3.2, 3.3, 3.4, and 3.5, the assumptions of Theorem 3.1 are satisfied. Therefore, the desired result follows, with the *a priori* estimate also stemming from the operators' stability (cf. (3.10)). □

4 The Galerkin scheme

We first let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of triangulations of $\overline{\Omega}$ made up of triangles K (when $n = 2$) or tetrahedra K (when $n = 3$) of diameter h_K, and set $h := \max\{h_K :\ K \in \mathcal{T}_h\}$. Given an integer $k \geqslant 0$ and a subset S of R^n, we denote by $\mathrm{P}_k(S)$ the space of polynomials of total degree at most k defined on S, and $\mathbf{P}_k(S)$ its vector counterpart. We also let $\mathbf{H}^{\boldsymbol{\eta}}_h$, $\mathbb{H}^{\boldsymbol{\xi}}_h$, Q_h, X^{φ}_h, $\mathbb{X}^{\boldsymbol{\sigma}}_h$, $\mathbf{Y}^{\mathbf{u}}_h$ and $\mathbb{Y}^{\gamma}_h$ be arbitrary finite element subspaces of $\mathbf{H}(\mathrm{div};\Omega)$, $\mathbb{L}^2(\Omega)$, $\mathrm{L}^2(\Omega)$, $\mathrm{H}^{1/2}_{00}(\Gamma)$, $\mathbb{H}_\Sigma(\mathbf{div},\Omega)$, $\mathbf{L}^2(\Omega)$ and $\mathbb{L}^2_{\mathrm{skew}}(\Omega)$, respectively, all of them endowed with the corresponding subspace topology. Similarly to the continuous case, we define

$$\mathbf{H}_h := \mathbf{H}^{\boldsymbol{\eta}}_h \times \mathbb{H}^{\boldsymbol{\xi}}_h\,, \qquad \mathbf{X}_h := \mathrm{X}^{\varphi}_h \times \mathbb{X}^{\boldsymbol{\sigma}}_h\,, \qquad \text{and} \qquad \mathbf{Y}_h := \mathbf{Y}^{\mathbf{u}}_h \times \mathbb{Y}^{\gamma}_h\,,$$

and introduce the notation

$$\begin{aligned} &\vec{\boldsymbol{\eta}}_h := (\boldsymbol{\eta}_h, \boldsymbol{\xi}_h) \in \mathbf{H}_h\,, && \vec{\varphi}_h := (\varphi_h, \boldsymbol{\sigma}_h) \in \mathbf{X}_h\,, && \vec{\mathbf{u}}_h := (\mathbf{u}_h, \gamma_h) \in \mathbf{Y}_h\,,\\ &\vec{\boldsymbol{\chi}}_h := (\boldsymbol{\chi}_h, \boldsymbol{\rho}_h) \in \mathbf{H}_h\,, && \vec{\psi}_h := (\psi_h, \boldsymbol{\tau}_h) \in \mathbf{X}_h\,, && \vec{\mathbf{v}}_h := (\mathbf{v}_h, \boldsymbol{\delta}_h) \in \mathbf{Y}_h\,, \end{aligned}$$

which enables us to state the Galerkin scheme associated with the continuous problem (3.9). It consists in finding $(\,\vec{\boldsymbol{\eta}}_h\,,\, p_h\,,\, \vec{\varphi}_h\,,\, \vec{\mathbf{u}}_h\,) \in \mathbf{H}_h \times \mathrm{Q}_h \times \mathbf{X}_h \times \mathbf{Y}_h$ such that

$$\begin{pmatrix} \mathcal{A}_{1,h} & \mathcal{B}^{\mathtt{t}}_{1,h} & \mathcal{B}^{\mathtt{t}}_{2,h} & \\ \mathcal{B}_{1,h} & -\mathcal{D}_{1,h} & & \\ \mathcal{B}_{2,h} & & & \mathcal{B}^{\mathtt{t}}_{3,h} \\ & & \mathcal{B}_{3,h} & \end{pmatrix} \begin{pmatrix} \vec{\boldsymbol{\eta}}_h \\ p_h \\ \vec{\varphi}_h \\ \vec{\mathbf{u}}_h \end{pmatrix} = \begin{pmatrix} \mathcal{F}_h \\ \mathcal{G}_h \\ \mathcal{H}_h \\ \mathcal{I}_h \end{pmatrix}, \tag{4.1}$$

where all the operators and functionals appearing above are understood as the restrictions of their continuous counterparts (see (3.9)) to the corresponding finite element subspaces. The discrete kernels associated with $\mathcal{B}_1$ and $\mathcal{B}_3$ are defined, respectively, as

$$\begin{gathered}\mathcal{K}_{1,h} := \ker(\mathcal{B}_{1,h}) = \left\{ \vec{\boldsymbol{\chi}}_h \in \mathbf{H}_h \;:\; \int_\Omega q_h \left(\operatorname{div}(\boldsymbol{\chi}_h) - \alpha\, \operatorname{tr}(\boldsymbol{\rho}_h)\right) = 0 \qquad \forall\, q_h \in \mathrm{Q}_h \right\} \\ \text{and} \quad \mathcal{K}_{3,h} := \ker(\mathcal{B}_{3,h}) = \mathrm{X}_h^{\varphi} \times \mathcal{K}_{3,h}^{(2)}, \end{gathered} \tag{4.2}$$

where

$$\mathcal{K}_{3,h}^{(2)} := \left\{ \boldsymbol{\tau}_h \in \mathbb{X}_h^{\boldsymbol{\sigma}} \;:\; \int_\Omega \mathbf{v}_h \cdot \mathbf{div}(\boldsymbol{\tau}_h) + \int_\Omega \boldsymbol{\delta}_h : \boldsymbol{\tau}_h = 0 \quad \forall\, (\mathbf{v}_h, \boldsymbol{\delta}_h) \in \mathbf{Y}_h \right\}.$$

In addition, similarly to the continuous case (cf. (3.19)), we introduce the spaces

$$\mathcal{V}_h := \left\{ \vec{\boldsymbol{\chi}}_h \in \mathbf{H}_h \;:\; [\mathcal{B}_{2,h}(\vec{\boldsymbol{\chi}}_h), \vec{\psi}_h] = 0 \qquad \forall\, \vec{\psi}_h \in \mathcal{K}_{3,h} \right\} \quad \text{and} \quad \mathcal{W}_h := \mathcal{K}_{1,h} \cap \mathcal{V}_h\,. \tag{4.3}$$

4.1 Discrete well-posedness

We begin this section by formulating suitable assumptions on the finite element spaces and on the discretization of the domain, providing sufficient conditions to ensure the well-posedness of the Galerkin scheme (4.1). To that end, we briefly review the conditions stated in Theorem 3.1, thereby motivating the need to impose specific assumptions on the finite element spaces. The first of these is the inf-sup condition associated with the operator $\mathcal{B}_{3,h}$, which, not involving the variable $\psi_h \in \mathrm{X}_h^{\varphi}$ in its definition, coincides with the bilinear form arising from the saddle-point formulation of the linear elasticity problem (see, for instance, [34]). Accordingly, the discrete spaces involved in this inf-sup condition—namely, $\mathbf{X}_h$ and $\mathbf{Y}_h$—must form a stable pair for the linear elasticity problem, as we shall see in Section 5. This motivates the following assumption:

Assumption 4.1. *There exists a positive constant $\beta_{3,\mathtt{d}}$, independent of h, such that*

$$\sup_{0 \neq \vec{\psi}_h \in \mathbf{X}_h} \frac{[\mathcal{B}_{3,h}(\vec{\psi}_h), \vec{\mathbf{v}}_h]}{\|\vec{\psi}_h\|_{\mathbf{X}_h}} \geqslant \beta_{3,\mathtt{d}}\, \|\vec{\mathbf{v}}_h\|_{\mathbf{Y}_h} \qquad \forall\, \vec{\mathbf{v}}_h \in \mathbf{Y}_h\,. \tag{4.4}$$

Now we aim to prove an inf-sup condition for the operator $\mathcal{B}_{2,h}$, which serves as the discrete counterpart of Lemma 3.3. Namely, we seek to establish the existence of $\beta_{2,\mathtt{d}} > 0$ such that

$$\sup_{0 \neq \vec{\boldsymbol{\chi}}_h \in \mathbf{H}_h} \frac{[\mathcal{B}_{2,h}(\vec{\boldsymbol{\chi}}_h), \vec{\psi}_h]}{\|\vec{\boldsymbol{\chi}}_h\|_{\mathbf{H}_h}} \geqslant \beta_{2,\mathtt{d}}\, \|\vec{\psi}_h\|_{\mathbf{X}_h} \qquad \forall\, \vec{\psi}_h \in \mathcal{K}_{3,h}\,. \tag{4.5}$$

To this end, we follow closely the approach of [6, Lemmas 3.2 and 3.3]. In order to retain the generality of the framework, we introduce a set of natural assumptions on the finite element subspaces, together with some auxiliary constructions. As a first step, we define an auxiliary space that will serve as a discretization of $\mathrm{H}_{00}^{-1/2}(\Gamma)$:

$$\mathrm{H}_h^{-1/2} := \left\{ \mu_h \in \mathrm{L}^2(\Gamma) \;:\; \mu_h|_{K \cap \Gamma} \in \mathrm{P}_0(K \cap \Gamma) \quad \forall\, K \in \mathcal{T}_h\,,\; K \cap \Gamma \neq \varnothing \right\}. \tag{4.6}$$

Certainly, it can be proved that $\mathrm{H}_h^{-1/2}$ is a subspace of $\mathrm{H}_{00}^{s}(\Gamma)$ for all $-1/2 \leqslant s < 1/2$, and this fact will be used in the subsequent analysis. In addition, we consider an independent simplicial discretization $\{\widetilde{\Gamma}_1, \ldots, \widetilde{\Gamma}_m\}$ of Γ, parametrized by $\widetilde{h} := \max\limits_{1 \leqslant j \leqslant m} |\widetilde{\Gamma}_j|$, which enables us to approximate the space $\mathrm{H}_{00}^{1/2}(\Gamma)$ in terms of $\widetilde{h}$ rather than h. Accordingly, we now denote by $\mathrm{X}_{\widetilde{h}}^{\varphi}$ the approximation space for φ, and denote by $\mathbf{X}_{\underline{h}} := \mathrm{X}_{\widetilde{h}}^{\varphi} \times \mathbb{X}_h^{\boldsymbol{\sigma}}$ the associated product space, where $\underline{h} := (h, \widetilde{h})$ is used to indicate dependence on both discretization parameters.

Assumption 4.2 (Regularity of the discretizations of Γ).

1. *The family of triangulations $\{\mathcal{T}_h\}_{h>0}$ is uniformly regular near Γ. That is, there exists a positive constant C, independent of h, such that $|K \cap \Gamma| \geqslant C\, h^{n-1}$, for all $K \in \mathcal{T}_h$ that intersect Γ.*

2. *$\widetilde{\Gamma}$ is uniformly regular. Namely, there exists $C > 0$, independent of $\widetilde{h}$, such that $|\widetilde{\Gamma}_j| \geqslant C\, \widetilde{h}^{n-1}$, for all $j \in \{1, \ldots, m\}$.*

Under this assumption, the auxiliary space $\mathrm{H}_h^{-1/2}$ satisfies an approximation property (see, for instance, [5]): there exists a positive constant C, independent of h, such that for all $s \in (-1/2, 1/2]$ and for all $\mu \in \mathrm{H}^s_{00}(\Gamma)$, there exists $\widehat{\mu}_h \in \mathrm{H}_h^{-1/2}$ such that

$$\|\mu - \widehat{\mu}_h\|_{-1/2,00;\Gamma} \;\leqslant\; C\, h^{s+1/2}\, \|\mu\|_{s,00,\Gamma}\,. \tag{4.7}$$

In turn, inverse inequalities for both $\mathrm{X}^{\varphi}_{\widetilde{h}}$ and $\mathrm{H}_h^{-1/2}$ will play a key role in the forthcoming analysis. The one for $\mathrm{H}_h^{-1/2}$ is valid under Assumption 4.2. Indeed, as mentioned in [36, Remark 4.4.4, (b)], whose corresponding proof actually follows from the more general results provided in [19, Theorems 4.2 and 4.6], there exists a positive constant $\widetilde{C}$, independent of h, such that for all the indexes $(t,s) \in \{0\} \times [0, 1/2) \cup [-1, 0] \times \{0\}$ there holds

$$\|\mu_h\|_{s,00;\Gamma} \;\leqslant\; \widetilde{C}\, h^{t-s}\, \|\mu_h\|_{t,00;\Gamma} \qquad \forall\, \mu_h \in \mathrm{H}_h^{-1/2}\,.$$

In particular, taking $s \in [0, 1/2)$ and $t = 0$, we obtain

$$\|\mu_h\|_{s,00,\Gamma} \;\leqslant\; \widetilde{C}\, h^{-s}\, \|\mu_h\|_{0,\Gamma} \qquad \forall\, \mu_h \in \mathrm{H}_h^{-1/2}\,,$$

whereas $s = 0$ and $t \in [-1, 0]$ yield

$$\|\mu_h\|_{0,\Gamma} \;\leqslant\; \widetilde{C}\, h^{t}\, \|\mu_h\|_{t,00;\Gamma} \qquad \forall\, \mu_h \in \mathrm{H}_h^{-1/2}\,,$$

so that, combining the foregoing inequalities, we deduce the existence of a positive constant C, independent of h, such that for all the indexes (t,s) such that $-1 \leqslant t \leqslant 0 \leqslant s < 1/2$, there holds

$$\|\mu_h\|_{s,00;\Gamma} \;\leqslant\; C\, h^{t-s}\, \|\mu_h\|_{t,00;\Gamma} \qquad \forall\, \mu_h \in \mathrm{H}_h^{-1/2}\,. \tag{4.8}$$

We shall assume an inverse inequality for the subspace $\mathrm{X}^{\varphi}_{\widetilde{h}}$. Nevertheless, it is worth mentioning that, in practice, this inequality can be derived from the choice of $\mathrm{X}^{\varphi}_{\widetilde{h}}$, together with the regularity of the triangulation (cf. Assumption 4.2). We will refer again to this point in Section 5 for a specific choice of spaces.

Assumption 4.3 (Inverse inequality for $\mathrm{X}^{\varphi}_{\widetilde{h}}$). *There exists a positive constant C, independent of $\widetilde{h}$, such that for all $0 \leqslant s \leqslant t \leqslant 1$, there holds*

$$\|\psi_{\widetilde{h}}\|_{t,00;\Gamma} \;\leqslant\; C\, \widetilde{h}^{s-t}\, \|\psi_{\widetilde{h}}\|_{s,00;\Gamma} \qquad \forall\, \psi_{\widetilde{h}} \in \mathrm{X}^{\varphi}_{\widetilde{h}}\,. \tag{4.9}$$

We now impose an additional regularity assumption on the boundary discretization, specifically on the space $\mathrm{X}^{\varphi}_{\widetilde{h}}$.

Assumption 4.4. $\mathrm{X}^{\varphi}_{\widetilde{h}} \subset \mathrm{H}^1_{00}(\Gamma)$.

Next, we introduce two additional assumptions, which guarantee the existence of a mixed finite element interpolation operator, also referred to as the equilibrium interpolation operator. In what follows, $\mathcal{P}_h$ and Ξ_h denote the standard L^2-orthogonal projections onto Q_h and onto the normal trace of functions in $\mathbf{H}^{\boldsymbol{\eta}}_h$, respectively.

Assumption 4.5. $\operatorname{div}(\mathbf{H}_h^{\boldsymbol{\eta}}) \subset \mathrm{Q}_h$.

Assumption 4.6 (Existence of an equilibrium interpolation operator). *For all $\delta > 1/2$, $\mathbf{H}_h^{\boldsymbol{\eta}}$ admits an operator $\mathcal{E}_h : \mathbf{H}^\delta(\Omega) \cap \mathbf{H}(\operatorname{div};\Omega) \to \mathbf{H}_h^{\boldsymbol{\eta}}$ such that*

$$\operatorname{div}(\mathcal{E}_h(\boldsymbol{\zeta})) = \mathcal{P}_h(\operatorname{div}(\boldsymbol{\zeta})) \quad \textit{in} \quad \Omega\,, \quad \textit{and} \quad \mathcal{E}_h(\boldsymbol{\zeta})\cdot\boldsymbol{\nu} = \Xi_h(\boldsymbol{\zeta}\cdot\boldsymbol{\nu}) \quad \textit{on} \quad \partial\Omega\,,$$

for all $\boldsymbol{\zeta} \in \mathbf{H}^\delta(\Omega) \cap \mathbf{H}(\operatorname{div};\Omega)$. Moreover, assume that there exists a positive constant C_{eq}, independent of h, such that

$$\|\boldsymbol{\zeta} - \mathcal{E}_h(\boldsymbol{\zeta})\|_{0,\Omega} \leqslant C_{\mathrm{eq}}\, h^\delta\, \|\boldsymbol{\zeta}\|_{\delta,\Omega} \qquad \forall\, \boldsymbol{\zeta} \in \mathbf{H}^\delta(\Omega) \cap \mathbf{H}(\operatorname{div};\Omega)\,.$$

We now introduce an additional assumption concerning the interaction between the finite element spaces and the auxiliary space $\mathrm{H}_h^{-1/2}$.

Assumption 4.7. *$\mathrm{H}_h^{-1/2}$ is contained in the restriction to Γ of the normal traces of functions in $\mathbf{H}_h^{\boldsymbol{\eta}}$.*

This assumption, in particular, implies that $\Xi_h(\mu_h) = \mu_h$ on Γ for all $\mu_h \in \mathrm{H}_h^{-1/2}$. We are now ready to establish the first of three steps in the proof of (4.5).

Lemma 4.1. *Assume that Ω is convex. Then, there exists a positive constant C_1, independent of h and $\widetilde{h}$, such that*

$$\sup_{0\,\neq\,\boldsymbol{\chi}_h \in \mathbf{H}_h^{\boldsymbol{\eta}}} \frac{\langle \boldsymbol{\chi}_h\cdot\boldsymbol{\nu}, \psi_{\widetilde{h}}\rangle_\Gamma}{\|\boldsymbol{\chi}_h\|_{\operatorname{div};\Omega}} \geqslant C_1 \sup_{0\,\neq\,\mu_h \in \mathrm{H}_h^{-1/2}} \frac{\langle \mu_h, \psi_{\widetilde{h}}\rangle_\Gamma}{\|\mu_h\|_{-1/2,00;\Gamma}} \qquad \forall\, \psi_{\widetilde{h}} \in \mathrm{X}_{\widetilde{h}}^{\varphi}\,. \tag{4.10}$$

Proof. Given $\mu_h \in \mathrm{H}_h^{-1/2}$, $\mu_h \neq 0$, we consider the mixed boundary value problem of finding z such that

$$-\Delta z = 0 \quad \text{in} \quad \Omega\,, \qquad z = 0 \quad \text{on} \quad \Sigma\,, \qquad \nabla z\cdot\boldsymbol{\nu} = \mu_h \quad \text{on} \quad \Gamma\,.$$

Recalling that, for every $-1/2 \leqslant s < 1/2$, $\mathrm{H}_h^{-1/2} \subset \mathrm{H}_{00}^s(\Gamma)$ (cf. (4.6)), it follows that $\mu_h \in \mathrm{H}_{00}^s(\Gamma)$. Then, by standard elliptic regularity theory (see, for instance, [25, 26]), the weak solution z to this problem belongs to $\mathrm{H}^{1+\delta}(\Omega)$ and satisfies, for some positive constant C, independent of h and μ_h,

$$\|z\|_{1+\delta,\Omega} \leqslant C\, \|\mu_h\|_{-1/2+\delta,00,\Gamma}\,, \tag{4.11}$$

for every $\delta \in [0,\bar{\delta}]$, where $\bar{\delta} := \min\{1, \pi/(2\omega)\}$ with ω the largest interior angle of Ω. Observe that, since Ω is convex, there holds $\bar{\delta} > 1/2$. Thus, by taking a fixed $\delta \in (1/2,\bar{\delta})$, and putting $\widetilde{\boldsymbol{\chi}} := \nabla z \in \mathbf{H}^\delta(\Omega) \cap \mathbf{H}(\operatorname{div};\Omega)$, we obtain $\operatorname{div}(\widetilde{\boldsymbol{\chi}}) = 0$ in Ω, $\widetilde{\boldsymbol{\chi}}\cdot\boldsymbol{\nu} = \mu_h$ on Γ, and $\|\widetilde{\boldsymbol{\chi}}\|_{\delta,\Omega} \leqslant C\|\mu_h\|_{-1/2+\delta,00,\Gamma}$. In turn, using (4.11) with $\delta = 0$,

$$\|\widetilde{\boldsymbol{\chi}}\|_{\operatorname{div};\Omega} = \|\widetilde{\boldsymbol{\chi}}\|_{0,\Omega} \leqslant \|z\|_{1,\Omega} \leqslant C\, \|\mu_h\|_{-1/2,00,\Gamma}\,. \tag{4.12}$$

Moreover, by Assumption 4.6 along with Assumption 4.7, we find that

$$\operatorname{div}(\mathcal{E}_h(\widetilde{\boldsymbol{\chi}})) = 0 \quad \text{in} \quad \Omega\,, \quad \mathcal{E}_h(\widetilde{\boldsymbol{\chi}})\cdot\boldsymbol{\nu} = \mu_h \quad \text{on} \quad \Gamma \quad \text{and} \quad \|\widetilde{\boldsymbol{\chi}} - \mathcal{E}_h(\widetilde{\boldsymbol{\chi}})\|_{0,\Omega} \leqslant C_{\mathrm{eq}}\, h^\delta\, \|\widetilde{\boldsymbol{\chi}}\|_{\delta,\Omega}\,,$$

which, together with (4.11) and (4.12), yields

$$\|\mathcal{E}_h(\widetilde{\boldsymbol{\chi}})\|_{\operatorname{div};\Omega} \leqslant \|\widetilde{\boldsymbol{\chi}} - \mathcal{E}_h(\widetilde{\boldsymbol{\chi}})\|_{0,\Omega} + \|\widetilde{\boldsymbol{\chi}}\|_{0,\Omega} \leqslant C\, C_{\mathrm{eq}}\, h^\delta\, \|\mu_h\|_{-1/2+\delta,00,\Gamma} + C\, \|\mu_h\|_{-1/2,00;\Gamma}\,.$$

Owing to the inverse inequality (4.8) with $t = -1/2$ and $s = \delta - 1/2$ into this estimate, we find that

$$\|\mathcal{E}_h(\widetilde{\boldsymbol{\chi}})\|_{\operatorname{div};\Omega} \leqslant \frac{1}{C_1}\, \|\mu_h\|_{-1/2,00;\Gamma}\,,$$

where $C_1 := (C\, \max\{C_{\mathrm{eq}}\, C_{\mathrm{inv}}, 1\})^{-1}$. Finally, bearing in mind that $\mathcal{E}_h(\widetilde{\boldsymbol{\chi}})\cdot\boldsymbol{\nu} = \mu_h$ on Γ, we take $\mathcal{E}_h(\widetilde{\boldsymbol{\chi}}) \in \mathbf{H}_h^{\boldsymbol{\eta}}$ in the supremum on the left-hand side of (4.10), leading to

$$\sup_{0\neq\boldsymbol{\chi}_h \in \mathbf{H}_h^{\boldsymbol{\eta}}} \frac{\langle \boldsymbol{\chi}_h\cdot\boldsymbol{\nu}, \psi_{\widetilde{h}}\rangle_\Gamma}{\|\boldsymbol{\chi}_h\|_{\operatorname{div};\Omega}} \geqslant \frac{\langle \mu_h, \psi_{\widetilde{h}}\rangle_\Gamma}{\|\mathcal{E}_h(\widetilde{\boldsymbol{\chi}})\|_{\operatorname{div};\Omega}} \geqslant C_1\, \frac{\langle \mu_h, \psi_{\widetilde{h}}\rangle_\Gamma}{\|\mu_h\|_{-1/2,00;\Gamma}} \qquad \forall\, \mu_h \in \mathrm{H}_h^{-1/2}\backslash\{0\}\,,$$

and for all $\psi_{\widetilde{h}} \in \mathrm{X}_{\widetilde{h}}^{\varphi}$. Hence, taking the supremum over $\mathrm{H}_h^{-1/2}$, we conclude (4.10). □

Having established this result, we are now in a position to present the next one, which constitutes the main part of the effort in proving the desired inf-sup condition.

Lemma 4.2. *There exist $C_0, C_2 > 0$, independent of h and $\widetilde{h}$, such that if $h \leqslant C_0\,\widetilde{h}$, then*

$$\sup_{0\neq\mu_h\in \mathrm{H}_h^{-1/2}} \frac{\langle \mu_h, \psi_{\widetilde{h}}\rangle_\Gamma}{\|\mu_h\|_{-1/2,00;\Gamma}} \geqslant C_2\,\|\psi_{\widetilde{h}}\|_{1/2,00;\Gamma} \qquad \forall\, \psi_{\widetilde{h}} \in \mathrm{X}_{\widetilde{h}}^{\varphi}\,. \tag{4.13}$$

Proof. Given $\psi_{\widetilde{h}} \in \mathrm{X}_{\widetilde{h}}^{\varphi}$, denote by z the unique solution of the problem

$$-\Delta z + z = 0 \quad \text{in} \quad \Omega\,, \qquad z = 0 \quad \text{on} \quad \Sigma\,, \qquad \text{and} \qquad z = \psi_{\widetilde{h}} \quad \text{on} \quad \Gamma\,. \tag{4.14}$$

Then, bearing in mind that $E_{\Gamma,0}(\psi_{\widetilde{h}}) \in \mathrm{H}^1(\partial\Omega)$ by Assumption 4.4, a classical result on elliptic regularity (see, for instance, [25, 26]) guarantees that z belongs to $\mathrm{H}^{1+\delta}(\Omega)$ and satisfies

$$\|z\|_{1+\delta,\Omega} \leqslant C\,\|\psi_{\widetilde{h}}\|_{1/2+\delta,00,\Gamma}\,, \tag{4.15}$$

for some $C > 0$ and for all $\delta \in [0, 1/2]$. In turn, since z solves (4.14), partial integration yields

$$\|z\|_{1,\Omega}^2 = \int_\Omega \big\{ z\,\Delta z + |\nabla z|^2 \big\} = \big\langle \nabla z\cdot\boldsymbol{\nu}, E_{\Gamma,0}(\psi_{\widetilde{h}})\big\rangle_{\partial\Omega} = \big\langle \nabla z\cdot\boldsymbol{\nu}, \psi_{\widetilde{h}}\big\rangle_\Gamma\,,$$

where we used that $z = E_{\Gamma,0}(\psi_{\widetilde{h}})$ on $\partial\Omega$, by construction. Now, according to the continuity of the canonical trace operator and the definition of the norm in $\mathrm{H}_{00}^{1/2}(\Omega)$, there holds

$$\|\psi_{\widetilde{h}}\|_{1/2,00;\Gamma} = \|E_{\Gamma,0}(\psi_{\widetilde{h}})\|_{1/2,\partial\Omega} \leqslant \|z\|_{1,\Omega}\,.$$

We have thus proved that

$$\big\langle \nabla z\cdot\boldsymbol{\nu}, \psi_{\widetilde{h}}\big\rangle_\Gamma \geqslant \|\psi_{\widetilde{h}}\|_{1/2,00;\Gamma}^2\,. \tag{4.16}$$

Fix $\delta \in [0, 1/2)$. Since $z \in \mathrm{H}^{1+\delta}(\Omega)$ and the canonical trace operator is continuous from $\mathrm{H}^{1-\delta}(\Omega)$ to $\mathrm{H}^{1/2-\delta}(\partial\Omega)$ (see, for instance, [35, Theorem 3.37]), one can verify that $(\nabla z\cdot\boldsymbol{\nu})|_\Gamma \in \mathrm{H}_{00}^{-1/2+\delta}(\Gamma)$ and

$$\|\nabla z\cdot\boldsymbol{\nu}\|_{-1/2+\delta,00,\Gamma} \leqslant C_{\boldsymbol{\nu}}\,\|z\|_{1+\delta,\Omega}\,, \tag{4.17}$$

for some positive constant $C_{\boldsymbol{\nu}}$, depending only on Ω and δ. In turn, by the approximation property of our auxiliary space (cf. (4.7)), there exists $\widehat{\mu}_h \in \mathrm{H}_h^{-1/2}$ such that

$$\|\nabla z\cdot\boldsymbol{\nu} - \widehat{\mu}_h\|_{-1/2,00;\Gamma} \leqslant C\,h^\delta\,\|\nabla z\cdot\boldsymbol{\nu}\|_{-1/2+\delta,00,\Gamma} \leqslant C\,C_{\boldsymbol{\nu}}\,h^\delta\,\|z\|_{1+\delta,\Omega} \leqslant C^2\,C_{\boldsymbol{\nu}}\,h^\delta\,\|\psi_{\widetilde{h}}\|_{1/2+\delta,00;\Gamma}\,,$$

where the last two inequalities come from (4.17) and (4.15), respectively. Furthermore, using the inverse inequality (4.9) with $t = 1/2 + \delta$ and $s = 1/2$, we get

$$\|\nabla z\cdot\boldsymbol{\nu} - \widehat{\mu}_h\|_{-1/2,00;\Gamma} \leqslant \bar{C}\,(h/\widetilde{h})^\delta\,\|\psi_{\widetilde{h}}\|_{1/2,00;\Gamma}\,. \tag{4.18}$$

In particular, by applying the triangle inequality together with (4.17) and (4.15) with $\delta = 0$, we obtain

$$\|\widehat{\mu}_h\|_{-1/2,00;\Gamma} \leqslant \bar{C}\,(h/\widetilde{h})^\delta\,\|\psi_{\widetilde{h}}\|_{1/2,00;\Gamma} + C_{\boldsymbol{\nu}}C\,\|\psi_{\widetilde{h}}\|_{1/2,00;\Gamma} \leqslant \widehat{C}\,\|\psi_{\widetilde{h}}\|_{1/2,00;\Gamma},$$

for $h \leqslant \widetilde{h}$ and $\widehat{C} = \max \bar{C}, C_{\boldsymbol{\nu}}C$. We now use this to bound the supremum in (4.13) with the particular choice $\mu_h = \widehat{\mu}_h$, arriving at

$$\begin{aligned}\sup_{0\neq\mu_h\in \mathrm{H}_h^{-1/2}} \frac{\langle \mu_h, \psi_{\widetilde{h}}\rangle_\Gamma}{\|\mu_h\|_{-1/2,00;\Gamma}} \geqslant \frac{\langle \widehat{\mu}_h, \psi_{\widetilde{h}}\rangle_\Gamma}{\|\widehat{\mu}_h\|_{-1/2,00;\Gamma}} &\geqslant \frac{\widehat{C}^{-1}}{\|\psi_{\widetilde{h}}\|_{1/2,00;\Gamma}}\Big(\big\langle \nabla z\cdot\boldsymbol{\nu}, \psi_{\widetilde{h}}\big\rangle_\Gamma - \big\langle \nabla z\cdot\boldsymbol{\nu} - \widehat{\mu}_h, \psi_{\widetilde{h}}\big\rangle_\Gamma\Big)\\ &\geqslant \widehat{C}^{-1}\big(1 - \bar{C}\,(h/\widetilde{h})^\delta\big)\,\|\psi_{\widetilde{h}}\|_{1/2,00;\Gamma},\end{aligned}$$

where the last inequality follows from (4.16) and (4.18). In this way, recalling that $h \leqslant \widetilde{h}$, it suffices to require $h \leqslant C_0\,\widetilde{h}$, with $C_0 := \min\{1, (2\bar{C})^{-1/\delta}\}$, in order to ensure a positive constant on the right-hand side. This concludes the proof with $C_2 = \widehat{C}^{-1}/2$. □

To finally prove the inf-sup condition for $\mathcal{B}_{2,\underline{h}}$, we require one last assumption.

Assumption 4.8. $\mathbf{div}\big(\mathbb{X}_h^{\boldsymbol{\sigma}}\big) \subset \mathbf{Y}_h^{\mathbf{u}}$ *and* $\mathcal{K}_{3,h}^{(2)} \subset \mathbb{H}_h^{\boldsymbol{\xi}}$.

Lemma 4.3. *Assume that Ω is convex. Then, there exists a positive constant $\beta_{2,\mathtt{d}}$, independent of h and $\widetilde{h}$, such that for all $h \leqslant C_0\,\widetilde{h}$ there holds*

$$\sup_{\mathbf{0}\neq\vec{\boldsymbol{\chi}}_h\in\mathbf{H}_h} \frac{[\mathcal{B}_{2,\underline{h}}(\vec{\boldsymbol{\chi}}_h),\vec{\psi}_{\underline{h}}]}{\|\vec{\boldsymbol{\chi}}_h\|_{\mathbf{H}_h}} \geqslant \beta_{2,\mathtt{d}}\,\|\vec{\psi}_{\underline{h}}\|_{\mathbf{X}_{\underline{h}}} \qquad \forall\,\vec{\psi}_{\underline{h}} \in \mathcal{K}_{3,\underline{h}}\,. \tag{4.19}$$

Proof. Given $\vec{\psi}_{\underline{h}} = (\psi_{\widetilde{h}},\boldsymbol{\tau}_h) \in \mathcal{K}_{3,\underline{h}}$, by the definition of this kernel (cf. (4.2)), we have that

$$\int_\Omega \mathbf{v}_h\cdot\mathbf{div}(\boldsymbol{\tau}_h) = 0 \qquad \forall\,\mathbf{v}_h \in \mathbf{Y}_h^{\mathbf{u}}\,,$$

which, by the first inclusion in Assumption 4.8, implies that $\mathbf{div}(\boldsymbol{\tau}_h) = 0$. Furthermore, by the second inclusion in the same assumption we have that $\boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\xi}}$, and hence we are able to take $\vec{\boldsymbol{\chi}}_h = (0,\boldsymbol{\tau}_h) \in \mathbf{H}_h$ in the supremum on the left-hand side of (4.19), thus arriving at

$$\sup_{\mathbf{0}\neq\vec{\boldsymbol{\chi}}_h\in\mathbf{H}_h} \frac{[\mathcal{B}_{2,\underline{h}}(\vec{\boldsymbol{\chi}}_h),\vec{\psi}_{\underline{h}}]}{\|\vec{\boldsymbol{\chi}}_h\|_{\mathbf{H}_h}} \geqslant \|\boldsymbol{\tau}_h\|_{\mathbf{div};\Omega}\,. \tag{4.20}$$

In turn, we bound the same supremum by choosing $\vec{\boldsymbol{\chi}}_h = (\boldsymbol{\chi}_h,0)$, with arbitrary $\boldsymbol{\chi}_h \in \mathbf{H}_h^{\boldsymbol{\eta}}$, and apply Lemmas 4.1 and 4.2 to obtain

$$\sup_{\mathbf{0}\neq\vec{\boldsymbol{\chi}}_h\in\mathbf{H}_h} \frac{[\mathcal{B}_{2,\underline{h}}(\vec{\boldsymbol{\chi}}_h),\vec{\psi}_{\underline{h}}]}{\|\vec{\boldsymbol{\chi}}_h\|_{\mathbf{H}_h}} \geqslant \sup_{\mathbf{0}\neq\boldsymbol{\chi}_h\in\mathbf{H}_h^{\boldsymbol{\eta}}} \frac{\big\langle \boldsymbol{\chi}_h\cdot\boldsymbol{\nu},\psi_{\widetilde{h}}\big\rangle_\Gamma}{\|\boldsymbol{\chi}_h\|_{\mathbf{H}_h^{\boldsymbol{\eta}}}} \geqslant C_1\,C_2\,\|\psi_{\widetilde{h}}\|_{1/2,00;\Gamma}\,. \tag{4.21}$$

Thus, by summing both (4.20) and (4.21), we obtain the discrete inf-sup condition (4.19), with constant $\beta_{2,\mathtt{d}} := \frac{1}{2}\min\{1,C_1C_2\}$. $\square$

Next, we establish the inf-sup condition for $\mathcal{B}_{1,h}$, which corresponds to the discrete counterpart of Lemma 3.4. Notice that no additional assumptions beyond those already established are required.

Lemma 4.4. *Suppose that Ω is convex. Then, there exists a positive constant $\beta_{1,\mathtt{d}}$ such that*

$$\sup_{0\neq\vec{\boldsymbol{\chi}}_h\in\mathcal{V}_h} \frac{[\mathcal{B}_{1,h}(\vec{\boldsymbol{\chi}}_h),q_h]}{\|\vec{\boldsymbol{\chi}}_h\|_{\mathbf{H}_h}} \geqslant \beta_{1,\mathtt{d}}\,\|q_h\|_{\mathrm{Q}_h} \qquad \forall\,q_h \in \mathrm{Q}_h\,. \tag{4.22}$$

Proof. Let $q_h \in \mathrm{Q}_h \subset \mathrm{L}^2(\Omega)$, and consider $q_h \neq 0$, as otherwise the inequality holds trivially. Then, we consider the following boundary value problem: Find z such that

$$-\Delta z = q_h \quad \text{in} \quad \Omega, \qquad z = 0 \quad \text{on} \quad \Sigma, \quad \text{and} \quad \nabla z\cdot\boldsymbol{\nu} = 0 \quad \text{on} \quad \Gamma,$$

whose weak solution, by standard elliptic regularity theory (see, for instance, [25, 26]), belongs to $\mathrm{H}^{1+\delta}(\Omega)$ and satisfies $\|z\|_{1+\delta,\Omega} \leqslant C\,\|q_h\|_{0,\Omega}$ for every $\delta \in [0,\bar{\delta}]$, where $\bar{\delta} := \pi/(2\omega)$ with ω the largest interior angle of Ω. Notice that $\bar{\delta} > 1/2$ as Ω is convex. Then, fixing $\delta \in (1/2,\bar{\delta})$, and noting that $\mathrm{div}(\nabla z) = -q_h$ in $\mathrm{L}^2(\Omega)$, we deduce that $\nabla z \in \mathbf{H}^\delta(\Omega) \cap \mathbf{H}(\mathrm{div};\Omega)$. Thus, we may define $\widetilde{\boldsymbol{\chi}}_h := \mathcal{E}_h(\nabla z) \in \mathbf{H}_h^{\boldsymbol{\eta}}$ (cf. Assumption 4.6). Furthermore, $\widetilde{\boldsymbol{\chi}}_h\cdot\boldsymbol{\nu} = \Xi_h(\nabla z\cdot\boldsymbol{\nu}) = 0$ on Γ, which implies that $\big\langle\widetilde{\boldsymbol{\chi}}_h\cdot\boldsymbol{\nu},\psi_h\big\rangle_\Gamma = 0$ for all $\psi_h \in \mathrm{X}_h^{\varphi}$, so that $(\widetilde{\boldsymbol{\chi}}_h,0) \in \mathcal{V}_h$. In turn,

$$\|\widetilde{\boldsymbol{\chi}}_h\|_{\mathrm{div};\Omega}^2 = \|\widetilde{\boldsymbol{\chi}}_h\|_{0,\Omega}^2 + \|\mathrm{div}(\widetilde{\boldsymbol{\chi}}_h)\|_{0,\Omega}^2 = \|\widetilde{\boldsymbol{\chi}}_h\|_{0,\Omega}^2 + \|q_h\|_{0,\Omega}^2\,. \tag{4.23}$$

Now, employing Assumption 4.6, the *a priori* estimate of the auxiliary problem, and assuming that $h \leqslant 1$ without loss of generality, we write

$$\begin{aligned}\|\widetilde{\boldsymbol{\chi}}_h\|_{0,\Omega} = \|\mathcal{E}_h(\nabla z)\|_{0,\Omega} &\leqslant \|\mathcal{E}_h(\nabla z) - \nabla z\|_{0,\Omega} + \|\nabla z\|_{0,\Omega} \leqslant C_{\rm eq}\, h^\delta\, \|\nabla z\|_{\delta,\Omega} + \|\nabla z\|_{0,\Omega} \\ &\leqslant C_{\rm eq}\, h^\delta\, \|z\|_{1+\delta,\Omega} + \|z\|_{1,\Omega} \leqslant (C_{\rm eq}+1)\, C\, \|q_h\|_{0,\Omega}\,.\end{aligned}$$

Using this into (4.23), we find that $\|\widetilde{\boldsymbol{\chi}}_h\|_{\mathrm{div};\Omega} \leqslant \widehat{C}\, \|q_h\|_{0,\Omega}$, where $\widehat{C} := \big((C_{\rm eq}+1)^2\, C^2 + 1\big)^{1/2}$. Finally, bounding the supremum in (4.22) with $\vec{\boldsymbol{\chi}}_h = (\widetilde{\boldsymbol{\chi}}_h, 0) \in \mathcal{V}_h$, we obtain

$$\sup_{0 \neq \vec{\boldsymbol{\chi}}_h \in \mathcal{V}_h} \frac{[\mathcal{B}_{1,h}(\vec{\boldsymbol{\chi}}_h), q_h]}{\|\vec{\boldsymbol{\chi}}_h\|_{\mathbf{H}_h}} \geqslant \frac{\left|\displaystyle\int_\Omega q_h\, \mathrm{div}(\widetilde{\boldsymbol{\chi}}_h)\right|}{\|\widetilde{\boldsymbol{\chi}}_h\|_{\mathrm{div};\Omega}} \geqslant \frac{\|q_h\|^2_{0,\Omega}}{\widehat{C}\, \|q_h\|_{0,\Omega}} = \widehat{C}^{-1}\, \|q_h\|_{0,\Omega}\,,$$

which proves (4.22) with $\beta_{1,\mathtt{d}} = \widehat{C}^{-1}$, as desired. □

In order to establish the discrete counterpart of Lemma 3.5, we first note that the symmetry and positive semi-definiteness of $\mathcal{A}_{1,h}$ and $\mathcal{D}_{1,h}$ are inherited from those of $\mathcal{A}_1$ and $\mathcal{D}_1$, respectively. However, the coercivity of $\mathcal{A}_{1,h}$ does not follow directly from the continuous case, as $\mathcal{W}_h$ is not necessarily contained in $\mathcal{W}$ (cf. (3.19) and (4.3)). Nevertheless, under Assumption 4.5, we are still able to establish it.

Lemma 4.5. *There exists a positive constant $\alpha_{1,\mathtt{d}}$ such that*

$$[\mathcal{A}_{1,h}(\vec{\boldsymbol{\chi}}_h), \vec{\boldsymbol{\chi}}_h] \geqslant \alpha_{1,\mathtt{d}}\, \|\vec{\boldsymbol{\chi}}_h\|_{\mathrm{Q}_h} \qquad \forall\, \vec{\boldsymbol{\chi}}_h \in \mathcal{W}_h\,.$$

Proof. Let $\vec{\boldsymbol{\chi}}_h \in \mathcal{W}_h$. As in the continuous case, algebraic manipulations yield (cf. (3.25))

$$[\mathcal{A}_{1,h}(\vec{\boldsymbol{\chi}}_h), \vec{\boldsymbol{\chi}}_h] \geqslant \kappa_0\, \|\boldsymbol{\chi}_h\|^2_{0,\Omega} + 2\mu\, \|\boldsymbol{\rho}_h\|^2_{0,\Omega} + \lambda\, \|\mathrm{tr}(\boldsymbol{\rho}_h)\|^2_{0,\Omega}\,. \tag{4.24}$$

Since $\vec{\boldsymbol{\chi}}_h \in \mathcal{W}_h \subset \mathcal{K}_{1,h}$ (cf. (4.2)) and noting that $\mathrm{div}(\boldsymbol{\chi}_h) \in \mathrm{Q}_h$ by Assumption 4.5, there holds

$$\|\mathrm{div}(\boldsymbol{\chi}_h)\|^2_{0,\Omega} = \alpha \int_\Omega \mathrm{div}(\boldsymbol{\chi}_h)\, \mathrm{tr}(\boldsymbol{\rho}_h)\,.$$

Hence, by the Cauchy–Schwarz inequality, we obtain that $\|\mathrm{div}(\boldsymbol{\chi}_h)\|_{0,\Omega} \leqslant \alpha\, \|\mathrm{tr}(\boldsymbol{\rho}_h)\|_{0,\Omega}$. Therefore, (4.24) becomes

$$\begin{aligned}[\mathcal{A}_{1,h}(\vec{\boldsymbol{\chi}}_h), \vec{\boldsymbol{\chi}}_h] &\geqslant \kappa_0\, \|\boldsymbol{\chi}_h\|^2_{0,\Omega} + 2\mu\, \|\boldsymbol{\rho}_h\|^2_{0,\Omega} + \frac{\lambda}{\alpha^2}\, \|\mathrm{div}(\boldsymbol{\chi}_h)\|^2_{0,\Omega} \\ &\geqslant \min\{\kappa_0, \lambda\, \alpha^{-2}, 2\mu\}\, \big(\|\boldsymbol{\chi}_h\|^2_{\mathrm{div};\Omega} + \|\boldsymbol{\rho}_h\|^2_{0,\Omega}\big),\end{aligned}$$

which allows us to conclude the desired result, with $\alpha_{1,\mathtt{d}} = \min\{\kappa_0, \lambda\, \alpha^{-2}, 2\mu\}$. □

Theorem 4.6. *Suppose that Assumptions 4.1 through 4.8 hold, and that Ω is convex. Then, the problem (4.1) has a unique solution $(\,\vec{\boldsymbol{\eta}}_h\,,\, p_h\,,\, \vec{\varphi}_{\underline{h}}\,,\, \vec{\mathbf{u}}_h\,) \in \mathbf{H}_h \times \mathrm{Q}_h \times \mathbf{X}_{\underline{h}} \times \mathbf{Y}_h$ and there exists a positive constant $C_{\mathtt{d}}$, depending only on $\beta_{1,\mathtt{d}}$, $\beta_{2,\mathtt{d}}$, $\beta_{3,\mathtt{d}}$, κ_0, κ_1, μ, λ, α, c_0, and Ω, such that*

$$\|(\,\vec{\boldsymbol{\eta}}_h\,,\, p_h\,,\, \vec{\varphi}_{\underline{h}}\,,\, \vec{\mathbf{u}}_h\,)\| \leqslant C_{\mathtt{d}} \Big\{ \|g\|_{0,\Omega} + \|g_{\rm N}\|_{-1/2,00;\Gamma} + \|\mathbf{u}_{\rm D}\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,\Omega} \Big\}\,.$$

Proof. Under Assumption 4.1, and using Lemmas 4.3, 4.4, and 4.5, together with the observation that $\mathcal{A}_{1,h}$ and $\mathcal{D}_{1,h}$ are symmetric and positive semi-definite, we conclude that the hypotheses of Theorem 3.1 are satisfied. Therefore, applying this result in the discrete setting completes the proof. □

4.2 *A priori* error analysis

Now we aim to derive a Céa-type estimate associated with the Galerkin scheme (4.1). As usual, given a subspace U of an arbitrary Banach space $(V, \|\cdot\|_V)$, we set

$$\mathrm{dist}(v,U) := \inf_{u\in U} \|v-u\|_V \qquad \forall\, v\in V\,.$$

In addition, let us denote the solutions of (3.9) and (4.1) (cf. Theorems 3.6 and 4.6) as

$$\Theta \;:=\; (\vec{\boldsymbol{\eta}}, p, \vec{\varphi}, \vec{\mathbf{u}}) \in \mathbf{H}\times \mathrm{Q}\times \mathbf{X}\times \mathbf{Y} \quad \text{and} \quad \Theta_{\underline{h}} \;:=\; (\vec{\boldsymbol{\eta}}_h, p_h, \vec{\varphi}_{\underline{h}}, \vec{\mathbf{u}}_h) \in \mathbf{H}_h\times \mathrm{Q}_h\times \mathbf{X}_{\underline{h}}\times \mathbf{Y}_h\,, \tag{4.25}$$

respectively. The following result establishes the desired Céa estimate associated with (4.1).

Theorem 4.7. *Suppose that the assumptions of Theorem 4.6 hold. Then, there exists a positive constant* $C_{\mathtt{Cea}}$*, depending only on* κ_0*,* κ_1*,* μ*,* λ*,* α*,* c_0*,* $\beta_{1,\mathtt{d}}$*,* $\beta_{2,\mathtt{d}}$*,* $\beta_{3,\mathtt{d}}$*, and* Ω*, such that*

$$\|\Theta - \Theta_{\underline{h}}\| \leqslant C_{\mathtt{Cea}}\,\mathrm{dist}\big(\Theta\,,\,\mathbf{H}_h\times\mathrm{Q}_h\times\mathbf{X}_{\underline{h}}\times\mathbf{Y}_h\big). \tag{4.26}$$

Proof. It follows from standard arguments concerning Céa estimates in the context of Galerkin schemes. Specifically, we subtract the discrete system (4.1) from the continuous one (3.9), and apply the inf-sup conditions (4.4), (4.19), and (4.22), together with the coercivity property of $\mathcal{A}_{1,h}$ (cf. Lemma 4.5). Further details are omitted and can be found, for instance, in [20, Lemma 26.14] or [22, Section 2.5]. □

5 Specific finite element spaces

In this section, we provide specific examples for the choice of the spaces $\mathbf{H}_h^{\boldsymbol{\eta}}$, $\mathbb{H}_h^{\boldsymbol{\xi}}$, Q_h, $\mathbf{X}_{\tilde{h}}^{\varphi}$, $\mathbb{X}_h^{\boldsymbol{\sigma}}$, $\mathbf{Y}_h^{\mathbf{u}}$, and $\mathbb{Y}_h^{\boldsymbol{\gamma}}$ which satisfy Assumptions 4.1 through 4.8, and we establish the corresponding rates of convergence. Certainly, Assumption 4.2 must be assumed since it concerns the mesh discretization rather than the finite element spaces. In this way, we first introduce preliminary notations. For a nonnegative integer k and $K\in\mathcal{T}_h$, we let $\mathrm{P}_k(K)$ be the space of polynomials of total degree at most k defined on K. Its vector and tensorial counterparts are denoted by $\mathbf{P}_k(K) := [\mathrm{P}_k(K)]^n$ and $\mathbb{P}_k(K) := [\mathrm{P}_k(K)]^{n\times n}$, respectively. In addition, we let $\mathbf{RT}_k(K) := \mathbf{P}_k(K) + \mathrm{P}_k(K)\,\mathbf{x}$ be the local Raviart–Thomas space of order k defined on K, where $\mathbf{x}$ stands for a generic vector in R^n. We denote by $\mathbb{RT}_k(K)$ the space of tensor-valued functions whose rows lie in $\mathbf{RT}_k(K)$. Furthermore, we let b_K be the bubble function on K, which is given by the product of its $n+1$ barycentric coordinates. The local bubble space of order k is then given by

$$\mathbf{B}_k(K) := \begin{cases} \mathbf{curl}\,(b_K\,\mathrm{P}_k(K)) & \text{if } n=2\,,\\ \mathbf{curl}\,(b_K\,\mathbf{P}_k(K)) & \text{if } n=3\,,\end{cases}$$

where the curl operators are defined as $\mathbf{curl}\,(v) := \big(\frac{\partial v}{\partial x_2}, -\frac{\partial v}{\partial x_1}\big)$ for $v: K\to\mathrm{R}$ (if $n=2$), and $\mathbf{curl}\,(\mathbf{v}) := \nabla\times\mathbf{v}$ for $\mathbf{v}: K\to\mathrm{R}^3$ (if $n=3$). Finally, $\mathbb{B}_k(K)$ denotes the space of tensor-valued functions whose rows belong to $\mathbf{B}_k(K)$.

As mentioned earlier in Section 4, Assumption 4.1 corresponds to the inf-sup condition associated with the classical bilinear form arising in the saddle-point formulation of the linear elasticity problem with weakly imposed symmetry. Accordingly, it is natural to consider stable finite element spaces for linear elasticity as a choice for the spaces involved in this inf-sup. More precisely, we define

$$\begin{aligned} \mathbb{X}_h^{\boldsymbol{\sigma}} &:= \Big\{\boldsymbol{\tau}_h\in\mathbb{H}_\Sigma(\mathbf{div},\Omega):\ \boldsymbol{\tau}_h|_K\in\mathbb{RT}_k(K)\oplus\mathbb{B}_k(K)\quad\forall\,K\in\mathcal{T}_h\Big\},\\ \mathbf{Y}_h^{\mathbf{u}} &:= \Big\{\mathbf{v}_h\in\mathbf{L}^2(\Omega):\ \mathbf{v}_h|_K\in\mathbf{P}_k(K)\quad\forall\,K\in\mathcal{T}_h\Big\},\\ \mathbb{Y}_h^{\boldsymbol{\gamma}} &:= \Big\{\boldsymbol{\delta}_h\in[C(\overline{\Omega})]^{n\times n}\cap\mathbb{L}^2_{\mathrm{skew}}(\Omega):\ \boldsymbol{\delta}_h|_K\in\mathbb{P}_{k+1}(K)\quad\forall\,K\in\mathcal{T}_h\Big\}. \end{aligned} \tag{5.1}$$

Observe that $(\mathbb{X}_h^{\boldsymbol{\sigma}}, \mathbf{Y}_h^{\mathbf{u}}, \mathbb{Y}_h^{\gamma})$ corresponds to the PEERS_{k+1} element. Under this choice, Assumption 4.1 is satisfied. Indeed, in [34, Theorem 4.5], the authors establish this inf-sup condition for the triple associated with the BDMS_{k+1} element, and further remark in Section 5 of the same article that the corresponding analysis remains valid for PEERS_{k+1}. We also emphasize that other combinations of stable finite element spaces for linear elasticity with reduced symmetry could be employed to fulfill Assumption 4.1. Examples include the Amara–Thomas element [2], Arnold–Falk–Winther [4], and Cockburn–Gopalakrishnan–Guzmán [16] families. However, to keep the exposition focused, in the remaining of this work, we restrict ourselves to the PEERS_{k+1} element.

Next, let us define the remaining finite element subspaces:

$$
\begin{aligned}
\mathbf{H}_h^{\boldsymbol{\eta}} &:= \Big\{ \boldsymbol{\chi}_h \in \mathbf{H}(\mathrm{div};\Omega) : \quad \boldsymbol{\chi}_h|_K \in \mathbf{RT}_k(K) \qquad \forall\, K \in \mathcal{T}_h \Big\}, \\
\mathbb{H}_h^{\boldsymbol{\xi}} &:= \Big\{ \boldsymbol{\rho}_h \in \mathbb{L}^2(\Omega) : \quad \boldsymbol{\rho}_h|_K \in \mathbb{P}_k(K) \oplus \mathbb{B}_k(K) \qquad \forall\, K \in \mathcal{T}_h \Big\}, \\
\mathrm{Q}_h &:= \Big\{ q_h \in \mathrm{L}^2(\Omega) : \quad q_h|_K \in \mathrm{P}_k(K) \qquad \forall\, K \in \mathcal{T}_h \Big\}, \\
\mathbf{X}_{\tilde{h}}^{\varphi} &:= \Big\{ \psi_{\tilde{h}} \in \mathrm{H}_{00}^{1/2}(\Gamma) : \quad \psi_{\tilde{h}}|_{\widetilde{\Gamma}_j} \in \mathrm{P}_{k+1}(\widetilde{\Gamma}_j) \qquad \forall\, j \in \{1, \ldots, m\} \Big\}.
\end{aligned}
\tag{5.2}
$$

We recall from [6, eq. 3.12] that, in the special case $k = 0$, $\mathbf{X}_{\tilde{h}}^{\varphi} \subset \mathrm{H}_{00}^1(\Gamma)$, and the inverse inequality (4.9) holds. This result can be extended to the case $k \geqslant 1$, so that Assumptions 4.3 and 4.4 are satisfied. The corresponding proof for $s,\, t \in \{0, 1\}$ is provided in [36, Theorem 4.4.3], whereas the extension to the whole range $0 \leqslant s \leqslant t \leqslant 1$ follows from the more general result given by [19, Theorem 4.1]. In turn, since $\mathrm{div}(\mathbf{RT}_k(K)) \subseteq \mathrm{P}_k(K)$ for each $K \in \mathcal{T}_h$, Assumption 4.5 holds as well.

Now, we consider $\mathcal{E}_h$ as the usual Raviart–Thomas interpolation operator (see, for instance, [22, Chapter 3.4]), which can be defined from the space $\mathbf{H}^{\delta}(\Omega) \cap \mathbf{H}(\mathrm{div};\Omega)$ with $\delta > 1/2$, since the moments of the Raviart–Thomas space are well-defined as linear and bounded functionals in $\mathbf{H}^{\delta}(\widehat{K}) \cap \mathbf{H}(\mathrm{div};\widehat{K})$, where $\widehat{K}$ is the reference element associated to the mesh. This can be viewed from a slight modification of the analysis made in [23, Lemma C.1]. In this way, Assumption 4.6 is also satisfied.

On the other hand, since the normal traces of functions in $\mathbf{H}_h^{\boldsymbol{\eta}}$ are contained in the space of piecewise polynomials of order k defined on $\partial\Omega$, it follows that Assumption 4.7 holds.

Next, by noticing that $\mathbf{div}(\mathbb{B}_k(K)) = 0$, it follows that $\mathbf{div}(\mathbb{X}_h^{\boldsymbol{\sigma}}) \subset \mathbf{Y}_h^{\mathbf{u}}$, which means that the first inclusion of Assumption 4.8 is fulfilled. In order to establish the second inclusion of this assumption, we notice that, for $\boldsymbol{\tau}_h \in \mathcal{K}_{3,h}^{(2)} \subset \mathbb{H}_h^{\boldsymbol{\sigma}}$, there holds

$$
\int_{\Omega} \mathbf{v}_h \cdot \mathbf{div}(\boldsymbol{\tau}_h) = 0 \qquad \forall\, \mathbf{v}_h \in \mathbf{Y}_h^{\mathbf{u}}.
$$

Then, by exploiting again that the bubble functions are divergence-free, this identity means that the Raviart–Thomas component of $\boldsymbol{\tau}_h$ is divergence-free as well. In this way, from the proof of [22, Theorem 3.3], we obtain that the Raviart–Thomas component belongs to $\mathbb{P}_k(K)$ in each element $K \in \mathcal{T}_h$. In addition, since the bubble component of $\boldsymbol{\tau}_h$ remains unaltered, we need to incorporate it in the definition of $\mathbb{H}_h^{\boldsymbol{\xi}}$ in order to guarantee that $\boldsymbol{\tau}_h$ belongs to this subspace, thus proving that the second inclusion in Assumption 4.8 does hold. The above explains the rather unusual definition of $\mathbb{H}_h^{\boldsymbol{\xi}}$ (cf. (5.2)).

As a result of the previous discussion, we conclude that this choice of finite element spaces yields a stable Galerkin scheme.

Now we aim to obtain the rates of convergence of our Galerkin scheme (4.1) with the specific finite element subspaces defined previously. To this end, approximation properties of the finite element subspaces $\mathbf{H}_h^{\boldsymbol{\eta}}$, $\mathbb{H}_h^{\boldsymbol{\xi}}$, Q_h, $\mathbf{X}_{\tilde{h}}^{\varphi}$, $\mathbb{X}_h^{\boldsymbol{\sigma}}$, $\mathbf{Y}_h^{\mathbf{u}}$, and $\mathbb{Y}_h^{\gamma}$ are presented below, which follow from interpolation estimates for Sobolev spaces and the approximation properties of the relevant orthogonal projectors and the interpolation operators (see, for instance, [11], [12], [17], [22]).

$(\mathbf{AP}_h^{\boldsymbol{\eta}})$ There exists a positive constant C, independent of h, such that for each $\ell \in (0, k+1]$ and for each $\boldsymbol{\chi} \in \mathbf{H}^\ell(\Omega)$, with $\operatorname{div}(\boldsymbol{\chi}) \in \mathrm{H}^\ell(\Omega)$, there holds

$$\operatorname{dist}(\boldsymbol{\chi}, \mathbf{H}_h^{\boldsymbol{\eta}}) \leqslant C\, h^\ell \left\{ \|\boldsymbol{\chi}\|_{\ell,\Omega} + \|\operatorname{div}(\boldsymbol{\chi})\|_{\ell,\Omega} \right\}.$$

$(\mathbf{AP}_h^{\boldsymbol{\xi}})$ There exists a positive constant C, independent of h, such that for each $\ell \in [0, k+1]$ and for each $\boldsymbol{\rho} \in \mathbb{H}^\ell(\Omega)$, there holds

$$\operatorname{dist}(\boldsymbol{\rho}, \mathbb{H}_h^{\boldsymbol{\xi}}) \leqslant C\, h^\ell \, \|\boldsymbol{\rho}\|_{\ell,\Omega}\,.$$

$(\mathbf{AP}_h^{p})$ There exists a positive constant C, independent of h, such that for each $\ell \in [0, k+1]$ and for each $q \in \mathrm{H}^\ell(\Omega)$, there holds

$$\operatorname{dist}(q, \mathrm{Q}_h) \leqslant C\, h^\ell \, \|q\|_{\ell,\Omega}\,.$$

$(\mathbf{AP}_h^{\varphi})$ There exists a positive constant C, independent of $\widetilde{h}$, such that for each $\ell \in (0, k+1]$ and for each $\psi \in \mathrm{H}_{00}^{1/2+\ell}(\Gamma)$, there holds

$$\operatorname{dist}(\psi, \mathrm{X}_{\widetilde{h}}^{\varphi}) \leqslant C\, \widetilde{h}^\ell \, \|\psi\|_{1/2+\ell,00,\Gamma}\,.$$

$(\mathbf{AP}_h^{\boldsymbol{\sigma}})$ There exists a positive constant C, independent of h, such that for each $\ell \in (0, k+1]$ and for each $\boldsymbol{\tau} \in \mathbb{H}^\ell(\Omega) \cap \mathbb{H}_\Sigma(\mathbf{div};\Omega)$, with $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{H}^\ell(\Omega)$, there holds

$$\operatorname{dist}(\boldsymbol{\tau}, \mathbb{X}_h^{\boldsymbol{\sigma}}) \leqslant C\, h^\ell \left\{ \|\boldsymbol{\tau}\|_{\ell,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{\ell,\Omega} \right\}.$$

$(\mathbf{AP}_h^{\mathbf{u}})$ There exists a positive constant C, independent of h, such that for each $\ell \in [0, k+1]$ and for each $\mathbf{v} \in \mathbf{H}^\ell(\Omega)$, there holds

$$\operatorname{dist}(\mathbf{v}, \mathbf{Y}_h^{\mathbf{u}}) \leqslant C\, h^\ell \, \|\mathbf{v}\|_{\ell,\Omega}\,.$$

$(\mathbf{AP}_h^{\gamma})$ There exists a positive constant C, independent of h, such that for each $\ell \in [0, k+1]$ and for each $\boldsymbol{\eta} \in \mathbb{H}^\ell(\Omega) \cap \mathbb{L}^2_{\mathrm{skew}}(\Omega)$, there holds

$$\operatorname{dist}(\boldsymbol{\eta}, \mathbb{Y}_h^{\gamma}) \leqslant C\, h^\ell \, \|\boldsymbol{\eta}\|_{\ell,\Omega}\,.$$

Theorem 5.1. *In addition to the hypotheses of Theorem 4.6, assume that there exists $\ell \in (0, k+1]$ such that $\boldsymbol{\eta} \in \mathbf{H}^\ell(\Omega)$, $\operatorname{div}(\boldsymbol{\eta}) \in \mathrm{H}^\ell(\Omega)$, $\boldsymbol{\xi} \in \mathbb{H}^\ell(\Omega)$, $p \in \mathrm{H}^\ell(\Omega)$, $\varphi \in \mathrm{H}_{00}^{1/2+\ell}(\Gamma)$, $\boldsymbol{\sigma} \in \mathbb{H}^\ell(\Omega)$, $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{H}^\ell(\Omega)$, $\mathbf{u} \in \mathbf{H}^\ell(\Omega)$, and $\boldsymbol{\gamma} \in \mathbb{H}^\ell(\Omega)$. Furthermore, let Θ and $\Theta_{\underline{h}}$ be the continuous and discrete solutions, respectively, as in (4.25). Then, there exists $C > 0$, independent of h and $\widetilde{h}$, such that for all $h \leqslant C_0\, \widetilde{h}$, there holds*

$$\|\Theta - \Theta_{\underline{h}}\| \leqslant C\, \Psi_\ell(\Theta) \left(h^\ell + \widetilde{h}^\ell \right),$$

where

$$\Psi_\ell(\Theta) := \|\boldsymbol{\eta}\|_{\ell,\Omega} + \|\operatorname{div}(\boldsymbol{\eta})\|_{\ell,\Omega} + \|\boldsymbol{\xi}\|_{\ell,\Omega} + \|p\|_{\ell,\Omega} + \|\boldsymbol{\sigma}\|_{\ell,\Omega} + \|\mathbf{div}(\boldsymbol{\sigma})\|_{\ell,\Omega} + \|\mathbf{u}\|_{\ell,\Omega} + \|\boldsymbol{\gamma}\|_{\ell,\Omega} + \|\varphi\|_{1/2+\ell,00,\Gamma}\,.$$

Proof. The result follows from a straightforward application of the Céa estimate (4.26) along with the foregoing approximation properties. We omit further details. □

We end this section by stressing that the convexity of Ω guaranteeing the stability and convergence of our mixed finite element method is forced by the corresponding elliptic regularity result for the Poisson equation with mixed boundary conditions (cf. [25], [26]), as required in the proofs of Lemmas 4.1, 4.3, and 4.4, and Theorem 4.6. Nevertheless, in the following section we illustrate that even for nonconvex domains we obtain the theoretical rates of convergence predicted by Theorem 5.1, which, on one hand, confirms that numerical essays are usually more generous than the abstract theory, and, on the other hand, suggests that perhaps only technical difficulties stop us of proving the well-posedness of the Galerkin scheme in an arbitrary region. In turn, needless to say, the convexity assumption is certainly not needed when either Dirichlet or Neumann boundary conditions are considered since in this case a regularity $\delta > 1/2$ is ensured for any Lipschitz-continuous domain with largest interior angle $\omega < 2\,\pi$ (cf. [26]).

6 Numerical results

In this section, we illustrate the performance of the mixed finite element method (4.1) using the specific choices of discrete spaces introduced in (5.1) and (5.2). The implementation was carried out with the open-source finite element library `FEniCS` [1] and, in particular, with the specialized module $\texttt{FEniCS}_{ii}$ [29], which is required to handle mixed-dimensional, non-conforming meshes and is also instrumental in the numerical realization of the $\mathrm{H}^{1/2}_{00}(\Gamma)$ norm. The first two examples are devoted to corroborating the convergence rates predicted by Theorem 5.1, both on the unit square and on a nonconvex domain. The third example examines the method's performance in a three-dimensional setting.

As usual, we introduce the following notation to denote the associated errors to each unknown in (4.1):

$$\mathrm{e}(\boldsymbol{\eta}) := \|\boldsymbol{\eta} - \boldsymbol{\eta}_h\|_{\mathrm{div};\Omega}\,, \quad \mathrm{e}(\boldsymbol{\xi}) := \|\boldsymbol{\xi} - \boldsymbol{\xi}_h\|_{0,\Omega}\,, \quad \mathrm{e}(p) := \|p - p_h\|_{0,\Omega}\,, \quad \mathrm{e}(\varphi) := \|\varphi - \varphi_{\widetilde{h}}\|_{1/2,00,\Gamma}\,,$$
$$\mathrm{e}(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div};\Omega}\,, \quad \mathrm{e}(\mathbf{u}) := \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}\,, \quad \text{and} \quad \mathrm{e}(\boldsymbol{\gamma}) := \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,\Omega}\,.$$

Furthermore, we recall that the experimental rates of convergence are computed as

$$\mathrm{r}(\diamond) := \frac{\log\big(\mathrm{e}(\diamond)/\mathrm{e}'(\diamond)\big)}{\log\big(h/h'\big)} \quad \text{for } \diamond \in \{\boldsymbol{\eta}, \boldsymbol{\xi}, p, \boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}\}\,, \quad \text{and} \quad \mathrm{r}(\varphi) := \frac{\log(\mathrm{e}(\varphi)/\mathrm{e}'(\varphi))}{\log(\widetilde{h}/\widetilde{h}')}\,,$$

where h and h' (resp. $\widetilde{h}$ and $\widetilde{h}'$) are consecutive mesh sizes with respective errors e and e$'$. In order to compute $\mathrm{e}(\varphi)$, we employ the characterization of $\mathrm{H}^{1/2}_{00}(\Gamma)$ in terms of the spectral decomposition of the Laplacian operator (see, for instance, [30]). More precisely, let $S : \mathrm{H}^1_0(\Gamma) \to \mathrm{H}^1_0(\Gamma)$ be the linear and bounded operator uniquely determined by the relation

$$(S(u), v)_{1,\Gamma} = (u, v)_{0,\Gamma} \qquad \forall\, u, v \in \mathrm{H}^1_0(\Gamma)\,,$$

where $(\cdot,\cdot)_{1,\Gamma}$ and $(\cdot,\cdot)_{0,\Gamma}$ denote the inner products of $\mathrm{H}^1_0(\Gamma)$ and $\mathrm{L}^2(\Gamma)$, respectively. Then, one can define a basis $\{z_i\}_{i=1}^{\infty}$ of eigenfunctions of S with a non-increasing sequence of positive eigenvalues λ_i, and for any $u = \sum_{i=1}^{\infty} c_i\, z_i$ there holds

$$\|u\|^2_{1/2,00,\Gamma} = \sum_{i=1}^{\infty} c_i^2\, \lambda_i^{1/2}\,,$$

so that $\mathrm{H}^{1/2}_{00}(\Gamma)$ becomes the closure of the span of the basis $\{z_i\}_{i=1}^{\infty}$ with respect to this norm. Naturally, for the practical computation of $\|u\|^2_{1/2,00,\Gamma}$ one employs a discrete approximation of the aforementioned spectral decomposition.

Example 1: Convergence against smooth exact solutions in a 2D domain

In this test, we analyze the convergence with respect to the spatial discretization using a manufactured solution. The computational domain is the square $\Omega := (0,1)^2$, which is meshed by successively refined regular triangles. In addition, the boundary $\partial\Omega$ is partitioned into two parts, Γ and Σ, where Γ corresponds to the left and bottom sides of the square, whereas Σ denotes the union of the top and right sides. We take the physical adimensional parameters as

$$\mu = \lambda = 1 \quad c_0 = \alpha = 0.1\,, \quad \text{and} \quad \kappa(x,y) := \exp(xy)\,, \tag{6.1}$$

and adjust the source terms $\mathbf{f}$ and g so that the following manufactured solutions coincide to the prescribed analytical solutions (cf. (2.6)),

$$\mathbf{u} = 0.05 \begin{pmatrix} \cos\big(1.5\,\pi(x+y)\big) \\ \sin\big(1.5\,\pi(x-y)\big) \end{pmatrix} \quad \text{and} \quad p = \sin(\pi x)\,\sin(\pi y)\,.$$

The system is complemented with suitable non-homogeneous boundary conditions, which generalize (2.7). The analysis made in the previous sections can be adapted to handle this framework by employing a lifting technique and adding some terms to the right-hand side of the weak formulation (3.9). As requested by the constraint $h \leqslant C_0 \widetilde{h}$ introduced in Lemma 4.2, the boundary mesh for the Lagrange multiplier associated with the pressure is constructed one level lower than a conforming mesh to the boundary of the bulk mesh. That is, we construct the former with $2^{j+1}+2$ segments per side and the latter with 2^j+1 segments per side, giving $\widetilde{h} \approx 2h$. Tables 6.1 and 6.2 show the convergence history of the method, with $k \in \{0,1\}$, and confirm the optimal rates of convergence predicted by Theorem 5.1. In addition, Figure 6.1 shows the solutions obtained by the mixed scheme with $k = 1$ and mesh-sizes $h = 0.021$ and $\widetilde{h} = 0.043$, thus using 436,790 DOF.

Discretization with $k=0$ for $\boldsymbol{\eta}$, $\boldsymbol{\xi}$, p and φ										
DOF	h	$\widetilde{h}$	e($\boldsymbol{\eta}$)	r($\boldsymbol{\eta}$)	e($\boldsymbol{\xi}$)	r($\boldsymbol{\xi}$)	e(p)	r(p)	e(φ)	r(φ)
550	0.354	0.707	4.11e+00	–	1.50e-01	–	1.78e-01	–	9.91e-03	–
1208	0.236	0.471	2.80e+00	0.95	1.04e-01	0.90	1.20e-01	0.97	5.99e-03	1.24
3292	0.141	0.283	1.70e+00	0.98	6.38e-02	0.95	7.24e-02	0.99	3.20e-03	1.23
10532	0.079	0.157	9.48e-01	0.99	3.59e-02	0.98	4.03e-02	1.00	1.34e-03	1.48
37300	0.042	0.083	5.03e-01	1.00	1.92e-02	0.99	2.13e-02	1.00	4.45e-04	1.73
139988	0.021	0.043	2.59e-01	1.00	9.90e-03	0.99	1.09e-02	1.00	1.29e-04	1.86

Discretization with $k=0$ for $\boldsymbol{\sigma}$, $\mathbf{u}$, and $\boldsymbol{\gamma}$							
DOF	h	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e($\mathbf{u}$)	r($\mathbf{u}$)	e($\boldsymbol{\gamma}$)	r($\boldsymbol{\gamma}$)
550	0.354	2.24e+00	–	2.98e-02	–	1.07e-01	–
1208	0.236	1.53e+00	0.95	1.96e-02	1.03	6.33e-02	1.30
3292	0.141	9.25e-01	0.98	1.17e-02	1.02	3.32e-02	1.26
10532	0.079	5.16e-01	0.99	6.45e-03	1.01	1.65e-02	1.18
37300	0.042	2.74e-01	1.00	3.41e-03	1.00	8.20e-03	1.10
139988	0.021	1.41e-01	1.00	1.76e-03	1.00	4.10e-03	1.04

Table 6.1: [Example 1, $k = 0$] Number of degrees of freedom, meshsizes, errors, and rates of convergence.

Discretization with $k=1$ for $\boldsymbol{\eta}$, $\boldsymbol{\xi}$, p and φ										
DOF	h	$\widetilde{h}$	e($\boldsymbol{\eta}$)	r($\boldsymbol{\eta}$)	e($\boldsymbol{\xi}$)	r($\boldsymbol{\xi}$)	e(p)	r(p)	e(φ)	r(φ)
1674	0.354	0.707	5.84e-01	–	3.18e-02	–	1.96e-02	–	2.58e-03	–
3710	0.236	0.471	2.62e-01	1.98	1.47e-02	1.90	8.78e-03	1.98	5.25e-04	3.92
10182	0.141	0.283	9.48e-02	1.99	5.46e-03	1.94	3.18e-03	1.99	1.25e-04	2.80
32726	0.079	0.157	2.93e-02	2.00	1.72e-03	1.97	9.82e-04	2.00	1.21e-05	3.98
116214	0.042	0.083	8.22e-03	2.00	4.85e-04	1.99	2.75e-04	2.00	1.70e-06	3.08
436790	0.021	0.043	2.18e-03	2.00	1.29e-04	2.00	7.31e-05	2.00	2.85e-07	2.69

Discretization with $k=1$ for $\boldsymbol{\sigma}$, $\mathbf{u}$, and $\boldsymbol{\gamma}$							
DOF	h	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e($\mathbf{u}$)	r($\mathbf{u}$)	e($\boldsymbol{\gamma}$)	r($\boldsymbol{\gamma}$)
1674	0.354	4.73e-01	–	6.40e-03	–	3.22e-02	–
3710	0.236	2.17e-01	1.92	3.32e-03	1.62	1.68e-02	1.60
10182	0.141	7.96e-02	1.97	1.29e-03	1.85	6.83e-03	1.76
32726	0.079	2.48e-02	1.99	4.08e-04	1.96	2.33e-03	1.83
116214	0.042	6.96e-03	2.00	1.15e-04	1.99	6.95e-04	1.90
436790	0.021	1.85e-03	2.00	3.06e-05	2.00	1.89e-04	1.96

Table 6.2: [Example 1, $k = 1$] Number of degrees of freedom, meshsizes, errors, and rates of convergence.

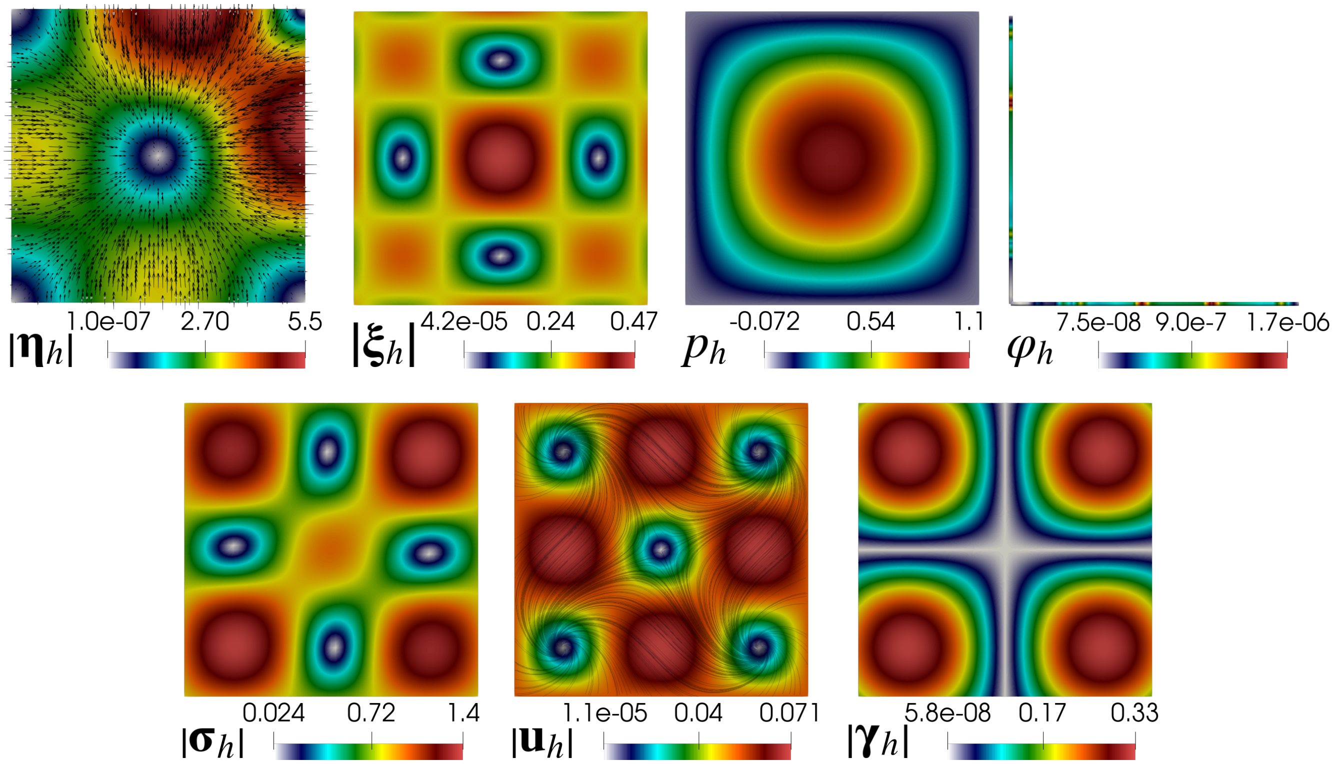

Figure 6.1: [Example 1] Computed solutions of the mixed scheme. The velocity field (center bottom) is displayed using line integral convolution (LIC).

Example 2: Convergence against smooth exact solutions in a nonconvex domain

We now consider the two-dimensional nonconvex domain Ω depicted in the first part of Figure 6.2. We use the same parameters as in the previous example (cf. (6.1)), while the manufactured solutions are given by

$$\mathbf{u} = 0.05 \begin{pmatrix} \cos\big(1.5\,\pi(x+y)\big) \\ \sin\big(1.5\,\pi(x-y)\big) \end{pmatrix} \quad \text{and} \quad p = (y-2x)^2\, y\, \cos(\pi x y)\,.$$

The boundary mesh is constructed in a similar manner as before. Furthermore, to obtain better convergence rates for φ, we introduce the additional term $-\varepsilon\,(\varphi_{\widetilde{h}}, \psi_{\widetilde{h}})_\Gamma$ in the third row of the left-hand side of (4.1). This term acts as a perturbation and helps the algebraic system become more stable. In our case, we set $\varepsilon = 10^{-12}$. In this context, and as anticipated at the end of Section 5, Tables 6.3 and 6.4 show that optimal convergence rates are achieved in most of the experiments. This observation suggests that Theorem 5.1 may be extended to a broader class of domains. Nevertheless, for the finest meshes we observe a deterioration in the convergence order, which is likely attributable to the non-convexity of the domain. Figure 6.2 depicts the numerical solutions obtained with the mixed scheme for $k = 1$, $h = 0.005$ and $\widetilde{h} = 0.012$, using 785,594 `DOF`.

Example 3: Convergence against smooth exact solutions in a 3D domain

In the final example, we consider the unit cube $\Omega = (0,1)^3$. The parameters μ, λ, c_0, and α are chosen as in (6.1), while the permeability κ is defined by

$$\kappa(x,y,z) := \exp(-xyz)\,.$$

In addition, the source terms $\mathbf{f}$ and g are defined in such a way that the manufactured solutions presented below coincide with the prescribed analytical solutions (cf. (2.6)). The corresponding velocity and pressure

Discretization with $k=0$ for $\boldsymbol{\eta}$, $\boldsymbol{\xi}$, p and φ										
DOF	h	$\widetilde{h}$	e($\boldsymbol{\eta}$)	r($\boldsymbol{\eta}$)	e($\boldsymbol{\xi}$)	r($\boldsymbol{\xi}$)	e(p)	r(p)	e(φ)	r(φ)
1002	0.147	0.280	6.16e-01	–	4.85e-02	–	1.57e-02	–	2.42e-02	–
4050	0.079	0.149	3.12e-01	1.10	2.53e-02	1.05	8.03e-03	1.08	8.66e-03	1.63
15682	0.041	0.079	1.51e-01	1.09	1.32e-02	0.97	3.70e-03	1.16	4.18e-03	1.14
62562	0.020	0.042	7.85e-02	0.94	6.84e-03	0.95	1.86e-03	0.99	2.04e-03	1.15
251714	0.010	0.024	4.07e-02	0.95	3.67e-03	0.91	9.32e-04	1.01	1.03e-03	1.20
999426	0.005	0.012	2.15e-02	0.92	2.06e-03	0.83	4.68e-04	1.00	5.14e-04	1.02

Discretization with $k=0$ for $\boldsymbol{\sigma}$, $\mathbf{u}$, and $\boldsymbol{\gamma}$							
DOF	h	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e($\mathbf{u}$)	r($\mathbf{u}$)	e($\boldsymbol{\gamma}$)	r($\boldsymbol{\gamma}$)
1002	0.147	5.49e-01	–	6.95e-03	–	5.74e-02	–
4050	0.079	2.84e-01	1.07	3.45e-03	1.13	2.43e-02	1.38
15682	0.041	1.43e-01	1.02	1.77e-03	0.99	1.20e-02	1.06
62562	0.020	7.12e-02	1.01	8.49e-04	1.06	6.16e-03	0.96
251714	0.010	3.61e-02	0.99	4.18e-04	1.03	3.21e-03	0.95
999426	0.005	1.86e-02	0.95	2.05e-04	1.03	1.80e-03	0.83

Table 6.3: [Example 2, $k=0$] Number of degrees of freedom, meshsizes, errors, and rates of convergence.

Discretization with $k=1$ for $\boldsymbol{\eta}$, $\boldsymbol{\xi}$, p and φ										
DOF	h	$\widetilde{h}$	e($\boldsymbol{\eta}$)	r($\boldsymbol{\eta}$)	e($\boldsymbol{\xi}$)	r($\boldsymbol{\xi}$)	e(p)	r(p)	e(φ)	r(φ)
3064	0.147	0.280	6.55e-02	–	3.60e-03	–	2.31e-03	–	5.73e-03	–
12526	0.079	0.149	1.75e-02	2.13	9.31e-04	2.18	5.73e-04	2.25	1.74e-03	1.89
48750	0.041	0.079	4.54e-03	2.02	2.30e-04	2.09	1.41e-04	2.09	5.13e-04	1.92
194998	0.020	0.042	1.20e-03	1.92	5.64e-05	2.03	3.48e-05	2.02	1.25e-04	2.27
785594	0.010	0.024	3.12e-04	1.96	1.42e-05	2.00	8.62e-06	2.03	3.24e-05	2.36
3121186	0.005	0.012	8.17e-05	1.93	3.60e-06	1.98	2.16e-06	2.00	1.16e-05	1.51

Discretization with $k=1$ for $\boldsymbol{\sigma}$, $\mathbf{u}$, and $\boldsymbol{\gamma}$							
DOF	h	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e($\mathbf{u}$)	r($\mathbf{u}$)	e($\boldsymbol{\gamma}$)	r($\boldsymbol{\gamma}$)
3064	0.147	4.44e-02	–	6.97e-04	–	4.19e-03	–
12526	0.079	1.09e-02	2.26	1.74e-04	2.24	1.07e-03	2.21
48750	0.041	2.74e-03	2.06	4.31e-05	2.08	2.62e-04	2.10
194998	0.020	6.85e-04	2.00	1.08e-05	2.00	6.73e-05	1.96
785594	0.010	1.69e-04	2.04	2.67e-06	2.03	1.67e-05	2.03
3121186	0.005	4.25e-05	1.99	6.71e-07	1.99	4.28e-06	1.96

Table 6.4: [Example 2, $k=1$] Number of degrees of freedom, meshsizes, errors, and rates of convergence.

fields are defined by

$$\mathbf{u} = 0.1 \begin{pmatrix} \sin(x)\,\cos(y)\,\cos(z) + 0.5\,x^2 \\ -2\,\cos(x)\,\sin(y)\,\cos(z) + 0.5\,y^2 \\ \cos(x)\,\cos(y)\,\sin(z) + 0.5\,z^2 \end{pmatrix} \quad \text{and} \quad p = \sin(\pi x)\,\sin(\pi y)\,\sin(\pi z)\,.$$

The independent boundary mesh is constructed as in the first example, now in the three-dimensional setting. The convergence history is reported in Table 6.5, where optimal convergence rates are observed for most variables, in agreement with the predictions of Theorem 5.1 for $k=0$. For one of the variables, however, the optimal rate is not fully attained on the meshes considered. This behavior is likely due to a pre-asymptotic effect, since the mesh refinements required to enter the asymptotic regime would be computationally prohibitive. In addition, the numerical solutions are depicted in Figure 6.3 for mesh sizes $h = 0.108$ and $\widetilde{h} = 0.192$, using 979,618 degrees of freedom.

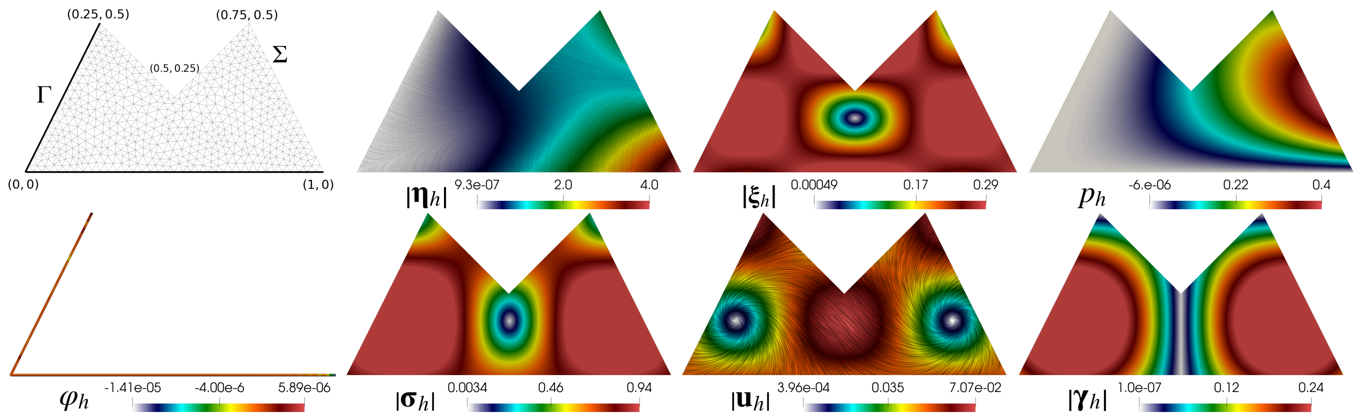

Figure 6.2: [Example 2] Nonconvex domain under consideration (top left). The black portion of the boundary represents Γ, while the remaining part corresponds to Σ. The mesh shown has size $h = 0.041$. Computed solutions of the mixed scheme (remaining panels). The pressure flux and the velocity fields are displayed using line integral convolution (LIC), colored by their respective magnitudes.

Discretization with $k = 0$ for $\boldsymbol{\eta}$, $\boldsymbol{\xi}$, p and φ										
DOF	h	$\widetilde{h}$	e($\boldsymbol{\eta}$)	r($\boldsymbol{\eta}$)	e($\boldsymbol{\xi}$)	r($\boldsymbol{\xi}$)	e(p)	r(p)	e(φ)	r(φ)
2068	0.866	0.866	5.63e+00	0.00	4.83e-02	0.00	2.28e-01	0.00	3.09e+01	0.00
15772	0.433	0.577	3.09e+00	0.86	2.83e-02	0.77	1.14e-01	0.99	2.33e-02	17.73
123622	0.217	0.346	1.59e+00	0.96	1.59e-02	0.83	5.85e-02	0.97	4.04e-03	3.43
979618	0.108	0.192	7.98e-01	0.99	8.98e-03	0.82	2.95e-02	0.99	1.23e-03	2.02

Discretization with $k = 0$ for $\boldsymbol{\sigma}$, $\mathbf{u}$, and $\boldsymbol{\gamma}$							
DOF	h	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e($\mathbf{u}$)	r($\mathbf{u}$)	e($\boldsymbol{\gamma}$)	r($\boldsymbol{\gamma}$)
2068	0.866	2.28e-01	0.00	2.38e-02	0.00	6.14e-02	0.00
15772	0.433	1.26e-01	0.86	1.16e-02	1.04	2.48e-02	1.31
123622	0.217	6.67e-02	0.92	5.77e-03	1.01	1.08e-02	1.20
979618	0.108	3.54e-02	0.91	2.88e-03	1.00	5.09e-03	1.09

Table 6.5: [Example 3, $k = 0$] Number of degrees of freedom, meshsizes, errors, and rates of convergence.

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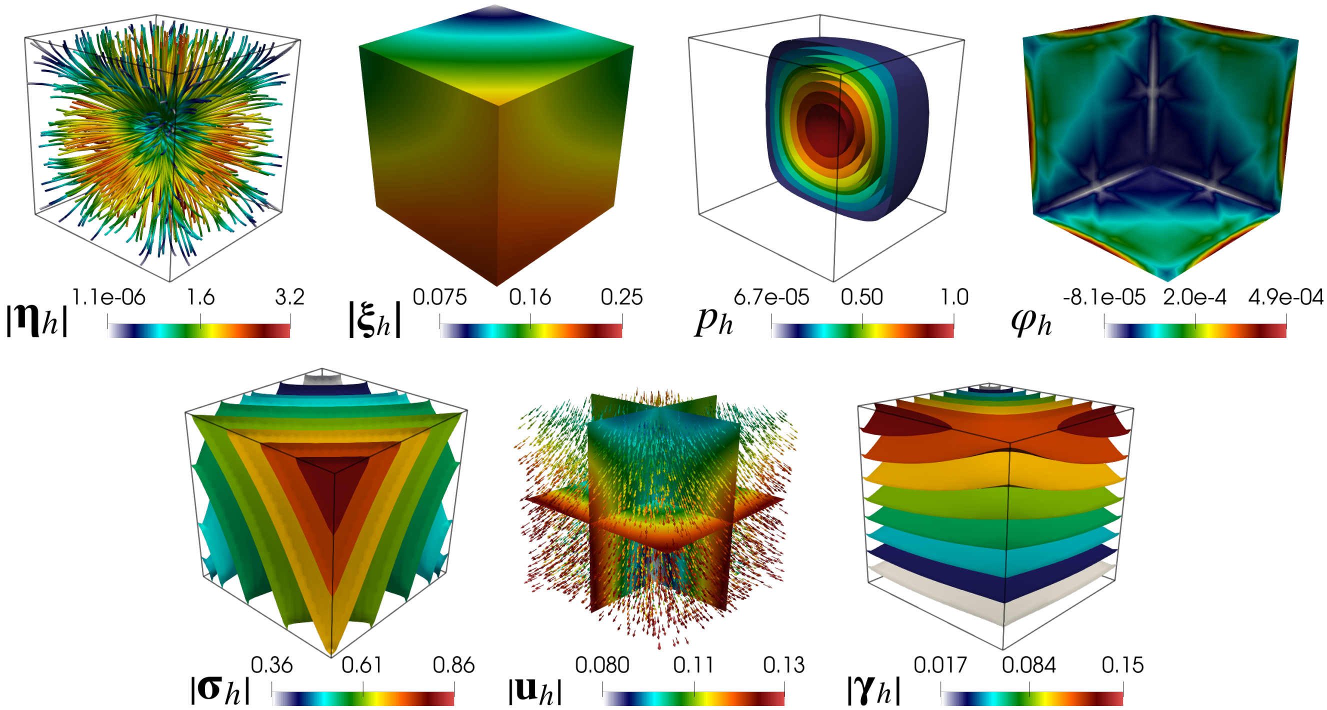

Figure 6.3: [Example 3] Computed solutions of the mixed scheme in the unit cube.

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