

A loosely coupled method for fluid-structure interaction in an Eulerian diffuse-interface framework

Martina Bukač^a, Ricardo Ruiz-Baier^{b,c}

^aApplied and Computational Mathematics and Statistics, University of Notre Dame, 201F Crowley Hall, Notre Dame, 46556, IN, United States

^bSchool of Mathematics, Monash University, 9 Rainforest Walk, 3800, VIC, Melbourne, Australia

^cUniversidad Adventista de Chile, Casilla 7-D, Chillán, Chile

ARTICLE INFO

Keywords:

fluid-structure interaction
phase-field method
Cahn–Hilliard equation
Eulerian formulation
energy stability
finite element method

ABSTRACT

We study a fully Eulerian, diffuse-interface formulation for the interaction between an incompressible viscous fluid and an incompressible neo-Hookean solid. The location of the solid is tracked implicitly by a Cahn–Hilliard-type phase field, while its elastic response is encoded by a symmetric positive-definite conformation tensor that evolves via an upper-convected-derivative transport equation, effectively recasting the solid as a viscoelastic fluid that is active only where the phase field indicates the solid phase. We derive the model starting from the classical Lagrangian and Eulerian elastodynamics equations, recall the corresponding sharp-interface Eulerian formulation, and present the resulting diffuse-interface system together with a continuous energy (free-energy dissipation) estimate. For the time discretization we propose a loosely coupled, fully implicit, three-step splitting algorithm that decouples the Cahn–Hilliard, Navier–Stokes, and conformation-tensor transport problems at each time step, and we prove that this scheme is energy stable under time-step restrictions relating the time step to the mobility and density parameters, and under a condition relating the interface width to the double-well potential’s concavity bound. The method is discretized in space using the finite element method. Numerical experiments confirm the predicted convergence rates and demonstrate the method’s ability to handle both large deformation (a channel flow past an elastic obstacle) and contact (a falling elastic ball against a rigid wall).

1. Introduction

Fluid-structure interaction (FSI) problems, in which an incompressible viscous fluid interacts with an elastic solid, arise throughout engineering and the life sciences, from the flow of blood through elastic vessels and heart valves to the flutter of aeroelastic structures and the design of soft robotic actuators. The numerical simulation of such problems is made difficult by the fact that the fluid and the solid are naturally described in different frames of reference: the fluid is most conveniently written in Eulerian (spatial) coordinates, while the solid is most naturally described in Lagrangian (material) coordinates, since its constitutive law is formulated relative to a fixed reference configuration.

The classical approach to this mismatch is the arbitrary Lagrangian–Eulerian (ALE) method, in which the fluid mesh is deformed to track the moving fluid-structure interface exactly [17]. While ALE methods are accurate and well established, they become problematic when the structure undergoes large deformations, self-contact, or topological changes (e.g., when two structures collide, or a structure fragments), since the underlying computational mesh may become excessively distorted or tangled, typically requiring costly remeshing procedures. This has motivated the development of *fully Eulerian* FSI formulations, in which both the fluid and the solid are described on a single, fixed computational mesh, and the location of the solid is tracked implicitly through an additional field defined on the whole domain, rather than through the mesh itself [15, 17].

In this work, we consider a fully Eulerian, diffuse-interface formulation for FSI between an incompressible Newtonian fluid and an incompressible neo-Hookean solid, following the modeling approach of [15]. The location of the solid is represented implicitly by a phase-field (order-parameter) function ϕ , governed by a Cahn–Hilliard-type equation, which takes the value $+1$ in the fluid and -1 in the solid and varies smoothly across a thin transition layer of width ε separating the two phases. This diffuse-interface, or phase-field, description avoids the need for an explicit parametrization of the fluid-structure interface and its motion, and naturally accommodates topological changes and

 mbukac@nd.edu (M. Bukač); ricardo.ruizbaier@monash.edu (R. Ruiz-Baier)

ORCID(s): 0000-0003-1956-9101 (M. Bukač); 0000-0003-3144-5822 (R. Ruiz-Baier)

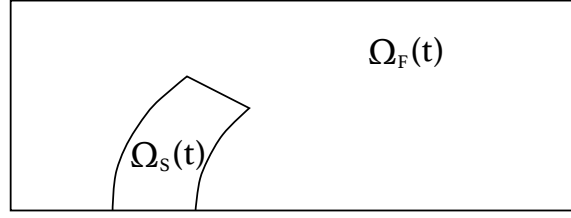


Figure 1: An example of fluid and structure domains at time t .

contact between elastic structures or between a structure and a rigid wall, which are difficult to handle with sharp-interface (ALE) methods. The elastic response of the solid is encoded, throughout the whole domain, by a symmetric positive-definite tensor $\mathbf{B}$ that plays the role of the left Cauchy–Green (conformation) tensor and evolves according to an upper-convected-derivative transport equation; it is coupled to the flow through a phase-field-weighted elastic stress and, in turn, is transported and stretched by the fluid velocity. This effectively recasts the incompressible neo-Hookean solid as a viscoelastic (Oldroyd-B-type) fluid that is only “active” where $\phi \approx -1$, with the phase field switching the elastic and relaxation coefficients on and off across the domain.

With respect to [15], where this diffuse-interface FSI model was introduced, the main contributions of this paper are the following: (i) a loosely coupled, fully implicit time-splitting scheme (Algorithm 1) that decouples, at each time step, the Cahn–Hilliard problem for the phase field and the chemical potential (ϕ, m) , the Navier–Stokes-type problem for the velocity and pressure $(\mathbf{v}, p)$, and the transport problem for the conformation tensor $\mathbf{B}$; (ii) a proof that this scheme is energy stable, in a sense that mirrors the energy balance of the continuous problem, under a condition relating the time step to the density and mobility parameters and a condition on the interface width; and (iii) its numerical validation on a convergence test and on two benchmarks involving large deformations and contact.

The remainder of the paper is organized as follows. In Section 2 we derive the governing equations: starting from the Lagrangian and Eulerian elastodynamics equations for an incompressible neo-Hookean solid (Section 2.1), we recall the sharp-interface Eulerian FSI system (Section 2.2) and then present its diffuse-interface counterpart, in which the solid is tracked by a Cahn–Hilliard phase field (Section 2.3). For the latter we show that the mass of the phase field is conserved and that smooth solutions satisfy an energy balance (Lemma 2.2), which is the continuous counterpart of the discrete stability result. Section 3 introduces the time discretization and Algorithm 1. Each of its three steps is implicit in the unknowns it solves for, so that each subproblem is a standard elliptic or saddle-point problem that can be handled with standard finite element solvers. In Section 4 we prove that Algorithm 1 is energy stable (Theorem 4.1) and discuss how the conditions of this result can be relaxed. In Section 5 we report numerical experiments, carried out with the finite element library Gridap [4]: a convergence test with manufactured solutions (Example 1), the flow past an elastic obstacle undergoing large deformations, compared against a reference ALE solution (Example 2), and a falling elastic ball that comes into contact with a rigid wall (Example 3). We close with some concluding remarks in Section 6.

2. Mathematical model

Let $\hat{\Omega}_S, \hat{\Omega}_F \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be disjoint, open, bounded sets with Lipschitz boundaries, corresponding to the structure and fluid regions in the reference configuration. We set $\hat{\Omega} = \hat{\Omega}_S \cup \hat{\Omega}_F$. Coordinates in the reference configuration $\hat{\Omega}$ are denoted by $\hat{\mathbf{x}}$, and coordinates in the physical configuration Ω by $\mathbf{x}$. Throughout, a hat indicates that a variable or operator is defined on the reference domain. Although the solid region may move in time, the overall domain is assumed to be fixed, that is, $\Omega(t) = \hat{\Omega}$ for all $t \geq 0$ (see Figure 1).

We begin by stating the structure problem in both the Lagrangian and Eulerian formulations, and then formulate the coupled FSI problem in Eulerian form.

2.1. Elastodynamics problem in the Lagrangian and Eulerian form

Following [17, p. 37], the elasticity equation in the Lagrangian formulation reads

$$\hat{J} \hat{\rho} \partial_{\hat{t}} \hat{\mathbf{u}} = \hat{\nabla} \cdot \hat{\mathbf{P}} + \hat{J} \hat{\rho} \hat{\mathbf{f}} \quad \text{in } \hat{\Omega}_S \times (0, T), \quad (2.1)$$

where $\hat{\mathbf{u}}$ denotes the displacement, $\hat{\rho}$ the density mapped to the reference configuration, $\hat{\mathbf{f}}$ a volume force, and $\hat{\mathbf{P}} = \hat{\mathbf{F}}\hat{\mathbf{\Sigma}}$ the first Piola–Kirchhoff stress, with $\hat{\mathbf{\Sigma}}$ the second Piola–Kirchhoff stress tensor. The deformation gradient $\hat{\mathbf{F}}$ and its determinant $\hat{J}$ are given by $\hat{\mathbf{F}} = \mathbf{I} + \hat{\nabla}\hat{\mathbf{u}}(\hat{\mathbf{x}}, t)$ and $\hat{J} = \det(\hat{\mathbf{F}})$. For gradients, we adopt the convention $(\nabla \mathbf{v})_{ij} = \frac{\partial v_i}{\partial x_j}$.

We note that using identity [17]

$$\hat{\nabla}\hat{\mathbf{v}} = (\nabla \mathbf{v})\hat{\mathbf{F}},$$

where $\hat{\mathbf{v}}(\hat{\mathbf{x}}, t) = \mathbf{v}(\mathbf{x}, t)$, we have

$$D_t \hat{\mathbf{F}} = D_t \hat{\nabla}\hat{\mathbf{u}} = \hat{\nabla}\hat{\mathbf{v}} = (\nabla \mathbf{v})\hat{\mathbf{F}}, \quad (2.2)$$

where $D_t = \partial_t + \mathbf{v} \cdot \nabla$ denotes the material derivative and $\hat{\mathbf{v}} = D_t \hat{\mathbf{u}}$ [14, 15]. We also define the left Cauchy–Green tensor as [17, p. 19]

$$\mathbf{b}^{-1} = \hat{\mathbf{F}}^{-\top} \hat{\mathbf{F}}^{-1} = \mathbf{F}^\top \mathbf{F}, \text{ so that } \mathbf{b} = \hat{\mathbf{F}}\hat{\mathbf{F}}^\top = \mathbf{F}^{-1} \mathbf{F}^{-\top},$$

where the inverse of $\hat{\mathbf{F}}$ is defined as [17, p. 19]

$$\mathbf{F} = \hat{\mathbf{F}}^{-1} := \mathbf{I} - \nabla \mathbf{u}.$$

Assuming that the material is hyperelastic, the second Piola–Kirchhoff stress tensor can be computed as

$$\hat{\mathbf{\Sigma}} = \partial_{\hat{\mathbf{E}}} \hat{\Psi} = \hat{\mathbf{F}}^{-1} \partial_{\hat{\mathbf{C}}} \hat{\Psi} = 2 \partial_{\hat{\mathbf{C}}} \hat{\Psi},$$

where $\hat{\Psi}$ is the strain energy density function, $\hat{\mathbf{E}} = \frac{1}{2}(\hat{\mathbf{C}} - \mathbf{I}) = \frac{1}{2}(\hat{\mathbf{F}}^\top \hat{\mathbf{F}} - \mathbf{I})$ is the Green–Lagrange strain tensor and $\hat{\mathbf{C}} = \hat{\mathbf{F}}^\top \hat{\mathbf{F}}$ is the right Cauchy–Green deformation tensor [17]. For an incompressible neo-Hookean constitutive model [11], the strain energy density function is given as

$$\hat{\Psi}(\hat{\mathbf{C}}) = \frac{G_S}{2}(\text{tr}(\hat{\mathbf{C}}) - d),$$

where G_S is the shear modulus. Note that G_S and μ_S (introduced in Section 2.2 below) are two different parameters attached to the solid: G_S enters the strain-energy density $\hat{\Psi}$ and hence the elastic (neo-Hookean) part of the stress derived in this section, while μ_S multiplies the viscous strain-rate term $2\mu(\phi)\mathbf{D}(\mathbf{v})$ in (2.5a), as an additional Kelvin–Voigt-type viscous damping in the solid. A purely (hyper)elastic solid has no extra viscous dissipation ($\mu_S = 0$) but does have elastic stress ($G_S \neq 0$) and a Kelvin–Voigt-type solid would instead have $\mu_S > 0$ as well. Note that $\text{tr}(\hat{\mathbf{C}}) = \text{tr}(\hat{\mathbf{F}}^\top \hat{\mathbf{F}}) = |\hat{\mathbf{F}}|^2$, and that $\partial_{\hat{\mathbf{F}}} \hat{\Psi}(\hat{\mathbf{F}}) = G_S \hat{\mathbf{F}}$. Therefore, we have

$$\hat{\mathbf{P}} = \hat{\mathbf{F}}\hat{\mathbf{\Sigma}} = G_S \hat{\mathbf{F}}.$$

In the Eulerian formulation, the elastodynamics equation is given by [17, p. 257]

$$\rho D_t \mathbf{v} = \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{f} \quad \text{in } \Omega_S(t) \times (0, T), \quad (2.3)$$

where $\mathbf{v}$ is the solid velocity, and $\boldsymbol{\sigma}$ is the Cauchy stress tensor, given by $\boldsymbol{\sigma} = J\mathbf{F}^{-1}\boldsymbol{\Sigma}\mathbf{F}^{-\top}$ [17, p. 258], where $J := \det \mathbf{F} = \det(\hat{\mathbf{F}}^{-1}) = \hat{J}^{-1}$. We define $\boldsymbol{\Sigma}$ in the Eulerian coordinates as

$$\boldsymbol{\Sigma} := \hat{\mathbf{\Sigma}} = G_S \mathbf{I}.$$

Taking into account that the model is incompressible, we can write the Cauchy stress tensor as

$$\boldsymbol{\sigma} = G_S \mathbf{F}^{-1} \mathbf{F}^{-\top} = G_S \mathbf{b}.$$

2.2. Sharp interface model

We recall that an FSI problem between an incompressible, viscous fluid and an incompressible, neo-Hookean solid, can be described using the following coupled problem in the Eulerian form [15]:

$$D_t(\rho \mathbf{v}) - \nabla \cdot (2\mu \mathbf{D}(\mathbf{v}) + G\mathbf{B}) + \nabla p = \rho \mathbf{f} \quad \text{in } \Omega \times (0, T), \quad (2.4a)$$

$$\nabla \cdot \mathbf{v} = 0 \quad \text{in } \Omega \times (0, T), \quad (2.4b)$$

$$G(D_t \mathbf{B} - (\nabla \mathbf{v})\mathbf{B} - \mathbf{B}(\nabla \mathbf{v})^\top) + \alpha(\mathbf{B} - \mathbf{I}) = \mathbf{0} \quad \text{in } \Omega \times (0, T), \quad (2.4c)$$

$$\mathbf{v} = \mathbf{0} \quad \text{on } \partial\Omega \times (0, T), \quad (2.4d)$$

where $\mathbf{v}$ is the velocity, p is the pressure, $D_t = \partial_t + \mathbf{v} \cdot \nabla$ denotes the material derivative, $\mathbf{D}(\mathbf{v}) = \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^\top)$ is the strain rate tensor, $\mathbf{B}$ is the left Cauchy-Green tensor, and

$$\mathbf{f} = \begin{cases} \mathbf{f}_F & \text{in } \Omega_F \\ \mathbf{f}_S & \text{in } \Omega_S \end{cases}$$

is the forcing term. Initially, at $t = 0$, we set $\mathbf{v} = \mathbf{0}$ and $\mathbf{B} = \mathbf{I}$ in Ω .

Equation (2.4c) contains the upper-convected Maxwell time derivative that rotates and stretches with the deformation [14]. The problem parameters are defined as

$$\rho = \begin{cases} \rho_F & \text{in } \Omega_F \\ \rho_S & \text{in } \Omega_S, \end{cases} \quad \mu = \begin{cases} \mu_F & \text{in } \Omega_F \\ \mu_S & \text{in } \Omega_S, \end{cases} \quad G = \begin{cases} 0 & \text{in } \Omega_F \\ G_S & \text{in } \Omega_S, \end{cases} \quad \alpha = \begin{cases} \alpha_F & \text{in } \Omega_F \\ 0 & \text{in } \Omega_S, \end{cases}$$

where ρ_F and ρ_S are the fluid and structure densities, μ_F and μ_S are the fluid and structure viscosities, respectively, and G_S is the shear modulus. We note that $\mu_S = 0$ for purely elastic structures. The parameter α_F is positive, and the ratio of G_S/α_F can be seen as the interface relaxation time which needs to scale with T . We can take, for example, $\alpha_F = G_S/T$.

We note that, since $\alpha \equiv 0$ in Ω_S , equation (2.4c) reduces there to $D_t \mathbf{B} = (\nabla \mathbf{v})\mathbf{B} + \mathbf{B}(\nabla \mathbf{v})^\top$. This is the transport equation satisfied by the left Cauchy-Green tensor $\mathbf{b} = \hat{\mathbf{F}}\hat{\mathbf{F}}^\top$ of the underlying deformation (differentiate $\mathbf{b} = \hat{\mathbf{F}}\hat{\mathbf{F}}^\top$ and use $D_t \hat{\mathbf{F}} = (\nabla \mathbf{v})\hat{\mathbf{F}}$ from (2.2) to get $D_t \mathbf{b} = (\nabla \mathbf{v})\mathbf{b} + \mathbf{b}(\nabla \mathbf{v})^\top$, an identical linear ODE along particle paths). Since $\mathbf{B} = \mathbf{I} = \mathbf{b}$ at $t = 0$ (the reference configuration is undeformed, so $\hat{\mathbf{F}} = \mathbf{I}$ there too), uniqueness of solutions of this ODE gives $\mathbf{B} \equiv \mathbf{b}$ in $\Omega_S(t)$ for all $t \geq 0$: $\mathbf{B}$ is the physical left Cauchy-Green tensor of the solid.

2.3. Diffuse interface model

In this section, we present a diffuse interface formulation of the Eulerian FSI problem. Rather than tracking the fluid-structure interface explicitly, we introduce a phase-field function $\phi = \phi(\mathbf{x}, t)$ that distinguishes the two regions: $\phi \approx 1$ in the fluid, $\phi \approx -1$ in the structure, and ϕ varies smoothly but rapidly between these two values across a transition layer whose width is proportional to a small parameter $\varepsilon > 0$. The sharp interface is thus replaced by a diffuse one, and its position can be recovered, e.g., as the zero level set of ϕ . This allows us to pose a single system of equations on the whole domain Ω , with material parameters defined as functions of ϕ that interpolate between the fluid and solid values.

Following [15], the diffuse interface FSI model reads:

$$D_t(\rho(\phi)\mathbf{v}) + \frac{\rho_F - \rho_S}{2}(M(\phi)\nabla m \cdot \nabla)\mathbf{v} - \nabla \cdot (2\mu(\phi)\mathbf{D}(\mathbf{v}) + G(\phi)\mathbf{B}) + \nabla p = -\gamma\varepsilon\nabla \cdot (\nabla\phi \otimes \nabla\phi) + \rho(\phi)\mathbf{f} \quad \text{in } \Omega, \quad (2.5a)$$

$$\nabla \cdot \mathbf{v} = 0 \quad \text{in } \Omega, \quad (2.5b)$$

$$G(\phi)(D_t \mathbf{B} - (\nabla \mathbf{v})\mathbf{B} - \mathbf{B}(\nabla \mathbf{v})^\top) + \alpha(\phi)(\mathbf{B} - \mathbf{I}) - \delta\Delta \mathbf{B} = \mathbf{0} \quad \text{in } \Omega, \quad (2.5c)$$

$$D_t \phi - \nabla \cdot (M(\phi)\nabla m) = 0 \quad \text{in } \Omega, \quad (2.5d)$$

$$\frac{1}{2}G'(\phi)\text{tr } \mathbf{B} + \gamma\left(\frac{1}{\varepsilon}W'(\phi) - \varepsilon\Delta\phi\right) = m \quad \text{in } \Omega, \quad (2.5e)$$

$$\mathbf{v} = \mathbf{0} \quad \text{on } \partial\Omega, \quad (2.5f)$$

$$\nabla \phi \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega, \quad (2.5g)$$

$$\nabla m \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega, \quad (2.5h)$$

$$\nabla \mathbf{B} \mathbf{n} = \mathbf{0} \quad \text{on } \partial\Omega. \quad (2.5i)$$

The diffuse version of problem (2.4a)–(2.4c) is given by equations (2.5a)–(2.5c). Equations (2.5d)–(2.5e) constitute the Cahn–Hilliard subproblem for the evolution of the phase-field variable ϕ . We write it as a system of two second-order equations for ϕ and the chemical potential m . The phase field is transported by the velocity $\mathbf{v}$ and, in addition, diffuses along the negative gradient of m , with a rate controlled by the mobility M , which may depend on ϕ ; we assume that $M : [-1, 1] \rightarrow \mathbb{R}$ is continuous and non-degenerate, i.e., $0 < M_* \leq M(\phi) \leq M^*$ for all $\phi \in [-1, 1]$ (the lower bound $M_* > 0$ is used in Section 4). The chemical potential is obtained from a free energy consisting of two competing contributions: a double-well potential $W(\phi) = \frac{1}{4}(\phi^2 - 1)^2$, which favors the pure phases $\phi = \pm 1$, and a gradient term proportional to $\epsilon |\nabla \phi|^2$, which penalizes steep transitions. The balance between these two terms produces an interface profile of width $\mathcal{O}(\epsilon)$. In the present model, m also contains a contribution from the elastic energy of the structure. Parameter γ is a scaled surface tension, related to the physical surface tension $\tilde{\gamma}$ by $\gamma = \frac{3}{2\sqrt{2}}\tilde{\gamma}$, and $\mathbf{n}$ is the outward unit normal to $\partial\Omega$. The term $-\gamma \epsilon \nabla \cdot (\nabla \phi \otimes \nabla \phi)$ in (2.5a) represents the capillary force due to surface tension. Since the Cahn–Hilliard equation is in divergence form and is supplemented with no-flux boundary conditions, the integral $\int_{\Omega} \phi \, d\mathbf{x}$ is conserved in time (see (2.9) below). The second term on the left-hand side of (2.5a) accounts for the additional momentum flux carried by the diffusive mass flux $\mathbf{J} = -\rho'(\phi)M(\phi)\nabla m = -\frac{\rho_F - \rho_S}{2}M(\phi)\nabla m$ when the two densities differ.

The phase-dependent coefficients are defined by

$$\rho(\phi) = \rho_F \frac{1 + \phi}{2} + \rho_S \frac{1 - \phi}{2}, \quad \mu(\phi) = \mu_F \frac{1 + \phi}{2} + \mu_S \frac{1 - \phi}{2}, \quad G(\phi) = G_S \frac{1 - \phi}{2}, \quad \alpha(\phi) = \alpha_F \frac{1 + \phi}{2}.$$

Here, ρ_F, ρ_S and μ_F, μ_S are the densities and viscosities of the fluid and the structure, G_S is the shear modulus of the structure, and α_F is the relaxation parameter in the fluid. Initially, we assume that $\mathbf{v} = \mathbf{0}$ and $\mathbf{B} = \mathbf{I}$ in Ω , and we set $\phi = \phi_0$, where $\phi_0 \in H^1(\Omega)$, with values in $[-1, 1]$, is a regularization, in a layer of width $\mathcal{O}(\epsilon)$ around $\partial\Omega_S(0)$, of the indicator function

$$\chi_0 = \begin{cases} 1 & \text{in } \Omega_F, \\ -1 & \text{in } \Omega_S. \end{cases}$$

A typical choice is $\phi_0(\mathbf{x}) = \tanh(\text{Sdist}(\mathbf{x})/(\sqrt{2}\epsilon))$, where Sdist is the signed distance to $\partial\Omega_S(0)$, positive in the fluid. The regularization is needed for the initial interfacial energy $\frac{\gamma\epsilon}{2}\|\nabla \phi_0\|_{L^2(\Omega)}^2$ to be finite.

The diffuse interface model presented in this section is similar to the one used in [15], which is based on a thermodynamically consistent derivation. In this work a stabilization term $\delta \Delta \mathbf{B}$, with $\delta \geq 0$ small, is added to the transport problem (2.5c). This term is commonly added to regularize and stabilize the transport problem, both in the analysis and the numerical simulations [5, 15].

The choice of the potential W in (2.5e) plays an important role in both the analysis and the numerical approximation of the Cahn–Hilliard equation. The free energy originally proposed by Cahn and Hilliard [8] involves the logarithmic potential

$$W_{\log}(\phi) = \frac{\theta}{2} \left((1 + \phi) \ln(1 + \phi) + (1 - \phi) \ln(1 - \phi) \right) - \frac{\theta_c}{2} \phi^2, \quad 0 < \theta < \theta_c, \quad (2.6)$$

where θ is the absolute temperature and θ_c is the critical temperature. Since $W'_{\log}(\phi) \rightarrow \pm\infty$ as $\phi \rightarrow \pm 1$, this singular potential confines the phase field to the physically meaningful interval. In this case, the Cahn–Hilliard equation has a unique solution satisfying $\phi \in (-1, 1)$ almost everywhere for all times [10]. However, $W''_{\log}(\phi) = \frac{\theta}{1 - \phi^2} - \theta_c$ is unbounded as $\phi \rightarrow \pm 1$, and this singularity makes $W_{\log}$ difficult to handle both analytically and numerically. For this reason, $W_{\log}$ is typically approximated by a smooth potential, most commonly the Ginzburg–Landau double-well potential $W(\phi) = \frac{1}{4}(\phi^2 - 1)^2$ introduced above. Due to the lack of a comparison principle for fourth-order diffusion equations, the values of ϕ are then no longer guaranteed to remain in $[-1, 1]$ (see, e.g., [1, 3]).

In this work, we use the Ginzburg–Landau potential and assume that the phase field remains within the physically meaningful range.

Assumption 2.1. The phase-field function satisfies $\phi(\mathbf{x}, t) \in [-1, 1]$ for almost every $\mathbf{x} \in \Omega$ and all $t \in [0, T]$.

Assumption 2.1 is consistent with the role of ϕ as an indicator of the fluid and solid phases, but it is a hypothesis on the solution, not a property of the model: with the Ginzburg–Landau potential, neither the continuous problem nor its time discretization is known to preserve the interval $[-1, 1]$. Assumption 2.1 ensures that the phase-dependent coefficients $\rho(\phi)$, $\mu(\phi)$, $G(\phi)$, and $\alpha(\phi)$ are nonnegative and bounded; more precisely, $\rho_* \leq \rho(\phi) \leq \rho^*$, $\mu_* \leq \mu(\phi) \leq \mu^*$ and $M_* \leq M(\phi) \leq M^*$, where $\rho_* = \min\{\rho_F, \rho_S\}$, $\rho^* = \max\{\rho_F, \rho_S\}$, and μ_* , μ^* are defined analogously. Moreover, since $W''(\phi) = 3\phi^2 - 1$, it implies

$$-\alpha_W \leq W''(\phi) \leq \beta_W \quad \text{with } \alpha_W = 1 \text{ and } \beta_W = 2. \quad (2.7)$$

The concavity bound α_W enters the stability analysis in Section 4, while the upper bound β_W is only needed for the stabilized variant discussed in Remark 4.2. An alternative approach, common in the analysis of the Allen–Cahn and Cahn–Hilliard equations [9, 12, 18], is to replace W by a truncated potential that coincides with W on $[-1, 1]$ and grows quadratically outside this interval, so that W'' is bounded on all of $\mathbb{R}$. Under Assumption 2.1, the two potentials coincide along the solution. We also note that the arguments below use only that $W \in C^2(\mathbb{R})$, $W \geq 0$, and that (2.7) holds on $[-1, 1]$; they therefore apply verbatim to other smooth double-well potentials, with α_W and β_W replaced by the corresponding bounds.

In the computations of Section 5, Assumption 2.1 is enforced by a cutoff of the phase field after STEP 1 of Algorithm 1. This cutoff does not increase the Cahn–Hilliard free energy, since $W(\pm 1) = 0 \leq W(s)$ for all $s \in \mathbb{R}$ and the gradient of the truncated function is pointwise bounded by $|\nabla \phi|$. However, it does not preserve the mass $\int_\Omega \phi$ exactly. Hence the conservation property (2.9) below, and Theorem 4.1, whose proof uses $\int_\Omega (\phi^{n+1} - \phi^n) = 0$, hold for the computed solutions only up to the mass change introduced by the cutoff.

2.3.1. Mass conservation and energy balance for the diffuse interface model

Given an open set $D \subset \mathbb{R}^d$, we denote by $L^2(D)$ and $H^k(D)$, $k \geq 0$, the standard Lebesgue and Sobolev spaces, and by $\|\cdot\|_{L^2(D)}$ the norm of $L^2(D)$, also for vector- and tensor-valued functions. Following [13], for a field $\mathbf{b}$ of symmetric positive semidefinite $d \times d$ tensors defined on D we write

$$\|\mathbf{b}\|_{*,D} = \int_D \text{tr } \mathbf{b}.$$

This is not a norm, since it is only defined on the cone of positive semidefinite tensors. However, it is nonnegative, additive and positively homogeneous, and it vanishes only if $\mathbf{b} = \mathbf{0}$ almost everywhere. Moreover, $|\mathbf{b}| \leq \text{tr } \mathbf{b} \leq \sqrt{d} |\mathbf{b}|$ for every symmetric positive semidefinite $\mathbf{b}$, where $|\cdot|$ denotes the Frobenius norm, so $\|\cdot\|_{*,D}$ is equivalent to the $L^1(D)$ norm on this cone. We use it for $\mathbf{b} = G(\phi)\mathbf{B}$ and $\mathbf{b} = \alpha(\phi)\mathbf{B}$, which are positive semidefinite whenever $\mathbf{B}$ is symmetric positive definite and Assumption 2.1 holds. We also denote by C_F and C_K the constants in the Poincaré–Friedrichs and Korn inequalities, so that $\|\mathbf{v}\|_{L^2(\Omega)} \leq C_F \|\nabla \mathbf{v}\|_{L^2(\Omega)} \leq C_F C_K \|\mathbf{D}(\mathbf{v})\|_{L^2(\Omega)}$ for all $\mathbf{v} \in H_0^1(\Omega)^d$, and we set $C = C_F^2 C_K^2$.

The total energy of the diffuse interface model is the sum of the kinetic energy, the elastic energy of the structure, and the Cahn–Hilliard (interfacial) free energy:

$$\mathcal{E}(t) = \frac{1}{2} \|\rho(\phi)^{\frac{1}{2}} \mathbf{v}\|_{L^2(\Omega)}^2 + \frac{1}{2} \|G(\phi)\mathbf{B}\|_{*,\Omega} + \frac{\gamma}{\varepsilon} \int_\Omega W(\phi) + \frac{\gamma\varepsilon}{2} \|\nabla \phi\|_{L^2(\Omega)}^2, \quad (2.8)$$

where all the fields are evaluated at time t . All four contributions are nonnegative under Assumption 2.1, provided that $\mathbf{B}$ is symmetric positive definite. The discrete energy $\mathcal{E}^n$ used in Section 4 is the time-discrete counterpart of (2.8).

Lemma 2.2 (Mass conservation and energy balance). *Let $(\mathbf{v}, p, \mathbf{B}, \phi, m)$ be a sufficiently smooth solution of (2.5a)–(2.5i) in $\Omega \times (0, T)$ with initial data $\mathbf{v}(0) = \mathbf{0}$, $\mathbf{B}(0) = \mathbf{I}$ and $\phi(0) = \phi_0$. Assume that Assumption 2.1 holds and that $\mathbf{B}(\mathbf{x}, t)$ is symmetric positive definite for all $(\mathbf{x}, t) \in \Omega \times [0, T]$. Then, for all $t \in [0, T]$:*

(i) *the mass of the phase field is conserved,*

$$\frac{d}{dt} \int_\Omega \phi = 0, \quad \text{i.e.,} \quad \int_\Omega \phi(t) = \int_\Omega \phi_0; \quad (2.9)$$

(ii) the following energy balance holds:

$$\frac{d}{dt} \mathcal{E} + 2\|\mu(\phi)^{\frac{1}{2}} \mathbf{D}(\mathbf{v})\|_{L^2(\Omega)}^2 + \|M(\phi)^{\frac{1}{2}} \nabla m\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\alpha(\phi) \mathbf{B}\|_{*,\Omega} = \int_{\Omega} \rho(\phi) \mathbf{f} \cdot \mathbf{v} + \frac{d}{2} \int_{\Omega} \alpha(\phi_0); \quad (2.10)$$

(iii) if, in addition, $\mu_* = \min\{\mu_F, \mu_S\} > 0$, then, with $\rho^* = \max\{\rho_F, \rho_S\}$,

$$\begin{aligned} \mathcal{E}(t) + \int_0^t \left(\mu_* \|\mathbf{D}(\mathbf{v})\|_{L^2(\Omega)}^2 + \|M(\phi)^{\frac{1}{2}} \nabla m\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\alpha(\phi) \mathbf{B}\|_{*,\Omega} \right) ds \\ \leq \mathcal{E}(0) + \frac{d}{2} \int_{\Omega} \alpha(\phi_0) + \frac{(\rho^*)^2 C}{4\mu_*} \int_0^t \|\mathbf{f}\|_{L^2(\Omega)}^2 ds. \end{aligned} \quad (2.11)$$

The arguments in the proof below are standard. Mass conservation and the treatment of the Cahn–Hilliard terms are classical (see, e.g., [1, 2]). The cancellation of the inertial terms involving the extra momentum flux due to non-matching densities is the one of [2]. The trace-based estimate for the conformation tensor follows the one for the Oldroyd-B model [5, 13]. See also [15] for the thermodynamically consistent derivation of the model. We include the details for completeness and because the proof of the discrete result, Theorem 4.1, follows the same steps.

Proof. Throughout the proof we use repeatedly that $\nabla \cdot \mathbf{v} = 0$ in Ω and $\mathbf{v} = \mathbf{0}$ on $\partial\Omega$, so that

$$\int_{\Omega} \mathbf{v} \cdot \nabla f = 0 \quad \text{for every sufficiently smooth scalar function } f. \quad (2.12)$$

We also write $\rho' = \frac{\rho_F - \rho_S}{2}$ for the (constant) derivative of the affine function $\rho(\phi)$.

Step 1: mass conservation. Integrating (2.5d) over Ω and using (2.12) with $f = \phi$ and the boundary condition $\nabla m \cdot \mathbf{n} = 0$ gives

$$\frac{d}{dt} \int_{\Omega} \phi = - \int_{\Omega} \mathbf{v} \cdot \nabla \phi + \int_{\partial\Omega} M(\phi) \nabla m \cdot \mathbf{n} = 0,$$

which is (2.9). Since $\alpha(\phi) = \frac{\alpha_F}{2}(1 + \phi)$ is affine, it follows that $\int_{\Omega} \alpha(\phi(t)) = \int_{\Omega} \alpha(\phi_0)$ for all t .

Step 2: momentum equation. We test (2.5a) with $\mathbf{v}$ and (2.5b) with p , and integrate by parts. The pressure term vanishes, $\int_{\Omega} \nabla p \cdot \mathbf{v} = - \int_{\Omega} p \nabla \cdot \mathbf{v} = 0$. By the symmetry of $\mathbf{D}(\mathbf{v})$, $\int_{\Omega} 2\mu(\phi) \mathbf{D}(\mathbf{v}) : \nabla \mathbf{v} = 2\|\mu(\phi)^{\frac{1}{2}} \mathbf{D}(\mathbf{v})\|_{L^2(\Omega)}^2$, and the elastic term gives $- \int_{\Omega} \nabla \cdot (G(\phi) \mathbf{B}) \cdot \mathbf{v} = \int_{\Omega} G(\phi) \mathbf{B} : \nabla \mathbf{v}$. For the inertial terms, recalling that $D_t(\rho \mathbf{v}) = \partial_t(\rho \mathbf{v}) + (\mathbf{v} \cdot \nabla)(\rho \mathbf{v})$, we have

$$\int_{\Omega} \partial_t(\rho(\phi) \mathbf{v}) \cdot \mathbf{v} = \frac{d}{dt} \int_{\Omega} \frac{\rho(\phi)}{2} |\mathbf{v}|^2 + \frac{1}{2} \int_{\Omega} \partial_t \rho(\phi) |\mathbf{v}|^2,$$

and, since $((\mathbf{v} \cdot \nabla)(\rho \mathbf{v})) \cdot \mathbf{v} = (\mathbf{v} \cdot \nabla \rho) |\mathbf{v}|^2 + \frac{1}{2} \rho \mathbf{v} \cdot \nabla |\mathbf{v}|^2$ and, by (2.5b) and $\mathbf{v}|_{\partial\Omega} = \mathbf{0}$, $\frac{1}{2} \int_{\Omega} \rho \mathbf{v} \cdot \nabla |\mathbf{v}|^2 = -\frac{1}{2} \int_{\Omega} (\mathbf{v} \cdot \nabla \rho) |\mathbf{v}|^2$,

$$\int_{\Omega} ((\mathbf{v} \cdot \nabla)(\rho(\phi) \mathbf{v})) \cdot \mathbf{v} = \frac{1}{2} \int_{\Omega} (\mathbf{v} \cdot \nabla \rho(\phi)) |\mathbf{v}|^2.$$

Adding these two identities, and using $D_t \rho(\phi) = \rho' D_t \phi = \rho' \nabla \cdot (M(\phi) \nabla m)$ (by (2.5d)) together with an integration by parts and $\nabla m \cdot \mathbf{n} = 0$, we obtain

$$\begin{aligned} \int_{\Omega} D_t(\rho(\phi) \mathbf{v}) \cdot \mathbf{v} &= \frac{d}{dt} \int_{\Omega} \frac{\rho(\phi)}{2} |\mathbf{v}|^2 + \frac{\rho'}{2} \int_{\Omega} \nabla \cdot (M(\phi) \nabla m) |\mathbf{v}|^2 \\ &= \frac{d}{dt} \int_{\Omega} \frac{\rho(\phi)}{2} |\mathbf{v}|^2 - \frac{\rho_F - \rho_S}{2} \int_{\Omega} ((M(\phi) \nabla m \cdot \nabla) \mathbf{v}) \cdot \mathbf{v}. \end{aligned} \quad (2.13)$$

The last term cancels exactly the contribution of the second term on the left-hand side of (2.5a). Therefore,

$$\frac{d}{dt} \int_{\Omega} \frac{\rho(\phi)}{2} |\mathbf{v}|^2 + 2\|\mu(\phi)^{\frac{1}{2}} \mathbf{D}(\mathbf{v})\|_{L^2(\Omega)}^2 + \int_{\Omega} G(\phi) \mathbf{B} : \nabla \mathbf{v} + \gamma \varepsilon \int_{\Omega} \nabla \cdot (\nabla \phi \otimes \nabla \phi) \cdot \mathbf{v} = \int_{\Omega} \rho(\phi) \mathbf{f} \cdot \mathbf{v}. \quad (2.14)$$

Step 3: Cahn–Hilliard equations. Testing (2.5d) with m and using $\nabla m \cdot \mathbf{n} = 0$ gives $\int_{\Omega} m D_t \phi + \|M(\phi)^{\frac{1}{2}} \nabla m\|_{L^2(\Omega)}^2 = 0$. Testing (2.5e) with $D_t \phi$ and using $\nabla \phi \cdot \mathbf{n} = 0$ gives

$$\int_{\Omega} m D_t \phi = \frac{1}{2} \int_{\Omega} G'(\phi) \operatorname{tr} \mathbf{B} D_t \phi + \frac{\gamma}{\varepsilon} \int_{\Omega} W'(\phi) D_t \phi + \gamma \varepsilon \int_{\Omega} \nabla \phi \cdot \nabla D_t \phi.$$

By the chain rule and (2.12) with $f = W(\phi)$,

$$\int_{\Omega} W'(\phi) D_t \phi = \int_{\Omega} \partial_t W(\phi) + \int_{\Omega} \mathbf{v} \cdot \nabla W(\phi) = \frac{d}{dt} \int_{\Omega} W(\phi).$$

Moreover, $\int_{\Omega} \nabla \phi \cdot \nabla D_t \phi = \frac{1}{2} \frac{d}{dt} \|\nabla \phi\|_{L^2(\Omega)}^2 + \int_{\Omega} \nabla \phi \cdot \nabla (\mathbf{v} \cdot \nabla \phi)$. Integrating by parts with $\nabla \phi \cdot \mathbf{n} = 0$, and using the identity $\nabla \cdot (\nabla \phi \otimes \nabla \phi) = \Delta \phi \nabla \phi + \frac{1}{2} \nabla |\nabla \phi|^2$ together with (2.12) with $f = |\nabla \phi|^2$, we get

$$\int_{\Omega} \nabla \phi \cdot \nabla (\mathbf{v} \cdot \nabla \phi) = - \int_{\Omega} \Delta \phi (\mathbf{v} \cdot \nabla \phi) = - \int_{\Omega} \nabla \cdot (\nabla \phi \otimes \nabla \phi) \cdot \mathbf{v}.$$

Combining these identities, we arrive at

$$\|M(\phi)^{\frac{1}{2}} \nabla m\|_{L^2(\Omega)}^2 + \frac{1}{2} \int_{\Omega} G'(\phi) \operatorname{tr} \mathbf{B} D_t \phi + \frac{d}{dt} \left(\frac{\gamma}{\varepsilon} \int_{\Omega} W(\phi) + \frac{\gamma \varepsilon}{2} \|\nabla \phi\|_{L^2(\Omega)}^2 \right) - \gamma \varepsilon \int_{\Omega} \nabla \cdot (\nabla \phi \otimes \nabla \phi) \cdot \mathbf{v} = 0. \quad (2.15)$$

Step 4: conformation tensor. Since $\mathbf{B}$ is symmetric, $\operatorname{tr} ((\nabla \mathbf{v}) \mathbf{B} + \mathbf{B} (\nabla \mathbf{v})^T) = 2 \mathbf{B} : \nabla \mathbf{v}$. Hence, taking the trace of (2.5c) gives

$$G(\phi) (D_t \operatorname{tr} \mathbf{B} - 2 \mathbf{B} : \nabla \mathbf{v}) + \alpha(\phi) (\operatorname{tr} \mathbf{B} - d) - \delta \Delta \operatorname{tr} \mathbf{B} = 0.$$

We multiply by $\frac{1}{2}$ and integrate over Ω . The diffusion term vanishes, since $\int_{\Omega} \Delta \operatorname{tr} \mathbf{B} = \int_{\partial \Omega} \operatorname{tr} (\nabla \mathbf{B} \mathbf{n}) = 0$ by (2.5i). Using Step 1 for the relaxation term, we obtain

$$\frac{1}{2} \int_{\Omega} G(\phi) D_t \operatorname{tr} \mathbf{B} - \int_{\Omega} G(\phi) \mathbf{B} : \nabla \mathbf{v} + \frac{1}{2} \|\alpha(\phi) \mathbf{B}\|_{*,\Omega} - \frac{d}{2} \int_{\Omega} \alpha(\phi_0) = 0. \quad (2.16)$$

Finally, we add (2.14), (2.15) and (2.16). The capillary terms cancel, and so do the terms $\pm \int_{\Omega} G(\phi) \mathbf{B} : \nabla \mathbf{v}$, which both come from the elastic stress, once through (2.5a) and once through the stretching terms in (2.5c). The two remaining elastic terms combine into a total derivative: by the product rule and (2.12) with $f = G(\phi) \operatorname{tr} \mathbf{B}$,

$$\frac{1}{2} \int_{\Omega} G(\phi) D_t \operatorname{tr} \mathbf{B} + \frac{1}{2} \int_{\Omega} G'(\phi) \operatorname{tr} \mathbf{B} D_t \phi = \frac{1}{2} \int_{\Omega} D_t (G(\phi) \operatorname{tr} \mathbf{B}) = \frac{1}{2} \frac{d}{dt} \|G(\phi) \mathbf{B}\|_{*,\Omega}.$$

This proves (2.10). Finally, by the Cauchy–Schwarz, Poincaré–Friedrichs, Korn and Young inequalities,

$$\int_{\Omega} \rho(\phi) \mathbf{f} \cdot \mathbf{v} \leq \rho^* \|\mathbf{f}\|_{L^2(\Omega)} C^{\frac{1}{2}} \|\mathbf{D}(\mathbf{v})\|_{L^2(\Omega)} \leq \frac{(\rho^*)^2 C}{4 \mu_*} \|\mathbf{f}\|_{L^2(\Omega)}^2 + \mu_* \|\mathbf{D}(\mathbf{v})\|_{L^2(\Omega)}^2.$$

Using $2 \|\mu(\phi)^{\frac{1}{2}} \mathbf{D}(\mathbf{v})\|_{L^2(\Omega)}^2 \geq 2 \mu_* \|\mathbf{D}(\mathbf{v})\|_{L^2(\Omega)}^2$ (by Assumption 2.1) and integrating (2.10) over $(0, t)$ yields (2.11). $\square$

Estimate (2.11) is the continuous counterpart of Theorem 4.1 below: the sum over n in the theorem replaces the time integral, and its right-hand side contains the same source term, with $\int_{\Omega} \alpha(\phi^{n+1})$ in place of $\int_{\Omega} \alpha(\phi_0)$. This source term is constant in time, and approximately equal to $\frac{d \alpha_F}{2} |\Omega_F(0)|$, since ϕ_0 is a regularization of χ_0 . It comes from the relaxation term $\alpha(\phi)(\mathbf{B} - \mathbf{I})$ combined with the choice of $\frac{1}{2} G(\phi) \operatorname{tr} \mathbf{B}$ as the elastic energy, so the energy can grow at most linearly in time. With the free energy of the Oldroyd-B model, $\frac{1}{2} G(\phi)(\operatorname{tr} \mathbf{B} - d - \ln \det \mathbf{B})$, the relaxation term would instead be dissipative, because $(\mathbf{B} - \mathbf{I}) : (\mathbf{I} - \mathbf{B}^{-1}) = \operatorname{tr}(\mathbf{B} + \mathbf{B}^{-1} - 2\mathbf{I}) \geq 0$; see, e.g., [5]. However, this choice changes the chemical potential in (2.5e), and controlling $\ln \det \mathbf{B}$ at the discrete level requires additional techniques that we do not pursue here.

3. Numerical method

3.1. Temporal discretization

Let Δt be the time step, $t^n = n\Delta t$, for all $0 \leq n \leq N$, where the final time is $T = N\Delta t$. Let z^n be the approximation of a time-dependent function z at time t^n . We denote the discrete time derivative by

$$d_t z^{n+1} := \frac{z^{n+1} - z^n}{\Delta t}.$$

To solve problem (2.5a)–(2.5i), we propose the following loosely-coupled algorithm.

Algorithm 1. Given $\mathbf{v}^0 = \mathbf{0}$, $\mathbf{B}^0 = \mathbf{I}$, and $\phi^0 = \phi_0$ in Ω , for all $n \geq 1$, compute the following steps:

STEP 1: CAHN–HILLIARD PROBLEM. Given $(\mathbf{v}^n, \mathbf{B}^n, \phi^n, m^n)$, solve for $(\phi^{n+1}, m^{n+1}) \in$ such that

$$d_t \phi^{n+1} + \mathbf{v}^n \cdot \nabla \phi^{n+1} = \nabla \cdot (M(\phi^{n+1}) \nabla m^{n+1}) \quad \text{in } \Omega, \quad (3.1a)$$

$$-\frac{G_S}{4} \text{tr } \mathbf{B}^n + \gamma \left(\frac{1}{\varepsilon} W'(\phi^{n+1}) - \varepsilon \Delta \phi^{n+1} \right) = m^{n+1} \quad \text{in } \Omega, \quad (3.1b)$$

$$\nabla \phi^{n+1} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega, \quad (3.1c)$$

$$\nabla m^{n+1} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega. \quad (3.1d)$$

STEP 2: NAVIER–STOKES PROBLEM. Given $(\mathbf{v}^n, p^n, \mathbf{B}^n)$ and (ϕ^{n+1}, m^{n+1}) , solve for $(\mathbf{v}^{n+1}, p^{n+1})$ such that

$$\begin{aligned} & \frac{\rho(\phi^{n+1})\mathbf{v}^{n+1} - \rho(\phi^n)\mathbf{v}^n}{\Delta t} + (\mathbf{v}^n \cdot \nabla)(\rho(\phi^{n+1})\mathbf{v}^{n+1}) + \frac{\rho_F - \rho_S}{2} M(\phi^{n+1})(\nabla m^{n+1} \cdot \nabla)\mathbf{v}^{n+1} \\ & - \nabla \cdot (2\mu(\phi^{n+1})\mathbf{D}(\mathbf{v}^{n+1})) + \nabla p^{n+1} \\ & = \nabla \cdot (G(\phi^{n+1})\mathbf{B}^n) - \gamma \varepsilon \nabla \cdot (\nabla \phi^{n+1} \otimes \nabla \phi^{n+1}) + \rho(\phi^{n+1})\mathbf{f} \end{aligned} \quad \text{in } \Omega, \quad (3.2a)$$

$$\nabla \cdot \mathbf{v}^{n+1} = 0 \quad \text{in } \Omega, \quad (3.2b)$$

$$\mathbf{v}^{n+1} = 0 \quad \text{on } \partial\Omega. \quad (3.2c)$$

STEP 3: TRANSPORT PROBLEM. Given $\mathbf{B}^n$ and $(\mathbf{v}^{n+1}, \phi^{n+1})$, solve for $\mathbf{B}^{n+1}$ such that

$$\begin{aligned} & G(\phi^{n+1}) (d_t \mathbf{B}^{n+1} + (\mathbf{v}^{n+1} \cdot \nabla)\mathbf{B}^n - (\nabla \mathbf{v}^{n+1})\mathbf{B}^n - \mathbf{B}^n(\nabla \mathbf{v}^{n+1})^\top) \\ & + \alpha(\phi^{n+1})(\mathbf{B}^{n+1} - \mathbf{I}) - \delta \Delta \mathbf{B}^{n+1} = 0 \end{aligned} \quad \text{in } \Omega \quad (3.3a)$$

$$\nabla \mathbf{B}^{n+1} \mathbf{n} = 0 \quad \text{on } \partial\Omega. \quad (3.3b)$$

Set $n = n + 1$ and go back to STEP 1.

4. Stability analysis

Let

$$\mathcal{E}^n = \frac{1}{2} \|\rho(\phi^n)^{\frac{1}{2}} \mathbf{v}^n\|_{L^2(\Omega)}^2 + \frac{\varepsilon \gamma}{2} \|\nabla \phi^n\|_{L^2(\Omega)}^2 + \frac{1}{2} \|G(\phi^n) \mathbf{B}^n\|_{*,\Omega} + \frac{\gamma}{\varepsilon} \int_{\Omega} W(\phi^n).$$

We also denote

$$\begin{aligned} \mu_* &= \min\{\mu_F, \mu_S\}, & \mu^* &= \max\{\mu_F, \mu_S\} & \rho_* &= \min\{\rho_F, \rho_S\}, & \rho^* &= \max\{\rho_F, \rho_S\}, \\ M_* &= \min_{\phi \in [-1,1]} M(\phi), & M^* &= \max_{\phi \in [-1,1]} M(\phi). \end{aligned}$$

Theorem 4.1. Let $\{(\phi^n, m^n, \mathbf{v}^n, p^n, \mathbf{B}^n)\}_{1 \leq n \leq N}$ be the solution of Algorithm 1. Assume that Assumption 2.1 is satisfied, that $\mu_S > 0$ (so that $\mu_* > 0$), and that

$$\Delta t < \rho_* M_*, \quad \varepsilon^2 \geq \alpha_W C_{PW}^2,$$

where C_{PW} is the Poincaré–Wirtinger constant of Ω , i.e., the smallest constant such that $\|f - \bar{f}\|_{L^2(\Omega)} \leq C_{PW} \|\nabla f\|_{L^2(\Omega)}$ for all $f \in H^1(\Omega)$, with $\bar{f} = \frac{1}{|\Omega|} \int_{\Omega} f$. Then, the following estimate holds

$$\begin{aligned} \mathcal{E}^N + \frac{\Delta t M_*}{2} \sum_{n=0}^{N-1} \|\nabla m^{n+1}\|_{L^2(\Omega)}^2 + \frac{\gamma}{2} \left(\varepsilon - \frac{\alpha_W C_{PW}^2}{\varepsilon} \right) \sum_{n=0}^{N-1} \|\nabla \phi^{n+1} - \nabla \phi^n\|_{L^2(\Omega)}^2 + \Delta t \mu_* \sum_{n=0}^{N-1} \|\mathbf{D}(\mathbf{v}^{n+1})\|_{L^2(\Omega)}^2 \\ + \frac{\Delta t}{2} \sum_{n=0}^{N-1} \|\alpha(\phi^{n+1}) \mathbf{B}^{n+1}\|_{*,\Omega} \leq \mathcal{E}^0 + \frac{d \Delta t}{2} \sum_{n=0}^{N-1} \int_{\Omega} \alpha(\phi^{n+1}) + \frac{\Delta t (\rho^*)^2 C}{4 \mu_*} \sum_{n=0}^{N-1} \|\mathbf{f}^{n+1}\|_{L^2(\Omega)}^2. \end{aligned}$$

Proof. We multiply (3.1a) by $\Delta t m^{n+1}$ and (3.1b) by $\phi^{n+1} - \phi^n + \Delta t \mathbf{v}^{n+1} \cdot \nabla \phi^{n+1}$ and integrate over Ω . Adding the results together and using $2a(a-b) = a^2 - b^2 + (a-b)^2$, we obtain

$$\begin{aligned} \Delta t \|M(\phi^{n+1})^{\frac{1}{2}} \nabla m^{n+1}\|_{L^2(\Omega)}^2 + \frac{\varepsilon \gamma}{2} \|\nabla \phi^{n+1}\|_{L^2(\Omega)}^2 - \frac{\varepsilon \gamma}{2} \|\nabla \phi^n\|_{L^2(\Omega)}^2 + \frac{\varepsilon \gamma}{2} \|\nabla \phi^{n+1} - \nabla \phi^n\|_{L^2(\Omega)}^2 \\ - \Delta t \gamma \varepsilon \int_{\Omega} \nabla \cdot (\nabla \phi^{n+1} \otimes \nabla \phi^{n+1}) \cdot \mathbf{v}^{n+1} + \frac{\gamma}{\varepsilon} \int_{\Omega} W'(\phi^{n+1})(\phi^{n+1} - \phi^n) \\ - \frac{G_S}{4} \int_{\Omega} \text{tr} \mathbf{B}^n(\phi^{n+1} - \phi^n + \Delta t \mathbf{v}^{n+1} \cdot \nabla \phi^{n+1}) = \Delta t \int_{\Omega} m^{n+1}(\mathbf{v}^{n+1} - \mathbf{v}^n) \cdot \nabla \phi^{n+1}, \end{aligned} \quad (4.1)$$

where we have used

$$\frac{\gamma}{\varepsilon} \int_{\Omega} W'(\phi^{n+1}) \nabla \phi^{n+1} \cdot \mathbf{v}^{n+1} = \frac{\gamma}{\varepsilon} \int_{\Omega} \nabla W(\phi^{n+1}) \cdot \mathbf{v}^{n+1} = -\frac{\gamma}{\varepsilon} \int_{\Omega} W(\phi^{n+1}) \nabla \cdot \mathbf{v}^{n+1} = 0,$$

Supposing that W is sufficiently smooth, we can apply Taylor's expansion around ϕ^{n+1} to assert that there is η between ϕ^{n+1} and ϕ^n such that

$$W'(\phi^{n+1})(\phi^{n+1} - \phi^n) = W(\phi^{n+1}) - W(\phi^n) + \frac{1}{2} W''(\eta)(\phi^{n+1} - \phi^n)^2. \quad (4.2)$$

Then, multiplying (4.2) by γ/ε and integrating over Ω ,

$$\frac{\gamma}{\varepsilon} \int_{\Omega} W'(\phi^{n+1})(\phi^{n+1} - \phi^n) = \frac{\gamma}{\varepsilon} \int_{\Omega} [W(\phi^{n+1}) - W(\phi^n) + \frac{1}{2} W''(\eta)(\phi^{n+1} - \phi^n)^2].$$

Assumption 2.1 (see (2.7)) only gives $W''(\eta) \geq -\alpha_W$ (it need not be nonnegative), so instead we bound the remainder as follows:

$$\begin{aligned} \frac{\gamma}{\varepsilon} \int_{\Omega} W'(\phi^{n+1})(\phi^{n+1} - \phi^n) &= \frac{\gamma}{\varepsilon} \int_{\Omega} [W(\phi^{n+1}) - W(\phi^n)] + \frac{\gamma}{2\varepsilon} \int_{\Omega} W''(\eta)(\phi^{n+1} - \phi^n)^2 \\ &\geq \frac{\gamma}{\varepsilon} \int_{\Omega} [W(\phi^{n+1}) - W(\phi^n)] - \frac{\gamma \alpha_W}{2\varepsilon} \int_{\Omega} (\phi^{n+1} - \phi^n)^2. \end{aligned}$$

Integrating (3.1a) over Ω and using that $\mathbf{v}^n$ is divergence-free with $\mathbf{v}^n = \mathbf{0}$ on $\partial\Omega$ (so $\int_{\Omega} \mathbf{v}^n \cdot \nabla \phi^{n+1} = -\int_{\Omega} (\nabla \cdot \mathbf{v}^n) \phi^{n+1} + \int_{\partial\Omega} (\mathbf{v}^n \cdot \mathbf{n}) \phi^{n+1} = 0$), together with the homogeneous Neumann condition on m^{n+1} , gives $\int_{\Omega} (\phi^{n+1} - \phi^n) = 0$, i.e., $\phi^{n+1} - \phi^n$ has zero mean. The Poincaré–Wirtinger inequality therefore gives $\int_{\Omega} (\phi^{n+1} - \phi^n)^2 \leq C_{PW}^2 \|\nabla \phi^{n+1} - \nabla \phi^n\|_{L^2(\Omega)}^2$, and hence

$$\frac{\gamma}{\varepsilon} \int_{\Omega} W'(\phi^{n+1})(\phi^{n+1} - \phi^n) \geq \frac{\gamma}{\varepsilon} \int_{\Omega} [W(\phi^{n+1}) - W(\phi^n)] - \frac{\gamma \alpha_W C_{PW}^2}{2\varepsilon} \|\nabla \phi^{n+1} - \nabla \phi^n\|_{L^2(\Omega)}^2.$$

obtaining the following estimate

$$\Delta t \|M(\phi^{n+1})^{\frac{1}{2}} \nabla m^{n+1}\|_{L^2(\Omega)}^2 + \frac{\varepsilon \gamma}{2} \|\nabla \phi^{n+1}\|_{L^2(\Omega)}^2 - \frac{\varepsilon \gamma}{2} \|\nabla \phi^n\|_{L^2(\Omega)}^2 + \frac{\gamma}{\varepsilon} \int_{\Omega} [W(\phi^{n+1}) - W(\phi^n)]$$

$$\begin{aligned}
 & + \frac{\gamma}{2} \left(\varepsilon - \frac{\alpha_W C_{PW}^2}{\varepsilon} \right) \|\nabla \phi^{n+1} - \nabla \phi^n\|_{L^2(\Omega)}^2 - \Delta t \gamma \varepsilon \int_{\Omega} \nabla \cdot (\nabla \phi^{n+1} \otimes \nabla \phi^{n+1}) \cdot \mathbf{v}^{n+1} \\
 & - \frac{G_S}{4} \int_{\Omega} \text{tr} \mathbf{B}^n (\phi^{n+1} - \phi^n + \Delta t \mathbf{v}^{n+1} \cdot \nabla \phi^{n+1}) \\
 & \leq \Delta t \int_{\Omega} m^{n+1} (\mathbf{v}^{n+1} - \mathbf{v}^n) \cdot \nabla \phi^{n+1}.
 \end{aligned} \tag{4.3}$$

For the fluid problem, multiplying (3.2a) by $\Delta t \mathbf{v}^{n+1}$ and (3.2b) by $\Delta t p^{n+1}$, integrating over Ω , and adding the equations together, we have

$$\begin{aligned}
 & \underbrace{\int_{\Omega} (\rho(\phi^{n+1}) \mathbf{v}^{n+1} - \rho(\phi^n) \mathbf{v}^n) \cdot \mathbf{v}^{n+1}}_{I_1} + \underbrace{\Delta t \int_{\Omega} (\mathbf{v}^n \cdot \nabla) (\rho(\phi^{n+1}) \mathbf{v}^{n+1}) \cdot \mathbf{v}^{n+1}}_{I_2} \\
 & + \frac{\rho_F - \rho_S}{2} \Delta t \int_{\Omega} [M(\phi^{n+1}) (\nabla m^{n+1} \cdot \nabla) \mathbf{v}^{n+1}] \cdot \mathbf{v}^{n+1} + 2 \Delta t \int_{\Omega} \mu(\phi^{n+1}) |\mathbf{D}(\mathbf{v}^{n+1})|^2 \\
 & + \Delta t \int_{\Omega} G(\phi^{n+1}) \mathbf{B}^n : \nabla \mathbf{v}^{n+1} = -\Delta t \gamma \varepsilon \int_{\Omega} \nabla \cdot (\nabla \phi^{n+1} \otimes \nabla \phi^{n+1}) \cdot \mathbf{v}^{n+1} + \Delta t \int_{\Omega} \rho(\phi^{n+1}) \mathbf{f} \cdot \mathbf{v}^{n+1}.
 \end{aligned} \tag{4.4}$$

For I_1 , the following equality holds:

$$I_1 = \frac{1}{2} \int_{\Omega} \rho(\phi^{n+1}) |\mathbf{v}^{n+1}|^2 - \frac{1}{2} \int_{\Omega} \rho(\phi^n) |\mathbf{v}^n|^2 + \underbrace{\frac{1}{2} \int_{\Omega} (\rho(\phi^{n+1}) - \rho(\phi^n)) |\mathbf{v}^{n+1}|^2 + \frac{1}{2} \int_{\Omega} \rho(\phi^n) |\mathbf{v}^{n+1} - \mathbf{v}^n|^2}_{I_3}.$$

Using (3.1a), we have

$$\begin{aligned}
 I_3 & = \frac{\rho_F - \rho_S}{4} \int_{\Omega} (\phi^{n+1} - \phi^n) |\mathbf{v}^{n+1}|^2 \\
 & = -\frac{\rho_F - \rho_S}{4} \Delta t \int_{\Omega} \mathbf{v}^n \cdot \nabla \phi^{n+1} |\mathbf{v}^{n+1}|^2 + \frac{\rho_F - \rho_S}{4} \Delta t \int_{\Omega} \nabla \cdot (M(\phi^{n+1}) \nabla m^{n+1}) |\mathbf{v}^{n+1}|^2 \\
 & = \underbrace{-\frac{\Delta t}{2} \int_{\Omega} \mathbf{v}^n \cdot \nabla \rho(\phi^{n+1}) |\mathbf{v}^{n+1}|^2}_{I_4} - \frac{\rho_F - \rho_S}{2} \Delta t \int_{\Omega} ((M(\phi^{n+1}) \nabla m^{n+1} \cdot \nabla) \mathbf{v}^{n+1}) \cdot \mathbf{v}^{n+1}.
 \end{aligned}$$

For integral I_4 , after integration by parts, we have

$$I_4 = \Delta t \int_{\Omega} \rho(\phi^{n+1}) ((\mathbf{v}^n \cdot \nabla) \mathbf{v}^{n+1}) \cdot \mathbf{v}^{n+1}.$$

Similarly, integral I_2 can be written as

$$I_2 = -\Delta t \int_{\Omega} \rho(\phi^{n+1}) ((\mathbf{v}^n \cdot \nabla) \mathbf{v}^{n+1}) \cdot \mathbf{v}^{n+1},$$

obtaining the following estimate

$$\begin{aligned}
 & \frac{1}{2} \int_{\Omega} \rho(\phi^{n+1}) |\mathbf{v}^{n+1}|^2 - \frac{1}{2} \int_{\Omega} \rho(\phi^n) |\mathbf{v}^n|^2 + \frac{1}{2} \int_{\Omega} \rho(\phi^n) |\mathbf{v}^{n+1} - \mathbf{v}^n|^2 + 2 \Delta t \int_{\Omega} \mu(\phi^{n+1}) |\mathbf{D}(\mathbf{v}^{n+1})|^2 \\
 & + \Delta t \int_{\Omega} G(\phi^{n+1}) \mathbf{B}^n : \nabla \mathbf{v}^{n+1} = -\Delta t \gamma \varepsilon \int_{\Omega} \nabla \cdot (\nabla \phi^{n+1} \otimes \nabla \phi^{n+1}) \cdot \mathbf{v}^{n+1} + \Delta t \int_{\Omega} \rho(\phi^{n+1}) \mathbf{f} \cdot \mathbf{v}^{n+1}.
 \end{aligned} \tag{4.5}$$

Next, we take the trace of (3.3a), multiply by $\frac{\Delta t}{2}$ and integrate over Ω , obtaining the following:

$$\begin{aligned} & \frac{1}{2} \|G(\phi^{n+1})\mathbf{B}^{n+1}\|_{*,\Omega} - \frac{1}{2} \|G(\phi^n)\mathbf{B}^n\|_{*,\Omega} - \frac{1}{2} \int_{\Omega} (G(\phi^{n+1}) - G(\phi^n)) \text{tr}(\mathbf{B}^n) \\ & + \frac{1}{2} \Delta t \int_{\Omega} G(\phi^{n+1})(\mathbf{v}^{n+1} \cdot \nabla) \text{tr} \mathbf{B}^n - \Delta t \int_{\Omega} G(\phi^{n+1}) \nabla \mathbf{v}^{n+1} : \mathbf{B}^n \\ & + \frac{\Delta t}{2} \|\alpha(\phi^{n+1})\mathbf{B}^{n+1}\|_{*,\Omega} - \frac{d\Delta t}{2} \int_{\Omega} \alpha(\phi^{n+1}) = 0. \end{aligned} \quad (4.6)$$

Adding (4.3), (4.5) and (4.6) together, and taking into account the estimates above, we obtain:

$$\begin{aligned} & \frac{1}{2} \|\rho(\phi^{n+1})^{\frac{1}{2}} \mathbf{v}^{n+1}\|_{L^2(\Omega)}^2 - \frac{1}{2} \|\rho(\phi^n)^{\frac{1}{2}} \mathbf{v}^n\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\rho(\phi^n)^{\frac{1}{2}} (\mathbf{v}^{n+1} - \mathbf{v}^n)\|_{L^2(\Omega)}^2 + \Delta t \|M(\phi^{n+1})^{\frac{1}{2}} \nabla m^{n+1}\|_{L^2(\Omega)}^2 \\ & + \frac{\gamma}{\varepsilon} \int_{\Omega} (W(\phi^{n+1}) - W(\phi^n)) + \frac{\varepsilon\gamma}{2} \|\nabla \phi^{n+1}\|_{L^2(\Omega)}^2 - \frac{\varepsilon\gamma}{2} \|\nabla \phi^n\|_{L^2(\Omega)}^2 + \frac{\gamma}{2} \left(\varepsilon - \frac{\alpha_W C_{PW}^2}{\varepsilon} \right) \|\nabla \phi^{n+1} - \nabla \phi^n\|_{L^2(\Omega)}^2 \\ & + 2\Delta t \int_{\Omega} \mu(\phi^{n+1}) |\mathbf{D}(\mathbf{v}^{n+1})|^2 + \frac{1}{2} \|G(\phi^{n+1})\mathbf{B}^{n+1}\|_{*,\Omega} - \frac{1}{2} \|G(\phi^n)\mathbf{B}^n\|_{*,\Omega} + \frac{\Delta t}{2} \|\alpha(\phi^{n+1})\mathbf{B}^{n+1}\|_{*,\Omega} \\ & - \frac{d\Delta t}{2} \int_{\Omega} \alpha(\phi^{n+1}) = \Delta t \int_{\Omega} m^{n+1} \nabla \phi^{n+1} \cdot (\mathbf{v}^{n+1} - \mathbf{v}^n) + \Delta t \int_{\Omega} \rho(\phi^{n+1}) \mathbf{f}^{n+1} \cdot \mathbf{v}^{n+1}. \end{aligned} \quad (4.7)$$

To estimate the terms on the right hand side, we proceed as follows. For the first term, using Assumption 2.1 we have the following

$$\begin{aligned} \Delta t \int_{\Omega} m^{n+1} \nabla \phi^{n+1} \cdot (\mathbf{v}^{n+1} - \mathbf{v}^n) & = -\Delta t \int_{\Omega} \nabla \cdot (m^{n+1}(\mathbf{v}^{n+1} - \mathbf{v}^n)) \phi^{n+1} + \Delta t \int_{\Omega} \nabla \cdot (m^{n+1} \phi^{n+1} (\mathbf{v}^{n+1} - \mathbf{v}^n)) \\ & = -\Delta t \int_{\Omega} \underbrace{\nabla \cdot (\mathbf{v}^{n+1} - \mathbf{v}^n) m^{n+1} \phi^{n+1}}_{=0} - \Delta t \int_{\Omega} (\mathbf{v}^{n+1} - \mathbf{v}^n) \cdot \nabla m^{n+1} \phi^{n+1} \\ & \leq \Delta t \|\phi^{n+1}\|_{L^\infty(\Omega)} \|\mathbf{v}^{n+1} - \mathbf{v}^n\|_{L^2(\Omega)} \|\nabla m^{n+1}\|_{L^2(\Omega)} \\ & \leq \Delta t \|\mathbf{v}^{n+1} - \mathbf{v}^n\|_{L^2(\Omega)} \|\nabla m^{n+1}\|_{L^2(\Omega)} \\ & \leq \frac{\Delta t}{2M_*} \int_{\Omega} |\mathbf{v}^{n+1} - \mathbf{v}^n|^2 + \frac{\Delta t M_*}{2} \|\nabla m^{n+1}\|_{L^2(\Omega)}^2. \end{aligned} \quad (4.8)$$

We also bound

$$\Delta t \int_{\Omega} \rho(\phi^{n+1}) \mathbf{f} \cdot \mathbf{v}^{n+1} \leq \frac{\Delta t (\rho^*)^2 C}{4\mu_*} \|\mathbf{f}^{n+1}\|_{L^2(\Omega)}^2 + \Delta t \mu_* \|\mathbf{D}(\mathbf{v}^{n+1})\|_{L^2(\Omega)}^2, \quad (4.9)$$

where C is a constant resulting from the Poincare-Friedrichs and Korn inequalities. Incorporating (4.8) and (4.9) in (4.7), we obtain

$$\begin{aligned} & \mathcal{E}^{n+1} + \frac{1}{2} \left(\rho_* - \frac{\Delta t}{M_*} \right) \|\mathbf{v}^{n+1} - \mathbf{v}^n\|_{L^2(\Omega)}^2 + \frac{\Delta t M_*}{2} \|\nabla m^{n+1}\|_{L^2(\Omega)}^2 + \frac{\gamma}{2} \left(\varepsilon - \frac{\alpha_W C_{PW}^2}{\varepsilon} \right) \|\nabla \phi^{n+1} - \nabla \phi^n\|_{L^2(\Omega)}^2 \\ & + \Delta t \mu_* \|\mathbf{D}(\mathbf{v}^{n+1})\|_{L^2(\Omega)}^2 + \frac{\Delta t}{2} \|\alpha(\phi^{n+1})\mathbf{B}^{n+1}\|_{*,\Omega} \\ & \leq \frac{d\Delta t}{2} \int_{\Omega} \alpha(\phi^{n+1}) + \frac{\Delta t (\rho^*)^2 C}{4\mu_*} \|\mathbf{f}^{n+1}\|_{L^2(\Omega)}^2. \end{aligned}$$

Summing from $n = 0$ to $N - 1$, we obtain the desired estimate. $\square$

Remark 4.2. Even though parameter condition

$$\varepsilon^2 \geq \alpha_W C_{PW}^2$$

in Theorem 4.1 seems restrictive, it can be avoided. It arises from the need to control the remainder term involving W'' in the energy estimate. Since C_{PW} scales like the diameter of Ω , it is a restriction on the interface width relative to the size of the domain. We note two alternative approaches, although other options are possible.

First, instead of imposing a condition on the model parameters, one can assume a restriction on the time step of the form $\Delta t \leq C\alpha_W$ and absorb the remainder term into the dissipation term associated with the velocity differences $\|\mathbf{v}^{n+1} - \mathbf{v}^n\|^2$. Combined with a discrete Grönwall argument, this yields a stability bound under a time-step restriction depending on α_W . The price is that the resulting bound may grow exponentially in the final time T .

Second, a common approach in the numerical analysis of the Cahn–Hilliard equation is to add a stabilization term $L(\phi^{n+1} - \phi^n)$ to the equation for the chemical potential, see, e.g., [18]. In our setting, (3.1b) would be replaced by

$$-\frac{G_S}{4} \operatorname{tr} \mathbf{B}^n + \gamma \left(\frac{1}{\varepsilon} W'(\phi^n) - \varepsilon \Delta \phi^{n+1} \right) + L(\phi^{n+1} - \phi^n) = m^{n+1}.$$

For L sufficiently large relative to $\gamma\beta_W/\varepsilon$, the stabilization term absorbs $\frac{\gamma\beta_W}{2\varepsilon} \|\phi^{n+1} - \phi^n\|_{L^2(\Omega)}^2$ (here the upper bound β_W in (2.7) is needed, since W' is evaluated at ϕ^n), which typically leads to energy stability without a restriction on Δt . However, in our numerical simulations, we did not experience any need to include this stabilization.

A third option is to exploit the structure of the Cahn–Hilliard step: by (3.1a), $\phi^{n+1} - \phi^n$ can be bounded in $H^{-1}(\Omega)$ in terms of $\Delta t \|\nabla m^{n+1}\|_{L^2(\Omega)}$ and $\Delta t \|\mathbf{v}^n\|_{L^2(\Omega)}$. An interpolation between $H^{-1}(\Omega)$ and $H^1(\Omega)$ then allows the remainder to be absorbed under a time-step restriction depending on ε , as is classical for implicit discretizations of the Cahn–Hilliard equation.

5. Numerical results

In this section, we investigate the rates of convergence and the performance of Algorithm 1. In Example 1, we use the method of manufactured solutions to compute convergence rates of Algorithm 1 in the case of both matching and non-matching densities. A classical benchmark problem of a channel flow over an elastic obstacle is considered in Example 2, and a contact problem of a falling elastic ball is simulated in Example 3.

The proposed numerical scheme is discretized in space using the finite element method and implemented in the high-performance programming language Julia, using the finite element package Gridap [4]. In all examples, we use the Taylor–Hood elements for the velocity and pressure, the $\mathbb{P}_2$ elements for the phase-field variables and the $\mathbb{P}_1$ elements for $\mathbf{B}$. In all numerical examples below, we use the double-well potential $W(\phi) = \frac{1}{4}(1 - \phi^2)^2$, for which $W'(\phi) = \phi^3 - \phi$. Although $W'(\phi)$ is treated implicitly in Algorithm 1, in the implementation we use the linearization

$$W'(\phi^{n+1}) \approx ((\phi^n)^2 - 1) \phi^{n+1}.$$

With this choice, the phase-field subproblem is linear in ϕ^{n+1} at each time step, so no nonlinear iterations (e.g., Newton’s method) are needed. For sufficiently smooth solutions, $(\phi^n)^2 = (\phi^{n+1})^2 + \mathcal{O}(\Delta t)$, so the linearization adds a consistency error of $\mathcal{O}(\Delta t)$. This does not reduce the first-order accuracy of the scheme. After solving the Cahn–Hilliard problem in STEP 1 of Algorithm 1, we apply a cutoff to the computed phase field, setting $\phi^{n+1} = 1$ wherever $\phi^{n+1} > 1$ and $\phi^{n+1} = -1$ wherever $\phi^{n+1} < -1$. This ensures that $\phi^{n+1} \in [-1, 1]$, so that Assumption 2.1 is satisfied. As discussed in Section 2.3, this cutoff does not preserve the mass $\int_{\Omega} \phi$ exactly. However, we reported the relative mass variation $(\int_{\Omega} \phi^n - \int_{\Omega} \phi^0) / \int_{\Omega} \phi^0$, showing that the mass loss stayed well below 1 % for the examples we considered.

5.1. Example 1: convergence against smooth manufactured solutions

In this example, we compute the convergence rates using the method of manufactured solutions. We set the computational domain to be the unit square. The exact solution is given by:

$$\begin{aligned} \mathbf{v}_{\text{ref}} &= 0.2 \sin(\pi t) \begin{bmatrix} \sin(\pi x) \cos(\pi y) \\ -\cos(\pi x) \sin(\pi y) \end{bmatrix}, & \mathbf{B}_{\text{ref}} &= 10^{-5} \sin(\pi t) (x+10)^2 (y+10)^2 \begin{bmatrix} 6 & 1 \\ 1 & 5 \end{bmatrix}, \\ \phi_{\text{ref}} &= \sin(\pi t) \cos(\pi x) \cos(\pi y), \end{aligned}$$

and p_{ref} is assumed to be a constant. The forcing term $\mathbf{f}$ in (2.5a) is computed using the exact solution. Forcing terms based on the exact solutions are also added to equations (2.5d) and (2.5c). Dirichlet boundary conditions are imposed

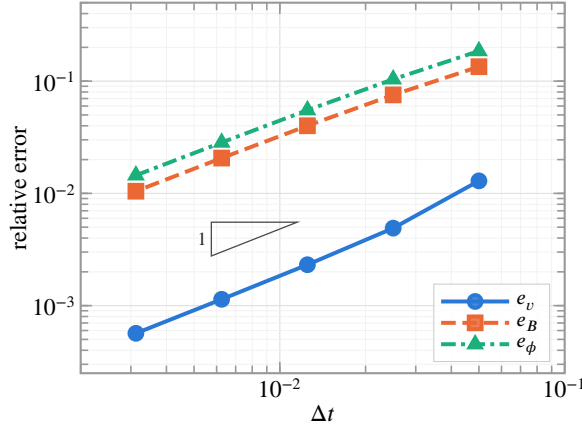


Figure 2: Example 1. Relative errors computed at the final time.

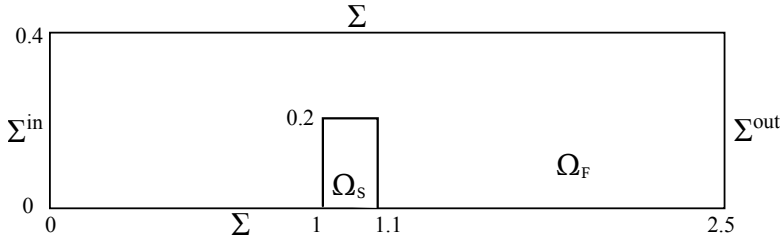


Figure 3: Example 2: fluid and structure reference domains used in Section 5.2.

for the velocity, and the homogeneous Neumann conditions are used in the Cahn–Hilliard problem. The final time is $T = 0.8$ s.

The physical parameters used in this example are: $\rho_F = \mu_F = G_S = 1$, $\rho_S = 2$, $\mu_S = 0$, $\gamma = 10^{-4}$, $\alpha_F = 1/T$, and we set $M(\phi) = 1$. No stabilization is needed in this example, so we set $\delta_{stab} = 0$. We use $\mathbb{P}_2 - \mathbb{P}_1$ elements for the $\mathbf{v}$ and p , $\mathbb{P}_1$ elements for $\mathbf{B}$, and $\mathbb{P}_2$ elements for ϕ and m . We define the relative errors as

$$e_v = \frac{\|\mathbf{v}_{\text{ref}}^N - \mathbf{v}_h^N\|_{L^2(\Omega)^2}}{\|\mathbf{v}_{\text{ref}}^N\|_{L^2(\Omega)^2}}, \quad e_B = \frac{\|\mathbf{B}_{\text{ref}}^N - \mathbf{B}_h^N\|_{\Omega}}{\|\mathbf{B}_{\text{ref}}^N\|_{\Omega}}, \quad e_\phi = \frac{\|\phi_{\text{ref}}^N - \phi_h^N\|_{L^2(\Omega)}}{\|\phi_{\text{ref}}^N\|_{L^2(\Omega)}},$$

evaluated at the final time. To compute the rates of convergence, we use the following set of discretization parameters

$$\{\Delta t, \Delta x\} = \left\{ \frac{5 \times 10^{-2}}{2^i}, \frac{2.5 \times 10^{-1}}{2^i} \right\}_{i=0}^3.$$

Parameter ε changes with the mesh size and is set to $\varepsilon = \Delta x$. The relative errors and rates of convergence are shown in Figure 2, where we observe that the error decay rate approaches first order for all variables. This is as anticipated, since the error is dominated by the temporal discretization error, which is expected to be first order.

5.2. Example 2: flow past an elastic step

We consider the channel flow past an elastic obstacle. Initially, the phase-field function is set to 1 in the fluid domain, which is a rectangle of length 2.5 cm and height $H = 0.4$ cm. It is equal to -1 in the region occupied by an elastic structure of length 0.1 cm and height 0.2 cm, positioned in the channel so that its bottom-left corner is at the point (1, 0). The fluid and structure reference domains are given in Figure 3.

Parameter	Value	Parameter	Value
ρ_F (g/cm ³)	1	ρ_S (g/cm ³)	1
μ_F (g/(cm s))	0.01	μ_S (g/(cm s))	1
G_S (dyne/cm ²)	500	G_F (dyne/cm ²)	0
γ	5×10^{-3}	δ	10^{-3}

Table 1

Example 2: parameter values used in Section 5.2.

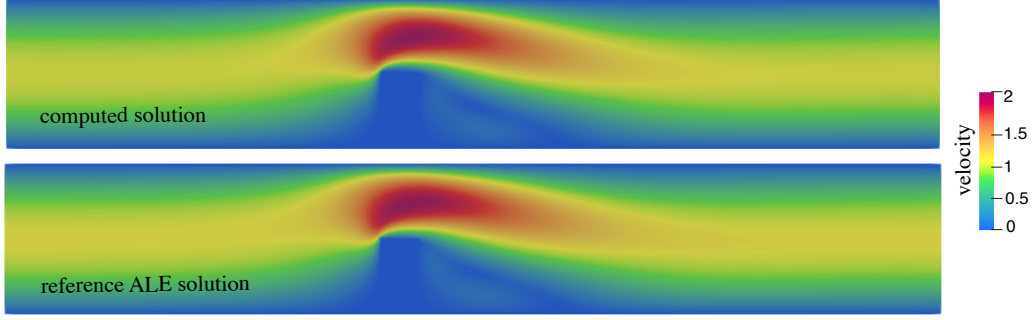


Figure 4: Example 2: approximate fluid velocity and structure displacement obtained using Algorithm 1 (top), compared to the reference solution from [6, 7] (bottom).

At the inflow and outflow boundaries, we impose

$$\mathbf{v} = \begin{cases} \left(\frac{1}{2}(1 - \cos(\pi t)) \frac{y(H-y)}{(H/2)^2}, 0 \right), & t \leq 1, \\ \left(\frac{y(H-y)}{(H/2)^2}, 0 \right), & \text{otherwise,} \end{cases} \quad \text{on } \Sigma_F^{\text{in}},$$

$$\sigma_F \mathbf{n}_F = \mathbf{0} \quad \text{on } \Sigma_F^{\text{out}},$$

while $\mathbf{v} = \mathbf{0}$ on $\partial\Omega_F \setminus (\Sigma_F^{\text{in}} \cup \Sigma_F^{\text{out}})$. The parameter values used in this example are given in Table 1.

Initially, we set $\mathbf{v} = \mathbf{0}$, $\mathbf{B} = \mathbf{I}$, and

$$\phi = \begin{cases} -1 & \text{if } 1 \leq x \leq 1.1 \text{ and } y \leq 0.2, \\ 1 & \text{otherwise.} \end{cases}$$

The external force is set to zero, $\mathbf{f} = \mathbf{0}$.

All meshes in this example are initially uniform with mesh size h and are locally refined around the fluid–structure interface every 0.1 s of simulated time, where the diffuse interface is identified as the region in which $\nabla\phi$ is nonzero. The smallest element size after the mesh is refined is denoted by $h_{\min}$. We set $\varepsilon = 2h_{\min}$ and $M = 10^{-4}\varepsilon$.

The simulations are performed until $T = 3$ s, using $\Delta t = 5 \times 10^{-4}$ and $\alpha_F = G_S/T$. The solution at the final time, obtained with $h = 0.013$, is shown in Figure 4, where it is compared against the reference solution obtained using a sharp interface, ALE approach [6, 7].

We further compare the outline of the deformed structure obtained using different values of h against the reference solution. The results are shown in Figure 5. As the mesh size decreases, the solution converges to the reference solution. Overall, an excellent agreement is observed.

Finally, we compute the relative mass variation over time, defined as $(\int_{\Omega} \phi^n - \int_{\Omega} \phi^0) / \int_{\Omega} \phi^0$. Figure 6 shows that the relative mass variation stays below 0.12% for the duration of the simulation and decreases as the mesh is refined. On the finest mesh, it remains below 0.02%.

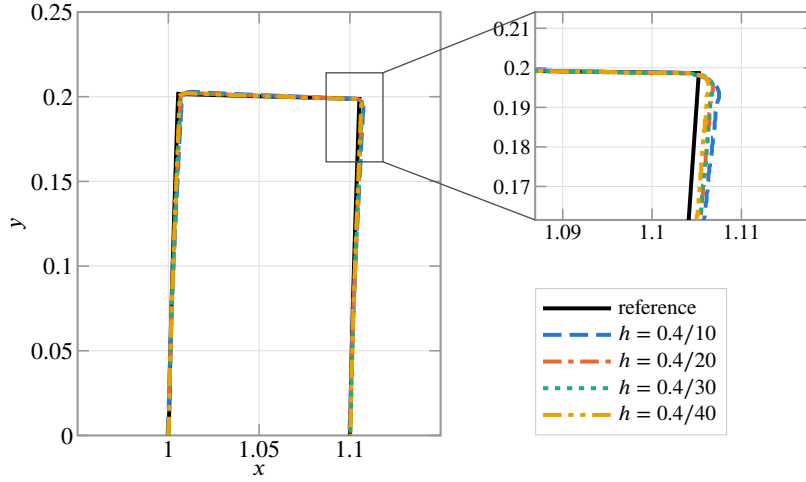


Figure 5: Example 2: outline of the elastic structure obtained using Algorithm 1 with various values of h , compared to the reference solution from [6, 7].

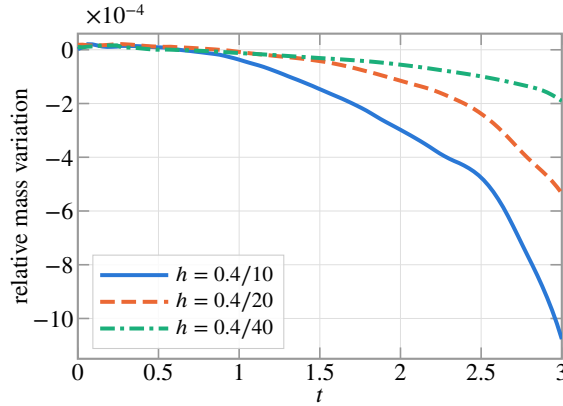


Figure 6: Example 2: the relative mass variation over time, computed using different meshes. The mass loss decreases as the mesh size is refined.

Parameter	Value	Parameter	Value
ρ_F (kg/m ³)	10^3	ρ_S (kg/m ³)	5×10^3
μ_F (Pa s)	10	μ_S (Pa s)	100
G_S (Pa)	10^4	G_F (Pa)	0
γ (N/m)	5×10^{-3}	δ (N/s)	10^{-1}

Table 2

Example 3: parameter values used in Section 5.3.

5.3. Example 3: bouncing ball

We consider a contact problem between an elastic solid immersed in a fluid and a rigid wall. The elastic solid is placed at the center of a closed tank containing a viscous incompressible fluid. The fluid domain is a square $[-1, 1] \times [-1, 1]$ m. The elastic solid is a circle with the center placed at $(0, 0)$ and the radius of 0.4 m. The fluid and solid parameters are given in Table 2. We impose no-slip boundary conditions for the velocity and the homogeneous Neumann boundary conditions for the left Cauchy–Green tensor. To handle contact, we impose the

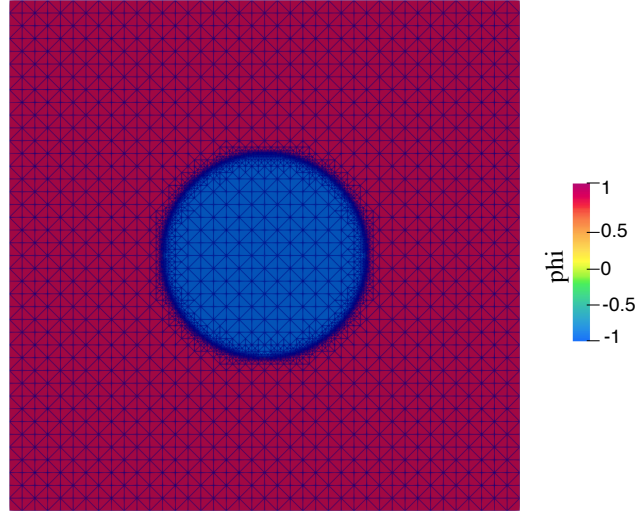


Figure 7: Example 3: initial value of ϕ and the computational mesh used in Section 5.3.

no-wetting condition

$$\phi = 1, \quad m = 0 \quad \text{on } \partial\Omega \times (0, T).$$

The ball starts falling due to gravity, which is imposed by setting $\mathbf{f} = (0, -0.98)$. Initially, we set $\mathbf{v} = \mathbf{0}$, $\mathbf{B} = \mathbf{I}$, and

$$\phi = \begin{cases} -1 & \text{if } x^2 + y^2 \leq 0.4^2, \\ 1 & \text{otherwise.} \end{cases}$$

As in the previous example, the meshes considered in this example are first constructed as uniform meshes of size h , which are then refined around the fluid-structure interface every 0.02 s, where the smallest element size after the mesh is refined is denoted by h_{min} . The computational mesh and ϕ at $t = 0$ are shown in Figure 7. We set $\varepsilon = 2h_{min}$ and $M = 10^{-5}$. The final time is set to $T = 5$ s, and $\alpha_F = G_S/5$.

Figure 8 shows the solution obtained using $\Delta t = 5 \times 10^{-4}$ and $h = 0.05$ at different points in time. First, the ball begins to move down due to the action of gravity, with increasing speed. It comes into contact with the bottom wall and begins to compress. Then, it rebounds due to the stored energy and rises up again, with the peak position after rebound occurring around $t = 2.86$ s.

We also compute the y -component of the center of mass as

$$y_{COM} = \frac{\int_{\Omega} y(1 - \phi)}{\int_{\Omega} (1 - \phi)}.$$

Figure 9 shows y_{COM} obtained using different values of Δt and h . The left panel shows results for three different time steps with $h = 0.1$, while the right panel shows results for three different meshes with $\Delta t = 2.5 \times 10^{-4}$. As expected with a diffuse-interface method, the ball does not make sharp contact with the boundary. However, decreasing the time step allows the ball to approach the bottom boundary more closely. We note that our simulations are in general agreement with the ones in [16]. However, a direct comparison is not possible since some parameters are not identified in [16].

The relative mass variation over time for this example is shown in Figure 10, for different time steps (left) and different mesh sizes (right). In all cases, the relative mass variation at the final time is below 0.2%, and the smallest variation is obtained with the finest mesh and the smallest time step. With the smallest mesh size, the variation does not exceed 0.05%. Overall, the mass variation is more sensitive to the mesh size than to the time step.

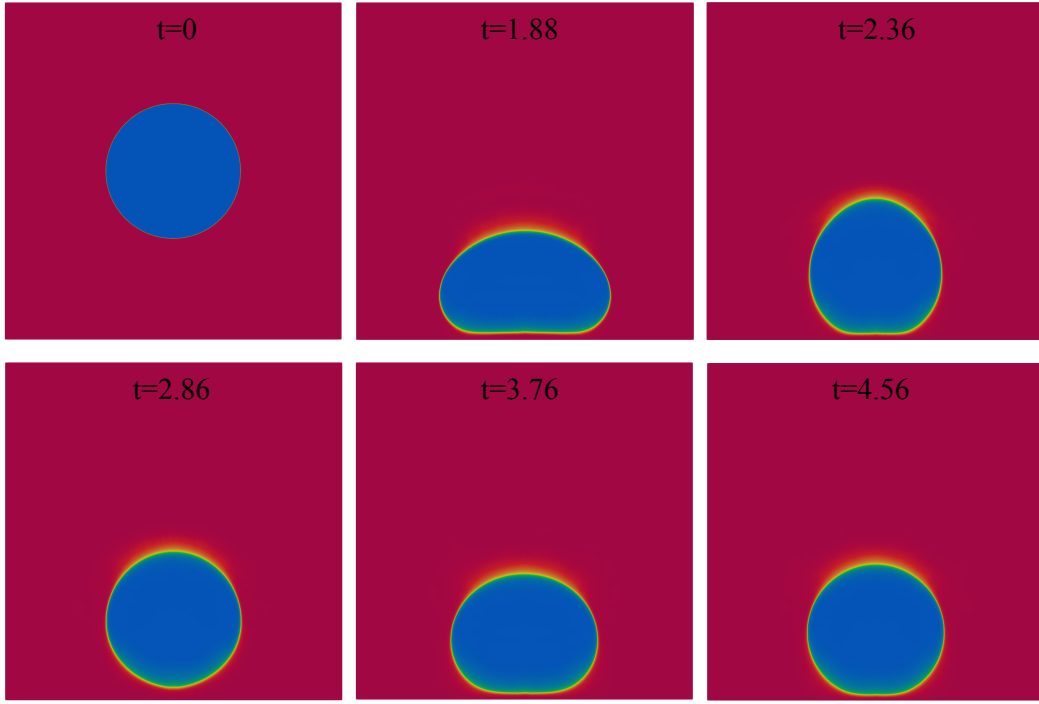


Figure 8: Example 3: an elastic ball falling in a viscous fluid. The ball touches the bottom wall and rebounds.

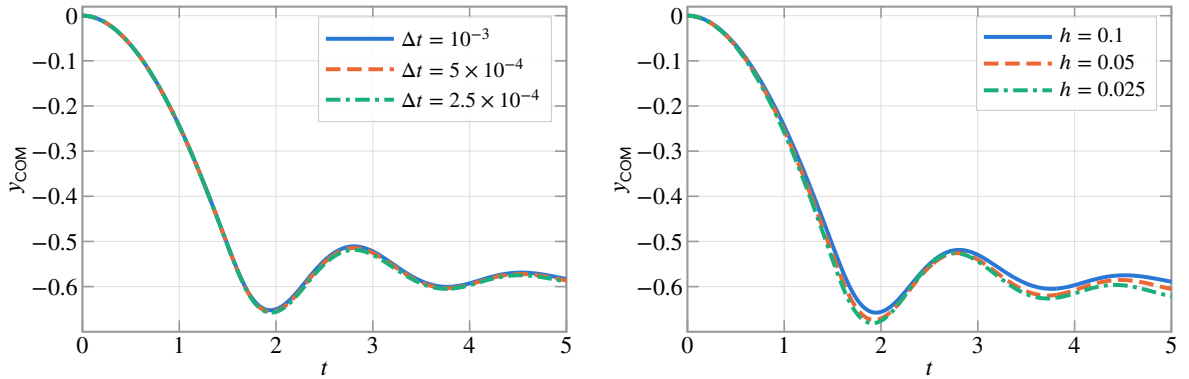


Figure 9: Example 3: approximate y -component of the center of mass obtained using different time steps (left) and mesh sizes (right).

6. Concluding remarks

We presented a loosely coupled time-discretization scheme for a fully Eulerian, diffuse-interface model of the interaction between an incompressible viscous fluid and an incompressible neo-Hookean solid, following the modeling approach of [15]. In this formulation, the location of the solid is tracked by a Cahn–Hilliard-type phase field, and its elastic response is described by a conformation tensor that satisfies an upper-convected transport equation and is active only in the solid phase. Starting from the Lagrangian and Eulerian elastodynamics equations, we recalled the sharp-interface Eulerian FSI system and the corresponding diffuse-interface model, and derived an energy estimate for the continuous problem.

The proposed algorithm decouples, at each time step, the Cahn–Hilliard problem, the Navier–Stokes problem, and the transport problem for the conformation tensor. Each subproblem is implicit in the variable it solves for, so it can be

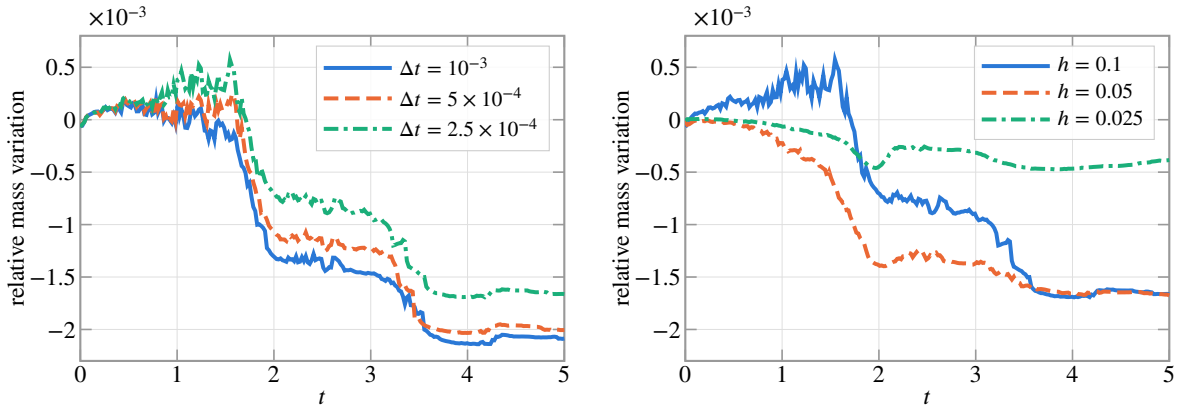


Figure 10: Example 3: the relative mass variation over time, computed using different time steps (left) and meshes (right). The mass loss decreases as the time step and the mesh size are refined.

handled with standard finite element solvers. We proved that the scheme satisfies a discrete energy estimate that mirrors the continuous one, under a condition relating the time step to the density and mobility parameters and a condition relating the interface width to the concavity bound of the double-well potential. The numerical results confirmed first-order convergence in time for a manufactured solution. For the flow past an elastic obstacle, the deformed structure agreed well with a reference ALE solution and converged to it under mesh refinement. For the falling elastic ball, the scheme captured the approach to the wall, contact and compression, and rebound without remeshing. As is expected with a diffuse-interface formulation, the ball does not make sharp contact with the wall, although it approaches the wall more closely as the time step decreases.

Further directions include a convergence analysis of the fully discrete scheme, second-order extensions in time, and three-dimensional problems involving contact between several elastic structures.

Acknowledgement

Martina Bukač has been partially supported by the United States National Science Foundation via grants DMS-2205695 and DMS-2602215. Ricardo Ruiz-Baier has been partially supported by the Australian Research Council through the Future Fellowship FT220100496. We also acknowledge the support by the Swedish Research Council under grant no. 2021-06594 while both authors were in residence at Institut Mittag-Leffler in Djursholm, Sweden during the fall semester of 2025.

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