# Multiresolution schemes for an extended clarifier-thickener model

Raimund Bürger<sup>\*1</sup>, Ricardo Ruiz<sup>\*\*1</sup>, Kai Schneider<sup>\*\*\*2</sup>, and Mauricio Sepúlveda<sup>†1</sup>

Departamento de Ingeniería Matemática, Facultad de Ciencias Físicas y Matemáticas, Universidad de Concepción, Casilla 160-C, Concepción, Chile.

<sup>2</sup> Centre de Mathématiques et d'Informatique, Université de Provence, 39 rue Joliot-Curie, 13453 Marseille cedex 13, France.

A fully adaptive finite volume multiresolution scheme for one-dimensional strongly degenerate parabolic equations with discontinuous flux modelling an extended clarifier-thickener, is presented. The numerical scheme is based on a finite volume discretization using the approximation of Engquist-Osher for the flux and explicit time stepping. Cell averages multiresolution scheme speeds up CPU time and memory requirements. A particular feature of our scheme is the storage of the multiresolution representation of the solution in a dynamic graded tree.

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## 1 Introduction

High resolution finite volume schemes for the approximation of discontinuous solutions to conservation laws are of at least second-order accuracy in regions where the solution is smooth and resolve discontinuities sharply and without spurious oscillations. Methods of this type include the schemes described in [1, 2, 3]. In standard situations, the solution u(x,t) of a conservation law  $u_t + f(u)_x = 0$ , exhibits strong variations (shocks) in small regions but behaves smoothly on the major portion of the computational domain. The multiresolution technique adaptively concentrates computational effort associated with a high resolution scheme on the regions of strong variation. It goes back to Harten [4] for hyperbolic equations and was used by Roussel et al. [5] for parabolic equations. Important contributions to the analysis of multiresolution methods for conservation laws include [6, 7]. In this note, we present a multiresolution scheme and corresponding numerical experiments for strongly degenerate parabolic equations with discontinuous flux modelling an extended clarifier-thickener. Specifically, we consider equations of the type

$$u_t + f(\boldsymbol{\gamma}(x), u)_x = \gamma^3(x)u_x \quad \text{for } x \in \Pi_T := \mathbb{R} \times (0, T], \tag{1}$$

where  $f : \mathbb{R}^{n+1} \to \mathbb{R}$  is a piecewise smooth and Lipschitz continuous function, and  $\gamma(x)$  is a vector of scalar parameters, which are discontinuous at most in a finite number of points. The unknown is the solids concentration u as function of time tand depth x. The extended model for the clarifier-thickener is given by (1) with  $f(\gamma(x), u) = \gamma^2(x)(u - u_F) + \gamma^1(x)b(u)$ , where  $b(u) = v_{\infty}u(1-u)^C$  for  $u \in (0, u_{\max})$ , and 0 otherwise. The discontinuities are given by:  $\gamma^1(x) = 1$ , if  $x \in [x_L, x_R]$ , and 0 otherwise;  $\gamma^2(x) = \tilde{q}_R - q_F$ , if  $x \leq 0$ , and  $\gamma^2(x) = \tilde{q}_R$ , if x > 0;  $\gamma^3(x) = 0$ , if  $x < x_D$ , and  $\gamma^3(x) = -q_D$ , if  $x > x_D$ . Assume an initial concentration  $u(x, 0) = u_0(x)$ , for  $x \in \mathbb{R}$  and  $u_0(x) \in [0, u_{\max}]$ .

## 2 Numerical scheme and multiresolution discretization

The numerical scheme for the solution of (1) is essentially described in [8]. We discretize  $\mathbb{R}$  into cells  $I_j := [x_{j-1/2}, x_{j+1/2})$ , where  $x_{j+1/2} = (j+1/2)\Delta x$  with  $j \in \mathbb{Z}$ . Let  $\lambda = \Delta t/\Delta x$  and  $U_j^0 = u_0(x_j)$ . We define the approximations according to

$$U_{j}^{n+1} = U_{j}^{n} - \lambda \Delta_{-} h(\gamma_{j+1/2}, U_{j+1}^{n}, U_{j}^{n}) - \lambda \gamma_{j}^{3} \Delta_{+} U_{j}^{n},$$
(2)

where  $\gamma_{j+1/2} := \gamma(x_{j+1/2}^-)$ , and  $\gamma_j^3 := \gamma^3(x_j^-)$ . The symbols  $\Delta_{\pm}$  are spatial difference operators:  $\Delta_- V_j := V_j - V_{j-1}$ and  $\Delta_+ V_j := V_{j+1} - V_j$ , and we use the Engquist-Osher flux  $h(\gamma, v, u) := \frac{1}{2} \left[ f(\gamma, u) + f(\gamma, v) - \int_u^v |f_u(\gamma, w)| dw \right]$ . For stability we need to satisfy a CFL condition requiring that  $\Delta t / \Delta x$  be bounded (see [8]). In our scheme, the differences and the solution at different levels, are organized in a tree structure. Whenever an element is included in the tree, all other elements corresponding to the same spatial region in coarser resolutions, are also included. The adaptive grid corresponds to a set of nested dyadic grids generated by refining recursively a given cell depending on the local regularity of the solution. See [9] for more details of the multiresolution strategy.

<sup>\*</sup> e-mail: rburger@ing-mat.udec.cl, Phone: +56412203126, Fax: +56412522055

<sup>\*\*</sup> e-mail: rruiz@ing-mat.udec.cl, Phone: +56412203455, Fax: +56412522055

<sup>\*\*\*</sup> e-mail: kschneid@cmi.univ-mrs.fr, Phone: +33 4 91 11 85 29 Fax: +33 4 91 11 35 02

<sup>&</sup>lt;sup>†</sup> Corresponding author: e-mail: mauricio@ing-mat.udec.cl, Phone: +56 41 2203118, Fax: +56 41 2522055

#### Numerical results 3

In this example we use the fully adaptive multiresolution scheme to solve a problem modelled by a conservation law with discontinuous flux. The corresponding equation possesses a nonconservative term, and despite this fact we see that with our adaptive scheme we obtain correct and highly accurated solutions. Let us consider a suspension characterized by  $v_{\infty}=$  $1.0 \times 10^{-4} \text{ m/s}, C = 5 \text{ and } u_{\text{max}} = 1$ . Furthermore  $x_{\text{L}} = -2 \text{ m}$  and  $x_{\text{R}} = 1 \text{ m}$ , the device being initially empty ( $u_0 \equiv 0$ ). These parameters and the control variables  $q_{\rm L} = 0.0 \,{\rm m/s}$ ,  $q_{\rm R} = 0.6 \,{\rm m/s}$ ,  $q_{\rm D} = -1.0 \,{\rm m/s}$  and  $u_{\rm F} = 0.7$  are chosen as in Case 5 of [8]. The reference tolerance used for this example is  $\varepsilon = 4.6 \times 10^{-4}$ . Figures 1 show the numerical solution using multiresolution. In every case the figures on the right side show that the multiresolution effectively detects the stationary shocks corresponding to the flux discontinuities and the differences of gradients in the solution.



Fig. 1 Left side: numerical solution (asterises), initial condition (dashed); right side: position of the leaves; t = 1 s and t = 4 s.

	MR Method	V	$\mu$	$L^1$ -error	$L^2$ -error	$L^{\infty}$ -error
$t = 1 \mathrm{s}$	r = 3	7.46	10.6391	$8.56  imes 10^{-5}$	$2.35  imes 10^{-7}$	$3.83 \times 10^{-5}$
	r = 5	7.25	10.1450	$2.16  imes 10^{-5}$	$7.48\times10^{-8}$	$1.29  imes 10^{-5}$
$t = 4 \mathrm{s}$	r = 3	8.29	11.3463	$2.83 \times 10^{-4}$	$5.32 \times 10^{-9}$	$9.51 \times 10^{-8}$
	r = 5	8.01	11.2871	$7.55\times10^{-5}$	$1.39\times10^{-9}$	$3.62\times 10^{-8}$

**Table 1** Speed–up rate  $V = \frac{(\text{CPU time})_{\text{FV}}}{(\text{CPU time})_{\text{MR}}}$ , compression rate  $\mu$ , and normalized errors. L = 12 (more details can be found in [9]).

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