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A discontinuous method for oil-water flow in heterogeneous porous media

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An oil-water flow-transport problem in a heterogeneous porous medium (HPM), given by the Brinkman model coupled with a fractional flow formulation, is solved numerically by a discontinuous finite volume element (DFVE) method coupled with a Runge-Kutta Discontinuous Galerkin (RKDG) scheme. A new (original) numerical example is presented.

1 Introduction and numerical scheme

This contribution is focused on the accurate simulation of viscous two-phase flow (e.g., of oil (o) and water (w)) in an HPM. The flow of the mixture is governed by the Brinkman model while the interaction of the two phases is described by the fractional flow formalism. Specifically, we consider the saturations $\phi_{\mathrm{w}} = \phi$ and $\phi_{\mathrm{o}} = 1 - \phi$, respectively, in a domain $\Omega \subset \mathbb{R}^2$. If the fluids are incompressible and capillary forces are negligible, then the model

$$\text{(a) } \mathbf{K}^{-1}(\boldsymbol{x})\boldsymbol{u} - \mathbf{div}\big(\mu(\phi)\boldsymbol{\varepsilon}(\boldsymbol{u}) - p\mathbf{I}\big) - \phi\boldsymbol{g} = \mathbf{0}, \quad \text{(b) } \operatorname{div}\boldsymbol{u} = 0, \quad \text{(c) } \partial_t\phi + \operatorname{div}\mathbf{F}(\phi,\boldsymbol{u},\boldsymbol{x}) = 0, \tag{1}$$

along with initial and boundary conditions, describes the viscous motion of the mixture in a porous medium. The unknowns are the volume average flow velocity $\boldsymbol{u}$, the saturation ϕ, and the pressure p. Here $\mu(\phi)\boldsymbol{\varepsilon}(\boldsymbol{u}) - p\mathbf{I}$ is the Cauchy stress tensor, $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^{\mathrm{T}})$, $\mu = \mu(\phi)$ is the viscosity, $\mathbf{F}(\phi,\boldsymbol{u},\boldsymbol{x}) = f(\phi)\boldsymbol{u} + b(\phi)\mathbf{K}(\boldsymbol{x})\boldsymbol{g}$ is a nonlinear flux vector, $\mathbf{K}$ is the permeability tensor of the medium, and $\boldsymbol{g}$ is the gravity acceleration. The fractional flow functions f and b are specified later.

Numerical methods for (1) should accurately capture flux discontinuities, satisfy discrete maximum principles, produce divergence-free approximations of $\boldsymbol{u}$, and ensure local mass conservation. Methods for (1 a, b) should permit a natural development of error estimates (associated with finite element formulations), be mass conservative by construction (therefore be related to mixed formulations, or to pure finite volume schemes), and handle unstructured meshes. We therefore propose DFVE methods for the approximation of (1 a, b); that alternates with an RKDG approximation of (1 c), see [3] for details.

The finite volume element (FVE) method goes back to [2, 4]. To explain the main idea, consider a primal mesh $\mathcal{T}_h$ that is a regular, quasi-uniform partition of Ω by closed triangles. The dual mesh $\mathcal{T}_h^\sharp$ consists of all sub-triangles obtained by dividing each $K \in \mathcal{T}_h$ into sub-triangles by connecting its barycenter to its vertices, cf. Figure 1 (top). The FVE method is a special class of Petrov-Galerkin methods where the trial function spaces for $\boldsymbol{u}$ and p are associated with $\mathcal{T}_h$ while the test functions are associated with the dual partition $\mathcal{T}_h^\sharp$ induced by the control volumes. A particular projection operator then ensures that the final DFVE scheme can be written as a Galerkin scheme where trial and test spaces, namely $\{\boldsymbol{v} \in \mathbf{L}^2(\Omega) : \boldsymbol{v}|_K \in \mathbb{P}_1(K)^2\,\forall K \in \mathcal{T}_h\}$ for $\boldsymbol{u}$ and $\{q \in L_0^2(\Omega) : q|_K \in \mathbb{P}_0(K)\,\forall K \in \mathcal{T}_h\}$ for p, coincide. Thus local conservativity is inherited from the finite volume part of the method, while maintaining the versatility and systematic L^2 error analysis.

The novel ingredient of the RKDG part is the numerical flux J that has to be defined along the boundary σ between two finite volumes, say $L, K \in \mathcal{T}_h$, with normal $\boldsymbol{n} = (n_1, n_2)$. The difficulty is that $\mathbf{F}$ is non-monotone in ϕ and discontinuous in $\boldsymbol{x}$. Let α and β denote the approximate values of ϕ on K and L, respectively, $\boldsymbol{u}_h(\boldsymbol{x})$ the velocity (continuous across each edge), and $\boldsymbol{n}$ the normal to σ. Then the extension to the situation at hand of the DFLU flux [1], a modified Godunov numerical flux appropriate for fluxes that depend discontinuously on $\boldsymbol{x}$, gives $J = J(\alpha,\beta,\boldsymbol{u}_h(\boldsymbol{x}),\boldsymbol{x},\boldsymbol{n}) = \hat{F}_1(\alpha,\beta,n_1) + \hat{F}_2(\alpha,\beta,n_2)$, where $\hat{F}_1$ and $\hat{F}_2$ are now computed as follows, where we use the notation $F_i^- n_i(\cdot) = F_i(\cdot,\boldsymbol{u}_h(\boldsymbol{x}),\boldsymbol{x}^-)n_i$ and $F_i^+ n_i(\cdot) = F_i(\cdot,\boldsymbol{u}_h(\boldsymbol{x}),\boldsymbol{x}^+)n_i$. The the numerical fluxes are here given as follows, where s_i denotes the sign of n_i:

$$\hat{F}_i(\alpha,\beta,n_i) = s_i \max\{s_i F_i^- n_i(s_i \max\{s_i\alpha, s_i\theta(F_i^- n_i)\}), s_i F_i^+ n_i(s_i \min\{s_i\beta, s_i\theta(F_i^+ n_i)\})\}, \quad i = 1,2, \tag{2}$$

where $\theta(F_i^\pm n_i)$ are such that $F_i^\pm n_i(\theta(F_i^\pm n_i)) = \min_{\phi\in[0,1]}(s_i F_i^\pm n_i(\phi))$. For fixed values of $\boldsymbol{u}_h(\boldsymbol{x})$, $\boldsymbol{x}$ and $\boldsymbol{n}$, the numerical flux J is monotone, that is, for given $\boldsymbol{n}$, $J(\alpha,\beta,\boldsymbol{u}_h(\boldsymbol{x}),\boldsymbol{x},\boldsymbol{n})$ is a non-decreasing function of α and a non-increasing function of β; and it is conservative, i.e., $J(\alpha,\beta,\boldsymbol{u}_h(\boldsymbol{x}),\boldsymbol{x},\boldsymbol{n}) = -J(\beta,\alpha,\boldsymbol{u}_h(\boldsymbol{x}),\boldsymbol{x},-\boldsymbol{n})$. Finally, we mention that plain high-order DG formulations ($k \geq 1$) typically lack general stability properties exhibited by low-order methods, and spurious oscillations may develop. This issue is resolved following [5] by applying a limiter to the new solution at time t^{n+1}.

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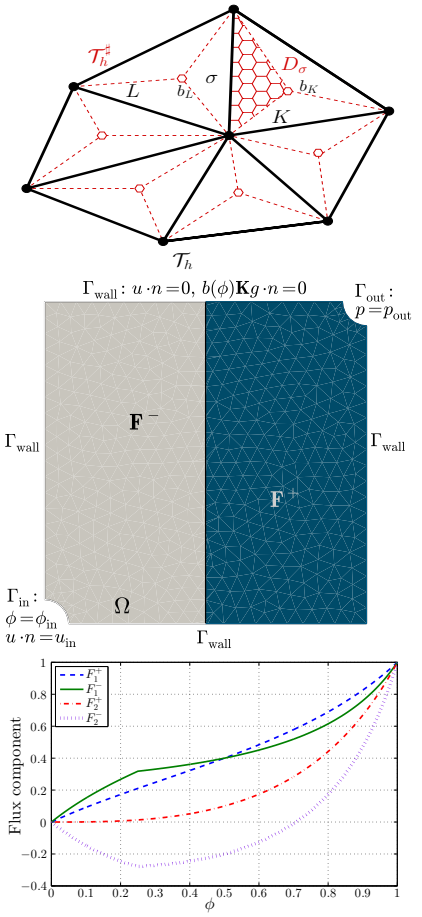

Fig. 1: Top: triangular elements in $\mathcal{T}_h$ (solid lines) and diamonds in $\mathcal{T}_h^\sharp$ (dashed), middle: domain of the example, bottom: $\mathbf{F}^\pm = (F_1^\pm, F_2^\pm)$ for $\boldsymbol{u} = (1,1)^{\mathrm{T}}$ (bottom).

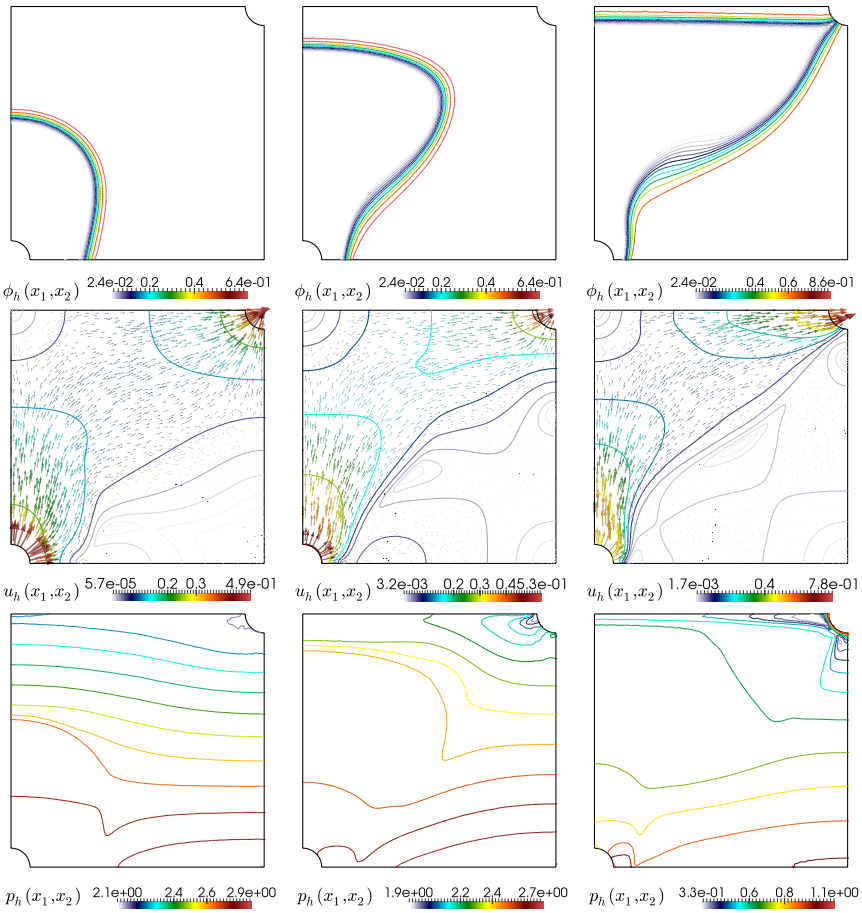

Fig. 2: Approximate saturation (top), vectors and magnitude contours of $\boldsymbol{u}$ (middle), and contours of p (bottom panels), on an unstructured mesh (7809 vertices 15616 primal triangular elements), at early (left), middle (center), and advanced (right) times.

2 Numerical example

We consider the upper right quarter of the well-known five spot geometry (cf. Figure 1 (middle)), based on the unit square. The domain is initially at rest and with constant saturation $\phi = \phi_0$. An inlet section (or "injector site") Γ_{in} is located at the bottom left corner, where we inject material with $\phi = \phi_{\text{in}}$ at a normal inflow velocity of $\boldsymbol{u}\cdot\boldsymbol{n} = u_{\text{in}}$. At the outlet Γ_{out} (the "producer well") we impose $p = p_{\text{out}}$, whereas on the walls $\Gamma_{\text{wall}} = \partial\Omega \setminus (\Gamma_{\text{in}} \cup \Gamma_{\text{out}})$, slip velocities ($\boldsymbol{u}\cdot\boldsymbol{n} = 0$) are allowed and ϕ satisfies a zero-flux condition (i.e., $b(\phi)\mathbf{K}\boldsymbol{g}\cdot\boldsymbol{n} = 0$). We choose $\mu(\phi) = 0.01(1-\frac{\phi}{2})^{-2.5}$, $\mu_{\text{w}} = \mu_{\text{n}} = 1$, $\mathbf{K}(\boldsymbol{x}) = (2\mathbf{I}$ if $x_1 < \frac{1}{2}$, $\mathbf{I}$ if $x_1 > \frac{1}{2})$, $K_{\text{w}}^-(\phi) = (1.75\phi$ if $\phi \le \frac{1}{4}$, $0.25\phi + 0.375$ if $\phi > \frac{1}{4})$, $K_{\text{w}}^+(\phi) = \phi$, $K_{\text{n}}(\phi) = 1-\phi^2$, $f(\phi) = K_{\text{w}}(\phi)/(K_{\text{w}}(\phi)+K_{\text{n}}(\phi))$, $b(\phi) = f(\phi)K_{\text{n}}(\phi)\Delta\rho$, and $\mathbf{F}(\phi,\boldsymbol{u}) = f(\phi)\boldsymbol{u} + b(\phi)\mathbf{K}\boldsymbol{g}$ with $\phi_{\text{in}} = 0.02$, $\phi_0 = 0.75$, $u_{\text{in}} = -0.1$, $p_{\text{out}} = 2$, $\rho_{\text{w}} = 20$, and $\rho_{\text{n}} = 10$. We have plotted the branches of the horizontal and vertical flux components in Figure 1 (bottom). The numerical solution is shown in Figure 2. We observe that the low-saturation fluid is flowing from the inlet towards the outlet following an unsymmetric pattern influenced by gravity and by the discontinuity of permeabilities. The method is suitable for long-time simulations of realistic scenarios. In particular saturation fronts are sharply resolved, with saturations assuming physically relevant values between zero and one.

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