Mixed formulations for poroelasticity/free-flow using total pressure

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(joint work with Wietse M. Boon, Martin Hørnkjol, Miroslav Kuchta, Kent-André Mardal, Matteo Taffetani, Hans D. Westermeyer, Ivan Yotov)

We consider a multiphysics model for the flow of Newtonian fluid coupled with Biot consolidation equations through an interface Σ (see, e.g., [3]). Let $t \in (0, t_{\mathrm{end}}]$ and take a bounded connected Lipschitz spatial domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$:

$$
\begin{aligned}
-\mathbf{div}[2\mu_f \boldsymbol{\epsilon}(\boldsymbol{u}) - p_F \mathbf{I}] = \rho_f \boldsymbol{g}; \quad \operatorname{div} \boldsymbol{u} = 0 \qquad & \text{in } \Omega_F \times (0, t_{\mathrm{end}}],\\
-\mathbf{div}[2\mu_s \boldsymbol{\epsilon}(\boldsymbol{d}) - \varphi \mathbf{I}] = \rho_s \boldsymbol{f}; \quad \varphi - \alpha p_P + \lambda \operatorname{div} \boldsymbol{d} = 0 \qquad & \text{in } \Omega_P \times (0, t_{\mathrm{end}}],\\
\big(C_0 + \frac{\alpha^2}{\lambda}\big)\frac{1}{\Delta t} p_P - \frac{\alpha}{(\Delta t)\lambda}\varphi - \operatorname{div}\left(\frac{\kappa}{\mu_f}\nabla p_P - \rho_f \boldsymbol{g}\right) = m_P \qquad & \text{in } \Omega_P \times (0, t_{\mathrm{end}}].
\end{aligned}
\tag{1}
$$

We consider mixed boundary conditions on both subdomains and the transmission conditions on Σ are (where $T_{\boldsymbol{n}}$, $T_{\boldsymbol{t}}$ denote normal and tangential trace operators)

$$
T_{\boldsymbol{n}} \boldsymbol{u} = T_{\boldsymbol{n}}\big(\frac{1}{\Delta t}\boldsymbol{d} - \frac{\kappa}{\mu_f}\nabla p_P\big); \quad T_{\boldsymbol{n}}(2\mu_f \boldsymbol{\epsilon}(\boldsymbol{u}) - p_F \mathbf{I}) = T_{\boldsymbol{n}}(2\mu_s \boldsymbol{\epsilon}(\boldsymbol{d}) - \varphi \mathbf{I}),
$$

$$
-T_{\boldsymbol{n}} T_{\boldsymbol{n}}(2\mu_f \boldsymbol{\epsilon}(\boldsymbol{u}) - p_F \mathbf{I}) = p_P; \quad -T_{\boldsymbol{n}} T_{\boldsymbol{t}}(2\mu_f \boldsymbol{\epsilon}(\boldsymbol{u}) - p_F \mathbf{I}) = \frac{\gamma \mu_f}{\sqrt{\kappa}} T_{\boldsymbol{t}}\big(\boldsymbol{u} - \frac{1}{\Delta t}\boldsymbol{d}\big).
$$

The stability and well-posedness of the semi-discrete problem are derived, and we also obtain the following result (see [5]).

Theorem 1. *For each $\boldsymbol{f} \in H^1(0, t_{\mathrm{end}}; \mathbf{L}^2(\Omega_P))$ and $p_{P,0} \in H^1_\star(\Omega_p)$, there exist initial data $\boldsymbol{u}_0 \in \mathbf{H}^1_\star(\Omega_F)$, $p_{F,0} \in L^2(\Omega_F)$, $\boldsymbol{d}_0 \in \mathbf{H}^1_\star(\Omega_P)$, and $\varphi_0 \in L^2(\Omega_P)$ such that the weak formulation of* (1) *complemented with the initial conditions $p_P(0) = p_{P,0}$, $\boldsymbol{d}(0) = \boldsymbol{d}_0$, and $\varphi(0) = \varphi_0$, has a unique solution.*

A new mixed-primal finite element scheme is proposed solving for the pairs fluid velocity - pressure and displacement - total poroelastic pressure using Stokes-stable elements. Optimal convergence rates are established, which are robust with respect to λ (see Figure 1). Upon time-discretisation, we are left with the Biot–Stokes equations written in the operator form $\mathcal{A}(\boldsymbol{u}, \boldsymbol{d}, p_F, \varphi, p_P)^{\mathtt{t}} = \mathcal{F}$, where

$$
\left(\begin{array}{cc|ccc}
-2\mu_f \,\mathbf{div}\,\boldsymbol{\epsilon} + \frac{\gamma\mu_f}{\sqrt{\kappa}} T_{\boldsymbol{t}}' T_{\boldsymbol{t}} & -\frac{\gamma\mu_f}{\sqrt{\kappa}} T_{\boldsymbol{t}}' & \nabla & & T_{\boldsymbol{n}}' \\
-\frac{\gamma\mu_f}{\sqrt{\kappa}} T_{\boldsymbol{t}} & -2\mu_s \,\mathbf{div}\,\boldsymbol{\epsilon} + \frac{\gamma\mu_f}{\sqrt{\kappa}} T_{\boldsymbol{t}}' T_{\boldsymbol{t}} & & \nabla & -T_{\boldsymbol{n}}' \\
\hline
-\operatorname{div} & & & & \\
 & -\operatorname{div} & & -\frac{1}{\lambda} I & \frac{\alpha}{\lambda} I \\
T_{\boldsymbol{n}} & -T_{\boldsymbol{n}} & & \frac{\alpha}{\lambda} I & -\big(C_0 + \frac{\alpha^2}{\lambda}\big) I + \frac{\kappa}{\mu_f}\Delta
\end{array}\right).
\tag{2}
$$

A main challenge for these equations is the construction of solvers that scale properly for nearly incompressible solids where λ tends to infinity, as well as in the case of nearly incompressible fluids, for which C_0 approaches zero, or the nearly impermeable regime where κ is very small. These scenarios entail not only a complication at the practical and implementation level, but also a difficulty inherent to the functional setting of the abstract formulation [4].

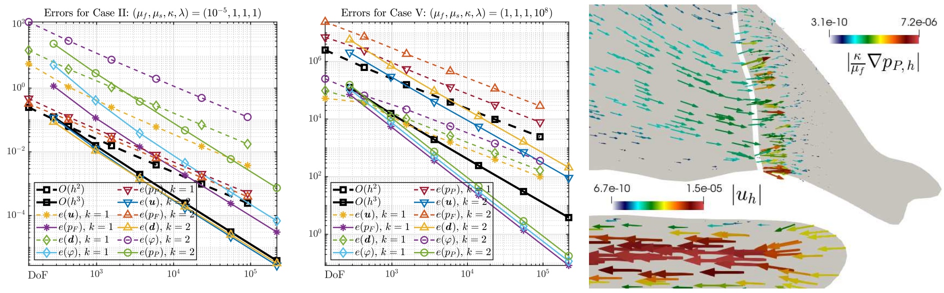

FIGURE 1. Interfacial flow in the eye, between trabecular meshwork and anterior chamber. Experimental error history and sample of axisymmetric numerical solution.

The problem defined by (2) can be shown to be well-posed using the usual space $\boldsymbol{H}$ and its natural metric (for instance, following the analysis performed in Theorem 1). Alternatively, consider the weighted product space $\boldsymbol{H}_\epsilon$, where ϵ encodes the weighting parameters $\kappa, \alpha, \gamma, \mu_f, \mu_s, C_0, \lambda$. Let us group the variables as $\vec{\boldsymbol{u}} = (\boldsymbol{u}, \boldsymbol{d})$ and $\vec{p} = (p_F, \varphi, p_P)$ and introduce the weighted norm

$$
\begin{aligned}
\|(\vec{\boldsymbol{u}}, \vec{p})\|^2_{\mathbf{H}_\epsilon} := {} & 2\mu_f \|\boldsymbol{\epsilon}(\boldsymbol{u})\|^2_{0,\Omega_F} + \frac{\gamma\mu_f}{\sqrt{\kappa}} \|T_{\boldsymbol{t}}(\boldsymbol{u} - \boldsymbol{d})\|^2_{0,\Sigma} + 2\mu_s \|\boldsymbol{\epsilon}(\boldsymbol{d})\|^2_{0,\Omega_P} \\
& + \frac{1}{2\mu_f}\|p_F\|^2_{0,\Omega_F} + \frac{1}{2\mu_s}\|\varphi\|^2_{0,\Omega_P} + \left(\frac{1}{2\mu_f} + \frac{1}{2\mu_s}\right)\|p_P|_\Sigma\|^2_{-\frac{1}{2},01,\Sigma} \\
& + \frac{1}{\lambda}\|\varphi - \alpha p_P\|^2_{0,\Omega_P} + C_0\|p_P\|^2_{0,\Omega_P} + \frac{\kappa}{\mu_f}\|\nabla p_P\|^2_{0,\Omega_P},
\end{aligned}
\tag{3}
$$

which hinges on a fractional norm of the restriction of the Biot pressure to Σ. The space $\mathbf{H}_\epsilon$ is such that contains all $(\vec{\boldsymbol{u}}, \vec{p})$ that are bounded in this norm (see [1]).

Theorem 2. *The problem defined by the solution operator* (2) *is well-posed in the space* $\mathbf{H}_\epsilon$ *equipped with the norm* (3).

A natural block-diagonal preconditioner for the Biot-Stokes problem is therefore the Riesz map with respect to the inner product in $\mathbf{H}_\epsilon$

$$
\mathcal{R} = \left(\begin{array}{cc:ccc}
\mathcal{A}_{FF} & \mathcal{A}_{FP} & & & \\
\mathcal{A}_{PF} & \mathcal{A}_{PP} & & & \\
\hdashline
& & \frac{1}{2\mu_f} I & & \\
& & & \left(\frac{1}{\lambda} + \frac{1}{2\mu_s}\right) I & -\frac{\alpha}{\lambda} I \\
& & & -\frac{\alpha}{\lambda} I & \left(C_0 + \frac{\alpha^2}{\lambda}\right) I - \frac{\kappa}{\mu_f}\Delta + \frac{1}{\mu}(-\Delta_{\Sigma,01})^{-\frac{1}{2}}
\end{array}\right)^{-1},
\tag{4}
$$

where $\mu^{-1} := (2\mu_s)^{-1} + (2\mu_f)^{-1}$. This preconditioner yields robustness with respect to a wide range of material parameters, as reported in Figure 2.

Several open problems and challenges arise as an extension to the results in [1, 5]. For example, the efficient realisation of the preconditioners using algebraic

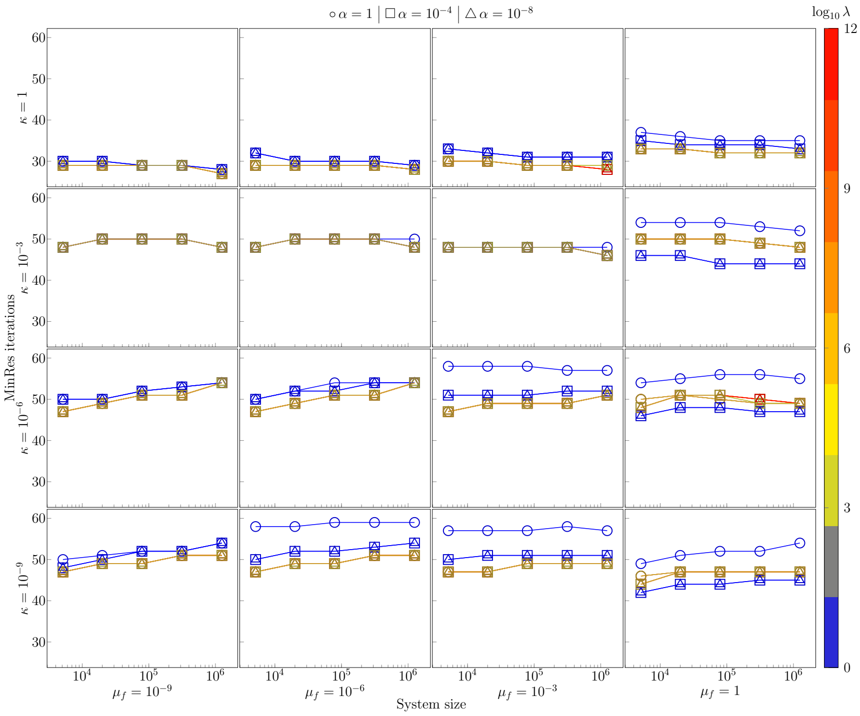

FIGURE 2. Performance of the Biot-Stokes preconditioner (4) setting μ_s, γ to 1 and $C_0 = 0$ and using TH_1 elements.

or geometric multigrid, setting up appropriate scalable solvers that maintain robustness with respect to the timestep, extending the current analysis of robust preconditioners to formulations based on four-field Biot equations (including both total pressure and Darcy flux), and generalising the model to the regime of large deformations and the incorporation of remodelling mechanisms better describing the consolidation of the interface and choking phenomena in eye poromechanics.

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