Fully mixed finite element methods for thermoelastic contact

Sergio Carrasco[a,*], Franz Chouly[b], Alberto F. Martín[c] and Ricardo Ruiz-Baier[a,d]

[a] *School of Mathematics, Monash University, 9 Rainforest Walk, 3800, Melbourne, Victoria, Australia*

[b] *Centro de Matemática (CMAT), Universidad de la República, IRL CNRS 2030 Ifumi, Montevideo, Uruguay*

[c] *School of Computing, Australian National University, 2601, Acton, ACT, Australia*

[d] *Universidad Adventista de Chile, Casilla 7-D, Chillán, Chile*

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ABSTRACT

We investigate the unique solvability of nonlinear static contact problems incorporating thermal effects and propose appropriate mixed finite element spaces for their numerical approximation. The fully mixed formulation simultaneously resolves the Cauchy stress, displacement, rotation tensor, temperature and heat flux, with nonlinearities arising from the bulk and contact thermomechanical coupling. The weak formulation is analysed within the framework of perturbed saddle-point problems in Banach spaces, employing fixed-point techniques alongside suitably defined variational inequalities. Error estimates are derived for general Galerkin discretisations, and convergence rates are established for Arnold–Falk–Winther elements for the mechanical variables, coupled with Brezzi–Douglas–Marini elements and piecewise polynomials for the thermal unknowns. Finally, several numerical experiments are presented, confirming the theoretical convergence results and demonstrating the robust performance of the proposed method.

1. Introduction

First we present the scope and the outline of the paper, then we describe the equations in strong form for thermoelastic unilateral contact.

1.1. Scope

Thermoelastic contact problems arise in a wide range of engineering and scientific contexts where deformable bodies interact under both mechanical and thermal effects. These processes are prevalent in high-precision manufacturing of composite materials, geomechanical systems, metal forming, drilling, and the modelling of biological tissues, where both temperature fluctuations and mechanical stresses influence the evolution of contact regions. Mathematically, these problems involve the coupling of elasticity and heat equations with nonlinear boundary conditions that model the contact behaviour between deformable bodies. In the literature, it is possible to find works which studied and developed different models of thermoelastic contact. Thermomechanical contact laws based on microscopic interface laws were introduced in [46], and stability analyses for a problem with contact pressure-dependent thermal resistance were carried out in [32]. Further contributions including frictional settings can be found in [38, 41].

A central challenge in the variational formulation of contact problems is the enforcement of contact constraints, which are inherently nonlinear and involve inequality-type conditions. Among various approaches, here we use the so-called Barber contact conditions [45], which entail a thermodynamically consistent framework that relates the normal stress and temperature jump across the interface, capturing the physical interaction between heat conduction and mechanical compression at the contact surface. These conditions generalise classical Signorini-type constraints by introducing nonlinear relations not only in the mechanical but also in the thermal components.

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[*] Corresponding author.

✉ `Sergio.CarrascoHidalgo@monash.edu` (S. Carrasco); `FChouly@cmat.edu.uy` (F. Chouly); `Alberto.F.Martin@anu.edu.au` (A.F. Martín); `Ricardo.RuizBaier@monash.edu` (R. Ruiz-Baier)

ORCID(s): 0009-0008-4786-7855 (S. Carrasco); 0000-0002-9087-3301 (F. Chouly); 0000-0001-5751-4561 (A.F. Martín); 0000-0003-3144-5822 (R. Ruiz-Baier)

Mathematical and numerical analysis aspects of thermo-mechanical contact problems have been studied extensively in references [6, 15, 16, 35, 38, 41]. From a numerical perspective, several strategies have been developed to approximate contact problems. Penalisation methods [30], augmented Lagrangian approaches [1], projection-based techniques [26] and Nitsche-type methods [12, 28, 29] are widely used to enforce the contact constraints. These methods have been successfully applied in various settings, including frictional and thermo-mechanical contact [35, 13, 16], and in two-body contact scenarios [44, 12]. However, most of these approaches rely on primal or primal-mixed finite element formulations, where displacement and temperature are the primary unknowns. Here we focus on fully mixed finite element methods, which provide a natural and flexible framework for discretising such coupled systems. Fully mixed methods for thermoelasticity allow for the simultaneous approximation of stress, displacement, temperature, and heat flux, facilitating accurate enforcement of incompressibility or balance laws. On the other hand, it is well-known that dealing with problems involving coupling and the introduction of additional variables as in mixed formulations, often leads naturally to variational settings posed in Banach spaces. This feature has been recently exploited for a wide class of models (see, e.g. [9, 14] and the references therein), giving rise to formulations that can be interpreted, for instance, as saddle-point or perturbed saddle-point problems. An advantage of this approach is that it avoids the need for augmentation procedures and, from a theoretical viewpoint, it allows the unknowns to be sought in the natural Banach spaces arising from the underlying equations after testing and integration by parts. This is particularly relevant in the present work due to the nonlinear coupling and the functional structure of the governing equations. Our analysis is framed within a rigorous functional analytic setting. We employ tools from fixed-point theory, perturbed saddle-point problems in Banach spaces, and establish well-posedness of the continuous problem. While the theory of solvability of mixed variational inequalities in Hilbert spaces (see, e.g., [2, 39]) provides the backbone of our analysis, the present problem involves a Banach-space setting that falls outside the scope of those results. Hence, a suitable adaptation of the abstract theory is required. In fact, a key part of those proofs are the elliptic regularisation which consists in introduce a perturbation to the problem, which involves the inner product of the dual space. This is clearly not possible in a Banach setting. In this way, we establish that under certain conditions provided in [31], the saddle-point is still uniquely solvable even if not in Hilbert spaces, and continuous dependence on data can be derived as usual. The numerical scheme is then constructed using stable conforming finite element pairs for elasticity and temperature, along with suitable discrete Lagrange multiplier spaces to capture the nonlinear contact constraints. We show appropriate discrete inf-sup conditions tailored to the L^p structure of the problem and show that the coupled problem is well-posed. Regarding the a priori error estimates, owing to the coupled nature of the problem, the analysis necessarily involves discretised functionals. Consequently, a Falk-type estimate is established to properly account for the consistency and approximation error. We emphasise that such estimates are often omitted in related works [2, 15, 39], so our contribution provides an additional step to the error analysis, providing error estimates under reasonable regularity assumptions.

The mechanical part of the analysis is inspired by the abstract framework developed in [39], which provides the theoretical foundation for the a priori error analysis that we follow in detail. In particular, we adapt this approach to the present mixed and Banach-space setting and the discrepancy between the right-hand side of the continuous and discrete formulations, extending it to account for the coupled structure of our problem. In addition, the numerical treatment of the contact constraints is inspired by the active set strategy introduced in [42], which provides an efficient framework for handling inequality constraints and forms the basis of the discrete algorithm proposed in this work. We also refer to [43] for a comprehensive overview of consistent discretisation schemes and numerical algorithms for contact problems. This combination allows us to connect the abstract functional analytic setting with a robust and practical computational strategy. Another distinctive aspect of the present work concerns the discrete inf-sup condition associated with weak symmetry, which is not standard in this setting. In addition to the correction technique introduced in [7], we propose a further modification to ensure the enforcement of a zero-trace condition on a portion of the boundary of the admissible stress space. This additional correction is essential to guarantee the stability of the mixed formulation and the proper handling of the constrained stress variables.

1.2. Outline

We have organised the contents of this article in the following manner. In the remainder of this section we state the equations describing a model of thermoelastic contact, making precise the boundary and contact conditions including Barber effects. In Section 2 we derive a weak formulation for the problem in mixed form, and provide a fixed-point scheme for its solvability. In section 3 we gather the abstract results in order to solve the variational inequality arising from the previous section. In Section 4 we apply this framework on the decoupled problems related to the

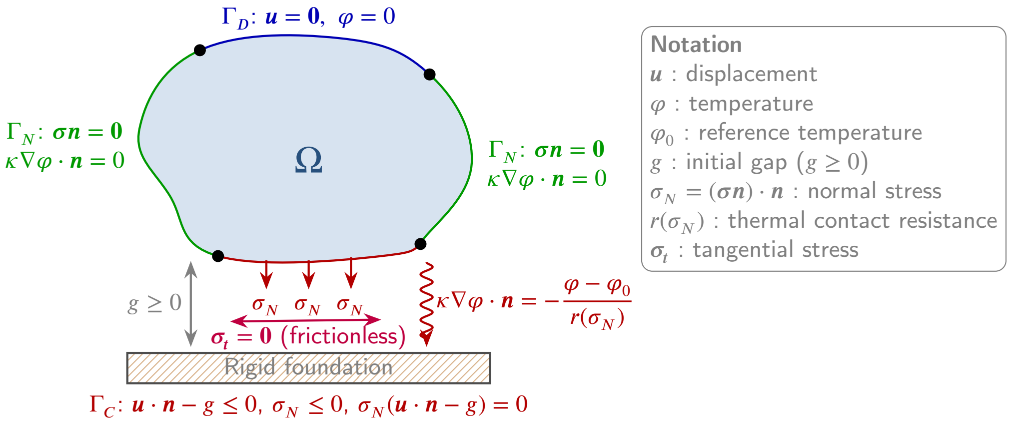


Figure 1.1: Sketch of the domain configuration, sub-boundaries, boundary conditions, and thermoelastic contact conditions. Here, g represents the initial gap between the elastic body and the rigid foundation (see, e.g. [30]).

solvability operators composing the fixed-point operator, proving their well–posedness, and the well–posedness of the fixed-point scheme. Section 5 is devoted to the construction and unique solvability analysis of the Galerkin discrete problem. In Section 6 we derive Falk estimates, and provide specific convergence rates for the finite element family using Arnold–Falk–Winther and Brezzi–Douglas–Marini elements for the mechanical and thermal sub-problems, respectively. Finally, in Section 7, we provide numerical experiments that confirm the theoretical convergence rates in the lowest-order case, and demonstrate the robustness of the method in practical scenarios.

1.3. Governing equations

Throughout the text, $d \in \{2,3\}$ denotes the spatial dimension. Given a normed space S, we denote by $\mathbf{S}$ and $\mathbb{S}$ the vector and tensor extensions S^d and $S^{d\times d}$, respectively. Next, for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,d}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,d}$, we let $\mathbf{div}(\boldsymbol{\tau})$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the deviatoric tensor, and the tensor inner product, respectively, as

$$\boldsymbol{\tau}^{\mathtt{t}} := (\tau_{ji})_{i,j=1,d}, \quad \operatorname{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{d} \tau_{ii}, \quad \boldsymbol{\tau}^{\mathtt{d}} := \boldsymbol{\tau} - \frac{1}{d}\operatorname{tr}(\boldsymbol{\tau})\,\mathbf{I}, \quad \text{and} \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{d} \tau_{ij}\,\zeta_{ij}\,.$$

Let $\Omega \subset \mathbb{R}^d$, be a simply connected, bounded Lipschitz domain occupied by a linearly elastic body. The boundary is decomposed into disjoint portions

$$\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N} \cup \overline{\Gamma_C},$$

where homogeneous displacement, traction, and contact boundary conditions are imposed, respectively, with all sub-boundaries non-empty: $|\Gamma_D| \cdot |\Gamma_N| \cdot |\Gamma_C| > 0$ (see a sketch in Figure 1.1).

We consider heat conduction influencing the elastic response. Neglecting inertial effects, the governing equations consist of the constitutive law, momentum balance, stress symmetry and energy balance equation respectively, namely

$$\begin{aligned}
\boldsymbol{\sigma} &= \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \alpha\varphi\mathbf{I} = 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) + \big(\lambda(\operatorname{div}\boldsymbol{u}) - \alpha\varphi\big)\mathbf{I} && \text{in } \Omega, && (1.1a)\\
-\mathbf{div}\,\boldsymbol{\sigma} &= \varrho\boldsymbol{f} && \text{in } \Omega, && (1.1b)\\
\boldsymbol{\sigma} &= \boldsymbol{\sigma}^{\mathtt{t}} && \text{in } \Omega, && (1.1c)\\
\ell\varphi + \boldsymbol{u}\cdot\nabla\varphi - \operatorname{div}(\kappa\nabla\varphi) &= m && \text{in } \Omega, && (1.1d)
\end{aligned}$$

where $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^{\mathtt{t}})$ is the infinitesimal strain, $\boldsymbol{\sigma}$ the Cauchy stress, $\boldsymbol{u}$ the displacement vector, φ the temperature, $\boldsymbol{f}$ the body force, ϱ the density, $\mathcal{C}$ the Hooke tensor, λ, μ are the Lamé parameters, α the thermal expansion coefficient, κ the (rescaled) thermal conductivity, ℓ a scaling parameter, and m a prescribed heat source. The thermal equation (1.1d) is obtained via backward Euler discretisation of the time-dependent energy balance, which explains the advecting term written with displacement rather than velocity. Following [13], the boundary conditions are

$$\boldsymbol{u} = \mathbf{0}, \quad \varphi = 0 \qquad \text{on } \Gamma_D, \tag{1.2a}$$

$$\boldsymbol{\sigma n} = \mathbf{0}, \quad \kappa \nabla \varphi \cdot \boldsymbol{n} = 0 \qquad \text{on } \Gamma_N, \tag{1.2b}$$

$$\boldsymbol{u} \cdot \boldsymbol{n} - g \le 0, \quad \sigma_N \le 0, \quad \sigma_N(\boldsymbol{u} \cdot \boldsymbol{n} - g) = 0 \qquad \text{on } \Gamma_C, \tag{1.2c}$$

$$\kappa \nabla \varphi \cdot \boldsymbol{n} = -\frac{\varphi - \varphi_0}{r(\sigma_N)} \qquad \text{on } \Gamma_C, \tag{1.2d}$$

where $g : \Gamma_C \to [0, +\infty)$ is the initial gap, $\boldsymbol{n}$ the outward unit normal, $\sigma_N := (\boldsymbol{\sigma n}) \cdot \boldsymbol{n}$ the normal stress, $\boldsymbol{\sigma}_t := \boldsymbol{\sigma n} - \sigma_N \boldsymbol{n}$ the tangential stress [3, 12, 30, 37], and φ_0 is a prescribed reference temperature at the rigid foundation. Frictionless contact is represented by

$$\boldsymbol{\sigma}_t = \mathbf{0} \quad \text{on } \Gamma_C. \tag{1.3}$$

The conditions in (1.2c) are the classical Signorini conditions (also referred to as complementarity conditions), which imply: if $\boldsymbol{u} \cdot \boldsymbol{n} < g$, then $\sigma_N = 0$ (no contact); if $\sigma_N < 0$, then $\boldsymbol{u} \cdot \boldsymbol{n} = g$ (active contact). Notably, the complementarity conditions prevent unphysical states with positive normal stress in absence of contact. Moreover, they allow both $\sigma_N = 0$ and $\boldsymbol{u} \cdot \boldsymbol{n} = g$ simultaneously, corresponding to grazing contact, where the body touches the foundation without developing compressive pressure and consequently with no heat transfer through the interface, see, e.g., [12, 30].

The thermal equivalence (1.2d) constitutes the so-called Barber condition [45] where the thermal contact resistance $r(\sigma_N)$ is assumed to be uniformly bounded and Lipschitz continuous on $(-\infty, -\delta]$, $\delta > 0$: there exist $0 < r_0 \le r_1$ and $L_r > 0$ such that

$$r_0 \le |r(\zeta)| \le r_1, \quad \forall \zeta \le -\delta, \tag{1.4a}$$

$$\|r(\zeta_N) - r(\tilde{\zeta}_N)\|_{0,\infty;\Gamma_C} \le L_r \|\boldsymbol{\zeta} - \tilde{\boldsymbol{\zeta}}\|_{0,\Omega} \quad \forall \boldsymbol{\zeta}, \tilde{\boldsymbol{\zeta}} \in L^2(\Omega). \tag{1.4b}$$

Experimental models include (see [41, 44])

$$r(\zeta_N) = b_0(\kappa)\, |\zeta_N|^{-b_1}, \quad r(\zeta_N) = c_0 \left(\frac{|\zeta_N|}{H_v} \right)^{-\omega}, \tag{1.5}$$

where the constants $b_0(\kappa)$, b_1, c_0 and ω reflect material properties, hardness, and surface roughness. These models satisfy (1.4a) and (1.4b) in the regime $\zeta_N \le \delta < 0$.

1.4. Strong mixed form

To obtain a fully mixed strong formulation for the problem (1.1)–(1.3), we first recall that for any smooth tensor $\boldsymbol{\zeta}$, we have

$$\mathcal{C}^{-1}\boldsymbol{\zeta} := \frac{1}{\mu}\, \boldsymbol{\zeta}^{\mathrm{d}} + \frac{1}{d(d\lambda + (d+1)\mu)}\, \operatorname{tr}(\boldsymbol{\zeta})\, \mathbf{I}, \tag{1.6}$$

so that the constitutive equation for stress can be written as

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = \mathcal{C}^{-1}\boldsymbol{\sigma} + \alpha\, \gamma(\lambda)\, \varphi\, \mathbf{I},$$

with

$$\gamma(\lambda) = \frac{1}{d\lambda + (d+1)\mu}. \tag{1.7}$$

We remark that, although $\gamma(\lambda)$ also depends on μ and d as defined in (1.7), we highlight the dependence on λ since it plays a crucial role in the nearly incompressible limit $\lambda \to +\infty$, where $\gamma(\lambda) \to 0$ and $\dfrac{1}{\gamma(\lambda)} \to +\infty$.

We now introduce the tensor of body rotations $\boldsymbol{\rho} := \frac{1}{2}(\boldsymbol{\nabla u} - (\boldsymbol{\nabla u})^{\mathtt{t}})$ as an additional unknown, yielding

$$\begin{aligned} \mathcal{C}^{-1}\boldsymbol{\sigma} + \alpha\,\gamma(\lambda)\,\varphi\,\mathbf{I} + \boldsymbol{\rho} &= \boldsymbol{\nabla u} && \text{in } \Omega, \\ -\,\mathbf{div}\,\boldsymbol{\sigma} &= \varrho\boldsymbol{f} && \text{in } \Omega, \\ \boldsymbol{\sigma} &= \boldsymbol{\sigma}^{\mathtt{t}} && \text{in } \Omega. \end{aligned}$$

For the thermal equation, we introduce the heat flux $\boldsymbol{\theta} := \kappa\nabla\varphi$, leading to

$$\begin{aligned} \kappa^{-1}\boldsymbol{\theta} &= \nabla\varphi && \text{in } \Omega, \\ \ell\varphi + \boldsymbol{u}\cdot\kappa^{-1}\boldsymbol{\theta} - \operatorname{div}\boldsymbol{\theta} &= m && \text{in } \Omega. \end{aligned}$$

Collecting all unknowns and rewriting appropriately the boundary conditions (1.2), the fully mixed strong form of the thermoelastic contact problem reads: find $(\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{\rho})$ and $(\varphi, \boldsymbol{\theta})$ such that

$$\mathcal{C}^{-1}\boldsymbol{\sigma} - \alpha\,\gamma(\lambda)\,\varphi\,\mathbf{I} + \boldsymbol{\rho} = \boldsymbol{\nabla u} \quad \text{in } \Omega, \tag{1.8a}$$

$$\mathbf{div}\,\boldsymbol{\sigma} = \varrho\boldsymbol{f} \quad \text{in } \Omega, \tag{1.8b}$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathtt{t}} \quad \text{in } \Omega, \tag{1.8c}$$

$$\kappa^{-1}\boldsymbol{\theta} = \nabla\varphi \quad \text{in } \Omega, \tag{1.8d}$$

$$\ell\varphi + \boldsymbol{u}\cdot\kappa^{-1}\boldsymbol{\theta} - \operatorname{div}\boldsymbol{\theta} = m \quad \text{in } \Omega, \tag{1.8e}$$

$$\boldsymbol{u} = \mathbf{0}, \quad \varphi = 0 \quad \text{on } \Gamma_D, \tag{1.8f}$$

$$\boldsymbol{\sigma n} = \mathbf{0}, \quad \boldsymbol{\theta}\cdot\boldsymbol{n} = 0 \quad \text{on } \Gamma_N, \tag{1.8g}$$

$$\begin{cases} \boldsymbol{u}\cdot\boldsymbol{n} \le g, \quad \sigma_N \le 0, \quad \sigma_N(\boldsymbol{u}\cdot\boldsymbol{n} - g) = 0, \\ \boldsymbol{\sigma}_t = \mathbf{0}, \\ \boldsymbol{\theta}\cdot\boldsymbol{n} = -\dfrac{\varphi - \varphi_0}{r(\sigma_N)}, \end{cases} \quad \text{on } \Gamma_C. \tag{1.8h}$$

2. Weak formulation in mixed form

2.1. Preliminaries

For each $t \in [1, +\infty)$ we introduce the Banach spaces

$$\mathbf{H}(\operatorname{div}_t;\Omega) := \Big\{ \boldsymbol{\xi} \in \mathbf{L}^2(\Omega) : \quad \operatorname{div}(\boldsymbol{\xi}) \in \mathrm{L}^t(\Omega) \Big\} \quad \text{and} \quad \mathbb{H}(\mathbf{div}_t;\Omega) := \Big\{ \boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \quad \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^t(\Omega) \Big\},$$

equipped with their natural norms

$$\|\boldsymbol{\xi}\|_{\operatorname{div}_t;\Omega} := \|\boldsymbol{\xi}\|_{0,\Omega} + \|\operatorname{div}(\boldsymbol{\xi})\|_{0,t;\Omega} \quad \text{and} \quad \|\boldsymbol{\tau}\|_{\mathbf{div}_t;\Omega} := \|\boldsymbol{\tau}\|_{0,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,t;\Omega},$$

for all $\boldsymbol{\xi} \in \mathbf{H}(\operatorname{div}_t;\Omega)$ and $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_t;\Omega)$, respectively. Additionally, we recall from [23, eq. (1.43), Section 1.3.4], that for $t \in \left\{ \begin{array}{l} (1, +\infty] \text{ in } \mathbb{R}^2, \\ \left[\frac{6}{5}, +\infty\right] \text{ in } \mathbb{R}^3, \end{array} \right.$ the following integration by parts formulae

$$\langle \boldsymbol{\xi}\cdot\boldsymbol{n}, v\rangle_{\partial\Omega} = \int_\Omega \Big\{ \boldsymbol{\xi}\cdot\nabla v + v\,\operatorname{div}(\boldsymbol{\xi}) \Big\} \quad \forall\,(\boldsymbol{\xi}, v) \in \mathbf{H}(\operatorname{div}_t;\Omega) \times \mathrm{H}^1(\Omega), \tag{2.1a}$$

$$\langle \boldsymbol{\tau n}, \boldsymbol{v}\rangle_{\partial\Omega} = \int_\Omega \Big\{ \boldsymbol{\tau} : \boldsymbol{\nabla v} + \boldsymbol{v}\cdot\mathbf{div}(\boldsymbol{\tau}) \Big\} \quad \forall\,(\boldsymbol{\tau}, \boldsymbol{v}) \in \mathbb{H}(\mathbf{div}_t;\Omega) \times \mathbf{H}^1(\Omega), \tag{2.1b}$$

hold true, where $\langle \bullet, \bullet\rangle_{\partial\Omega}$ denotes in (2.1a) the duality pairing between $\mathrm{H}^{-1/2}(\partial\Omega)$ and $\mathrm{H}^{1/2}(\partial\Omega)$, whereas in (2.1b) the duality pairing between $\mathbf{H}^{-1/2}(\partial\Omega)$ and $\mathbf{H}^{1/2}(\partial\Omega)$. For $t = 2$ we have the Hilbert space $\mathbb{H}(\mathbf{div};\Omega) := \{\boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \mathbf{div}(\boldsymbol{\tau}) \in \mathbb{L}^2(\Omega)\}$. In addition, for $\star \in \{D, N, C\}$ we define the following vectorial space and its tensorial counterpart

$$\mathbf{H}_\star(\operatorname{div},\Omega) := \{\boldsymbol{\xi} \in \mathbf{H}(\operatorname{div},\Omega) : \boldsymbol{\xi}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma_\star\}, \quad \mathbb{H}_\star(\mathbf{div},\Omega) := \{\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div},\Omega) : \boldsymbol{\tau n} = \mathbf{0} \text{ on } \Gamma_\star\},$$

and the space of $\mathbb{L}^2(\Omega)$ skew-symmetric tensors as

$$\mathbb{L}^2_{\rm skew}(\Omega) := \{\boldsymbol{\eta} \in \mathbb{L}^2(\Omega) : \boldsymbol{\eta} = -\boldsymbol{\eta}^{\tt t}\}\,.$$

For a given subset $\Sigma \subset \partial\Omega$, let $\mathrm{E}_{0,\Sigma} : \mathrm{H}^{1/2}(\Sigma) \to \mathrm{L}^2(\partial\Omega)$ denote the extension operator

$$\mathrm{E}_{0,\Sigma}(\psi) := \begin{cases} \psi & \text{on } \Sigma, \\ 0 & \text{on } \partial\Omega \setminus \overline{\Sigma}, \end{cases} \qquad \forall \psi \in \mathrm{H}^{1/2}(\Sigma).$$

The Lions–Magenes space in the sense of Grisvard [27] (see also [24, 37]) is then defined as

$$\mathrm{H}^{1/2}_{00}(\Sigma) := \big\{\psi \in \mathrm{H}^{1/2}(\Sigma) : \mathrm{E}_{0,\Sigma}(\psi) \in \mathrm{H}^{1/2}(\partial\Omega)\big\},$$

with norm

$$\|\psi\|_{1/2,00;\Sigma} := \|\mathrm{E}_{0,\Sigma}(\psi)\|_{1/2,\partial\Omega}.$$

Its vector-valued counterpart is

$$\mathbf{H}^{1/2}_{00}(\Sigma) := [\mathrm{H}^{1/2}_{00}(\Sigma)]^n,$$

and the corresponding dual spaces are denoted by $\left(\mathrm{H}^{1/2}_{00}(\Sigma)\right)'$ and $\left(\mathbf{H}^{1/2}_{00}(\Sigma)\right)'$.

Let $\partial\Omega = \overline{\Gamma}_0 \cup \overline{\Gamma}_1 \cup \overline{\Sigma}$ with $\mathrm{dist}(\Gamma_0, \Sigma) > 0$. Following [12], we define the subspace $\mathbf{W}_C \subseteq \mathbf{H}^{1/2}(\Sigma)$ by

$$\mathbf{W}_C := \big\{\boldsymbol{\gamma}_0(\boldsymbol{w})|_\Sigma \ : \ \boldsymbol{w} \in \mathbf{H}^1_{\Gamma_0}(\Omega)\big\}.$$

Observe that $\mathbf{H}^{1/2}_{00}(\Sigma) \subseteq \mathbf{W}_C$. In particular, if $|\Gamma_1| = 0$, then $\mathbf{W}_C = \mathbf{H}^{1/2}_{00}(\Sigma)$, whereas if Γ_0 is compactly embedded into Γ_1, we have $\mathbf{W}_C = \mathbf{H}^{1/2}(\Sigma)$. Setting $\widetilde{\Gamma} := \mathrm{int}(\overline{\Sigma} \cup \overline{\Gamma}_1)$, the dual of $\mathbf{W}_C$ is defined as

$$\mathbf{W}'_C := \left\{\boldsymbol{\xi}|_\Sigma \ : \ \boldsymbol{\xi} \in \left(\mathbf{H}^{1/2}_{00}(\widetilde{\Gamma})\right)',\ \boldsymbol{\xi} = \mathbf{0} \text{ on } \Gamma_1\right\} \subseteq \left(\mathbf{H}^{1/2}_{00}(\widetilde{\Gamma})\right)'.$$

In this case, if $|\Gamma_1| = 0$, then $\mathbf{W}'_C = \left(\mathbf{H}^{1/2}_{00}(\Sigma)\right)'$, while if Γ_0 is compactly embedded in Γ_1, we have $\boldsymbol{\xi} = \mathbf{0}$ on $\partial\Omega \setminus \overline{\Sigma}$, so that $\boldsymbol{\xi}|_\Sigma \in \mathbf{H}^{-1/2}(\Sigma)$ [22, Remark 2.6], and hence $\mathbf{W}'_C = \mathbf{H}^{-1/2}(\Sigma)$.

The duality pairing on $\mathbf{W}'_C \times \mathbf{W}_C$ is denoted by $\langle \cdot, \cdot \rangle_{\mathbf{W}'_C \times \mathbf{W}_C}$, or simply $\langle \cdot, \cdot \rangle_\Sigma$ when unambiguous:

$$\langle \boldsymbol{\psi}, \boldsymbol{\varphi} \rangle_\Sigma := \langle \boldsymbol{\xi}, \boldsymbol{\gamma}_0(\boldsymbol{w})|_{\widetilde{\Gamma}} \rangle_{(\mathbf{H}^{1/2}_{00}(\widetilde{\Gamma}))' \times \mathbf{H}^{1/2}_{00}(\widetilde{\Gamma})}, \quad \forall \boldsymbol{\varphi} \in \mathbf{W}_C,$$

where $\boldsymbol{w} \in \mathbf{H}^1_{\Gamma_0}(\Omega)$ satisfies $\boldsymbol{\varphi} = \boldsymbol{\gamma}_0(\boldsymbol{w})|_\Sigma$ and $\boldsymbol{\xi} \in (\mathbf{H}^{1/2}_{00}(\widetilde{\Gamma}))'$ satisfies $\boldsymbol{\xi}|_\Sigma = \boldsymbol{\psi}$.

Finally, the following decomposition holds [12, 30, 40]:

$$\mathbf{W}_C = \mathrm{W}^{\mathbf{n}}_C \oplus \mathbf{W}_{C,\mathbf{t}}, \quad \mathbf{W}'_C = (\mathrm{W}^{\mathbf{n}}_C)' \oplus \mathbf{W}'_{C,\mathbf{t}}, \tag{2.2}$$

where $\mathrm{W}^{\mathbf{n}}_C$ and $\mathbf{W}_{C,\mathbf{t}}$ denote the normal and tangential components of elements in $\mathbf{W}_C$, respectively:

$$\mathrm{W}^{\mathbf{n}}_C := \{\boldsymbol{\varphi} \cdot \boldsymbol{n} \ : \ \boldsymbol{\varphi} \in \mathbf{W}_C\}, \quad \mathbf{W}_{C,\mathbf{t}} := \{\boldsymbol{\varphi} \in \mathbf{W}_C \ : \ \boldsymbol{\varphi} \cdot \boldsymbol{n} = 0\}.$$

2.2. Weak mixed formulation

We now derive the weak mixed form corresponding to (1.8). To this end, we first consider the first two rows of (1.8). Let $\boldsymbol{\tau}$ and $\boldsymbol{v}$ denote test functions associated with the unknowns $\boldsymbol{\sigma}$ and $\boldsymbol{u}$, respectively. Testing the first equation with $\boldsymbol{\tau}-\boldsymbol{\sigma}$ and the second with $\boldsymbol{v}$ gives

$$\int_\Omega \mathcal{C}^{-1}\boldsymbol{\sigma} : (\boldsymbol{\tau}-\boldsymbol{\sigma}) - \alpha\gamma(\lambda)\int_\Omega \varphi\,\mathrm{tr}(\boldsymbol{\tau}-\boldsymbol{\sigma}) + \int_\Omega \boldsymbol{\rho} : (\boldsymbol{\tau}-\boldsymbol{\sigma}) = \int_\Omega \nabla\boldsymbol{u} : (\boldsymbol{\tau}-\boldsymbol{\sigma}), \tag{2.3a}$$

$$\int_\Omega \boldsymbol{v}\cdot\mathbf{div}\,\boldsymbol{\sigma} = \int_\Omega \varrho\boldsymbol{f}\cdot\boldsymbol{v}. \tag{2.3b}$$

Assuming $\boldsymbol{u}\in\mathbf{H}^1(\Omega)$ and $\boldsymbol{\sigma},\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div}_t;\Omega)$ for t fitting in the ranges of Section 2.1, we apply the integration by parts formula (2.1b) to the right-hand side of (2.3a), yielding

$$\int_\Omega \nabla\boldsymbol{u} : (\boldsymbol{\tau}-\boldsymbol{\sigma}) = -\int_\Omega \boldsymbol{u}\cdot\mathbf{div}(\boldsymbol{\tau}-\boldsymbol{\sigma}) + \langle(\boldsymbol{\tau}-\boldsymbol{\sigma})\boldsymbol{n},\boldsymbol{u}\rangle_{\partial\Omega}, \quad \forall\,\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div}_t;\Omega). \tag{2.4}$$

The boundary term is decomposed using the linearity of the trace operator, (1.3), and the boundary conditions on Γ_D and Γ_N ((1.8f) and (1.8g), respectively), leading to

$$\langle(\boldsymbol{\tau}-\boldsymbol{\sigma})\boldsymbol{n},\boldsymbol{u}\rangle_{\partial\Omega} = \langle\tau_N-\sigma_N,\boldsymbol{u}\cdot\boldsymbol{n}\rangle_{\Gamma_C} + \langle\boldsymbol{\tau}_t-\boldsymbol{\sigma}_t,\boldsymbol{u}_t\rangle_{\Gamma_C} \geq \langle\tau_N-\sigma_N,g\rangle_{\Gamma_C}. \tag{2.5}$$

Here, $\boldsymbol{\sigma n}$ and $\boldsymbol{\tau n}$ belong to $\mathbf{W}_C'$, while $\boldsymbol{u}|_{\Gamma_C}\in\mathbf{W}_C$. Invoking (2.2) gives the first equality, and using [30, eq. (5.38)] we deduce that $g\in\mathrm{W}_C$.

Decomposing $\boldsymbol{\tau n}=\boldsymbol{\tau}_t+\tau_N\boldsymbol{n}$, we impose $\boldsymbol{\tau}_t=0$ and $\tau_N\leq 0$ to validate the inequality in (2.5). Collecting terms, the weak mixed formulation of the mechanical problem reads

$$\int_\Omega \mathcal{C}^{-1}\boldsymbol{\sigma} : (\boldsymbol{\tau}-\boldsymbol{\sigma}) + \int_\Omega \boldsymbol{u}\cdot\mathbf{div}(\boldsymbol{\tau}-\boldsymbol{\sigma}) + \int_\Omega \boldsymbol{\rho} : (\boldsymbol{\tau}-\boldsymbol{\sigma}) \geq \langle\tau_N-\sigma_N,g\rangle_{\Gamma_C} + \alpha\gamma(\lambda)\int_\Omega \varphi\,\mathrm{tr}(\boldsymbol{\tau}-\boldsymbol{\sigma}), \tag{2.6a}$$

$$\int_\Omega \boldsymbol{v}\cdot\mathbf{div}\,\boldsymbol{\sigma} = \int_\Omega \varrho\boldsymbol{f}\cdot\boldsymbol{v}. \tag{2.6b}$$

The first term in (2.6a) is well-defined for $\boldsymbol{\sigma},\boldsymbol{\tau}\in\mathbb{L}^2(\Omega)$, and taking $\boldsymbol{\rho}\in\mathbb{L}^2_{\mathrm{skew}}(\Omega)$, the third term is also well-defined. Applying Hölder's inequality to the second term shows that $\boldsymbol{u},\boldsymbol{v}\in\mathbf{L}^{t'}(\Omega)$ provided $\varrho\boldsymbol{f}\in\mathbf{L}^t(\Omega)$. The second term on the right-hand side is finite if $\varphi\in\mathrm{L}^2(\Omega)$ due to $\|\,\mathrm{tr}(\boldsymbol{\zeta})\|_{0,\Omega}\leq d^{1/2}\|\boldsymbol{\zeta}\|_{0,\Omega}$. The symmetry of $\boldsymbol{\sigma}$ is imposed weakly as

$$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\delta} = 0, \quad \forall\,\boldsymbol{\delta}\in\mathbb{L}^2_{\mathrm{skew}}(\Omega). \tag{2.7}$$

For the thermal equations (1.8d)-(1.8e), testing with $\boldsymbol{\xi}$ and ψ associated with $\boldsymbol{\theta}$ and φ yields

$$\int_\Omega \kappa^{-1}\boldsymbol{\theta}\cdot\boldsymbol{\xi} = \int_\Omega \nabla\varphi\cdot\boldsymbol{\xi}, \tag{2.8a}$$

$$\int_\Omega \psi\,\mathrm{div}\,\boldsymbol{\theta} - \ell\int_\Omega \varphi\psi - \int_\Omega (\boldsymbol{u}\cdot\kappa^{-1}\boldsymbol{\theta})\,\psi = -\int_\Omega m\,\psi. \tag{2.8b}$$

Assuming $\varphi\in\mathrm{H}^1(\Omega)$ and $\boldsymbol{\xi}\in\mathbf{H}(\mathrm{div}_s;\Omega)$, where s fits in the ranges provided in Section 2.1 as well as t. The integration by parts in (2.8a) gives

$$\int_\Omega \kappa^{-1}\boldsymbol{\theta}\cdot\boldsymbol{\xi} + \int_\Omega \varphi\,\mathrm{div}\,\boldsymbol{\xi} = \langle\boldsymbol{\xi}\cdot\boldsymbol{n},\varphi\rangle_{\partial\Omega}. \tag{2.9}$$

Applying the boundary conditions (1.8f), (1.8g) and second one in (1.8h), allows us to rewrite the right-hand side of (2.9) as

$$\langle\boldsymbol{\xi}\cdot\boldsymbol{n},\varphi\rangle_{\partial\Omega} = \langle\boldsymbol{\xi}\cdot\boldsymbol{n},\varphi_0\rangle_{\Gamma_C} - \langle\boldsymbol{\xi}\cdot\boldsymbol{n},r(\sigma_N)\boldsymbol{\theta}\cdot\boldsymbol{n}\rangle_{\Gamma_C}. \tag{2.10}$$

Substituting back, we obtain the final weak form for the thermal flux:

$$\int_\Omega \kappa^{-1}\boldsymbol{\theta}\cdot\boldsymbol{\xi} + \langle \boldsymbol{\xi}\cdot\boldsymbol{n}, r(\sigma_N)\boldsymbol{\theta}\cdot\boldsymbol{n}\rangle_{\Gamma_C} + \int_\Omega \varphi\ \mathrm{div}\,\boldsymbol{\xi} = \langle \boldsymbol{\xi}\cdot\boldsymbol{n}, \varphi_0\rangle_{\Gamma_C}. \tag{2.11}$$

The nonlinear term in (2.8b) can be bounded using Hölder's inequality as

$$\left|\int_\Omega (\boldsymbol{u}\cdot\kappa^{-1}\boldsymbol{\theta})\,\psi\right| \le \|\kappa^{-1}\|_{0,\infty;\Omega}\,\|\boldsymbol{u}\|_{0,t';\Omega}\,\|\boldsymbol{\theta}\|_{0,q;\Omega}\,\|\psi\|_{0,s';\Omega}, \quad \frac{1}{t}+\frac{1}{q}=\frac{1}{s}. \tag{2.12}$$

Choosing $q=2$ for $\boldsymbol{\theta}\in\mathbf{L}^2(\Omega)$ implies $t'=2s/(2-s)$. Taking $s=3/2$ gives $t'=6$, yielding the functional framework

$$\boldsymbol{u}\in\mathbf{L}^6(\Omega),\quad \boldsymbol{\sigma}\in\mathbb{H}_N(\mathbf{div}_{6/5};\Omega),\quad \varphi\in\mathrm{L}^3(\Omega),\quad \boldsymbol{\theta}\in\mathbf{H}_N(\mathrm{div}_{3/2};\Omega).$$

Finally, we define the closed convex set for stresses

$$\mathbb{K} := \{\boldsymbol{\tau}\in\mathbb{H}_N(\mathbf{div}_{6/5};\Omega) \,:\, \boldsymbol{\tau}_t = 0,\ \tau_N\le 0 \text{ on } \Gamma_C\}, \tag{2.13}$$

where $\tau_N\le 0$ is understood weakly, see, e.g., [37, Section 15.4]. The thermal flux space is

$$\mathbf{X} := \{\boldsymbol{\xi}\in\mathbf{H}_N(\mathrm{div}_{3/2};\Omega) \,:\, \boldsymbol{\xi}\cdot\boldsymbol{n}|_{\Gamma_C}\in\mathrm{L}^2(\Gamma_C)\},\quad \|\boldsymbol{\xi}\|_{\mathbf{X}} := \|\boldsymbol{\xi}\|_{\mathrm{div}_{3/2};\Omega} + \|\boldsymbol{\xi}\cdot\boldsymbol{n}\|_{0,\Gamma_C}. \tag{2.14}$$

Summarising the previous analysis, and defining

$$\mathbf{Q} := \mathbf{L}^6(\Omega)\times\mathbb{L}^2_{\mathrm{skew}}(\Omega),\qquad \mathrm{Y} := \mathrm{L}^3(\Omega),$$

we obtain the following weak mixed formulation: find $(\boldsymbol{\sigma},(\boldsymbol{u},\boldsymbol{\rho}),\boldsymbol{\theta},\varphi)\in\mathbb{K}\times\mathbf{Q}\times\mathbf{X}\times\mathrm{Y}$ such that

$$\mathbf{A}(\boldsymbol{\sigma},\boldsymbol{\tau}-\boldsymbol{\sigma}) + \mathbf{B}(\boldsymbol{\tau}-\boldsymbol{\sigma},(\boldsymbol{u},\boldsymbol{\rho})) \ge \mathbf{F}_\varphi(\boldsymbol{\tau}-\boldsymbol{\sigma}), \tag{2.15a}$$

$$\mathbf{B}(\boldsymbol{\sigma},(\boldsymbol{v},\boldsymbol{\delta})) = \mathbf{G}(\boldsymbol{v},\boldsymbol{\delta}), \tag{2.15b}$$

$$a_{\boldsymbol{\sigma}}(\boldsymbol{\theta},\boldsymbol{\xi}) + b(\boldsymbol{\xi},\varphi) = F(\boldsymbol{\xi}), \tag{2.15c}$$

$$b(\boldsymbol{\theta},\psi) - c(\varphi,\psi) - \int_\Omega (\boldsymbol{u}\cdot\kappa^{-1}\boldsymbol{\theta})\,\psi = G(\psi), \tag{2.15d}$$

for all $(\boldsymbol{\tau},(\boldsymbol{v},\boldsymbol{\delta}),\boldsymbol{\xi},\psi)\in\mathbb{K}\times\mathbf{Q}\times\mathbf{X}\times\mathrm{Y}$. For a given $\widehat{\boldsymbol{\sigma}}\in\mathbb{K}$, the bilinear forms above are defined as follows:

$$\mathbf{A}(\boldsymbol{\sigma},\boldsymbol{\tau}) := \int_\Omega \mathcal{C}^{-1}\boldsymbol{\sigma} : \boldsymbol{\tau}, \qquad \forall\boldsymbol{\sigma},\boldsymbol{\tau}\in\mathbb{K}, \tag{2.16a}$$

$$\mathbf{B}(\boldsymbol{\tau},(\boldsymbol{v},\boldsymbol{\delta})) := \int_\Omega \boldsymbol{v}\cdot\mathbf{div}\,\boldsymbol{\tau} + \int_\Omega \boldsymbol{\tau} : \boldsymbol{\delta}, \qquad \forall(\boldsymbol{\tau},(\boldsymbol{v},\boldsymbol{\delta}))\in\mathbb{K}\times\mathbf{Q}, \tag{2.16b}$$

$$a_{\widehat{\boldsymbol{\sigma}}}(\boldsymbol{\theta},\boldsymbol{\xi}) := \int_\Omega \kappa^{-1}\boldsymbol{\theta}\cdot\boldsymbol{\xi} + \left\langle \boldsymbol{\xi}\cdot\boldsymbol{n}, r(\widehat{\sigma}_N)\,\boldsymbol{\theta}\cdot\boldsymbol{n}\right\rangle_{0,\Gamma_C}, \qquad \forall\boldsymbol{\theta},\boldsymbol{\xi}\in\mathbf{X}, \tag{2.16c}$$

$$b(\boldsymbol{\xi},\psi) := \int_\Omega \psi\ \mathrm{div}\,\boldsymbol{\xi}, \qquad \forall(\boldsymbol{\xi},\psi)\in\mathbf{X}\times\mathrm{Y}, \tag{2.16d}$$

$$c(\phi,\psi) := \int_\Omega \ell\,\phi\psi, \qquad \forall\phi,\psi\in\mathrm{Y}, \tag{2.16e}$$

and, for a given $\widehat{\varphi}\in\mathrm{Y}$ the linear functionals in (2.15) are specified as

$$\begin{aligned}
\mathbf{F}_{\widehat{\varphi}}(\boldsymbol{\tau}) &:= \alpha\gamma(\lambda)\int_\Omega \widehat{\varphi}\ \mathrm{tr}(\boldsymbol{\tau}) + \langle\tau_N, g\rangle_{\Gamma_C}, && \forall\boldsymbol{\tau}\in\mathbb{K},\\
\mathbf{G}(\boldsymbol{v},\boldsymbol{\delta}) &:= -\int_\Omega \varrho\boldsymbol{f}\cdot\boldsymbol{v}, && \forall(\boldsymbol{v},\boldsymbol{\delta})\in\mathbf{Q},\\
F(\boldsymbol{\xi}) &:= \langle\boldsymbol{\xi}\cdot\boldsymbol{n},\varphi_0\rangle_{0,\Gamma_C}, && \forall\boldsymbol{\xi}\in\mathbf{X},\\
G(\psi) &:= -\int_\Omega m\psi, && \forall\psi\in\mathrm{Y}.
\end{aligned} \tag{2.17}$$

Regarding the well-posedness of the stress- and temperature-dependent functional $\mathbf{F}_{\widehat{\varphi}}$ in (2.17), applying Cauchy–Schwarz and Hölder inequalities together with the bounded embedding of $\mathrm{L}^3(\Omega)$ into $\mathrm{L}^2(\Omega)$, we obtain

$$\left|\alpha\gamma(\lambda)\int_\Omega \widehat{\varphi}\ \mathrm{tr}(\boldsymbol{\tau})\right| \le \alpha\gamma(\lambda)\,|\Omega|^{1/6} d^{1/2}\,\|\widehat{\varphi}\|_{0,3;\Omega}\,\|\boldsymbol{\tau}\|_{\mathbf{div}_{6/5};\Omega},$$

which ensures that $\mathbf{F}_{\widehat{\varphi}} \in \mathbf{H}'$ for each $\widehat{\varphi} \in \mathrm{L}^3(\Omega)$.

2.3. Fixed-point scheme

In this section we develop a fixed-point scheme in order to establish the well-posedness of (2.15). For this goal we adopt a decoupling strategy, first defining the operator $\mathbf{S} : \mathrm{L}^3(\Omega) \to \mathbb{K} \times \mathrm{L}^6(\Omega)$ as

$$\mathbf{S}(\phi) := (\boldsymbol{\sigma}, \boldsymbol{u}) \in \mathbb{K} \times \mathbf{L}^6(\Omega)\,, \tag{2.18}$$

where $(\boldsymbol{\sigma}, (\boldsymbol{u}, \boldsymbol{\rho})) \in \mathbb{K} \times \mathbf{Q}$ is the unique solution (to be proved below) of the problem composed by (2.15a)-(2.15b), when φ is replaced by ϕ in the right-hand side of (2.15a), that is the problem to find $(\boldsymbol{\sigma}, (\boldsymbol{u}, \boldsymbol{\rho})) \in \mathbb{K} \times \mathbf{Q}$ such that

$$\begin{aligned} \mathbf{A}(\boldsymbol{\sigma}, \boldsymbol{\tau}-\boldsymbol{\sigma}) + \mathbf{B}(\boldsymbol{\tau}-\boldsymbol{\sigma}, (\boldsymbol{u}, \boldsymbol{\rho})) &\ \ge\ \mathbf{F}_\phi(\boldsymbol{\tau}-\boldsymbol{\sigma}) && \forall\, \boldsymbol{\tau} \in \mathbb{K}\,, \\ \mathbf{B}(\boldsymbol{\sigma}, (\boldsymbol{v}, \boldsymbol{\delta})) &\ =\ \mathbf{G}(\boldsymbol{v}, \boldsymbol{\delta}) && \forall\, (\boldsymbol{v}, \boldsymbol{\delta}) \in \mathbf{Q}\,. \end{aligned} \tag{2.19}$$

On the other hand, we define the operator $\mathrm{S} : \mathbb{K} \times \mathbf{L}^6(\Omega) \to \mathrm{L}^3(\Omega)$ as

$$\mathrm{S}(\boldsymbol{\zeta}, \boldsymbol{w}) := \varphi \in \mathrm{L}^3(\Omega)\,, \tag{2.20}$$

where $(\boldsymbol{\theta}, \varphi) \in \mathbf{X} \times \mathrm{Y}$ is the unique solution (to be proved below) to the problem composed for the last two rows of (2.15) replacing $\boldsymbol{\sigma}$ by $\boldsymbol{\zeta}$, that is

$$\begin{aligned} a_{\boldsymbol{\zeta}}(\boldsymbol{\theta}, \boldsymbol{\xi}) + b(\boldsymbol{\xi}, \varphi) &\ =\ F(\boldsymbol{\xi}) && \forall\, \boldsymbol{\xi} \in \mathbf{X}\,, \\ b(\boldsymbol{\theta}, \psi) - c(\varphi, \psi) - \int_\Omega (\boldsymbol{w}\cdot\kappa^{-1}\,\boldsymbol{\theta})\,\psi &\ =\ G(\psi) && \forall\, \psi \in \mathrm{Y}\,. \end{aligned} \tag{2.21}$$

Finally, we define the operator $\mathbf{T} : \mathbb{K} \times \mathbf{L}^6(\Omega) \to \mathbb{K} \times \mathbf{L}^6(\Omega)$ as the composition between $\mathbf{S}$ (cf. (2.18)) and S (cf. (2.20)), namely

$$\mathbf{T}(\boldsymbol{\zeta}, \boldsymbol{w}) := \mathbf{S}(\mathrm{S}(\boldsymbol{\zeta}, \boldsymbol{w})) \qquad \forall\, (\boldsymbol{\zeta}, \boldsymbol{w}) \in \mathbb{K} \times \mathbf{Q}\,, \tag{2.22}$$

and observe that solving the coupled problem (2.15) is equivalent to finding a fixed point of $\mathbf{T}$. That is

$$\mathbf{T}(\boldsymbol{\sigma}, \boldsymbol{u}) = (\boldsymbol{\sigma}, \boldsymbol{u})\,. \tag{2.23}$$

3. Abstract framework

In this section, we analyse the abstract framework for the solvability of the variational inequality (2.19). To this end, we adopt to the present context the framework developed in [39], which relies on classical saddle-point arguments: namely, the ellipticity of the principal bilinear form $a(\bullet,\bullet)$ over the kernel of $b(\bullet,\bullet)$ and the well-known inf-sup condition for $b(\bullet,\bullet)$. Here, these conditions are imposed on a subspace $\mathrm{W} \subset \mathbb{K}$ (cf. (2.13)). We first recall a preliminary result from [39].

Lemma 3.1. *Let E be a normed space, $K \subset E$ a closed convex subset, and $S \subseteq K$ a subspace. Then $K + S \subseteq K$. Moreover, if L is a linear form on E such that $L(v-w) \ge 0$ for all $v, w \in K$, then $L|_S = 0$.*

Consider a Banach space H, a closed convex subset $\mathrm{K} \subset \mathrm{H}$, and a Banach space Q. We study the general linear mixed variational inequality: find $(\sigma, u) \in \mathrm{K} \times \mathrm{Q}$ such that

$$\begin{aligned} a(\sigma, \tau-\sigma) + b(\tau-\sigma, u) \ge F(\tau-\sigma) &\quad \forall \tau \in \mathrm{K}, \\ b(\sigma, v) = G(v) &\quad \forall v \in \mathrm{Q}, \end{aligned} \tag{3.1}$$

where $a : \mathrm{K} \times \mathrm{K} \to \mathbb{R}$ and $b : \mathrm{K} \times \mathrm{Q} \to \mathbb{R}$ are bilinear forms with boundedness constants $\|a\|$ and $\|b\|$, respectively. Let $\mathrm{W} \subset \mathrm{K}$ be a closed subspace such that $b(\bullet, \bullet)$ satisfies the inf-sup condition

$$\sup_{\substack{\tau \in \mathrm{W} \\ \tau \neq 0}} \frac{b(\tau, v)}{\|\tau\|_{\mathrm{H}}} \geq \beta \, \|v\|_{\mathrm{Q}} \quad \forall v \in \mathrm{Q}, \tag{3.2}$$

for some $\beta > 0$. Equivalently, the induced operator is surjective, i.e., for each $G \in \mathrm{Q}'$ there exists $\sigma_G \in \mathrm{W}$ such that

$$b(\sigma_G, v) = G(v) \quad \forall v \in \mathrm{Q}, \qquad \|\sigma_G\|_{\mathrm{H}} \leq C \|G\|_{\mathrm{Q}'}.$$

Defining $\mathrm{V} := \{\tau \in \mathrm{H} : b(\tau, v) = 0, \quad \forall v \in \mathrm{Q}\}$, and assuming that $a(\bullet, \bullet)$ is V-elliptic, i.e.,

$$a(\tau, \tau) \geq \alpha \|\tau\|_{\mathrm{H}}^2 \quad \forall \tau \in \mathrm{V}, \tag{3.3}$$

one can derive the bound

$$\|\sigma\|_{\mathrm{H}} \leq \frac{1}{\alpha} \|F\|_{\mathrm{H}'} + \Big(\frac{C}{\alpha}\|a\| + 1\Big) \|G\|_{\mathrm{Q}'}. \tag{3.4}$$

Similarly, using Lemma 3.1 and the inf-sup condition (3.2), one obtains

$$\|u\|_{\mathrm{Q}} \leq \frac{1}{\beta}\left(\frac{\|a\|}{\alpha} + 1\right) \|F\|_{\mathrm{H}'} + \frac{\|a\|}{\beta}\Big(\frac{C}{\alpha}\|a\| + 1\Big) \|G\|_{\mathrm{Q}'}. \tag{3.5}$$

The existence of a solution follows from the general solvability result for variational inequalities in reflexive Banach spaces [31, Chapter III, Corollary 1.8]:

Lemma 3.2. *Let* H *be a reflexive Banach space,* $\mathrm{K} \subset \mathrm{H}$ *non-empty, bounded, closed, and convex. Let* $\mathcal{F} \in \mathrm{H}'$ *and* $\mathcal{A} : \mathrm{K} \to \mathrm{H}'$ *be monotone, coercive, linear, and bounded. Then, there exists* $u \in \mathrm{K}$ *such that*

$$(\mathcal{A}(u) - \mathcal{F})(v - u) \geq 0 \quad \forall v \in \mathrm{K}.$$

Applying this result to the augmented operator

$$\mathcal{A}((\sigma, u), (\tau, v)) := a(\sigma, \tau) + b(\tau, u) - b(\sigma, v),$$

with $\mathcal{F}((\tau, v)) := F(\tau) - G(v)$, one verifies that a solution of the mixed problem (3.1) exists and satisfies the bounds (3.4)–(3.5). We deduce from all these previous considerations the following result.

Theorem 3.1 (Solvability of the mixed variational inequality). *Let* H *and* Q *be reflexive Banach spaces,* $\mathrm{K} \subset \mathrm{H}$ *a non-empty closed convex set. Assume* $a : \mathrm{K} \times \mathrm{K} \to \mathbb{R}$ *and* $b : \mathrm{K} \times \mathrm{Q} \to \mathbb{R}$ *are bounded bilinear forms with constants* $\|a\|$ *and* $\|b\|$*, and that*

1. $a(\bullet, \bullet)$ *is positive semidefinite:* $a(\tau, \tau) \geq 0$ *for all* $\tau \in \mathrm{H}$,
2. $a(\bullet, \bullet)$ *is elliptic on* $\mathrm{V} := \{\tau \in \mathrm{H} : b(\tau, v) = 0 \; \forall v \in \mathrm{Q}\}$*: there exists* $\alpha > 0$ *such that* $a(\tau, \tau) \geq \alpha \|\tau\|_{\mathrm{H}}^2$ *for all* $\tau \in \mathrm{V}$,
3. *there exists a subspace* $\mathrm{W} \subset \mathrm{K}$ *such that* $b(\bullet, \bullet)$ *satisfies the inf-sup condition* (3.2) *with constant* $\beta > 0$.

Then, for any $F \in \mathrm{H}'$ *and* $G \in \mathrm{Q}'$*, there exists a unique* $(\sigma, u) \in \mathrm{K} \times \mathrm{Q}$ *satisfying* (3.1)*, and the continuous dependence estimates*

$$\|\sigma\|_{\mathrm{H}} \leq \frac{1}{\alpha} \|F\|_{\mathrm{H}'} + \Big(\frac{C}{\alpha}\|a\| + 1\Big) \|G\|_{\mathrm{Q}'},$$
$$\|u\|_{\mathrm{Q}} \leq \frac{1}{\beta}\left(\frac{\|a\|}{\alpha} + 1\right) \|F\|_{\mathrm{H}'} + \frac{\|a\|}{\beta}\Big(\frac{C}{\alpha}\|a\| + 1\Big) \|G\|_{\mathrm{Q}'}.$$

4. Well-posedness analysis

In this section we establish the well-posedness of the coupled problem (2.15) through the fixed-point scheme (2.23). The approach follows the classical two-step strategy: first, we prove the well-posedness of the operators $\mathbf{S}$ (cf. (2.18)) and S (cf. (2.20)); second, we verify that the global operator $\mathbf{T}$ satisfies the hypotheses of the Banach Fixed-Point Theorem, namely, Lipschitz continuity and contraction.

We begin with a preliminary observation. Owing to the uniform convexity and separability of $\mathrm{L}^p(\Omega)$, $p \in (1,\infty)$, all function spaces appearing in (2.15) inherit these properties. In particular, uniform convexity implies reflexivity, and any closed subspace of a reflexive Banach space is itself reflexive.

4.1. Preliminaries

We first collect stability bounds for the bilinear forms and functionals in (2.15). Using Cauchy–Schwarz and Hölder inequalities, the continuity of the embeddings $\mathrm{i}_\rho : \mathrm{H}^1(\Omega) \hookrightarrow \mathrm{L}^\rho(\Omega)$ (vectorial version $\mathbf{i}_\rho$), and the trace theorem, from the definition of the bilinear forms and linear functionals we can directly denote

$$\begin{aligned}&\|\mathbf{A}\| := \max\left\{\frac{1}{\mu}, \frac{\gamma(\lambda)}{d}\right\}, \quad \|\mathbf{B}\| := 1, \quad \|\mathbf{G}\| := \|\varrho\mathbf{f}\|_{0,6/5;\Omega},\\ &\|b\| := 1, \quad \|c\| := \ell|\Omega|^{1/6}, \quad \|F\| := \|\varphi_0\|_{0,\Gamma_C}, \quad \|G\| := \|m\|_{0,3/2;\Omega},\end{aligned}$$

and for $\phi \in \mathrm{L}^3(\Omega)$ and $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{6/5};\Omega)$

$$\|\mathbf{F}_\phi\| := \alpha\,\gamma(\lambda)\,|\Omega|^{1/6} d^{1/2}\|\phi\|_{\mathrm{Y}} + \|g\|_{\mathrm{W}_C}, \quad \|a_{\boldsymbol{\sigma}}\| := \|\kappa^{-1}\|_{0,\infty;\Omega} + r_1.$$

Consequently, all bilinear forms and functionals satisfy the following standard boundedness estimates:

$$\begin{aligned}|\mathbf{A}(\boldsymbol{\zeta},\boldsymbol{\tau})| &\le \|\mathbf{A}\|\,\|\boldsymbol{\zeta}\|_{\mathbf{H}}\,\|\boldsymbol{\tau}\|_{\mathbf{H}} && \forall\,\boldsymbol{\zeta},\boldsymbol{\tau} \in \mathbb{K},\\ |\mathbf{B}(\boldsymbol{\tau},(\boldsymbol{v},\boldsymbol{\delta}))| &\le \|\mathbf{B}\|\,\|\boldsymbol{\tau}\|_{\mathbf{H}}\,\|(\boldsymbol{v},\boldsymbol{\delta})\|_{\mathbf{Q}} && \forall\,(\boldsymbol{\tau},(\boldsymbol{v},\boldsymbol{\delta})) \in \mathbb{K}\times\mathbf{Q},\\ |\mathbf{F}_\phi(\boldsymbol{\tau})| &\le \|\mathbf{F}_\phi\|\,\|\boldsymbol{\tau}\|_{\mathbf{H}} && \forall\,\boldsymbol{\tau} \in \mathbb{K},\\ |\mathbf{G}(\boldsymbol{v})| &\le \|\mathbf{G}\|\,\|\boldsymbol{v}\|_{\mathbf{Q}} && \forall\,\boldsymbol{v} \in \mathbf{Q},\\ |a_{\boldsymbol{\sigma}}(\boldsymbol{\chi},\boldsymbol{\xi})| &\le \|a_{\boldsymbol{\sigma}}\|\,\|\boldsymbol{\chi}\|_{\mathbf{X}}\,\|\boldsymbol{\xi}\|_{\mathbf{X}} && \forall\,\boldsymbol{\chi},\boldsymbol{\xi} \in \mathbf{X},\\ |b(\boldsymbol{\xi},\psi)| &\le \|b\|\,\|\boldsymbol{\xi}\|_{\mathbf{X}}\,\|\psi\|_{\mathrm{Y}} && \forall\,(\boldsymbol{\xi},\psi) \in \mathbf{X}\times\mathrm{Y},\\ |c(\phi,\psi)| &\le \|c\|\,\|\phi\|_{\mathrm{Y}}\,\|\psi\|_{\mathrm{Y}} && \forall\,\phi,\psi \in \mathrm{Y},\\ |F(\boldsymbol{\xi})| &\le \|F\|\,\|\boldsymbol{\xi}\|_{\mathbf{X}} && \forall\,\boldsymbol{\xi} \in \mathbf{X},\\ |G(\psi)| &\le \|G\|\,\|\psi\|_{\mathbf{Y}} && \forall\,\psi \in \mathrm{Y}.\end{aligned} \tag{4.1}$$

4.2. Well-posedness of S

Let us define the kernel of the linear operator induced by $\mathbf{B}$:

$$\mathbf{V} := \{\boldsymbol{\tau} \in \mathbf{H} : \mathbf{div}\,\boldsymbol{\tau} = 0, \quad \boldsymbol{\tau} = \boldsymbol{\tau}^{\mathrm{t}} \text{ in } \Omega\}.$$

From (1.6) and the definition of $\mathbf{A}$, we have

$$\mathbf{A}(\boldsymbol{\tau},\boldsymbol{\tau}) = \int_\Omega \mathcal{C}^{-1}\boldsymbol{\tau} : \boldsymbol{\tau} = \frac{1}{\mu}\|\boldsymbol{\tau}^{\mathrm{d}}\|^2_{0,\Omega} + \frac{\gamma(\lambda)}{d}\|\operatorname{tr}(\boldsymbol{\tau})\|^2_{0,\Omega}.$$

It is immediate that $\mathbf{A}$ is positive semidefinite. Moreover, using the decomposition $\boldsymbol{\tau} = \boldsymbol{\tau}^{\mathrm{d}} + \frac{1}{d}\operatorname{tr}(\boldsymbol{\tau})\mathbf{I}$, we get

$$\mathbf{A}(\boldsymbol{\tau},\boldsymbol{\tau}) \ge \min\left\{\frac{1}{\mu}, \gamma(\lambda)\right\}\|\boldsymbol{\tau}\|^2_{\mathbf{H}} \quad \forall\,\boldsymbol{\tau} \in \mathbf{V},$$

so $\mathbf{A}$ is $\mathbf{V}$-elliptic with constant $\alpha_{\mathbf{A}} := \min\{\frac{1}{\mu}, \gamma(\lambda)\}$. Next, define the subspace $\mathbf{W} := \{\boldsymbol{\tau} \in \mathbb{K} : \tau_N = 0 \text{ on } \Gamma_C\}$.

Lemma 4.1 (Inf-sup condition for B). *There exists $\beta > 0$ such that*

$$\sup_{\boldsymbol{\tau}\in\mathbf{W},\boldsymbol{\tau}\neq 0} \frac{\mathbf{B}(\boldsymbol{\tau},(\boldsymbol{v},\boldsymbol{\delta}))}{\|\boldsymbol{\tau}\|_{\mathbf{H}}} \geq \beta\,\|(\boldsymbol{v},\boldsymbol{\delta})\|_{\mathbf{Q}}, \quad \forall (\boldsymbol{v},\boldsymbol{\delta}) \in \mathbf{Q}.$$

Proof. The proof follows by constructing a suitable lifting $\boldsymbol{w} \in \mathbf{H}^1_{\Gamma_D}(\Omega)$ and setting $\widetilde{\boldsymbol{\tau}} = \mathbf{e}(\boldsymbol{w})$, $\widehat{\boldsymbol{\tau}} = \mathbf{e}(\boldsymbol{w}) + \boldsymbol{\delta}$, then applying Korn and Poincaré inequalities, as well as Lax–Milgram's lemma. Details are omitted for brevity. ☐

With these ingredients, Theorem 3.1 applies, yielding:

Theorem 4.1. *For each $\phi \in \mathrm{L}^3(\Omega)$, the variational problem (2.19) has a unique solution $(\boldsymbol{\sigma},\boldsymbol{u}) \in \mathbb{K}\times\mathbf{Q}$, so that the operator $\mathbf{S}$ is well-defined. Moreover, there exist constants $C_1, C_2, C_3, C_4 > 0$ (depending on $\alpha, \alpha_{\mathbf{A}}, \beta, \|\mathbf{A}\|, |\Omega|, \gamma(\lambda)$) such that*

$$\begin{aligned} \|\boldsymbol{\sigma}\|_{\mathbf{H}} &\leq C_1(\|\phi\|_{\mathrm{Y}} + \|g\|_{\mathrm{W}_C}) + C_2\|\varrho\mathbf{f}\|_{0,6/5;\Omega}, \\ \|(\boldsymbol{u},\boldsymbol{\rho})\|_{\mathbf{Q}} &\leq C_3(\|\phi\|_{\mathrm{Y}} + \|g\|_{\mathrm{W}_C}) + C_4\|\varrho\mathbf{f}\|_{0,6/5;\Omega}. \end{aligned}$$

As a corollary, we have the global bound

$$\|\mathbf{S}(\phi)\|_{\mathbf{div}_{6/5}\times\mathbf{L}^6;\Omega} := \|\boldsymbol{\sigma}\|_{\mathbf{H}} + \|\boldsymbol{u}\|_{0,6;\Omega} \leq C_{\mathbf{S}}\left(\|\phi\|_{\mathrm{Y}} + \|g\|_{\mathrm{W}_C} + \|\varrho\mathbf{f}\|_{0,6/5;\Omega}\right),$$

with $C_{\mathbf{S}} := \max\{C_1 + C_3, C_2 + C_4\}$.

4.3. Well-posedness of S

Let $\widetilde{\mathbf{V}} := \{\xi \in \mathbf{X} : \operatorname{div}\xi = 0 \text{ in } \Omega\}$ denote the kernel of $b(\bullet,\bullet)$.

Lemma 4.2 ($a_{\boldsymbol{\sigma}}$-ellipticity). *For each $\boldsymbol{\sigma} \in \mathbb{K}$, the bilinear form $a_{\boldsymbol{\sigma}}(\bullet,\bullet)$ is symmetric, positive definite, and $\widetilde{\mathbf{V}}$-elliptic. That is, there exists a constant $\widetilde{\alpha} > 0$, depending on κ_0 and r_0, such that*

$$a_{\boldsymbol{\sigma}}(\xi,\xi) \geq \widetilde{\alpha}\,\|\xi\|^2_{\mathbf{X}}, \quad \forall \xi \in \widetilde{\mathbf{V}}.$$

Proof. The symmetry and positive definiteness follow directly from the definition. For $\boldsymbol{\xi} \in \widetilde{\mathbf{V}}$,

$$a_{\boldsymbol{\sigma}}(\xi,\xi) = \int_\Omega \kappa^{-1}|\xi|^2 + \int_{\Gamma_C} r(\sigma_N)|\xi\cdot\boldsymbol{n}|^2 \geq \min\{\kappa_0, r_0\}\|\xi\|^2_{\mathbf{X}},$$

so we can take $\widetilde{\alpha} := \min\{\kappa_0, r_0\}$. ☐

Lemma 4.3 (Inf-sup condition for b). *There exists $\beta > 0$ (depending only on $|\Omega|$) such that*

$$\sup_{\xi\in\mathbf{X},\xi\neq 0} \frac{b(\xi,\psi)}{\|\xi\|_{\mathbf{X}}} \geq \beta\,\|\psi\|_{\mathrm{Y}}, \quad \forall \psi \in \mathrm{Y}.$$

Proof. This follows directly from [25, Lemma 2.7] with $r = 3$. ☐

Now we are in a position to announce a preliminary step for proving the well-posedness of S, namely a global inf-sup condition (which is in turn a consequence of [17, Theorem 3.4]).

Lemma 4.4. *Given $\boldsymbol{\zeta} \in \mathbb{K}$, we define global bilinear form $A_{\boldsymbol{\zeta}} : (\mathbf{X}\times\mathrm{Y})\times(\mathbf{X}\times\mathrm{Y}) \to \mathbb{R}$ as*

$$A_{\boldsymbol{\zeta}}\big((\chi,\phi),(\xi,\psi)\big) := a_{\boldsymbol{\zeta}}(\chi,\xi) + b(\xi,\phi) + b(\chi,\psi) - c(\phi,\psi) \qquad \forall\,(\chi,\phi),(\xi,\psi) \in \mathbf{X}\times\mathrm{Y}\,. \tag{4.2}$$

Then, there exists a positive constant $\widetilde{\gamma}$ depending only on $\|a_{\boldsymbol{\zeta}}\|$, $\|c\|$, $\widetilde{\alpha}$ and $\widetilde{\beta}$, such that

$$\sup_{\substack{(\xi,\psi)\in\mathbf{X}\times\mathrm{Y}\\(\xi,\psi)\neq\mathbf{0}}} \frac{A_{\boldsymbol{\zeta}}\big((\chi,\phi),(\xi,\psi)\big)}{\|(\xi,\psi)\|_{\mathbf{X}\times\mathrm{Y}}} \geq \widetilde{\gamma}\,\|(\chi,\phi)\|_{\mathbf{X}\times\mathrm{Y}} \quad \forall\,(\chi,\phi) \in \mathbf{X}\times\mathrm{Y}\,. \tag{4.3}$$

Proof. We first observe that for each $\boldsymbol{\zeta} \in \mathbb{K}$, $a_{\boldsymbol{\zeta}}(\bullet,\bullet)$, $b(\bullet,\bullet)$ and $c(\bullet,\bullet)$ are bounded bilinear forms and $G(\bullet)$ and $F(\bullet)$ are linear and bounded functionals. Now, it is clear from the definition of $c(\bullet,\bullet)$ in (2.16e), that it is positive definite and symmetric. Furthermore, Lemmas 4.2 and 4.3 allow us to apply [17, Theorem 3.4], which states the existence of a unique $(\boldsymbol{\theta},\psi) \in \mathbf{X} \times \mathrm{Y}$ such that

$$\begin{aligned} a_{\sigma}(\boldsymbol{\theta},\xi) + b(\xi,\varphi) &= F(\xi) \qquad \forall\, \xi \in \mathbf{X}\,, \\ b(\boldsymbol{\theta},\psi) - c(\varphi,\psi) &= G(\psi) \qquad \forall\, \psi \in \mathrm{Y}\,. \end{aligned}$$

And this implies the global inf-sup condition (4.3), finishing the proof. □

Finally, we show that S (cf. (2.20)) is well-posed. The proof is based on the global inf–sup condition (4.3) from Lemma 4.4 and on the boundedness of the perturbation (cf. (2.21)).

Theorem 4.2. *Let $\boldsymbol{w} \in \mathbf{L}^6(\Omega)$ and $\boldsymbol{\zeta} \in \mathbb{K}$ with $\|\boldsymbol{w}\|_{0,6;\Omega} \le \frac{\widetilde{\gamma}}{2\|\kappa^{-1}\|_{0,\infty;\Omega}}$. Then, problem (2.21) has a unique solution $(\boldsymbol{\theta},\psi) \in \mathbf{X} \times \mathrm{Y}$ and hence one can define $\mathrm{S}(\boldsymbol{\zeta},\boldsymbol{w}) := \varphi$. Moreover, there exists a positive constant C_{S}, depending on $\|a_{\boldsymbol{\zeta}}\|$, $\|c\|$ and $\widetilde{\gamma}$ such that*

$$\|\mathrm{S}(\boldsymbol{\zeta},\boldsymbol{w})\|_{0,3;\Omega} := \|\varphi\|_{\mathrm{Y}} \le \|(\boldsymbol{\theta},\varphi)\|_{\mathbf{X}\times\mathrm{Y}} \le C_{\mathrm{S}} \left\{ \|\varphi_0\|_{0,\Gamma_C} + \|m\|_{0,3/2;\Omega} \right\}. \tag{4.4}$$

Proof. We make use of Lemma 4.4 and the bound in (2.12) fixing the values of t' and s' as provided, yielding

$$\sup_{\substack{(\xi,\psi)\in\mathbf{X}\times\mathrm{Y}\\(\xi,\psi)\neq\mathbf{0}}} \frac{A_{\boldsymbol{\zeta}}\big((\chi,\phi),(\xi,\psi)\big) - \displaystyle\int_\Omega (\boldsymbol{w}\cdot\kappa^{-1}\chi)\,\psi}{\|(\xi,\psi)\|_{\mathbf{X}\times\mathrm{Y}}} \ge \big(\widetilde{\gamma} - \|\kappa^{-1}\|_{0,\infty;\Omega}\|\boldsymbol{w}\|_{0,6;\Omega}\big)\, \|(\chi,\phi)\|_{\mathbf{X}\times\mathrm{Y}},$$

for each $(\chi,\phi) \in \mathbf{X}\times\mathrm{Y}$. Clearly, we have many options for bounding $\|\boldsymbol{w}\|_{0,6;\Omega}$ in order to maintain the inf-sup property for the full operator. We simply choose $\|\boldsymbol{w}\|_{0,6;\Omega} \le \frac{\widetilde{\gamma}}{2\|\kappa^{-1}\|_{0,\infty;\Omega}}$, obtaining

$$\sup_{\substack{(\xi,\psi)\in\mathbf{X}\times\mathrm{Y}\\(\xi,\psi)\neq\mathbf{0}}} \frac{A_{\boldsymbol{\zeta}}\big((\chi,\phi),(\xi,\psi)\big) - \displaystyle\int_\Omega (\boldsymbol{w}\cdot\kappa^{-1}\chi)\,\psi}{\|(\xi,\psi)\|_{\mathbf{X}\times\mathrm{Y}}} \ge \frac{\widetilde{\gamma}}{2}\, \|(\chi,\phi)\|_{\mathbf{X}\times\mathrm{Y}} \quad \forall\,(\chi,\phi) \in \mathbf{X}\times\mathrm{Y}\,.$$

Now, thanks to the positive definiteness of $r(\bullet)$ (cf. (1.4a)), $a_{\boldsymbol{\zeta}}(\bullet,\bullet)$ and $c(\bullet,\bullet)$ – defined in (2.16c),(2.16e) – are symmetric. Therefore, we can deduce that $A_{\boldsymbol{\zeta}}(\bullet,\bullet)$ (cf. (4.2)) is symmetric, thus we can use the same argument as before to prove that

$$\sup_{\substack{(\chi,\phi)\in\mathbf{X}\times\mathrm{Y}\\(\chi,\phi)\neq\mathbf{0}}} \frac{A_{\boldsymbol{\zeta}}\big((\chi,\phi),(\xi,\psi)\big) - \displaystyle\int_\Omega (\boldsymbol{w}\cdot\kappa^{-1}\chi)\,\psi}{\|(\chi,\phi)\|_{\mathbf{X}\times\mathrm{Y}}} \ge \frac{\widetilde{\gamma}}{2}\, \|(\xi,\psi)\|_{\mathbf{X}\times\mathrm{Y}} \quad \forall\,(\xi,\psi) \in \mathbf{X}\times\mathrm{Y}\,. \tag{4.5}$$

These conditions imply that the assumptions of the Banach–Nečas–Babuška theorem [20, Theorem 2.6] are satisfied, and then we can guarantee the existence of a unique solution $(\boldsymbol{\theta},\varphi) \in \mathbf{X}\times\mathrm{Y}$. Moreover, the a priori bound (4.4) follows from [20, Theorem 2.6, eq. (2.5)] and the bounds for $\|F\|_{\mathbf{X}'}$ and $\|G\|_{\mathrm{Y}'}$. □

From Theorems 4.1 and 4.2, we can establish that $\mathbf{T}$ (cf. (2.22)) is well-posed. Moreover, we have that

$$\|\mathbf{T}(\boldsymbol{\zeta},\boldsymbol{w})\|_{\mathbf{div}_{6/5}\times\mathbf{L}^6;\Omega} \le C_{\mathbf{T}}\Big(\|\varphi_0\|_{0,\Gamma_C} + \|m\|_{0,3/2;\Omega} + \|g\|_{\mathrm{W}_C} + \|\varrho \boldsymbol{f}\|_{0,6/5;\Omega} \Big),$$

for each $(\boldsymbol{\zeta},\boldsymbol{w}) \in \mathbb{K}\times\mathbf{L}^6(\Omega)$ satisfying $\|\boldsymbol{w}\|_{0,6;\Omega} \le \dfrac{\widetilde{\gamma}}{2\,\|\kappa^{-1}\|_{0,\infty;\Omega}}$ and $C_{\mathbf{T}} := C_{\mathbf{S}}\, \max\{C_{\mathrm{S}}, 1\}$.

4.4. Well-posedness of the fixed-point scheme

Our next goal is to prove that $\mathbf{T}$ (cf. (2.22)) satisfies the hypotheses of the Banach Fixed-Point Theorem: the operator $\mathbf{T}$ maps a set into itself and $\mathbf{T}$ is a contraction mapping (i.e., the Lipschitz continuity constant $L > 0$ must be strictly less than 1). Nevertheless, we note that the first component of the solution must be in a closed convex subset of $\mathbb{H}(\mathbf{div}_{6/5};\Omega)$, say $\mathbb{K}$ (cf. (2.13)), therefore, we require that $\mathbf{T}$ maps the first component of the input from a closed convex subset of $\mathbb{K}$ into the same set. For that end, given $R > 0$, we define the following bounded closed convex subset of $\mathbb{K} \times \mathbf{L}^6(\Omega)$

$$\mathcal{K}_R := \{(\boldsymbol{\zeta}, \boldsymbol{w}) \in \mathbb{K} \times \mathbf{L}^6(\Omega) \,:\, \|\boldsymbol{\zeta}\|_{\mathbf{H}} \leq R \quad \text{and} \quad \|\boldsymbol{w}\|_{0,6;\Omega} \leq R\}\,, \tag{4.6}$$

and we have the following result.

Lemma 4.5. *Assume that*

$$C_{\mathbf{T}}\Big(\|\varphi_0\|_{0,\Gamma_C} + \|m\|_{0,3/2;\Omega} + \|g\|_{\mathrm{W}_C} + \|\varrho \boldsymbol{f}\|_{0,6/5;\Omega}\Big) < R\,.$$

Then, $\mathbf{T}(\mathcal{K}_R) \subseteq \mathcal{K}_R$.

Now, in order to prove the Lipschitz continuity of $\mathbf{T}$ (cf. (2.22)), we need to prove the Lipschitz continuity of $\mathbf{S}$ (cf. (2.18)) and S (cf. (2.20)). We start with the first property.

Lemma 4.6. *There exists a positive constant* $L_{\mathbf{S}}$*, depending on* α*,* $\alpha_{\mathbf{A}}$*,* β *and* $\|\mathbf{A}\|$ *such that*

$$\|\mathbf{S}(\phi_1) - \mathbf{S}(\phi_2)\|_{\mathbf{div}_{6/5}\times\mathbf{L}^6;\Omega} \leq L_{\mathbf{S}}\,\gamma(\lambda)\,\|\phi_1 - \phi_2\|_{0,3;\Omega}\,.$$

Proof. Let ϕ, $\underset{\sim}{\phi} \in \mathrm{L}^3(\Omega)$ such that $\mathbf{S}(\phi) = (\boldsymbol{\zeta}, \boldsymbol{w})$ and $\mathbf{S}(\underset{\sim}{\phi}) = (\underset{\sim}{\boldsymbol{\zeta}}, \underset{\sim}{\boldsymbol{w}})$, where $(\boldsymbol{\zeta}, (\boldsymbol{w}, \boldsymbol{\rho}))$, $(\underset{\sim}{\boldsymbol{\zeta}}, (\underset{\sim}{\boldsymbol{w}}, \underset{\sim}{\boldsymbol{\rho}}))$ are solutions of (2.19) with ϕ and $\underset{\sim}{\phi}$, respectively. It is easy to see, after algebraic manipulations, that

$$\mathbf{A}(\boldsymbol{\zeta} - \underset{\sim}{\boldsymbol{\zeta}}, \boldsymbol{\zeta} - \underset{\sim}{\boldsymbol{\zeta}}) \leq \big(\mathbf{F}_{\phi} - \mathbf{F}_{\underset{\sim}{\phi}}\big)(\boldsymbol{\zeta} - \underset{\sim}{\boldsymbol{\zeta}}). \tag{4.7}$$

Next, from the definition of $\mathbf{F}_{\phi}$ (cf. (2.17)), we notice that

$$|\mathbf{F}_{\phi}(\boldsymbol{\tau}) - \mathbf{F}_{\underset{\sim}{\phi}}(\boldsymbol{\tau})| = \left|\alpha \int_{\Omega} \gamma(\lambda)(\phi - \underset{\sim}{\phi})\,\mathrm{tr}(\boldsymbol{\tau})\right| \leq C_{\mathbf{F}}\,\gamma(\lambda)\|\phi - \underset{\sim}{\phi}\|_{0,3;\Omega}\,\|\boldsymbol{\tau}\|_{\mathbf{H}}, \tag{4.8}$$

for each $\boldsymbol{\tau} \in \mathbf{H}$, where $C_{\mathbf{F}} := \alpha\, d^{1/2}|\Omega|^{1/6}$. This point, together with the fact that $\boldsymbol{\zeta} - \underset{\sim}{\boldsymbol{\zeta}} \in \mathbf{V}$, allows us to bound both sides of (4.7), and therefore get

$$\|\boldsymbol{\zeta} - \underset{\sim}{\boldsymbol{\zeta}}\|_{\mathbf{H}} \leq \frac{\alpha}{\alpha_{\mathbf{A}}} d^{1/2}|\Omega|^{1/6}\,\gamma(\lambda)\|\phi - \underset{\sim}{\phi}\|_{0,3;\Omega}\,. \tag{4.9}$$

Now, from the inf-sup condition we have that

$$\|(\boldsymbol{w}, \boldsymbol{\rho}) - (\underset{\sim}{\boldsymbol{w}}, \underset{\sim}{\boldsymbol{\rho}})\|_{\mathbf{Q}} \leq \frac{1}{\beta} \sup_{\substack{\boldsymbol{\tau}\in\mathbf{W} \\ \boldsymbol{\tau}\neq 0}} \frac{\mathbf{B}\big(\boldsymbol{\tau}, (\boldsymbol{w}, \boldsymbol{\rho}) - (\underset{\sim}{\boldsymbol{w}}, \underset{\sim}{\boldsymbol{\rho}})\big)}{\|\boldsymbol{\tau}\|_{\mathbf{H}}}\,.$$

From the first equation of (2.19) and testing with $\boldsymbol{\tau} = \mathbf{0}$ and $\boldsymbol{\tau} = 2\boldsymbol{\zeta}$, we observe that

$$\mathbf{A}(\boldsymbol{\zeta}, \boldsymbol{\zeta}) + \mathbf{B}(\boldsymbol{\zeta}, (\boldsymbol{w}, \boldsymbol{\rho})) = \mathbf{F}_{\phi}(\boldsymbol{\zeta})\,, \tag{4.10}$$

and hence, replacing (4.10) in the first row of (2.19) results in

$$\mathbf{B}(\boldsymbol{\tau}, (\boldsymbol{w}, \boldsymbol{\rho})) \geq \mathbf{F}_{\phi}(\boldsymbol{\tau} - \boldsymbol{\zeta}) - \mathbf{A}(\boldsymbol{\zeta}, \boldsymbol{\tau} - \boldsymbol{\zeta}) + \mathbf{B}(\boldsymbol{\zeta}, (\boldsymbol{w}, \boldsymbol{\rho})) = \mathbf{F}_{\phi}(\boldsymbol{\tau}) - \mathbf{A}(\boldsymbol{\zeta}, \boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in \mathbb{K}\,. \tag{4.11}$$

Analogously for $\mathbf{B}(\boldsymbol{\tau},(\underset{\sim}{\boldsymbol{w}},\underset{\sim}{\rho}))$, we can assert that

$$\mathbf{B}(\boldsymbol{\tau},(\underset{\sim}{\boldsymbol{w}},\underset{\sim}{\rho})) \geq \mathbf{F}_{\underset{\sim}{\phi}}(\boldsymbol{\tau}) - \mathbf{A}(\underset{\sim}{\boldsymbol{\zeta}},\boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in \mathbb{K}\,.$$

Then, owing to the fact that $\mathbf{A}(\underset{\sim}{\boldsymbol{\zeta}},\bullet) - \mathbf{F}_{\underset{\sim}{\phi}} + \mathbf{B}(\bullet,(\boldsymbol{w},\rho)) \in \mathbf{H}'$ we can write

$$\|(\boldsymbol{w},\rho) - (\underset{\sim}{\boldsymbol{w}},\underset{\sim}{\rho})\|_{\mathbf{Q}} \leq \frac{1}{\beta} \sup_{\substack{\boldsymbol{\tau}\in\mathbf{W}\\ \boldsymbol{\tau}\neq 0}} \frac{\mathbf{F}_{\underset{\sim}{\phi}}(\boldsymbol{\tau}) - \mathbf{A}(\underset{\sim}{\boldsymbol{\zeta}},\boldsymbol{\tau}) - \mathbf{B}(\boldsymbol{\tau},(\boldsymbol{w},\rho))}{\|\boldsymbol{\tau}\|_{\mathbf{H}}}\,. \tag{4.12}$$

Then, putting (4.11) back into (4.12), using the bound (4.9), and rearranging terms, we arrive at

$$\begin{aligned} \|\boldsymbol{w} - \underset{\sim}{\boldsymbol{w}}\|_{0,6;\Omega} \leq \|(\boldsymbol{w},\rho) - (\underset{\sim}{\boldsymbol{w}},\underset{\sim}{\rho})\|_{\mathbf{Q}} &\leq \frac{1}{\beta} \sup_{\substack{\boldsymbol{\tau}\in\mathbf{W}\\ \boldsymbol{\tau}\neq 0}} \frac{(\mathbf{F}_{\underset{\sim}{\phi}} - \mathbf{F}_{\phi})(\boldsymbol{\tau}) + \mathbf{A}(\boldsymbol{\zeta} - \underset{\sim}{\boldsymbol{\zeta}},\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\mathbf{H}}} \\ &\leq \frac{\alpha}{\beta}\, d^{1/2} |\Omega|^{1/6}\, \gamma(\lambda) \Big(1 + \frac{\|\mathbf{A}\|}{\alpha_{\mathbf{A}}}\Big)\, \|\phi - \underset{\sim}{\phi}\|_{0,3;\Omega}\,. \end{aligned} \tag{4.13}$$

Finally, adding (4.9) and (4.13) we are left with the bound

$$\|\mathbf{S}(\phi) - \mathbf{S}(\underset{\sim}{\phi})\|_{\mathbf{div}_{6/5}\times \mathbf{L}^6;\Omega} = \|\boldsymbol{\zeta} - \underset{\sim}{\boldsymbol{\zeta}}\|_{\mathbf{H}} + \|\boldsymbol{w} - \underset{\sim}{\boldsymbol{w}}\|_{0,6;\Omega} \leq L_{\mathbf{S}}\, \gamma(\lambda) \|\phi - \underset{\sim}{\phi}\|_{0,3;\Omega}\,,$$

where $L_{\mathbf{S}} := \alpha d^{1/2} |\Omega|^{1/6} \left(\dfrac{1}{\alpha_{\mathbf{A}}} + \dfrac{1}{\beta}\left(1 + \dfrac{\|\mathbf{A}\|}{\alpha_{\mathbf{A}}}\right)\right)$, finishing the proof. □

We next verify the Lipschitz continuity of S (cf. (2.20)) using similar steps as in the previous lemma.

Lemma 4.7. *There exists a positive constant L_{S}, depending on L_r, C_{S}, $\|\kappa^{-1}\|_{0,\infty;\Omega}$ and $\widetilde{\gamma}$ (cf (4.3)) such that*

$$\|\mathrm{S}(\boldsymbol{\zeta},\boldsymbol{w}) - \mathrm{S}(\underset{\sim}{\boldsymbol{\zeta}},\underset{\sim}{\boldsymbol{w}})\|_{0,3;\Omega} \leq L_{\mathrm{S}} \left(\|\varphi_0\|_{0,\Gamma_C} + \|m\|_{0,3/2;\Omega} \right) \|(\boldsymbol{\zeta},\boldsymbol{w}) - (\underset{\sim}{\boldsymbol{\zeta}},\underset{\sim}{\boldsymbol{w}})\|_{\mathbf{X}\times\mathrm{Y}}. \tag{4.14}$$

Proof. We denote by $\mathrm{S}(\boldsymbol{\zeta},\boldsymbol{w}) = \varphi$ and $\mathrm{S}(\underset{\sim}{\boldsymbol{\zeta}},\underset{\sim}{\boldsymbol{w}}) = \underset{\sim}{\varphi}$ the second components of the unique solutions $(\boldsymbol{\theta},\varphi)$ and $(\underset{\sim}{\boldsymbol{\theta}},\underset{\sim}{\varphi})$ in $\mathbf{X}\times\mathrm{Y}$ to the variational problems

$$A_{\boldsymbol{\zeta}}((\boldsymbol{\theta},\varphi),(\boldsymbol{\xi},\psi)) - \int_\Omega (\boldsymbol{w}\cdot\kappa^{-1}\boldsymbol{\theta})\psi = F(\boldsymbol{\xi}) + G(\psi) \quad \forall\,(\boldsymbol{\xi},\psi)\in\mathbf{X}\times\mathrm{Y}\,, \tag{4.15a}$$

$$A_{\underset{\sim}{\boldsymbol{\zeta}}}((\underset{\sim}{\boldsymbol{\theta}},\underset{\sim}{\varphi}),(\boldsymbol{\xi},\psi)) - \int_\Omega (\underset{\sim}{\boldsymbol{w}}\cdot\kappa^{-1}\underset{\sim}{\boldsymbol{\theta}})\psi = F(\boldsymbol{\xi}) + G(\psi) \quad \forall\,(\boldsymbol{\xi},\psi)\in\mathbf{X}\times\mathrm{Y}\,, \tag{4.15b}$$

respectively. Now, applying the global inf-sup condition (4.5) with $(\boldsymbol{\theta},\varphi) - (\underset{\sim}{\boldsymbol{\theta}},\underset{\sim}{\varphi})$ we have

$$\begin{aligned} \|\mathrm{S}(\boldsymbol{\zeta},\boldsymbol{w}) - \mathrm{S}(\underset{\sim}{\boldsymbol{\zeta}},\underset{\sim}{\boldsymbol{w}})\|_{0,3;\Omega} = \|\varphi - \underset{\sim}{\varphi}\|_{0,3;\Omega} &\leq \|(\boldsymbol{\theta},\varphi) - (\underset{\sim}{\boldsymbol{\theta}},\underset{\sim}{\varphi})\|_{\mathbf{X}\times\mathrm{Y}} \\ &\leq \frac{2}{\widetilde{\gamma}} \sup_{\substack{(\boldsymbol{\xi},\psi)\in\mathbf{X}\times\mathrm{Y}\\ (\boldsymbol{\xi},\psi)\neq\mathbf{0}}} \frac{A_{\boldsymbol{\zeta}}\big((\boldsymbol{\theta},\varphi) - (\underset{\sim}{\boldsymbol{\theta}},\underset{\sim}{\varphi}),(\boldsymbol{\xi},\psi)\big) - \displaystyle\int_\Omega \big(\boldsymbol{w}\cdot\kappa^{-1}(\boldsymbol{\theta} - \underset{\sim}{\boldsymbol{\theta}})\big)\,\psi}{\|(\boldsymbol{\xi},\psi)\|_{\mathbf{X}\times\mathrm{Y}}}\,. \end{aligned}$$

Next, performing some algebraic manipulations, since $A_{\boldsymbol{\zeta}}(\bullet,\bullet)$ (cf. (4.2)) is bilinear, bearing in mind (4.15a) and (4.15b), along with the a priori bound for $\|\underset{\sim}{\boldsymbol{\theta}}\|_{\mathbf{X}}$ (cf. (4.4)), and adding and subtracting $\int_\Omega (\underset{\sim}{\boldsymbol{w}}\cdot\kappa^{-1}\underset{\sim}{\boldsymbol{\theta}})\,\psi$ we realise that

$$\|(\boldsymbol{\theta},\varphi) - (\underset{\sim}{\boldsymbol{\theta}},\underset{\sim}{\varphi})\|_{\mathbf{X}\times\mathrm{Y}} \leq \frac{2}{\widetilde{\gamma}} \sup_{\substack{(\boldsymbol{\xi},\psi)\in\mathbf{X}\times\mathrm{Y}\\ (\boldsymbol{\xi},\psi)\neq\mathbf{0}}} \frac{(a_{\boldsymbol{\zeta}} - a_{\underset{\sim}{\boldsymbol{\zeta}}})(\underset{\sim}{\boldsymbol{\theta}},\boldsymbol{\xi}) + \displaystyle\int_\Omega \big((\boldsymbol{w} - \underset{\sim}{\boldsymbol{w}})\cdot\kappa^{-1}\underset{\sim}{\boldsymbol{\theta}}\big)\,\psi}{\|(\boldsymbol{\xi},\psi)\|_{\mathbf{X}\times\mathrm{Y}}} \tag{4.16}$$

$$\leq \|(a_{\boldsymbol{\zeta}} - a_{\underset{\sim}{\boldsymbol{\zeta}}})(\underset{\sim}{\boldsymbol{\theta}},\bullet)\|_{\mathbf{X}'} + C_{\mathrm{S}}\|\kappa^{-1}\|_{0,\infty;\Omega}\left\{\|\varphi_0\|_{0,\Gamma_C} + \|m\|_{0,3/2;\Omega}\right\}\|\boldsymbol{w} - \underset{\sim}{\boldsymbol{w}}\|_{0,6;\Omega}\,.$$

Now we need to bound $\|(a_{\boldsymbol{\zeta}} - a_{\underset{\sim}{\boldsymbol{\zeta}}})(\underset{\sim}{\boldsymbol{\theta}},\bullet)\|_{\mathbf{X}'}$. To this purpose, we observe that, for $\boldsymbol{\xi} \in \mathbf{X}$ there holds

$$(a_{\boldsymbol{\zeta}} - a_{\underset{\sim}{\boldsymbol{\zeta}}})(\underset{\sim}{\boldsymbol{\theta}},\boldsymbol{\xi}) = \left\langle \boldsymbol{\xi}\cdot\boldsymbol{n}, \left(r(\zeta_N) - r(\underset{\sim}{\zeta}_N)\right)\underset{\sim}{\boldsymbol{\theta}}\cdot\boldsymbol{n}\right\rangle_{0,\Gamma_C} \leq \|\boldsymbol{\xi}\|_{\mathbf{X}}\,\|\underset{\sim}{\boldsymbol{\theta}}\|_{\mathbf{X}}\,\|r(\zeta_N) - r(\underset{\sim}{\zeta}_N)\|_{0,\infty;\Gamma_C}\,.$$

Then, using the Lipschitz continuity of r (cf. (1.4a)), and using again the bound for $\|\underset{\sim}{\boldsymbol{\theta}}\|_{\mathbf{X}}$ (cf. (4.4)), leads to

$$\|(a_{\boldsymbol{\zeta}} - a_{\underset{\sim}{\boldsymbol{\zeta}}})(\underset{\sim}{\boldsymbol{\theta}},\bullet)\|_{\mathbf{X}'} \leq L_r\left\{\|\varphi_0\|_{0,\Gamma_C} + \|m\|_{0,3/2;\Omega}\right\}\|\boldsymbol{\zeta} - \underset{\sim}{\boldsymbol{\zeta}}\|_{\mathbf{H}}\,. \tag{4.17}$$

Finally, putting (4.17) back into (4.16), and performing some minor algebraic manipulations, we arrive at (4.14) with $L_{\mathrm{S}} := \dfrac{2}{\widetilde{\gamma}}\max\left\{L_r, C_{\mathrm{S}}\|\kappa^{-1}\|_{0,\infty;\Omega}\right\}$. ☐

As a direct consequence of the previous lemmas, we can address the Lipschitz continuity of $\mathbf{T}$ (cf. (2.22)) which is established in the following result.

Lemma 4.8. *There exists a positive constant* $L_{\mathbf{T}} := L_{\mathbf{S}}\,L_{\mathrm{S}}$ *such that*

$$\|\mathbf{T}(\boldsymbol{\zeta},\boldsymbol{w}) - \mathbf{T}(\underset{\sim}{\boldsymbol{\zeta}},\underset{\sim}{\boldsymbol{w}})\|_{\mathbf{H}\times\mathbf{L}^6(\Omega)} \leq L_{\mathbf{T}}\,\gamma(\lambda)\big(\|\varphi_0\|_{0,\Gamma_C} + \|m\|_{0,3/2;\Omega}\big)\|(\boldsymbol{\zeta},\boldsymbol{w}) - (\underset{\sim}{\boldsymbol{\zeta}},\underset{\sim}{\boldsymbol{w}})\|_{\mathbf{div}_{6/5}\times\mathbf{L}^6;\Omega}\,.$$

Proof. It is a direct consequence of Lemmas 4.6 and 4.7. In fact we notice that

$$\begin{aligned}\|\mathbf{T}(\boldsymbol{\zeta},\boldsymbol{w}) - \mathbf{T}(\underset{\sim}{\boldsymbol{\zeta}},\underset{\sim}{\boldsymbol{w}})\| &= \|\mathbf{S}(\mathrm{S}(\boldsymbol{\zeta},\boldsymbol{w}) - \mathbf{S}(\mathrm{S}(\underset{\sim}{\boldsymbol{\zeta}},\underset{\sim}{\boldsymbol{w}}))\| \leq L_{\mathbf{S}}\gamma(\lambda)\,\|\mathrm{S}(\boldsymbol{\zeta},\boldsymbol{w}) - \mathrm{S}(\underset{\sim}{\boldsymbol{\zeta}},\underset{\sim}{\boldsymbol{w}})\|\\ &\leq L_{\mathbf{S}}\,L_{\mathrm{S}}\,\gamma(\lambda)\big(\|\varphi_0\|_{0,\Gamma_C} + \|m\|_{0,3/2;\Omega}\big)\|(\boldsymbol{\zeta},\boldsymbol{w}) - (\underset{\sim}{\boldsymbol{\zeta}},\underset{\sim}{\boldsymbol{w}})\|_{\mathbf{div}_{6/5}\times\mathbf{L}^6;\Omega}\,.\end{aligned}$$

☐

Now, we are in a position to prove the main result of this section.

Theorem 4.3. *Assume that in addition to the hypothesis of Lemma 4.5, the data satisfies*

$$L_{\mathbf{T}}\,\gamma(\lambda)\big(\|\varphi_0\|_{0,\Gamma_C} + \|m\|_{0,3/2;\Omega}\big) \leq 1\,. \tag{4.18}$$

Then, the operator $\mathbf{T}$ *(cf.* (2.22)*) has a unique fixed point* $(\boldsymbol{\sigma},\boldsymbol{u}) \in \mathcal{K}_R$ *(cf.* (4.6)*). Equivalently, the coupled problem* (2.15) *has a unique solution* $(\boldsymbol{\sigma},(\boldsymbol{u},\boldsymbol{\rho})) \in \mathbb{K}\times\mathbf{Q}$, *with* $\|\boldsymbol{u}\|_{0,6;\Omega} \leq \frac{\widetilde{\gamma}}{2\,\|\kappa^{-1}\|_{0,\infty;\Omega}}$. *Moreover, there exists positive constants* C_1 *and* C_2 *such that*

$$\|(\boldsymbol{\sigma},(\boldsymbol{u},\boldsymbol{\rho}))\|_{\mathbf{H}\times\mathbf{Q}} \leq C_1\left(\|\varphi_0\|_{0,\Gamma_C} + \|m\|_{0,3/2;\Omega} + \|g\|_{\mathrm{W}_C} + \|\varrho\boldsymbol{f}\|_{0,6/5;\Omega}\right), \tag{4.19a}$$

$$\|(\boldsymbol{\theta},\varphi)\|_{\mathbf{X}\times\mathrm{Y}} \leq C_2\left(\|\varphi_0\|_{0,\Gamma_C} + \|m\|_{0,3/2;\Omega}\right)\,. \tag{4.19b}$$

Proof. Thanks to Lemma 4.5, $\mathbf{T}$ (cf. (2.22)) maps a closed bounded convex subset $\mathcal{K}_R$ into itself. Moreover, Lemma 4.8 combined with (4.18) implies that $\mathbf{T}$ is a contraction, so then a straightforward application of the Banach Fixed-Point Theorem yields the existence of a unique fixed point $(\boldsymbol{\sigma},\boldsymbol{u}) \in \mathcal{K}_R$, and hence a unique solution of the problem (2.15). Moreover, due $(\boldsymbol{\sigma},\boldsymbol{u}) = \mathbf{T}(\boldsymbol{\sigma},\boldsymbol{u}) = \mathbf{S}(\mathrm{S}(\boldsymbol{\sigma},\boldsymbol{u}))$, we deduce that $\varphi = \mathrm{S}(\boldsymbol{\sigma},\boldsymbol{u})$ and hence, the bounds (4.19a) and (4.19b) arise from (3.4), (3.5), and (4.4). ☐

As a final remark on the continuous analysis, we note that the quantity $L_{\mathbf{T}}\,\gamma(\lambda)$ is independent of λ. Consequently, the small data assumption (4.18) depends primarily on the prescribed temperature φ_0 and the heat source m, without imposing further restrictions on the initial gap g or the body force $\varrho\boldsymbol{f}$.

Nevertheless, in the nearly incompressible limit, i.e., as the Lamé parameter $\lambda \to +\infty$, we have $\alpha_{\mathbf{A}} \approx \gamma(\lambda)$ and $\|\mathbf{A}\| \approx \mu^{-1}$. In this regime, the continuous dependence of the mechanical variables can be quantified explicitly:

$$\|\boldsymbol{\sigma}\|_{\mathbf{H}} \leq \alpha\,|\Omega|^{1/6}d^{1/2}\|\phi\|_{\mathrm{Y}} + \frac{1}{\gamma(\lambda)}\left(\|g\|_{\mathrm{W}_C} + \frac{C}{\mu}\|\varrho\boldsymbol{f}\|_{0,6/5;\Omega}\right) + \gamma(\lambda)\,\|\varrho\boldsymbol{f}\|_{0,6/5;\Omega},$$

$$\|(\boldsymbol{u},\boldsymbol{\rho})\|_{\mathbf{Q}} \leq \frac{\alpha\,|\Omega|^{1/6}d^{1/2}}{\beta}\left(\frac{1}{\mu}+1\right)\|\phi\|_{\mathrm{Y}} + \frac{1}{\beta}\left(\frac{1}{\mu\gamma(\lambda)}+1\right)\|g\|_{\mathrm{W}_C} + \frac{1}{\mu\beta}\left(\frac{C}{\mu\gamma(\lambda)}+1\right)\|\varrho\boldsymbol{f}\|_{0,6/5;\Omega}.$$

These estimates highlight that, for stability, both the initial gap g and the body force $\varrho\boldsymbol{f}$ must remain sufficiently small. In particular, the nearly incompressible limit amplifies the influence of the Lamé parameter on the mechanical response, indicating that careful scaling of the data is crucial to ensure bounded stresses and displacements.

5. Galerkin discretisation

In this section, we address the Galerkin approximation of (2.15) and analyse its solvability. Specifically, we consider the discrete analogues of the decoupled variational inequality (2.19) and the discrete heat problem (2.21). For the thermal subproblem, solvability follows from the discrete counterpart of [17, Theorem 3.4], i.e., [17, Theorem 3.5]. For the variational inequality, we invoke a discrete version of Theorem 3.1, which will be formulated below.

5.1. Preliminaries

Let $\{\mathcal{T}_h\}_{h>0}$ be a shape-regular family of triangulations of Ω, consisting of triangles ($d=2$) or tetrahedra ($d=3$) with element diameters h_K and global mesh size $h := \max_{K\in\mathcal{T}_h} h_K$. For $k \geq 0$, let $\mathrm{P}_k(S)$ denote the space of polynomials of degree $\leq k$ on a set $S \subset \Omega$, with vector and tensor counterparts

$$\mathbf{P}_k(S) := [\mathrm{P}_k(S)]^d, \quad \mathbb{P}_k(S) := [\mathrm{P}_k(S)]^{d\times d}.$$

We also define their corresponding global discrete spaces

$$\begin{aligned}
\mathrm{P}_k(\Omega) &:= \left\{v_h \in L^2(\Omega) : v_h|_K \in \mathrm{P}_k(K)\ \forall K \in \mathcal{T}_h\right\},\\
\mathbf{P}_k(\Omega) &:= \left\{\boldsymbol{v}_h \in \mathbf{L}^2(\Omega) : \boldsymbol{v}_h|_K \in \mathbf{P}_k(K)\ \forall K \in \mathcal{T}_h\right\},\\
\mathbb{P}_k(\Omega) &:= \left\{\boldsymbol{\delta}_h \in \mathbb{L}^2(\Omega) : \boldsymbol{\delta}_h|_K \in \mathbb{P}_k(K)\ \forall K \in \mathcal{T}_h\right\}.
\end{aligned}$$

For the mechanical subproblem, we employ the Arnold–Falk–Winther (AFW) element of order $k \geq 0$ [4, 9]. The discrete spaces for stress, displacement, and body rotation are

$$\mathbb{H}_h^{\boldsymbol{\sigma}} := \mathbb{P}_{k+1}(\Omega) \cap \mathbb{H}_N(\mathbf{div}_{6/5};\Omega), \quad \mathbf{H}_h^{\boldsymbol{u}} := \mathbf{P}_k(\Omega), \quad \mathbb{H}_h^{\boldsymbol{\rho}} := \mathbb{P}_k(\Omega) \cap \mathbb{L}^2_{\mathrm{skew}}(\Omega), \tag{5.1}$$

where $\mathbf{H}_h^{\boldsymbol{u}}$ and $\mathbb{H}_h^{\boldsymbol{\rho}}$ approximate $\boldsymbol{u} \in \mathbf{L}^6(\Omega)$ and $\boldsymbol{\rho} \in \mathbb{L}^2_{\mathrm{skew}}(\Omega)$, respectively, while $\mathbb{H}_h^{\boldsymbol{\sigma}}$ approximates the stress $\boldsymbol{\sigma} \in \mathbb{K}$. The discrete convex set enforcing the contact constraints reads

$$\mathbb{K}_h := \left\{\boldsymbol{\zeta}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}} :\ \boldsymbol{\zeta}_{h,t} = \mathbf{0},\ \zeta_{h,N} \leq 0 \text{ on } \Gamma_C\right\}.$$

For the thermal subproblem, we employ the Brezzi–Douglas–Marini (BDM) element of order $k+1$ for the heat flux and P_k for the temperature:

$$\mathbf{H}_h^{\boldsymbol{\theta}} := \mathbf{P}_{k+1}(\Omega) \cap \mathbf{X}, \quad \mathrm{H}_h^{\varphi} := \mathrm{P}_k(\Omega). \tag{5.2}$$

Setting $\mathbf{Q}_h := \mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}}$, $\mathbf{X}_h := \mathbf{H}_h^{\boldsymbol{\theta}}$, and $\mathrm{Y}_h := \mathrm{H}_h^{\varphi}$, the fully discrete Galerkin problem reads: find

$$(\boldsymbol{\sigma}_h, (\boldsymbol{u}_h, \boldsymbol{\rho}_h)) \in \mathbb{K}_h \times \mathbf{Q}_h, \quad (\boldsymbol{\theta}_h, \varphi_h) \in \mathbf{X}_h \times \mathrm{Y}_h,$$

such that

$$\begin{aligned}
\mathbf{A}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h - \boldsymbol{\sigma}_h) + \mathbf{B}(\boldsymbol{\tau}_h - \boldsymbol{\sigma}_h, (\boldsymbol{u}_h, \boldsymbol{\rho}_h)) &\geq \mathbf{F}_{\varphi_h}(\boldsymbol{\tau}_h - \boldsymbol{\sigma}_h),\\
\mathbf{B}(\boldsymbol{\sigma}_h, (\boldsymbol{v}_h, \boldsymbol{\delta}_h)) &= \mathbf{G}(\boldsymbol{v}_h, \boldsymbol{\delta}_h),\\
a_{\boldsymbol{\sigma}_h}(\boldsymbol{\theta}_h, \boldsymbol{\xi}_h) + b(\boldsymbol{\xi}_h, \varphi_h) &= F(\boldsymbol{\xi}_h),\\
b(\boldsymbol{\theta}_h, \psi_h) - c(\varphi_h, \psi_h) - \int_\Omega (\boldsymbol{u}_h \cdot \kappa^{-1}\boldsymbol{\theta}_h)\,\psi_h &= G(\psi_h),
\end{aligned} \tag{5.3}$$

for all $(\boldsymbol{\tau}_h, (\boldsymbol{v}_h, \boldsymbol{\delta}_h)) \in \mathbb{K}_h \times \mathbf{Q}_h$ and $(\boldsymbol{\xi}_h, \psi_h) \in \mathbf{X}_h \times \mathrm{Y}_h$.

5.2. The discrete fixed-point strategy

Analogously to Section 2.3, we propose a fixed-point approach to establish the solvability of the discrete problem (5.3). To this end, we introduce the discrete counterparts of the solution operators $\mathbf{S}$ (cf. (2.18)), S (cf. (2.20)), and $\mathbf{T}$ (cf. (2.22)).

First, define the operator

$$\mathbf{S}_h \,:\, \mathrm{H}^{\varphi}_h \longrightarrow \mathbb{K}_h \times \mathbf{H}^{\boldsymbol{u}}_h, \qquad \mathbf{S}_h(\phi_h) := (\boldsymbol{\sigma}_h, \boldsymbol{u}_h), \tag{5.4}$$

where $(\boldsymbol{\sigma}_h, (\boldsymbol{u}_h, \boldsymbol{\rho}_h)) \in \mathbb{K}_h \times \mathbf{Q}_h$ is the unique solution (to be established below) of the discrete decoupled variational inequality

$$\begin{aligned} \mathbf{A}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h - \boldsymbol{\sigma}_h) + \mathbf{B}(\boldsymbol{\tau}_h - \boldsymbol{\sigma}_h, (\boldsymbol{u}_h, \boldsymbol{\rho}_h)) &\geq \mathbf{F}_{\phi_h}(\boldsymbol{\tau}_h - \boldsymbol{\sigma}_h), \quad \forall\, \boldsymbol{\tau}_h \in \mathbb{K}_h, \\ \mathbf{B}(\boldsymbol{\sigma}_h, (\boldsymbol{v}_h, \boldsymbol{\delta}_h)) &= \mathbf{G}(\boldsymbol{v}_h, \boldsymbol{\delta}_h), \qquad \forall\, (\boldsymbol{v}_h, \boldsymbol{\delta}_h) \in \mathbf{Q}_h. \end{aligned} \tag{5.5}$$

Next, we define the discrete thermal operator

$$\mathrm{S}_h \,:\, \mathbb{K}_h \times \mathbf{H}^{\boldsymbol{u}}_h \longrightarrow \mathrm{H}^{\varphi}_h, \qquad \mathrm{S}_h(\boldsymbol{\zeta}_h, \boldsymbol{w}_h) := \varphi_h, \tag{5.6}$$

where $(\boldsymbol{\theta}_h, \varphi_h)$ is the unique solution (to be confirmed) of the discrete decoupled thermal problem

$$\begin{aligned} a_{\ell}(\boldsymbol{\theta}_h, \boldsymbol{\xi}_h) + b(\boldsymbol{\xi}_h, \varphi_h) &= F_{\boldsymbol{\zeta}_h}(\boldsymbol{\xi}_h), \quad \forall\, \boldsymbol{\xi}_h \in \mathbf{X}_h, \\ b(\boldsymbol{\theta}_h, \psi_h) - c(\varphi_h, \psi_h) - \int_\Omega (\boldsymbol{w}_h \cdot \kappa^{-1} \boldsymbol{\theta}_h)\, \psi_h &= G(\psi_h), \qquad \forall\, \psi_h \in \mathrm{Y}_h. \end{aligned} \tag{5.7}$$

Finally, we introduce the discrete fixed-point operator

$$\mathbf{T}_h \,:\, \mathbb{K}_h \times \mathbf{H}^{\boldsymbol{u}}_h \longrightarrow \mathbb{K}_h \times \mathbf{H}^{\boldsymbol{u}}_h, \qquad \mathbf{T}_h(\boldsymbol{\zeta}_h, \boldsymbol{w}_h) := \mathbf{S}_h(\mathrm{S}_h(\boldsymbol{\zeta}_h, \boldsymbol{w}_h)). \tag{5.8}$$

With this definition, showing unique solvability of the fully discrete mixed problem (5.3) reduces to proving that $\mathbf{T}_h$ admits a unique fixed point:

$$(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) = \mathbf{T}_h(\boldsymbol{\sigma}_h, \boldsymbol{u}_h). \tag{5.9}$$

5.3. Discrete solvability analysis

In this section we address the well–posedness of $\mathbf{S}_h$ (cf. (5.4)) and S_h (cf. (5.6)), which are equivalent to prove the unique solvability of problems (5.5) and (5.7), respectively.

We begin by defining the kernel of the linear and bounded operator induced by $\mathbf{B}$ over the discrete subspaces as

$$\mathbf{V}_h := \left\{ \boldsymbol{\zeta}_h \in \mathbb{H}^{\boldsymbol{\sigma}}_h \quad : \quad \mathbf{div}(\boldsymbol{\zeta}_h) = \mathbf{0} \quad \text{and} \quad \int_\Omega \boldsymbol{\zeta}_h : \boldsymbol{\delta}_h = 0 \quad \forall\, \boldsymbol{\delta}_h \in \mathbb{H}^{\boldsymbol{\rho}}_h \right\}, \tag{5.10}$$

and we also define a subspace $\mathbf{W}_h$ of $\mathbb{H}^{\boldsymbol{\sigma}}_h$ such that $\mathbf{W}_h \subseteq \mathbb{K}_h$, as follows

$$\mathbf{W}_h := \left\{ \boldsymbol{\zeta}_h \in \mathbb{K}_h \quad : \quad \zeta_{h,N} = 0 \quad \text{on} \quad \Gamma_C \right\}.$$

Clearly, the fact that elements in $\mathbf{V}_h$ are divergence-free implies the $\mathbf{V}_h$−ellipticity of $\mathbf{A}(\bullet, \bullet)$ with the same constant $\alpha_{\mathtt{d}} := \alpha_{\mathbf{A}}$ as

$$\mathbf{A}(\boldsymbol{\zeta}_h, \boldsymbol{\zeta}_h) \geq \alpha_{\mathtt{d}}\, \|\boldsymbol{\zeta}_h\|^2_{\mathbf{H}} \qquad \forall\, \boldsymbol{\zeta}_h \in \mathbf{V}_h\,.$$

Before analysing the discrete counterpart of the inf-sup condition of $\mathbf{B}(\bullet, \bullet)$, we resort to [7, Section 9.3.2] and we define, for every tensor $\boldsymbol{\zeta}$, the skew-symmetric part and the asymmetric part, respectively as

$$\operatorname{as}(\boldsymbol{\zeta}) := \frac{1}{2}(\boldsymbol{\zeta} - \boldsymbol{\zeta}^{\mathtt{t}}) \quad \text{and} \quad \operatorname{asp}(\boldsymbol{\zeta}) := \left\{ \begin{array}{ll} \zeta_{12} - \zeta_{21} & \text{in 2D} \\ \begin{pmatrix} \zeta_{23} - \zeta_{32} \\ \zeta_{31} - \zeta_{13} \\ \zeta_{12} - \zeta_{21} \end{pmatrix} & \text{in 3D} \end{array} \right. . \tag{5.11}$$

Now, we analyse the 2D case of the discrete inf-sup condition (5.12) by constructing $\boldsymbol{\tau}_h \in \mathbf{W}_h$ satisfying

$$\mathcal{G}(\boldsymbol{v}_h, \boldsymbol{\delta}_h) := \sup_{\substack{\boldsymbol{\tau}_h \in \mathbf{W}_h \\ \boldsymbol{\tau}_h \neq 0}} \frac{\mathbf{B}(\boldsymbol{\tau}_h, (\boldsymbol{v}_h, \boldsymbol{\delta}_h))}{\|\boldsymbol{\tau}_h\|_{\mathbf{H}}} \geq \beta_{\mathrm{d}} \, \|(\boldsymbol{v}_h, \boldsymbol{\delta}_h)\|_{\mathbf{Q}} \quad \forall\, (\boldsymbol{v}_h, \boldsymbol{\delta}_h) \in \mathbf{Q}_h \,, \tag{5.12}$$

with $\beta_{\mathrm{d}} > 0$ and independent of h, in the following manner: First, given $\boldsymbol{v}_h \in \mathbf{H}^{\boldsymbol{u}}_h$, we let $\widetilde{\boldsymbol{v}}_h := |\boldsymbol{v}_h|^4 \boldsymbol{v}_h \in \mathbf{L}^{6/5}(\Omega)$. Let $\mathcal{O} \subset \mathbb{R}^d$, be a convex polyhedral domain such that $\overline{\Omega} \subseteq \mathcal{O}$ and $\partial\Omega \cap \partial\mathcal{O} = \widetilde{\Gamma} := \mathrm{int}(\overline{\Gamma}_{\mathrm{N}} \cup \overline{\Gamma}_{\mathrm{C}})$. We consider $\boldsymbol{w} \in \mathbf{W}^{1,6/5}(\mathcal{O})$ the unique solution (whose existence and uniqueness are guaranteed in [18, 34]) to the problem

$$-\boldsymbol{\Delta w} = \mathrm{E}_{0,\Omega}(\widetilde{\boldsymbol{v}}_{\mathrm{h}}) \quad \text{in} \quad \Omega\,, \quad (\boldsymbol{\nabla w})\boldsymbol{n} = \mathbf{0} \quad \text{on} \quad \widetilde{\Gamma}, \quad \text{and} \quad \boldsymbol{w} = \mathbf{0} \quad \text{on} \quad \partial\mathcal{O} \setminus \widetilde{\Gamma}\,, \tag{5.13}$$

where $\mathrm{E}_{0,\Omega}$ stands for the extension by zero outside Ω. We claim that $\boldsymbol{w} \in \mathbf{W}^{2,6/5}(\mathcal{O})$. To get this, we invoke [18, Corollary 3.7], with $k = 0$, $p = 6/5$, which guarantees the required regularity of $\boldsymbol{w}$ providing the following spectral conditions hold at every singular point of $\mathcal{O}$

$$\lambda_1(e) = \frac{\pi}{2\omega_e} > \frac{1}{3} \quad \forall e \in E \quad \text{and} \quad \widetilde{\lambda}_1(v) = \min\{\lambda_1(v), 2\} > -\frac{1}{2} \quad \forall v \in V\,. \tag{5.14}$$

where ω_e is the opening angle of a mixed edge (where the boundary condition changes). Both condition are satisfied in a convex polyhedra because $\omega_e \leq \pi$. Therefore, there exists a positive constant $C_{\mathtt{reg}}$, independent of h such that

$$\|\boldsymbol{w}\|_{2,6/5;\Omega} \leq C_{\mathtt{reg}} \, \|\widetilde{\boldsymbol{v}}_h\|_{0,6/5;\Omega}\,.$$

Next, we define $\overline{\boldsymbol{\tau}} = -\boldsymbol{\nabla w}$ in Ω. We have that $\overline{\boldsymbol{\tau}} \in \mathbb{W}^{1,6/5}(\Omega)$, $\mathbf{div}(\overline{\boldsymbol{\tau}}) = \widetilde{\boldsymbol{v}}_h$ in Ω, and $\overline{\boldsymbol{\tau}}\boldsymbol{n} = \mathbf{0}$ on $\widetilde{\Gamma}$. Moreover, the continuous dependence holds as follows

$$\|\overline{\boldsymbol{\tau}}\|_{1,6/5;\Omega} \leq C_{\mathtt{reg}} \|\widetilde{\boldsymbol{v}}_h\|_{0,6/5;\Omega}\,.$$

Now, letting $\boldsymbol{\tau}_h^{(1)} := \boldsymbol{\Pi}_h(\overline{\boldsymbol{\tau}})$, where $\boldsymbol{\Pi}_h$ is the global BDM interpolation operator, we make use of the commutative diagram and the stability properties (see, e.g., [7, 8, 23]) to get

$$\mathbf{div}(\boldsymbol{\tau}_h^{(1)}) = \boldsymbol{\mathcal{P}}_h^k(\widetilde{\boldsymbol{v}}_h)\,, \qquad \boldsymbol{\tau}_h^{(1)}\boldsymbol{n} = \boldsymbol{\mathcal{Q}}_h^k(\overline{\boldsymbol{\tau}}\boldsymbol{n}) \quad \text{and} \quad \|\boldsymbol{\tau}_h^{(1)}\|_{\mathbf{div}_{6/5;\Omega}} \leq \widetilde{C}\|\widetilde{\boldsymbol{v}}_h\|_{0,6/5;\Omega},$$

and with $\boldsymbol{\mathcal{P}}_h^k : \mathbf{L}^1(\Omega) \to \mathbf{P}_k(\Omega)$ and $\boldsymbol{\mathcal{Q}}_h^k : \mathbf{L}^1(\partial\Omega) \to \mathbf{P}_k(\mathcal{T}_h^\partial)$ are the projectors respect to the $\mathbf{L}^2(\Omega)$ and $\mathbf{L}^2(\partial\Omega)$ inner product, respectively, where

$$\mathbf{P}_k(\mathcal{T}_h^\partial) := \left\{\boldsymbol{v} \in \mathbf{L}^2(\partial\Omega) : \boldsymbol{v}|_F \in \mathbf{P}_k(F) \quad \forall F \in \mathcal{T}_h \cap \partial\Omega\right\}\,, \tag{5.15}$$

In the second step, we aim at constructing a divergence-free tensor $\boldsymbol{\tau}_h^{(2)}$ correcting $\boldsymbol{\tau}_h^{(1)}$ to obtain the asymmetry condition. That is, $\mathrm{as}(\boldsymbol{\tau}_h^{(2)}) = \boldsymbol{\delta}_h - \mathrm{as}(\boldsymbol{\tau}_h^{(1)})$ for every $\boldsymbol{\delta}_h \in \mathbb{H}_h^{\boldsymbol{\rho}}$. For that end we resort to [14] (see also [7, 21]) and let $\mathbf{U}_h \subseteq \mathbf{H}_0^1(\Omega)$, $\mathrm{L}_h \subseteq \mathrm{L}^2(\Omega)$ with $\mathrm{L}_h^0 := \mathrm{L}_h \cap \mathrm{L}_0^2(\Omega)$ such that $(\mathbf{U}_h, \mathrm{L}_h^0)$ is a stable pair approximating the Stokes problem. Next, define $(\boldsymbol{z}_h, p_h) \in \mathbf{U}_h \times \mathrm{L}_h^0$ the unique solution to the Stokes problem

$$\begin{array}{rclr}
\displaystyle\int_\Omega \boldsymbol{\nabla z}_h : \boldsymbol{\nabla w}_h + \int_\Omega p_h \,\mathrm{div}(\boldsymbol{w}_h) & = & 0 & \forall\, \boldsymbol{w}_h \in \mathbf{U}_h, \\
\displaystyle\int_\Omega q_h^0 \,\mathrm{div}(\boldsymbol{z}_h) & = & \displaystyle\int_\Omega q_h^0 \left(\mathrm{asp}(\boldsymbol{\delta}_h - \boldsymbol{\tau}_h^{(1)}) - \chi\right) & \forall\, q_h^0 \in \mathrm{L}_h^0\,,
\end{array}$$

where χ stands for the mean value of $\mathrm{asp}(\boldsymbol{\delta}_h - \boldsymbol{\tau}_h^{(1)})$. The candidate for this correction tensor is $\boldsymbol{\tau}_h^{(2)} := \mathbf{curl}(\boldsymbol{z}_h)$. Clearly, $\boldsymbol{\tau}_h^{(2)}$ is divergence-free, $\boldsymbol{\tau}_h^{(2)}\boldsymbol{n} = \mathbf{0}$ on $\partial\Omega$ and $\|\boldsymbol{\tau}_h^{(2)}\|_{0,\Omega} \leq C(\|\boldsymbol{\tau}_h^{(1)}\|_{\mathbf{div}_{6/5;\Omega}} + \|\boldsymbol{\delta}_h\|_{0,\Omega})$.

The third step is to analyse the remaining inf-sup condition on χ. Let $\phi_0 \in \mathrm{P}_{k+1}(\Omega)$ such that $\phi_0 = 0$ on $\widetilde{\Gamma}$ and $\phi_0 > 0$ in $\overline{\Omega} \setminus \widetilde{\Gamma}$. Let $\mathcal{O}$ a convex polyhedron such that $\overline{\Omega} \subseteq \mathcal{O}$ and $\partial\Omega \cap \partial\mathcal{O} = \widetilde{\Gamma}$. Consider $\boldsymbol{w}_0$ the solution to the following problem

$$-\,\mathbf{div}(\boldsymbol{\varepsilon}(\boldsymbol{w}_0)) = \mathbf{E}_{0,\Omega}(\mathrm{curl}(\phi_0)) \quad \text{in} \quad \Omega\,, \quad \boldsymbol{\varepsilon}(\boldsymbol{w}_0)\boldsymbol{n} = \mathbf{0} \quad \text{on} \quad \widetilde{\Gamma} \quad \text{and} \quad \boldsymbol{w}_0 = 0 \quad \text{on} \quad \partial\Omega \setminus \widetilde{\Gamma}\,. \tag{5.16}$$

Next, we define $\overline{\boldsymbol{\tau}}_0 := -\boldsymbol{\varepsilon}(\boldsymbol{w}_0)|_\Omega \in \mathbb{H}(\mathbf{div}_{6/5};\Omega)$ and

$$\boldsymbol{\tau}_0 := \phi_0 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \boldsymbol{\Pi}_h^k(\overline{\boldsymbol{\tau}}_0) \in \mathbf{W}_h\,. \tag{5.17}$$

Observe that $\boldsymbol{\tau}_0$ is divergence-free, $\operatorname{asp}(\boldsymbol{\tau}_0) = 2\phi_0 > 0$ in Ω, and, by construction of ϕ_0, $\boldsymbol{\tau}_0\boldsymbol{n} = \mathbf{0}$ on $\widetilde{\Gamma}$. The next correction of $\boldsymbol{\tau}_h^{(1)}$ is given by

$$\boldsymbol{\tau}_h^{(3)} := \frac{\chi|\Omega|}{\displaystyle\int_\Omega \operatorname{asp}(\boldsymbol{\tau}_0)}\boldsymbol{\tau}_0 \in \mathbf{W}_h\,, \tag{5.18}$$

and satisfies

$$\|\boldsymbol{\tau}_h^{(3)}\|_{\mathbf{div}_{6/5};\Omega} \leq \overline{C}\big(\|\boldsymbol{\delta}_h\|_{0,\Omega} + \|\boldsymbol{v}_h\|_{0,6;\Omega}^5\big)\,, \tag{5.19}$$

where

$$\overline{C} := 2\frac{\|\boldsymbol{\tau}_0\|_{\mathbf{div}_{6/5};\Omega}}{\left|\displaystyle\int_\Omega \operatorname{asp}(\boldsymbol{\tau}_0)\right|}\max\{1,\widetilde{C}\}.$$

The final step is to define $\widehat{\boldsymbol{\tau}}_h := \lambda\boldsymbol{\tau}_h^{(1)} + \boldsymbol{\tau}_h^{(2)} + \boldsymbol{\tau}_h^{(3)}$, with $\lambda := \frac{|\Omega|\min\phi_0}{\int_\Omega \phi_0} > 0$. The previous analysis shows that $\widehat{\boldsymbol{\tau}}_h\boldsymbol{n} = \boldsymbol{\tau}_h^{(1)}\boldsymbol{n} + \boldsymbol{\tau}_h^{(2)}\boldsymbol{n} + \boldsymbol{\tau}_h^{(3)}\boldsymbol{n} = \mathbf{0}$ on $\widetilde{\Gamma}$, hence $\widehat{\boldsymbol{\tau}}_h \in \mathbf{W}_h$. Moreover, we notice that $\mathbf{B}(\widehat{\boldsymbol{\tau}}_h,(\boldsymbol{v}_h,\boldsymbol{\delta}_h)) \geq \lambda\left(\|\boldsymbol{v}_h\|_{0,6;\Omega}^6 + \|\boldsymbol{\delta}_h\|_{0,\Omega}^2\right)$ and that there exists a constant $\widehat{C}$, depending on $\widetilde{C}$, C and $\overline{C}$, such that

$$\|\widehat{\boldsymbol{\tau}}_h\|_{\mathbf{div}_{6/5};\Omega} \leq \widehat{C}\big(\|\boldsymbol{v}_h\|_{0,6;\Omega}^5 + \|\boldsymbol{\delta}_h\|_{0,\Omega}\big)\,.$$

Hence

$$\mathcal{G}(\boldsymbol{v}_h,\boldsymbol{\delta}_h) \geq \frac{\lambda}{\widehat{C}}\frac{\|\boldsymbol{v}_h\|_{0,6;\Omega}^6 + \|\boldsymbol{\delta}\|_{0,\Omega}^2}{\|\boldsymbol{v}_h\|_{0,6;\Omega}^5 + \|\boldsymbol{\delta}\|_{0,\Omega}} \geq \beta_{\mathrm{d}}\big(\|\boldsymbol{v}_h\|_{0,6;\Omega} + \|\boldsymbol{\delta}_h\|_{0,\Omega}\big)\,,$$

with $\beta_{\mathrm{d}} := \frac{\widehat{C}}{\lambda}\widetilde{\beta}$, where $\widetilde{\beta}$ is a small parameter ensuring that

$$\frac{x^6+y^2}{x^5+y} \geq \widetilde{\beta}(x+y) \qquad \forall\, x,y>0\,.$$

Collecting the ellipticity of $\mathbf{A}$ on $\mathbf{V}_h$ and the uniform inf-sup bound (5.12), we obtain the following theorem of existence, uniqueness and stability for the discrete solution operator $\mathbf{S}_h$.

Theorem 5.1. *Given $\phi_h \in \mathrm{H}_h^\varphi$, problem (5.5) has a unique solution $(\boldsymbol{\sigma}_h,(\boldsymbol{u}_h,\boldsymbol{\rho}_h)) \in \mathbb{K}_h \times \mathbf{Q}_h$, and hence one can define $\mathbf{S}_h(\phi_h) := (\boldsymbol{\sigma}_h,\boldsymbol{u}_h)$. Moreover there exist positive constants $C_{i,\mathrm{d}}$, $i \in \{1,2,3,4\}$, depending on α_{d}, $\alpha_{\mathbf{A}}$, β_{d}, $\|\mathbf{A}\|$ (cf. (4.2)), $|\Omega|$ and $\gamma(\lambda)$, and independent of h such that*

$$\|\boldsymbol{\sigma}_h\|_{\mathbf{div}_{6/5};\Omega} \leq C_{1,\mathrm{d}}\left(\|\phi_h\|_{\mathrm{Y}} + \|g\|_{\mathrm{W}_C}\right) + C_{2,\mathrm{d}}\,\|\varrho\boldsymbol{f}\|_{0,6/5;\Omega},$$
$$\|(\boldsymbol{u}_h,\boldsymbol{\rho}_h)\|_{\mathbf{Q}} \leq C_{3,\mathrm{d}}\left(\|\phi_h\|_{\mathrm{Y}} + \|g\|_{\mathrm{W}_C}\right) + C_{4,\mathrm{d}}\,\|\varrho\boldsymbol{f}\|_{0,6/5;\Omega}\,.$$

Proof. The proof follows similarly as the proof of Theorem 4.1 applied to the discrete context. □

On the other hand, for the well-definedness of S_h (cf. (5.6)), we apply [17, Theorem 3.5] and we notice that along with the property $\operatorname{div}\left(\mathbf{H}_h^\theta\right) \subseteq \mathrm{H}_h^\varphi$, and the properties of the thermal conductivity κ and the thermal resistance r (cf. (1.4a)), the $\widetilde{\mathbf{V}}_h$-ellipticity of $a_{\zeta_h}(\bullet,\bullet)$ (cf. (2.16c)) holds with the same constant, say $\widetilde{\alpha}_{\mathrm{d}} = \widetilde{\alpha}$. Now, we provide the discrete counterpart of the inf-sup condition of b, whose proof is constructed as in the continuous level, and utilise the commutative properties of the interpolation operator of the selected finite element subspace for the heat flux (cf. (5.2)).

Lemma 5.1. *There exists a positive constant β, depending only on $|\Omega|$ such that*

$$\sup_{\substack{\boldsymbol{\xi}_h\in \mathbf{H}_h^{\boldsymbol{\theta}}\\ \boldsymbol{\xi}_h\neq \mathbf{0}}} \frac{b(\boldsymbol{\xi}_h,\psi_h)}{\|\boldsymbol{\xi}_h\|_{\mathbf{X}}} \geq \beta_{\mathrm{d}}\,\|\psi_h\|_{\mathrm{Y}} \qquad \forall\,\psi_h\in \mathrm{H}_h^{\varphi}\,. \tag{5.20}$$

Proof. We follow a similar argument as in the first part of the proof of (5.12). In fact, given $\psi_h \in \mathrm{H}_h^{\varphi} \subseteq \mathrm{L}^3(\Omega)$, we define $\widetilde{\psi} := |\psi_h|\psi_h \in \mathrm{L}^{3/2}(\Omega)$. Now, we consider a polyhedron $\mathcal{O} \subseteq \mathbb{R}^d$ such that $\overline{\Omega} \subseteq \mathcal{O}$ and $\partial\Omega \cap \partial\mathcal{O} = \widetilde{\Gamma}$. Next, we consider $w \in \mathrm{W}^{1,3/2}(\mathcal{O})$ the unique solution to the problem

$$-\Delta w = \mathrm{E}_{0,\Omega}(\widetilde{\psi}) \quad \text{in} \quad \mathcal{O}, \quad \nabla w\cdot \boldsymbol{n} = 0 \quad \text{on} \quad \widetilde{\Gamma}, \quad w = 0 \quad \text{on} \quad \partial\mathcal{O}\setminus\widetilde{\Gamma}.$$

Now, we invoke [18, Corollary 3.7] for $p = 3/2$ and $k = 0$. Now the spectral conditions become

$$\lambda_1(e) = \frac{\pi}{2\omega_e} > \frac{2}{3} \quad \text{and} \quad \widetilde{\lambda}_1(v) := \min\{\lambda_1(v), 2\} > 0\,. \tag{5.21}$$

While the condition on the vertex of the polyhedra is satisfied, the first condition implies $\omega_e < \dfrac{3\pi}{4}$ in order to get $w \in \mathrm{W}^{2,3/2}(\mathcal{O})$ and ensuring the existence of a constant $C > 0$ depending only on Ω such that

$$\|w\|_{2,3/2;\mathcal{O}} \leq C\|\mathrm{E}_{0,\Omega}(\widetilde{\psi})\|_{0,3/2;\mathcal{O}} = C\|\widetilde{\psi}\|_{0,3/2;\Omega}\,.$$

Next, we define $\bar{\boldsymbol{\xi}} = -\nabla w|_{\Omega} \in \mathbf{W}^{1,3/2}(\Omega)$ with $\|\bar{\boldsymbol{\xi}}\|_{1,3/2;\Omega} \leq C\|\widetilde{\psi}\|_{0,3/2;\Omega}$. Finally, we set $\widetilde{\boldsymbol{\xi}}_h := \Pi_h(\bar{\boldsymbol{\xi}}) \in \mathbf{H}_h^{\boldsymbol{\theta}}$, where Π_h is the vectorial BDM interpolation operator. Thanks to its commutative properties, there holds

$$\mathrm{div}(\widetilde{\boldsymbol{\xi}}_h) = \mathcal{P}_h^k(\widetilde{\psi}) \quad \text{in} \quad \Omega, \quad \widetilde{\boldsymbol{\xi}}_h\cdot\boldsymbol{n}|_{\widetilde{\Gamma}} = \mathcal{Q}_h^k(\bar{\boldsymbol{\xi}}\cdot\boldsymbol{n}|_{\widetilde{\Gamma}}) = 0 \quad \text{on} \quad \widetilde{\Gamma} \quad \text{and} \quad \|\widetilde{\boldsymbol{\xi}}_h\|_{\mathbf{X}} \leq C\|\widetilde{\psi}\|_{0,3/2;\Omega}\,.$$

Hence, replacing $\widetilde{\boldsymbol{\xi}}_h$ in (5.20), the result follows with $\widetilde{\beta}_{\mathrm{d}} := C^{-1}$. □

Note that the geometric condition $\omega_e < \frac{3\pi}{4}$ is imposed solely on the auxiliary domain $\mathcal{O}$ used in the regularity argument. However, since $\widetilde{\Gamma} \subseteq \partial\mathcal{O}$, the portion $\widetilde{\Gamma}$ needs to be polyhedral, which is assumed throughout the paper. For the particular case of 3D domains, we additionally assume that Ω is such that there exists a polyhedron $\mathcal{O}$ satisfying the hypotheses described in the proof of Lemma 5.1.

The estimates established above now allow us to state the discrete well-definedness of the operator S_h (cf. (5.6)), or equivalently, the well–posedness of the decoupled problem (5.7) in the following result.

Theorem 5.2. *Let $\boldsymbol{w}_h \in \mathbf{H}_h^{\boldsymbol{u}}$ and $\boldsymbol{\zeta}_h \in \mathbb{K}_h$ with $\|\boldsymbol{w}_h\|_{0,6;\Omega} \leq \frac{\widetilde{\gamma}_{\mathrm{d}}}{2\|\kappa^{-1}\|_{0,\infty;\Omega}}$. Then, problem (5.7) has a unique solution $(\boldsymbol{\theta}_h, \varphi_h) \in \mathbf{H}_h^{\boldsymbol{\theta}} \times \mathrm{H}_h^{\varphi}$ and hence one can define $\mathrm{S}_h(\boldsymbol{\zeta}_h, \boldsymbol{w}_h) := \varphi_h$. Moreover, there exists a positive constant $C_{\mathrm{S,d}}$, depending on $\|a_{\boldsymbol{\zeta}_h}\|$, $\|c\|$ and $\widetilde{\gamma}_{\mathrm{d}}$ such that*

$$\|\mathrm{S}_h(\boldsymbol{\zeta}_h, \boldsymbol{w}_h)\|_{0,3;\Omega} := \|\varphi_h\|_{\mathrm{Y}} \leq \|(\boldsymbol{\theta}_h, \varphi_h)\|_{\mathbf{X}\times\mathrm{Y}} \leq C_{\mathrm{S,d}} \left\{ \|\varphi_0\|_{0,\Gamma_C} + \|m\|_{0,3/2;\Omega} \right\}\,.$$

Proof. The proof follows similarly as in Theorem 4.2, considering the discrete counterpart of Lemma 4.4 with constant $\widetilde{\gamma}_{\mathrm{d}}$ which follows from [17, Theorem 3.5] and applying the same procedure as in Theorem 4.2 incorporating the off-diagonal perturbation. Further details are omitted. □

Finally, we notice that the well-definedness of $\mathbf{T}_h$ (cf. (5.8)) arises from the one of $\mathbf{S}_h$ and S_h. In addition, similarly to the continuous case, we have the uniform boundedness of $\mathbf{T}_h$ by the data as follows

$$\|\mathbf{T}_h(\boldsymbol{\zeta}_h, \boldsymbol{w}_h)\|_{\mathbf{div}_{6/5}\times\mathbf{L}^6;\Omega} \leq C_{\mathbf{T},\mathrm{d}}\Big(\|\varphi_0\|_{0,\Gamma_C} + \|m\|_{0,3/2;\Omega} + \|g\|_{\mathrm{W}_C} + \|\varrho\boldsymbol{f}\|_{0,6/5;\Omega} \Big)\,,$$

for each $(\boldsymbol{\zeta}_h, \boldsymbol{w}_h) \in \mathbb{K}_h \times \mathbf{H}_h^{\boldsymbol{u}}$ satisfying $\|\boldsymbol{w}_h\|_{0,6;\Omega} \leq \dfrac{\widetilde{\gamma}_{\mathrm{d}}}{2\,\|\kappa^{-1}\|_{0,\infty;\Omega}}$, and where $C_{\mathbf{T},\mathrm{d}} := C_{\mathbf{S},\mathrm{d}}\max\{C_{\mathrm{S,d}}, 1\}$ is independent of h.

Having established, thanks to Theorems 5.1 and 5.2, that $\mathbf{S}_h$ (cf. (5.4)) and S_h (cf. (5.6)) are well-defined, and hence the well-definedness $\mathbf{T}_h$ (cf. (5.8)), we now aim at showing that $\mathbf{T}_h$ has a unique fixed point. More precisely, analogously to the continuous case, in what follows, we prove that $\mathbf{T}_h$ verifies the hypothesis of the Banach fixed-point theorem. Before proving the discrete analogue of Lemma 4.5, we define

$$\mathcal{K}_{R,h} := \{(\boldsymbol{\zeta}_h, \boldsymbol{w}_h) \in \mathbb{K}_h \times \mathbf{H}_h^{\boldsymbol{u}} : \|\boldsymbol{\zeta}_h\|_{\mathbf{div}_{6/5};\Omega} \le R \quad \text{and} \quad \|\boldsymbol{w}_h\|_{0,6;\Omega} \le R\}\,, \tag{5.22}$$

and hence, announce the desired result.

Lemma 5.2. *Assume that*

$$C_{\mathbf{T},\mathrm{d}}\Big(\|\varphi_0\|_{0,\Gamma_C} + \|m\|_{0,3/2;\Omega} + \|g\|_{\mathrm{W}_C} + \|\varrho\boldsymbol{f}\|_{0,6/5;\Omega}\Big) < R\,.$$

Then, $\mathbf{T}_h(\mathcal{K}_{R,h}) \subseteq \mathcal{K}_{R,h}$.

Extending the methodology of Lemmas 4.6, 4.7, and 4.8 to the discrete setting yields analogous Lipschitz continuity results for $\mathbf{S}_h$, S_h, and $\mathbf{T}_h$. The proofs follow the same structure as their continuous versions and are therefore omitted; we state the discrete analogues below.

Lemma 5.3. *There exists a positive constant* $L_{\mathbf{S},\mathrm{d}}$, *depending on* α_{d}, $\alpha_{\mathbf{A},\mathrm{d}}$, β_{d} *and* $\|\mathbf{A}\|$ *such that*

$$\|\mathbf{S}_h(\phi_h) - \mathbf{S}_h(\underset{\sim}{\phi}_h)\|_{\mathbf{div}_{6/5}\times\mathbf{L}^6;\Omega} \le L_{\mathbf{S},\mathrm{d}}\,\gamma(\lambda)\,\|\phi_h - \underset{\sim}{\phi}_h\|_{0,3;\Omega}\,.$$

Lemma 5.4. *There exists a positive constant* $L_{\mathrm{S},\mathrm{d}}$, *depending on* L_r, $C_{\mathrm{S},\mathrm{d}}$, $\|\kappa^{-1}\|_{0,\infty;\Omega}$ *and* $\widetilde{\gamma}_{\mathrm{d}}$ *such that*

$$\|\mathrm{S}_h(\boldsymbol{\zeta}_h, \boldsymbol{w}_h) - \mathrm{S}_h(\underset{\sim}{\boldsymbol{\zeta}}_h, \underset{\sim}{\boldsymbol{w}}_h)\|_{0,3;\Omega} \le L_{\mathrm{S},\mathrm{d}}\left(\|\varphi_0\|_{0,\Gamma_C} + \|m\|_{0,3/2;\Omega}\right)\|(\boldsymbol{\zeta}_h, \boldsymbol{w}_h) - (\underset{\sim}{\boldsymbol{\zeta}}_h, \underset{\sim}{\boldsymbol{w}}_h)\|_{\mathbf{div}_{6/5}\times\mathbf{L}^6;\Omega}\,.$$

Lemma 5.5. *There exists a positive constant* $L_{\mathbf{T},\mathrm{d}} := L_{\mathbf{S},\mathrm{d}}\,L_{\mathrm{S},\mathrm{d}}$ *such that*

$$\begin{aligned}&\|\mathbf{T}_h(\boldsymbol{\zeta}_h, \boldsymbol{w}_h) - \mathbf{T}_h(\underset{\sim}{\boldsymbol{\zeta}}_h, \underset{\sim}{\boldsymbol{w}}_h)\|_{\mathbf{div}_{6/5}\times\mathbf{L}^6;\Omega}\\ &\le L_{\mathbf{T},\mathrm{d}}\,\gamma(\lambda)\big(\|\varphi_0\|_{0,\Gamma_C} + \|m\|_{0,3/2;\Omega}\big)\|(\boldsymbol{\zeta}_h, \boldsymbol{w}_h) - (\underset{\sim}{\boldsymbol{\zeta}}_h, \underset{\sim}{\boldsymbol{w}}_h)\|_{\mathbf{div}_{6/5}\times\mathbf{L}^6;\Omega}\,.\end{aligned}$$

We end this section with the discrete solvability result for (5.9), equivalently, the one for (5.3).

Theorem 5.3. *Assume that in addition to the hypothesis of Lemma 5.2, the data satisfies*

$$L_{\mathbf{T},\mathrm{d}}\,\gamma(\lambda)\big(\|\varphi_0\|_{0,\Gamma_C} + \|m\|_{0,3/2;\Omega}\big) < 1\,.$$

Then, the operator $\mathbf{T}_h$ *(cf. (5.8)) has a unique fixed point* $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in \mathcal{K}_{R,\mathrm{d}}$ *(cf. (5.22)). Equivalently, the discrete coupled problem (5.3) has a unique solution* $(\boldsymbol{\sigma}_h, (\boldsymbol{u}_h, \boldsymbol{\rho}_h)) \in \mathbb{K}_h \times \mathbf{Q}_h$, *with* $\|\boldsymbol{u}_h\|_{0,6;\Omega} \le \frac{\widetilde{\gamma}_{\mathrm{d}}}{2\,\|\kappa^{-1}\|_{0,\infty;\Omega}}$. *Moreover, there exist positive constants* $C_{1,\mathrm{d}}$ *and* $C_{2,\mathrm{d}}$ *– independent of h – such that*

$$\begin{aligned}\|(\boldsymbol{\sigma}_h, (\boldsymbol{u}_h, \boldsymbol{\rho}_h))\|_{\mathbf{div}_{6/5}\times\mathbf{Q}} &\le C_{1,\mathrm{d}}\left(\|\varphi_0\|_{0,\Gamma_C} + \|m\|_{0,3/2;\Omega} + \|g\|_{\mathrm{W}_C} + \|\varrho\boldsymbol{f}\|_{0,6/5;\Omega}\right),\\ \|(\boldsymbol{\theta}_h, \varphi_h)\|_{\mathbf{X}\times\mathrm{Y}} &\le C_{2,\mathrm{d}}\left(\|\varphi_0\|_{0,\Gamma_C} + \|m\|_{0,3/2;\Omega}\right).\end{aligned}$$

Proof. The proof follows almost verbatim as the proof of Theorem 4.3, applied to the discrete setting analysed in this section. Further details are omitted. □

6. Error analysis

This section presents an a priori error analysis of the Galerkin scheme (5.3) using the finite element spaces defined in (5.1) and (5.2).

6.1. Falk estimates

Our first aim is to establish an abstract, Falk-type estimate for the global error

$$\|(\boldsymbol{\sigma},(\boldsymbol{u},\boldsymbol{\rho}))-(\boldsymbol{\sigma}_h,(\boldsymbol{u}_h,\boldsymbol{\rho}_h))\|_{\mathbf{div}_{6/5}\times\mathbf{Q}}+\|(\theta,\varphi)-(\theta_h,\varphi_h)\|_{\mathrm{X}\times\mathrm{Y}},$$

where $(\boldsymbol{\sigma},(\boldsymbol{u},\boldsymbol{\rho}))\in\mathbb{K}$ and $(\boldsymbol{\sigma}_h,(\boldsymbol{u}_h,\boldsymbol{\rho}_h))\in\mathbb{K}_h$ denote the unique solutions of (2.19) and (5.5), respectively, with $\phi=\varphi\in\mathrm{L}^3(\Omega)$ and $\phi_h=\varphi_h\in\mathbf{H}_h^{\varphi}$.

Henceforth, for any subspace or closed subset U_h of a Banach space $(U,\|\cdot\|_U)$, we define $\operatorname{dist}(u,U_h):=\inf_{u_h\in U_h}\|u-u_h\|_U$.

We begin by exploiting the discrete inf-sup condition (5.12) to derive a Falk estimate for the displacement and body rotation approximation.

Lemma 6.1. *The following error bound holds:*

$$\|(\boldsymbol{u},\rho)-(\boldsymbol{u}_h,\rho_h)\|_{\mathbf{Q}}\leq\left(1+\frac{\|\mathbf{B}\|}{\beta_{\mathrm{d}}}\right)\operatorname{dist}((\boldsymbol{u},\rho),\mathbf{Q}_h)+\frac{\|\mathbf{A}\|}{\beta_{\mathrm{d}}}\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathbf{div}_{6/5};\Omega}+\frac{1}{\beta_{\mathrm{d}}}\|\mathbf{F}_{\varphi}-\mathbf{F}_{\varphi_h}\|.$$

Proof. The proof adapts [39, Lemma 2.9]. Applying the discrete inf-sup condition (5.12) to $(\boldsymbol{v}_h,\boldsymbol{\delta}_h)-(\boldsymbol{u}_h,\boldsymbol{\rho}_h)\in\mathbf{Q}_h$ yields

$$\beta_{\mathrm{d}}\|(\boldsymbol{v}_h,\boldsymbol{\delta}_h)-(\boldsymbol{u}_h,\boldsymbol{\rho}_h)\|_{\mathbf{Q}}\leq\sup_{\substack{\boldsymbol{\tau}_h\in\mathbf{W}_h\\ \boldsymbol{\tau}_h\neq 0}}\frac{\mathbf{B}(\boldsymbol{\tau}_h,(\boldsymbol{v}_h,\boldsymbol{\delta}_h)-(\boldsymbol{u}_h,\boldsymbol{\rho}_h))}{\|\boldsymbol{\tau}_h\|_{\mathbf{div}_{6/5};\Omega}}. \tag{6.1}$$

From the first row of (5.5) and (4.10), we obtain

$$\mathbf{B}(\boldsymbol{\tau}_h,(\boldsymbol{v}_h,\boldsymbol{\delta}_h)-(\boldsymbol{u}_h,\boldsymbol{\rho}_h))\leq\mathbf{B}(\boldsymbol{\tau}_h,(\boldsymbol{v}_h,\boldsymbol{\delta}_h))+\mathbf{A}(\boldsymbol{\sigma}_h,\boldsymbol{\tau}_h)-\mathbf{F}_{\varphi_h}(\boldsymbol{\tau}_h). \tag{6.2}$$

Since $(\mathbf{A}(\boldsymbol{\sigma},\cdot)-\mathbf{B}(\cdot,(\boldsymbol{u},\boldsymbol{\rho}))-\mathbf{F}_{\varphi})(\boldsymbol{\zeta}-\boldsymbol{\sigma})\geq 0$ for all $\boldsymbol{\zeta}\in\mathbb{K}$, Lemma 3.1 implies that this functional vanishes on $\mathbf{W}$, and hence on $\mathbf{W}_h\subseteq\mathbf{W}$. Thus, (6.2) becomes

$$\mathbf{B}(\boldsymbol{\tau}_h,(\boldsymbol{v}_h,\boldsymbol{\delta}_h)-(\boldsymbol{u}_h,\boldsymbol{\rho}_h))\leq\mathbf{B}(\boldsymbol{\tau}_h,(\boldsymbol{u},\rho)-(\boldsymbol{v}_h,\boldsymbol{\delta}_h))+\mathbf{A}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\boldsymbol{\tau}_h)-(\mathbf{F}_{\boldsymbol{\varphi}}-\mathbf{F}_{\boldsymbol{\varphi}_h})(\boldsymbol{\tau}_h).$$

Applying boundedness properties and substituting into (6.1) gives

$$\beta_{\mathrm{d}}\|(\boldsymbol{v}_h,\boldsymbol{\delta}_h)-(\boldsymbol{u}_h,\boldsymbol{\rho}_h)\|_{\mathbf{Q}}\leq\|\mathbf{A}\|\|\boldsymbol{\sigma}_h-\boldsymbol{\sigma}\|_{\mathbf{div}_{6/5};\Omega}+\|\mathbf{B}\|\|(\boldsymbol{v}_h,\boldsymbol{\delta}_h)-(\boldsymbol{u},\rho)\|_{\mathbf{Q}}+\|\mathbf{F}_{\varphi}-\mathbf{F}_{\varphi_h}\|,$$

for all $(\boldsymbol{v}_h,\boldsymbol{\delta}_h)\in\mathbf{Q}_h$. The triangle inequality completes the proof. □

Next we establish an a priori estimate for the stress approximation, modifying [39, Lemma 2.7] to account for the discrepancy between discrete and continuous right-hand sides.

Lemma 6.2. *Define the discrete space*

$$\mathbf{V}_h^{\mathbf{G}}:=\left\{\boldsymbol{\tau}_h\in\mathbb{K}_h\quad:\quad\mathbf{B}(\boldsymbol{\tau}_h,(\boldsymbol{v}_h,\boldsymbol{\delta}_h))=\mathbf{G}(\boldsymbol{v}_h,\boldsymbol{\delta}_h)\quad\forall\,(\boldsymbol{v}_h,\boldsymbol{\delta}_h)\in\mathbf{Q}_h\right\}.$$

Then there exists $C>0$ such that for all $\boldsymbol{\tau}_h\in\mathbf{V}_h^{\mathbf{G}}$ and $(\boldsymbol{v}_h,\boldsymbol{\delta}_h)\in\mathbf{Q}_h$,

$$\|\boldsymbol{\sigma}_h-\boldsymbol{\tau}_h\|^2_{\mathbf{div}_{6/5};\Omega}\leq C\left(A_1(\boldsymbol{\tau}_h)+\|\mathbf{F}_{\varphi}-\mathbf{F}_{\varphi_h}\|^2_{\mathrm{X}\times\mathrm{Y}}+\|(\boldsymbol{u},\rho)-(\boldsymbol{v}_h,\boldsymbol{\delta}_h)\|^2_{\mathbf{Q}}+\|\boldsymbol{\sigma}-\boldsymbol{\tau}_h\|^2_{\mathbf{div}_{6/5};\Omega}\right),$$

where

$$A_1(\boldsymbol{\tau}_h):=\mathbf{A}(\boldsymbol{\sigma},\boldsymbol{\tau}_h-\boldsymbol{\sigma})+\mathbf{B}(\boldsymbol{\tau}_h-\boldsymbol{\sigma},(\boldsymbol{u},\rho))-\mathbf{F}(\boldsymbol{\tau}_h-\boldsymbol{\sigma})\quad\forall\,\boldsymbol{\tau}_h\in\mathbb{K}_h.$$

Proof. For any $\boldsymbol{\tau}_h \in \mathbb{K}_h$, we decompose

$$\mathbf{A}(\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h, \boldsymbol{\sigma}_h - \boldsymbol{\tau}_h) = \mathbf{A}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}, \boldsymbol{\sigma}_h - \boldsymbol{\sigma}) + \mathbf{A}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}, \boldsymbol{\sigma} - \boldsymbol{\tau}_h) + \mathbf{A}(\boldsymbol{\sigma} - \boldsymbol{\tau}_h, \boldsymbol{\sigma}_h - \boldsymbol{\tau}_h). \tag{6.3}$$

Expanding the first term gives

$$\mathbf{A}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h) = \mathbf{A}(\boldsymbol{\sigma}_h, \boldsymbol{\sigma}_h) + \mathbf{A}(\boldsymbol{\sigma}, \boldsymbol{\sigma}) - \mathbf{A}(\boldsymbol{\sigma}_h, \boldsymbol{\sigma}) - \mathbf{A}(\boldsymbol{\sigma}, \boldsymbol{\sigma}_h). \tag{6.4}$$

From (2.15) and (5.3), we have

$$\begin{aligned}
\mathbf{A}(\boldsymbol{\sigma}, \boldsymbol{\sigma}) &\le \mathbf{F}(\boldsymbol{\sigma} - \boldsymbol{\tau}) - \mathbf{B}(\boldsymbol{\sigma} - \boldsymbol{\tau}, (\boldsymbol{u}, \boldsymbol{\rho})) + \mathbf{A}(\boldsymbol{\sigma}, \boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in \mathbb{K}, \\
\mathbf{A}(\boldsymbol{\sigma}_h, \boldsymbol{\sigma}_h) &\le \mathbf{F}_h(\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h) - \mathbf{B}(\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h, (\boldsymbol{u}_h, \boldsymbol{\rho}_h)) + \mathbf{A}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) \quad \forall\, \boldsymbol{\tau}_h \in \mathbb{K}_h.
\end{aligned}$$

Using the conformity $\mathbb{K}_h \subseteq \mathbb{K}$, we proceed to take $\boldsymbol{\tau} = \boldsymbol{\sigma}_h$. Noting that $\boldsymbol{\sigma} - \boldsymbol{\sigma}_h = (\boldsymbol{\sigma} - \boldsymbol{\tau}_h) + (\boldsymbol{\tau}_h - \boldsymbol{\sigma}_h)$ and $\mathbf{B}(\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h, (\boldsymbol{u}, \boldsymbol{\rho}) - (\boldsymbol{u}_h, \boldsymbol{\rho}_h)) = \mathbf{B}(\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h, (\boldsymbol{u}, \boldsymbol{\rho}) - (\boldsymbol{v}_h, \boldsymbol{\delta}_h))$ for every $(\boldsymbol{v}_h, \boldsymbol{\delta}_h) \in \mathbf{Q}_h$, we derive from (6.3):

$$\begin{aligned}
\mathbf{A}(\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h, \boldsymbol{\sigma}_h - \boldsymbol{\tau}_h) \le{}& A_1(\boldsymbol{\tau}_h) + (\mathbf{F}_\varphi - \mathbf{F}_{\varphi_h})(\boldsymbol{\tau}_h - \boldsymbol{\sigma}_h) \\
&+ \mathbf{B}(\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h, (\boldsymbol{u}, \boldsymbol{\rho}) - (\boldsymbol{v}_h, \boldsymbol{\delta}_h))) + \mathbf{A}(\boldsymbol{\sigma} - \boldsymbol{\tau}_h, \boldsymbol{\sigma}_h - \boldsymbol{\tau}_h).
\end{aligned}$$

For $\boldsymbol{\tau}_h \in \mathbf{V}_h^{\mathbf{G}}$, we have $\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h \in \mathbf{V}_h$ (cf. (5.10)). Using the $\mathbf{V}_h$-ellipticity of $\mathbf{A}(\bullet, \bullet)$ and Young's inequality yields

$$\begin{aligned}
(\alpha_{\mathrm{d}} - \varepsilon_1 - \varepsilon_2 \|\mathbf{B}\| - \varepsilon_3 \|\mathbf{A}\|) \|\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h\|^2_{\mathbf{div}_{6/5};\Omega} \le{}& A_1(\boldsymbol{\tau}_h) + \frac{1}{4\varepsilon_1} \|\mathbf{F}_\varphi - \mathbf{F}_{\varphi_h}\|^2 \\
&+ \frac{\|\mathbf{B}\|}{4\varepsilon_2} \|(\boldsymbol{u}, \boldsymbol{\rho}) - (\boldsymbol{v}_h, \boldsymbol{\delta}_h)\|^2_{\mathbf{Q}} + \frac{\|\mathbf{A}\|}{4\varepsilon_3} \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|^2_{\mathbf{div}_{6/5};\Omega}.
\end{aligned}$$

Finally, choosing $\varepsilon_1 = \alpha_{\mathrm{d}}/6$, $\varepsilon_2 = \alpha_{\mathrm{d}}/(6\|\mathbf{B}\|)$, and $\varepsilon_3 = \alpha_{\mathrm{d}}/(6\|\mathbf{A}\|)$ produces

$$\|\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h\|^2_{\mathbf{div}_{6/5};\Omega} \le \frac{2}{\alpha_{\mathrm{d}}} A_1(\boldsymbol{\tau}_h) + \frac{3}{\alpha_{\mathrm{d}}^2} \|\mathbf{F}_\varphi - \mathbf{F}_{\varphi_h}\|^2 + \frac{3\|\mathbf{B}\|^2}{\alpha_{\mathrm{d}}^2} \|(\boldsymbol{u}, \boldsymbol{\rho}) - (\boldsymbol{v}_h, \boldsymbol{\delta}_h)\|^2_{\mathbf{Q}} + \frac{3\|\mathbf{A}\|^2}{\alpha_{\mathrm{d}}^2} \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|^2_{\mathbf{div}_{6/5};\Omega}, \tag{6.5}$$

where the constant C emerges from the coefficients in (6.5). ☐

Now, we are in position to state the error estimate for $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div}_{6/5};\Omega}$.

Lemma 6.3. *Let* $(\boldsymbol{\sigma}, (\boldsymbol{u}, \boldsymbol{\rho})) \in \mathbb{H}_N(\mathbf{div}_{6/5}; \Omega) \times \mathbf{Q}$, $(\boldsymbol{\theta}, \varphi) \in \mathbf{X} \times \mathrm{Y}$ *and* $(\boldsymbol{\sigma}_h, (\boldsymbol{u}_h, \boldsymbol{\rho}_h)) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{Q}_h$, $(\boldsymbol{\theta}, \varphi_h) \in \mathbf{H}_h^{\boldsymbol{\theta}} \times \mathrm{H}_h^{\varphi}$ *solutions of* (2.15) *and* (5.3) *respectively. Then, there holds*

$$\begin{aligned}
&\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div}_{6/5};\Omega} \\
&\lesssim \inf_{\boldsymbol{\tau}_h \in \mathbb{K}_h} \left(A_1(\boldsymbol{\tau}_h)^{1/2} + \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{\mathbf{div}_{6/5};\Omega} \right) + \frac{3\|\mathbf{B}\|^2}{\alpha_{\mathrm{d}}^2} \operatorname{dist}((\boldsymbol{u}, \boldsymbol{\rho}), \mathbf{Q}_h) + \frac{3\, C_{\mathbf{F}}^2 \gamma(\lambda)}{\alpha_{\mathrm{d}}^2} \|(\boldsymbol{\theta}, \varphi) - (\boldsymbol{\theta}_h, \varphi_h)\|_{\mathbf{X} \times \mathrm{Y}}.
\end{aligned}$$

Proof. The bound for $\|\mathbf{F}_\varphi - \mathbf{F}_{\varphi_h}\|$ follows directly from (4.8). To complete the estimate, thanks to the conformity of the scheme we can invoke [39, Lemma 2.4] and ensure the existence of a operator $\boldsymbol{J}_h : \mathbb{K}_h \to \mathbf{V}_h^{\mathbf{G}}$ such that

$$\|\boldsymbol{\sigma} - \boldsymbol{J}_h(\boldsymbol{\tau}_h)\|_{\mathbf{div}_{6/5};\Omega} \le C \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{\mathbf{div}_{6/5};\Omega} \quad \text{and} \quad A_1(\boldsymbol{J}_h(\boldsymbol{\tau}_h)) = A_1(\boldsymbol{\tau}_h) \quad \forall \boldsymbol{\tau}_h \in \mathbb{K}_h\,, \tag{6.6}$$

where C depends on β_{d} and $\|\mathbf{B}\|$. Therefore, applying Cauchy–Schwarz inequality to $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div}_{6/5};\Omega}$ and plugging (6.6) into (6.5) we get the result. ☐

The thermal variables analysis begins by rewriting (2.21) and (5.7) as

$$\begin{aligned}
\mathcal{A}((\boldsymbol{\theta}, \varphi), (\boldsymbol{\xi}, \psi)) &= \mathcal{F}((\boldsymbol{\xi}, \psi)) \quad && \forall (\boldsymbol{\xi}, \psi) \in \mathbf{X} \times \mathrm{Y}, \\
\mathcal{A}_h((\boldsymbol{\theta}_h, \varphi_h), (\boldsymbol{\xi}_h, \psi_h)) &= \mathcal{F}_h((\boldsymbol{\xi}_h, \psi_h)) \quad && \forall (\boldsymbol{\xi}_h, \psi_h) \in \mathbf{H}_h^{\boldsymbol{\theta}} \times \mathrm{H}_h^{\varphi},
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{A}((\chi,\phi),(\xi,\psi)) &:= A_{\chi}((\chi,\phi),(\xi,\psi)) - \int_\Omega (\boldsymbol{u}\cdot\kappa^{-1}\chi)\psi,\\
\mathcal{A}_h((\chi_h,\phi_h),(\xi_h,\psi_h)) &:= A_{\chi_h}((\chi_h,\phi_h),(\xi_h,\psi_h)) - \int_\Omega (\boldsymbol{u}_h\cdot\kappa^{-1}\chi_h)\psi_h,\\
\mathcal{F}((\xi,\psi)) &:= F(\xi) + G(\psi), \qquad \text{and} \quad \mathcal{F}_h := \mathcal{F}|_{\mathbf{H}_h^{\theta}\times \mathrm{H}_h^{\varphi}},
\end{aligned}$$

for all $(\boldsymbol{\theta},\varphi),(\boldsymbol{\xi},\psi)\in \mathbf{X}\times\mathrm{Y}$ and $(\boldsymbol{\theta}_h,\varphi_h),(\boldsymbol{\xi}_h,\psi_h)\in \mathbf{H}_h^{\theta}\times\mathrm{H}_h^{\varphi}$.

Applying [36, Theorem 11.1] (see also [10, Lemma 5.1]) yields the estimate

$$\begin{aligned}
\|(\boldsymbol{\theta},\varphi)-(\boldsymbol{\theta}_h,\varphi_h)\|_{\mathbf{X}\times\mathrm{Y}} \le \left(1+\frac{4\|\mathcal{A}\|}{\widetilde{\gamma}_{\mathtt{d}}}+\frac{2\|\mathcal{A}_h\|}{\widetilde{\gamma}_{\mathtt{d}}}\right)\operatorname{dist}((\boldsymbol{\theta},\varphi),\mathbf{H}_h^{\theta}\times\mathrm{H}_h^{\varphi})\\
+\frac{2}{\widetilde{\gamma}_{\mathtt{d}}}\sup_{(\boldsymbol{\theta}_h,\psi_h)\in\mathbf{H}_h^{\theta}\times\mathrm{H}_h^{\varphi}\setminus\{\mathbf{0}\}}\frac{(a_{\boldsymbol{\sigma}}-a_{\boldsymbol{\sigma}_h})(\boldsymbol{\theta}_h,\boldsymbol{\xi}_h)-\int_\Omega((\boldsymbol{u}-\boldsymbol{u}_h)\cdot\kappa^{-1}\boldsymbol{\theta}_h)\psi_h}{\|(\boldsymbol{\xi}_h,\psi_h)\|_{\mathbf{X}\times\mathrm{Y}}}.
\end{aligned} \tag{6.7}$$

Here $\|\mathcal{A}\|$ scales with $\|a_{\boldsymbol{\sigma}}\| = \|\kappa^{-1}\|_{0,\infty;\Omega}+r_1$, $\|b\|=1$, $\|c\| = \ell|\Omega|^{1/6}$, and $\|\boldsymbol{u}\|_{0,6;\Omega}\le \frac{\widetilde{\gamma}}{2\|\kappa^{-1}\|_{0,\infty;\Omega}}$ (cf. Theorem 4.3). Similarly, $\|\mathcal{A}_h\|$ scales with the same quantities except that $\|\boldsymbol{u}\|_{0,6;\Omega}$ is replaced by $\|\boldsymbol{u}_h\|_{0,6;\Omega}$, which is bounded above by $\frac{\widetilde{\gamma}_{\mathtt{d}}}{2\|\kappa^{-1}\|_{0,\infty;\Omega}}$ (cf. Theorem 5.3).

The error estimate associated with the thermal variables follows from bounding the supremum term in (6.7) using the boundedness of the involved forms, the estimate (4.17), and the bound on $\|\boldsymbol{\theta}_h\|_{\mathbf{X}}$ from Theorem 5.3. We state this result in the following lemma.

Lemma 6.4. *There exist positive constants C_1, C_0 independent of h such that*

$$\begin{aligned}
\|(\boldsymbol{\theta},\varphi)-(\boldsymbol{\theta}_h,\varphi_h)\|_{\mathbf{X}\times\mathrm{Y}} \le C_1\operatorname{dist}((\boldsymbol{\theta},\varphi),\mathbf{H}_h^{\theta}\times\mathrm{H}_h^{\varphi})\\
+ C_0\big(\|\varphi_0\|_{0,\Gamma_C}+\|m\|_{0,3/2;\Omega}\big)\left(\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathbf{div}_{6/5};\Omega}+\|(\boldsymbol{u},\rho)-(\boldsymbol{u}_h,\rho_h)\|_{\mathbf{Q}}\right).
\end{aligned}$$

Proof. The proof only concentrates on C_0 since we addressed the dependence of C_1 in the analysis below (6.7). We notice that we use (4.17) and (2.12) to get

$$\begin{aligned}
&\left|(a_{\boldsymbol{\sigma}}-a_{\boldsymbol{\sigma}_h})(\boldsymbol{\theta}_h,\boldsymbol{\xi}_h)-\int_\Omega((\boldsymbol{u}-\boldsymbol{u}_h)\cdot\kappa^{-1}\boldsymbol{\theta}_h)\psi_h\right|\\
&\quad\le C_{\mathrm{S},\mathtt{d}}\max\{L_r,\|\kappa^{-1}\|_{0,\infty;\Omega}\}\left\{\|\varphi_0\|_{0,\Gamma_C}+\|m\|_{0,3/2;\Omega}\right\}\left\{\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathbf{div}_{6/5};\Omega}+\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,6;\Omega}\right\}.
\end{aligned}$$

Hence, $C_0 := C_{\mathrm{S},\mathtt{d}}\max\{L_r,\|\kappa^{-1}\|_{0,\infty;\Omega}\}$, finishing the proof. ☐

Finally, we can derive a Falk–type estimate in the theorem below.

Theorem 6.1. *Assume that the data satisfy*

$$C_0\,C_{\mathbf{F}}\gamma(\lambda)\big(\|m\|_{0,3/2;\Omega}+\|\varphi_0\|_{0,\Gamma_C}\big)\left(\frac{3^{1/2}}{\alpha_{\mathtt{d}}}\left(1+\frac{\|\mathbf{A}\|}{\beta_{\mathtt{d}}}\right)+\frac{1}{\beta_{\mathtt{d}}}\right)<1. \tag{6.8}$$

Then, there exists a positive constant C, independent of h such that

$$\begin{aligned}
&\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathbf{div}_{6/5};\Omega}+\|(\boldsymbol{u},\rho)-(\boldsymbol{u}_h,\rho_h)\|_{\mathbf{Q}}+\|(\boldsymbol{\theta},\varphi)-(\boldsymbol{\theta}_h,\varphi_h)\|_{\mathbf{X}\times\mathrm{Y}}\\
&\quad\le C\Bigg(\operatorname{dist}((\boldsymbol{u},\rho),\mathbf{Q}_h)+\operatorname{dist}\big((\boldsymbol{\theta},\varphi),\mathbf{H}_h^{\theta}\times\mathrm{H}_h^{\varphi}\big)+\inf_{\boldsymbol{\tau}_h\in\mathbb{K}_h}\Big(\|\boldsymbol{\sigma}-\boldsymbol{\tau}_h\|_{\mathbf{div}_{6/5};\Omega}+A_1(\boldsymbol{\tau}_h)^{1/2}\Big)\Bigg).
\end{aligned}$$

Proof. We first gather the results given in Lemmas 6.1, 6.2 and 6.4. For sake of simplicity, we define the variables

$$S_1 := \|(\boldsymbol{u},\boldsymbol{\rho}) - (\boldsymbol{u}_h,\boldsymbol{\rho}_h)\|_{\mathbf{Q}}\,, \qquad S_2 := \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathbf{div}_{6/5};\Omega} \quad \text{and} \quad S_3 := \|(\boldsymbol{\theta},\varphi)-(\boldsymbol{\theta}_h,\varphi_h)\|_{\mathbf{X}\times\mathrm{Y}}\,,$$

$$a_1 := \left(1+\frac{\|\mathbf{B}\|}{\beta_{\mathrm{d}}}\right)\operatorname{dist}(u,\mathbf{Q}_h)\,, \qquad a_3 := C_1\,\operatorname{dist}\left((\boldsymbol{\theta},\varphi),\mathbf{H}_h^{\boldsymbol{\theta}}\times\mathrm{H}_h^{\varphi}\right) \quad \text{and}$$

$$a_2 := \left(\frac{2}{\alpha_{\mathrm{d}}}A_1(\boldsymbol{\tau}_h) + \left(2+\frac{3\|\mathbf{A}\|^2}{\alpha_{\mathrm{d}}^2}\right)\left(1+\frac{\|\mathbf{B}\|}{\beta_{\mathrm{d}}}\right)^2\|\boldsymbol{\sigma}-\boldsymbol{\tau}_h\|^2_{\mathbf{div}_{6/5};\Omega} + \frac{3\|\mathbf{B}\|^2}{\alpha_{\mathrm{d}}^2}\operatorname{dist}((\boldsymbol{u},\boldsymbol{\rho}),\mathbf{Q}_h)^2\right)^{1/2}\,,$$

and as a consequence we have the following system of inequalities

$$\begin{aligned} S_1 &\leq a_1 + b_1\,S_2 + b_2\,S_3 && \text{with } b_1 = \frac{\|\mathbf{A}\|}{\beta_{\mathrm{d}}} \quad \text{and} \quad b_2 = \frac{C_{\mathbf{F}}\gamma(\lambda)}{\beta_{\mathrm{d}}}, \\ S_2 &\leq a_2 + b_3\,S_3 && \text{with } b_3 = \frac{3^{1/2}C_{\mathbf{F}}\gamma(\lambda)}{\alpha_{\mathrm{d}}}, \\ S_3 &\leq a_3 + b_4\,S_2 + b_4\,S_1 && \text{with } b_4 = C_0\big(\|\varphi_0\|_{0,\Gamma_C} + \|m\|_{0,3/2;\Omega}\big)\,. \end{aligned} \tag{6.9}$$

Using the second inequality of (6.9) combined with the first inequality, and then using both in the last inequality results in

$$S_3 \leq a_3 + b_4(a_1 + a_2(1+b_1)) + b_4(b_3(1+b_1)+b_2)S_3\,,$$

where we impose that $b_4(b_3(1+b_1)+b_2) < 1$, which is equivalent to (6.8). After that we use back substitution to bound S_j in terms of a_j times positive combinations of b_j, $j \in \{1,2,3\}$, finishing the proof. ☐

6.2. Convergence rates

We now turn our attention to the convergence rates of the Galerkin scheme. First of all we will provide approximation properties to the displacement, body rotation, temperature and heat flux in the following list, where we resort to the approximation properties of the finite element spaces defined in Section 5.1.

($\mathbf{AP}_{\boldsymbol{u}}$) There exists a positive constant C, independent of h, such that for each $s \in [0,k+1]$ and for each $\boldsymbol{v} \in \mathbf{W}^{s,6}(\Omega)$, there holds

$$\operatorname{dist}(\boldsymbol{v},\mathbf{H}_h^{\boldsymbol{u}}) \leq C\,h^s\,\|\boldsymbol{v}\|_{s,6;\Omega}\,,$$

($\mathbf{AP}_{\rho}$) There exists a positive constant C, independent of h, such that for each $s \in [0,k+1]$ and for each $\boldsymbol{\delta} \in \mathbb{H}^s(\Omega) \cap \mathbb{L}^2_{\text{skew}}(\Omega)$, there holds

$$\operatorname{dist}(\boldsymbol{\delta},\mathbb{H}_h^{\rho}) \leq C\,h^s\,\|\boldsymbol{\delta}\|_{s,\Omega}\,,$$

($\mathbf{AP}_{\theta}$) There exists a positive constant C, independent of h, such that for each $s \in (0,k+1]$ and for each $\boldsymbol{\xi} \in \mathbf{H}^s(\Omega) \cap \mathbf{H}_N(\mathbf{div}_{3/2};\Omega)$, with $\operatorname{div}(\boldsymbol{\xi}) \in \mathrm{W}^{s,3/2}(\Omega)$, there holds

$$\operatorname{dist}(\boldsymbol{\xi},\mathbf{H}_h^{\boldsymbol{\theta}}) \leq C\,h^s\left(\|\boldsymbol{\xi}\|_{s,\Omega} + \|\operatorname{div}(\boldsymbol{\xi})\|_{s,3/2;\Omega}\right), \quad \text{and}$$

($\mathbf{AP}_{\varphi}$) There exists a positive constant C, independent of h, such that for each $s \in [0,k+1]$ and for each $\psi \in \mathrm{W}^{s,3}(\Omega)$, there holds

$$\operatorname{dist}(\psi,\mathrm{H}_h^{\varphi}) \leq C\,h^s\,\|\psi\|_{s,3;\Omega}\,.$$

Since the analysis of the convergence rate of the stress requires approximation on the contact boundary Γ_C, we additionally assume that Γ_C is a flat portion of $\partial\Omega$. Under this geometric assumption, the spaces $\mathrm{H}^s(\Gamma_C)$ are well-defined for all $s \geq 0$. Hence, we get the approximation property of the $\mathrm{L}^2(\Gamma_C)$ projection operator.

($\mathbf{AP}_{\Gamma_C}$) There exists a positive constant C, independent of h, such that for each $s \in (0, k+1]$ and for each $w \in \mathrm{H}^{s+1/2}(\Gamma_C)$, there holds

$$\|\boldsymbol{w} - \mathcal{P}_h^k \boldsymbol{w}\|_{1/2;\Gamma_C} \le C\, h^s\, |\boldsymbol{w}|_{s+1/2;\Gamma_C}\,,$$

where Γ_C is a flat portion of $\partial\Omega$ and $\mathcal{P}_h^k$ is the $\mathrm{L}^2(\Gamma_C)$-projector onto $\mathrm{P}_k(\mathcal{T}_h^\partial)$.

Now we want to analyse the convergence rate of the stress, which is given by

$$\mathcal{I} := \inf_{\boldsymbol{\tau}_h \in \mathbb{K}_h} \left(\|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{\mathbf{div}_{6/5};\Omega} + A_1(\boldsymbol{\tau}_h)^{1/2} \right).$$

We begin by noticing that using integration by parts along with the first equation of the strong form (1.8), $A_1(\boldsymbol{\tau}_h)$ becomes

$$A_1(\boldsymbol{\tau}_h) = \langle (\boldsymbol{\tau}_h - \boldsymbol{\sigma})\boldsymbol{n}, \boldsymbol{u} \rangle_{\Gamma_C} \qquad \forall\, \boldsymbol{\tau}_h \in \mathbb{K}_h\,. \tag{6.10}$$

Assume that $\boldsymbol{\sigma} \in \mathbb{H}^{\ell+1/2}(\Omega) \cap \mathbb{H}_N(\mathbf{div}_{6/5};\Omega)$ for some $\ell \in (0, k+1]$, and that $\boldsymbol{u}|_{\Gamma_C} \in \mathbf{H}^{s+1/2}(\Gamma_C)$ for some $s \in (0, k+1]$. Under these conditions we can identify the right-hand side of (6.10) with the $\mathbf{L}^2(\Gamma_C)$ inner product. In turn, we take $\boldsymbol{\tau}_h := \boldsymbol{\Pi}_h^k \boldsymbol{\sigma} \in \mathbb{K}_h$, where $\boldsymbol{\Pi}_h^k$ stands for the global BDM interpolator, to get

$$\mathcal{I} \le \|\boldsymbol{\sigma} - \boldsymbol{\Pi}_h^k \boldsymbol{\sigma}\|_{\mathbf{div}_{6/5};\Omega} + \left(\int_{\Gamma_C} \boldsymbol{u} \cdot \big((\boldsymbol{\Pi}_h^k \boldsymbol{\sigma} - \boldsymbol{\sigma})\boldsymbol{n} \big) \right)^{1/2}. \tag{6.11}$$

At this point we remark that the degrees of freedom on the faces $F \in \partial\mathcal{T}_h$ of the BDM finite element are defined, for every $\boldsymbol{\tau} \in \mathbb{H}^1(\Omega)$, as

$$\int_F \boldsymbol{\mu}_h \cdot \boldsymbol{\tau}\boldsymbol{n} = \int_F \boldsymbol{\mu}_h \cdot (\boldsymbol{\Pi}_h^k \boldsymbol{\tau})\boldsymbol{n} \qquad \forall \boldsymbol{\mu}_h \in \mathbf{P}_k(\Omega)\,, \tag{6.12}$$

and hence, considering $\boldsymbol{\mu}_h := \mathcal{P}_h^k \boldsymbol{u}|_{\Gamma_C}$ and plugging (6.12) back into (6.11), we observe that

$$\int_{\Gamma_C} \boldsymbol{u} \cdot \big((\boldsymbol{\Pi}_h^k \boldsymbol{\sigma} - \boldsymbol{\sigma})\boldsymbol{n} \big) = \int_{\Gamma_C} \big(\boldsymbol{u} - \mathcal{P}_h^k \boldsymbol{u} \big) \cdot \big((\boldsymbol{\Pi}_h^k \boldsymbol{\sigma} - \boldsymbol{\sigma})\boldsymbol{n} \big)\,. \tag{6.13}$$

We bound the right-hand side by duality between $\mathbf{H}^{1/2}(\Gamma_C)$ and $\mathbf{H}^{-1/2}(\Gamma_C)$:

$$\int_{\Gamma_C} \boldsymbol{u} \cdot \big((\boldsymbol{\Pi}_h^k \boldsymbol{\sigma} - \boldsymbol{\sigma})\boldsymbol{n} \big) \le \|\boldsymbol{u} - \mathcal{P}_h^k \boldsymbol{u}\|_{1/2;\Gamma_C}\, \|(\boldsymbol{\Pi}_h^k \boldsymbol{\sigma} - \boldsymbol{\sigma})\boldsymbol{n}\|_{-1/2;\Gamma_C}\,.$$

Applying ($\mathbf{AP}_{\Gamma_C}$) to the first factor gives

$$\|\boldsymbol{u} - \mathcal{P}_h^k \boldsymbol{u}\|_{1/2,\Gamma_C} \le C h^s\, |\boldsymbol{u}|_{s+1/2,\Gamma_C}\,.$$

For the second factor, the continuity of the normal trace operator from $\mathbb{H}(\mathbf{div}_{6/5};\Omega)$ into $\mathbf{H}^{-1/2}(\Gamma_C)$ holds for Lipschitz domains, and combined with the approximation property of the BDM interpolator yields

$$\|(\boldsymbol{\Pi}_h^k \boldsymbol{\sigma} - \boldsymbol{\sigma})\boldsymbol{n}\|_{-1/2;\Gamma_C} \le C \|\boldsymbol{\Pi}_h^k \boldsymbol{\sigma} - \boldsymbol{\sigma}\|_{\mathbf{div}_{6/5};\Omega} \le C h^{\ell}\, \|\boldsymbol{\sigma}\|_{\ell+1/2;\Omega}\,.$$

Combining the two estimates we obtain

$$\int_{\Gamma_C} \boldsymbol{u} \cdot \big((\boldsymbol{\Pi}_h^k \boldsymbol{\sigma} - \boldsymbol{\sigma})\boldsymbol{n} \big) \le C h^{s+\ell}\, |\boldsymbol{u}|_{s+1/2;\Gamma_C}\, \|\boldsymbol{\sigma}\|_{\ell+1/2;\Omega}\,.$$

In addition, the first term in the definition of $\mathcal{I}$ is handled by the usual approximation property of the BDM finite element:

$$\|\boldsymbol{\sigma} - \boldsymbol{\Pi}_h^k \boldsymbol{\sigma}\|_{\mathbf{div}_{6/5};\Omega} \le C h^{\ell+1/2} \|\boldsymbol{\sigma}\|_{\ell+1/2;\Omega}\,.$$

Collecting the above estimates we conclude that

$$\mathcal{I} \le C h^{(s+\ell)/2}\Big(\|\boldsymbol{\sigma}\|_{\ell+1/2;\Omega} + \|\boldsymbol{\sigma}\|^{1/2}_{\ell+1/2;\Omega} |\boldsymbol{u}|^{1/2}_{s+1/2;\Gamma_C} \Big)\,.$$

We remark that, unlike primal formulations where an $\mathrm{H}^s(\Gamma_C)$-stable operator preserving non-positivity of the contact pressure is required (see e.g., [19]), the dual-mixed formulation avoids this construction. Indeed, the non-positivity of σ_N is enforced through the definition of the BDM degrees of freedom (6.12) and is sufficient to handle the boundary term on the right-hand-side of (6.13) via classical boundedness. Collecting all the above estimate, we are now in a position to state the main result of this section

Theorem 6.2. *In addition to the hyphotheses of the Theorems 4.3, 5.3 and 6.1, assume there exists* $v \in [0, k+1]$ *and* $s, \ell \in (0, k+1]$ *such that* $\boldsymbol{\sigma} \in \mathbb{H}^{\ell+1/2}(\Omega) \cap \mathbb{K}$ *with* $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{W}^{\ell,6/5}(\Omega)$, $\boldsymbol{u} \in \mathbf{W}^{s,6}(\Omega)$ *with* $\boldsymbol{u}|_{\Gamma_C} \in \mathbf{H}^{s+1/2}(\Gamma_C)$, $\boldsymbol{\rho} \in \mathbb{H}^{v}(\Omega) \cap \mathbb{L}^2_{\mathrm{skew}}(\Omega)$, $\boldsymbol{\theta} \in \mathbf{H}^s(\Omega) \cap \mathbf{H}_N(\mathbf{div}_{3/2};\Omega)$ *with* $\mathrm{div}(\boldsymbol{\theta}) \in \mathrm{W}^{s,3/2}(\Omega)$, *and* $\varphi \in \mathrm{W}^{v,3}(\Omega)$. *Then there exists a positive constant* C, *independent of* h, *such that*

$$\begin{aligned}
&\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div}_{6/5};\Omega} + \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,6;\Omega} + \|\boldsymbol{\rho} - \boldsymbol{\rho}_h\|_{0,\Omega} + \|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_{\mathbf{div}_{3/2};\Omega} + \|\varphi - \varphi_h\|_{0,3;\Omega} \\
&\le C\, h^{(s+\ell)/2}\Big(\|\boldsymbol{\sigma}\|_{\ell+1/2;\Omega} + \|\mathbf{div}(\boldsymbol{\sigma})\|_{\ell,6/5;\Omega} + |\boldsymbol{u}|_{s+1/2;\Gamma_C} \Big) \\
&\quad + C\, h^{s}\Big(\|\boldsymbol{u}\|_{s,6;\Omega} + \|\boldsymbol{\theta}\|_{s,\Omega} + \|\mathrm{div}(\boldsymbol{\theta})\|_{s,3/2;\Omega} \Big) + C\, h^{v}\Big(\|\boldsymbol{\rho}\|_{v,\Omega} + \|\varphi\|_{v,3;\Omega} \Big)\,.
\end{aligned}$$

7. Numerical results

In this section, we assess the performance of the proposed mixed finite element method for the coupled thermoelastic problem. First, we provide some details about the implementation. Then we verify the implementation using a manufactured solution. Finally we present a two dimensional test case and a three dimensional test case closer to applications.

7.1. Implementation details

All routines used in this section were implemented using the finite element library `Gridap` [5].

We now briefly discuss the implementation of the mechanical subproblem. Following [12] the normal stress σ_N on Γ_C can be expressed as

$$\sigma_N := \min\{0, \sigma_N - \rho(\boldsymbol{u}\cdot\boldsymbol{n} - g)\}\,, \quad \forall \rho > 0\,. \tag{7.1}$$

This reformulation is convenient for the design of an active-set strategy. Since $\boldsymbol{u}_h$ is a piecewise polynomial of degree k, its normal trace $\boldsymbol{u}_h|_F \cdot \boldsymbol{n}_F$ is well defined on each facet $F \in \Gamma_C$, where $\boldsymbol{n}_F$ is the outward unit normal to F, and we set $\zeta_F := \boldsymbol{u}_h|_F \cdot \boldsymbol{n}_F$, $F \in \Gamma_C$, which can be computed on the boundary facets and employed in the active-set update. Based on this, the active-set strategy is driven by the discrete indicator

$$\Phi_F := \sigma_{N,h}|_F - \rho(\zeta_F - \Pi_F(g))\,, \tag{7.2}$$

where Π_F denotes the $\mathrm{L}^2(F)$-projection onto $\mathrm{P}_k(F)$, used to ensure g is represented in the same discrete space as $\boldsymbol{u}_h|_F \cdot \boldsymbol{n}_F$. At the computational level, the contact boundary is partitioned into active and inactive facets according with the information carried by this quantity facet-wise. In order to get a conformal cone $\mathbb{K}_h \subseteq \mathbb{K}$, we use the following partition

$$\mathcal{A} := \{F \in \Gamma_C : \Phi_F(s) \le 0\,, \quad \forall s \in \mathbf{C}\}\,, \qquad \mathcal{I} := \Gamma_C \setminus \mathcal{A}\,, \tag{7.3}$$

where $\mathbf{C}$ is the smallest set of points that determines the sign of a polynomial. Another possibility is to use a mean criterion

$$\mathcal{A} := \left\{ F \in \Gamma_C : \int_F \Phi_F \le 0 \right\}\,, \qquad \mathcal{I} := \Gamma_C \setminus \mathcal{A}\,. \tag{7.4}$$

This option, however, does not constitute a conformal cone $\mathbb{K}_h$, therefore introduces an additional source of error, see for instance [12, 39].

We also mention that the results that follow are obtained by imposing the frictionless condition on the active set using a penalisation term $\beta_t \int_{\Gamma_C} \boldsymbol{\sigma}_t \boldsymbol{\tau}_t$ with $\beta_t = 10^3$. We use the classical fixed-point algorithm with a tolerance *tol* defined in each experiment and the iterations stop once the error between two consecutive iterates is sufficiently small, that is

$$\frac{\|\mathrm{coeff}^{n+1} - \mathrm{coeff}^{n}\|}{\|\mathrm{coeff}^{n+1}\|} \le tol,$$

where $\| \bullet \|$ stands for the Euclidean norm in $\mathbb{R}^{\mathrm{DoF}}$ with DoF representing the total numbers of degrees of freedom and $\mathrm{coeff}^{\mathrm{n}}$ is the coefficient vector of the solution in the iteration n.

7.2. Convergence against smooth manufactured solutions with contact

In this test we consider a elastic body whose geometry is the unit square $\Omega = [0,1]^2$ and its boundary is decomposed into $\Gamma_D = \{0\} \times [0,1]$, $\Gamma_N = ([0,1] \times \{1\}) \cup (\{1\} \times [0,1])$ the top and right edges, and the potential contact boundary Γ_C is the bottom edge, i.e., $\Gamma_C = [0,1] \times \{0\}$. The elastic body is resting in a flat rigid foundation so then $g(x) \equiv 0$. The material is assumed isotropic with parameters $E = 1000$, $\alpha = 0.5$, $\beta = 1$ and conductivity tensor $\kappa = \mathbf{I}$ and $c_0 = H_v = \vartheta \equiv 1$. To illustrate the performance of the method, we prescribe an exact displacement and exact temperature given by

$$\boldsymbol{u}_{\mathrm{ex}}(x,y) = \begin{pmatrix} \frac{1}{10}\sin(\frac{\pi}{2}x)y^2 \\ -\frac{1}{2}x^2 y \end{pmatrix} \quad \text{and} \quad \varphi_{\mathrm{ex}}(x,y) = \sin(\pi x)\sin(\pi y)\,,$$

from which the exact stress and body rotation tensor, the heat flux and forcing terms are constructed consistently. The domain is discretised by a sequence of uniformly refined triangular meshes and the problem is solved using the fixed point strategy proposed in Section 5.2 with $tol = 10^{-8}$ for the fixed point algorithm.

We also point out that the mechanical subproblem is solved by a primal-dual active set method with activation tolerance 10^{-12} and scaling parameter $\rho = 0.1$, starting from the initial guess that the body is not in contact with the rigid foundation. Since $\boldsymbol{u}_{\mathrm{ex}} \cdot \boldsymbol{n} = g$ on Γ_C, the algorithm simply updates the boundary classification in the first iteration and then confirms the final active contact boundary. Therefore, this test serves as validation for the contact algorithm. The results are displayed in Table 7.1 and confirm the theoretical convergence rates predicted in Theorem 6.2 along repeated experiments for $\nu \in \{0.3, 0.4, 0.49, 0.4999\}$ keeping all the other parameters fixed. All fields converge at the optimal rate $\mathcal{O}(h)$, which corresponds to $s = \ell = v = 1$ in the estimate of Theorem 6.2, consistent with the smoothness of the manufactured solution, which does not exhibit contact singularities, and confirming that the method is locking-free within this range of ν. A notable feature of the results is that the displacement and body rotation error converge at rate $\mathcal{O}(h^2)$ for $\nu \ge 0.499$.

7.3. Application: square with slip and separation

In this example we follow [11] and consider a coupled thermoelastic problem posed on the unit square $\Omega = [0,1]^2$. The body is clamped and with zero-temperature at the Dirichlet boundary (left edge), the top and bottom sides are assigned as with homogeneous Neumann boundary conditions for the mechanical and thermal models. The potential contact boundary Γ_C is the right edge, where the rigid foundation is located at $x = 1$. Initially, all the potential contact boundary is in effective contact. The material parameters are $E = 10^6$, $\nu = 0.3$, $\alpha = 10^5$, $\beta = 10$, and $\kappa = 5\mathbf{I}$. We apply a body force of $\boldsymbol{f} = (0, -76518)^{\mathrm{t}}$ and a heat source of $m \equiv 0$. The contact-heat transfer law is characterised by $c_0 = 10^4$, $H_v = 1.0$ and $\vartheta = 0.5$. The reference temperature at the foundation is prescribed as the Gaussian profile

$$\varphi_0(x,y) = 10\exp(-300(y-0.85)^2)\,,$$

concentrated in the region of non-zero contact pressure. This choice ensures that the thermal data is negligible at the binding point between active and inactive boundary, avoiding discontinuity in the normal heat flux $\boldsymbol{\theta}_h \cdot \boldsymbol{n}|_{\Gamma_C}$, and recovering optimal convergence rate for this field.

ν	DoF	h	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$e(\boldsymbol{u})$	$r(\boldsymbol{u})$	$e(\boldsymbol{\rho})$	$r(\boldsymbol{\rho})$	$e(\boldsymbol{\theta})$	$r(\boldsymbol{\theta})$	$e(\varphi)$	$r(\varphi)$	#FP
0.3	208	0.7071	1.29e+02	⋆	7.94e-02	⋆	4.69e-02	⋆	5.52e+00	⋆	3.10e-01	⋆	3
	832	0.3536	6.48e+01	0.993	3.91e-02	1.021	2.36e-02	0.988	2.77e+00	0.993	1.54e-01	1.009	3
	3328	0.1768	3.29e+01	0.979	1.93e-02	1.018	1.18e-02	1.001	1.35e+00	1.043	7.70e-02	1.000	4
	13312	0.0884	1.66e+01	0.986	9.62e-03	1.006	5.90e-03	1.003	6.53e-01	1.043	3.85e-02	1.000	4
	53248	0.0442	8.35e+00	0.992	4.80e-03	1.002	2.95e-03	1.001	3.26e-01	1.003	1.93e-02	1.000	4
	212992	0.0221	4.19e+00	0.995	2.40e-03	0.999	1.47e-03	1.000	1.72e-01	0.924	9.63e-03	1.000	4
0.4	208	0.7071	2.28e+02	⋆	8.08e-02	⋆	4.76e-02	⋆	5.50e+00	⋆	3.10e-01	⋆	3
	832	0.3536	1.09e+02	1.062	3.93e-02	1.041	2.37e-02	1.008	2.90e+00	0.924	1.54e-01	1.008	3
	3328	0.1768	5.43e+01	1.009	1.93e-02	1.022	1.18e-02	1.004	1.49e+00	0.962	7.70e-02	1.000	4
	13312	0.0884	2.72e+01	0.998	9.62e-03	1.007	5.89e-03	1.002	7.18e-01	1.052	3.85e-02	1.000	4
	53248	0.0442	1.36e+01	0.997	4.80e-03	1.002	2.94e-03	1.001	3.48e-01	1.044	1.93e-02	1.000	4
	212992	0.0221	6.82e+00	0.998	2.40e-03	1.000	1.47e-03	1.000	1.75e-01	0.994	9.63e-03	1.000	4
0.49	208	0.7071	2.23e+03	⋆	1.42e-01	⋆	1.49e-01	⋆	5.42e+00	⋆	3.09e-01	⋆	3
	832	0.3536	1.02e+03	1.123	5.10e-02	1.480	4.53e-02	1.722	2.61e+00	1.054	1.54e-01	1.005	3
	3328	0.1768	4.90e+02	1.065	2.12e-02	1.268	1.53e-02	1.567	1.44e+00	0.863	7.70e-02	0.998	3
	13312	0.0884	2.40e+02	1.031	9.88e-03	1.101	6.37e-03	1.264	8.06e-01	0.832	3.85e-02	1.000	3
	53248	0.0442	1.19e+02	1.013	4.84e-03	1.030	3.00e-03	1.084	3.96e-01	1.024	1.93e-02	1.000	3
	212992	0.0221	5.92e+01	1.004	2.41e-03	1.008	1.48e-03	1.022	1.88e-01	1.080	9.63e-03	1.000	3
0.4999	208	0.7071	2.24e+05	⋆	1.17e+01	⋆	1.48e+01	⋆	1.63e+01	⋆	1.18e+00	⋆	3
	832	0.3536	1.03e+05	1.124	3.23e+00	1.859	4.07e+00	1.858	6.35e+00	1.362	3.19e-01	1.888	3
	3328	0.1768	4.90e+04	1.067	8.27e-01	1.969	1.04e+00	1.972	2.00e+00	1.667	9.53e-02	1.743	3
	13312	0.0884	2.39e+04	1.035	2.08e-01	1.991	2.61e-01	1.993	7.48e-01	1.418	4.15e-02	1.200	3
	53248	0.0442	1.18e+04	1.018	5.22e-02	1.994	6.54e-02	1.997	3.38e-01	1.144	1.97e-02	1.076	3
	212992	0.0221	5.87e+03	1.009	1.32e-02	1.983	1.64e-02	1.995	1.78e-01	0.924	9.68e-03	1.022	3

Table 7.1
Convergence history for experiment 1: Error decay on each unknown and experimental rate of convergence, computed with the lowest-order scheme for increasing values of ν (equivalently $\lambda \to +\infty$). The ⋆ symbol denotes that no convergence rate is computed for the first mesh refinement.

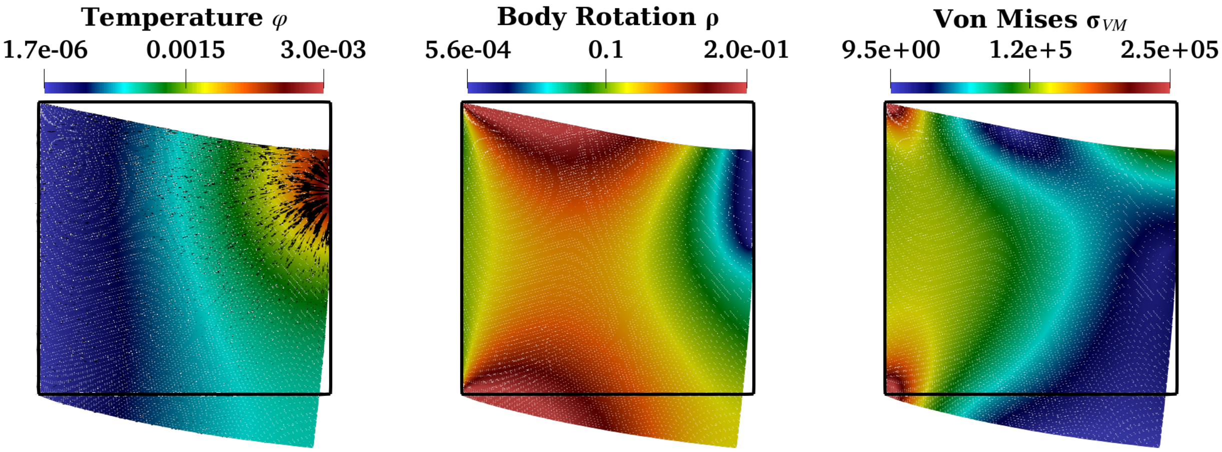
Figure 7.1 shows the numerical solution fields on the deformed domain for the finest mesh ($N = 131,072$ cells) for the temperature φ together with glyph arrows representing the thermal flux $-\boldsymbol{\theta}_h$, the arrows point inward at the active contact zone $\Gamma_A = \{1\} \times [0.668, 1]$ reflecting the heat entering the body from the warmer foundation through the reduced thermal resistance $r(\sigma_N)$. The center panel shows the body rotation $\boldsymbol{\rho}_h$ which captures the bending-like response induced by the asymmetric body force. The right panel shows the Von Mises stress, with a higher concentration at the active zone.

Figure 7.2 reports the global errors and convergence rates for all fields. Since no closed-form analytical solution is available for this benchmark, error are computed against a reference solution obtained on a fine mesh with 131,072 cells. The results confirm optimal convergence $\mathcal{O}(h)$ for all fields in the asymptotic regime (from $N = 2048$ elements).

7.4. Application: Hot flat-punch indentation in 3D

We consider a hot-flat punch indentation problem posed on the cube $\Omega = (-1, 1)^3$. The bottom face $z = -1$ is assigned as Γ_D, where the vertical displacement $\boldsymbol{g}_D = (0, 0, \frac{1}{10})^{\mathtt{t}}$ is prescribed together with zero temperature. The lateral faces constitute Γ_N, where no traction and zero normal heat flux are imposed. The top face is the potential contact boundary Γ_C, where the rigid punch is located. The punch occupies the square region $[-\frac{1}{5}, \frac{1}{5}]^2 \times [1, +\infty)$, posed at the centre of the contact face, and the footprint (i.e. the square $[-\frac{1}{5}, \frac{1}{5}]^2$) is maintained with a reference temperature of $\varphi_0 = 5.0$. The gap function is given by

$$g(x, y, z) = \begin{cases} 0 & \text{if } (x, y) \in [-\frac{1}{5}, \frac{1}{5}]^2, \\ +\infty & \text{otherwise,} \end{cases}$$



(a) Temperature. (b) Body rotation. (c) Von Mises stress.

Figure 7.1: Square with slip separation. Samples of the numerical solution fields plotted on the deformed configuration.

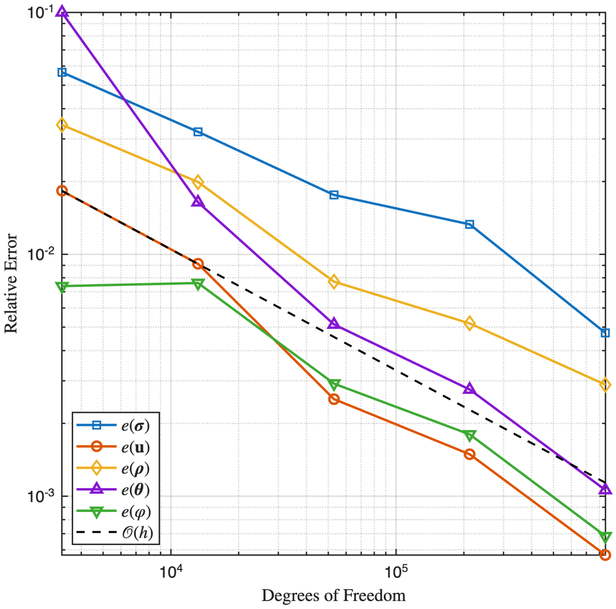


Figure 7.2: Square with slip separation. Convergence history (for all thermo-mechanical variables) against a fine-mesh reference solution.

so that the contact is only possible on the square $[-\frac{1}{5}, \frac{1}{5}]^2$ while the outside region of Γ_C remains permanently inactive. The material parameters are $E = 1$, $\nu = 0.3$, $\alpha = \beta = 1$, $\kappa = 10\mathbf{I}$, no body forces and volume heat source are applied to the body. The contact-heat transfer law is determined by $c_0 = 2.0$, $H_v = 3.0$ and $\vartheta = 0.95$, so the thermal resistance decreases sharply as the contact pressure increases outside the punch. The prescribed displacement on Γ_D compresses the body upward, bringing the top surface into contact with the punch over the footprint. The elevated punch temperature drives heat into the cooler body. The domain is discretised using a uniform tetrahedral mesh with $N = 32$ subdivision per edge, yielding 192,608 cells and corresponding discrete system has 10,536,352 degrees of freedom in total.

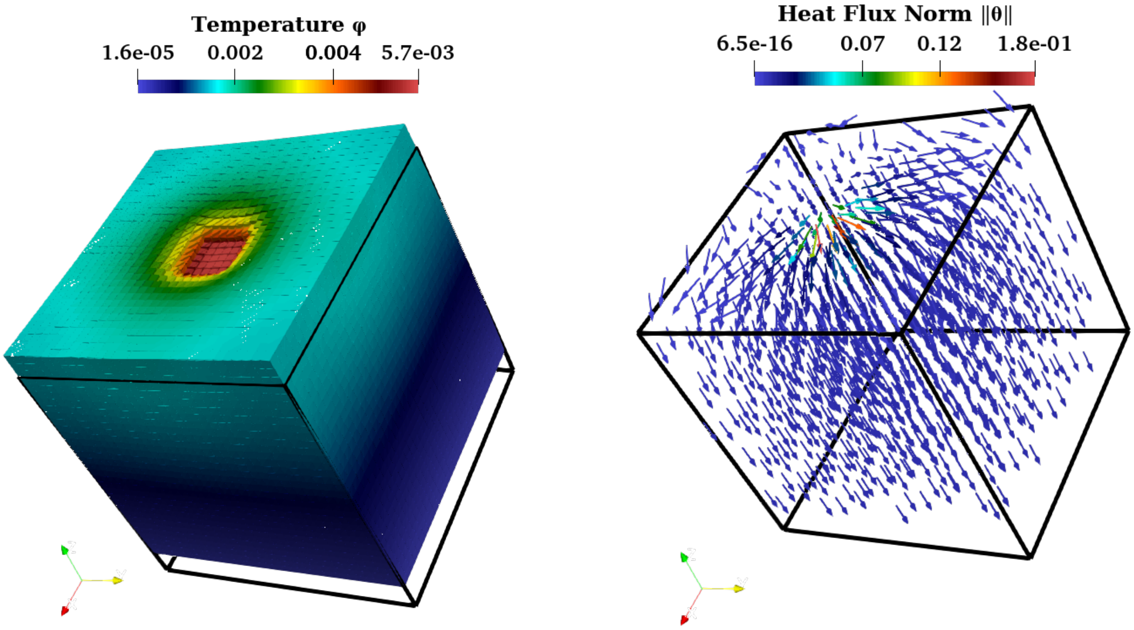


Figure 7.3: Hot flat punch. Numerical approximation of thermal variables: temperature distribution (left) and heat flux glyphs (right, not resized by magnitude). Both are plotted on the deformed 3D domain, showing also an outline of the undeformed configuration.

Figures 7.3–7.4 illustrate the computed solution fields on the deformed configuration for $N = 32$. Figure 7.3 displays the thermal fields: the temperature φ_h (left) shows heat diffusing from the punch footprint, decaying smoothly toward the lateral and bottom boundaries. In turn, the heat flux $-\boldsymbol{\theta}_h$ (right) confirms the expected flow source from the hot contact zone. Figure 7.4 shows stress and rotation. The Von Mises stress $\boldsymbol{\sigma}_{VM}$ is depicted in the volume view (left) indicating stress concentration along the boundary of the active set, i.e., the boundary of the footprint, with peak values at the binding zone. The slice through the ZY-plane (centre) exposes the stress bulb propagating into the bulk, this behaviour is expected and also spotted for other shapes of flat punch, see, for instance [33]. Finally, Figure 7.4 (right) presents the body rotation vector asp($\boldsymbol{\rho}$) (cf. (5.11)) on the contact face $z = 1$, coloured by its magnitude. The vortical pattern along the contact zone captures the material rotation induced by the kinematic and thermal constraints at the punch contact.

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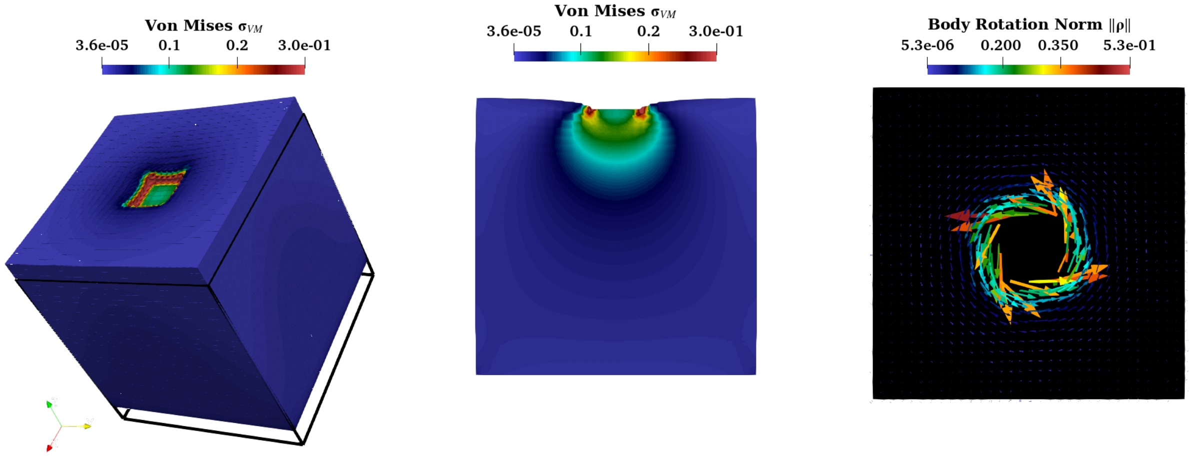

Figure 7.4: Hot flat punch. Numerical approximation of mechanical fields depicted on the deformed 3D domain (left and centre), and body rotation rendered on the plane $z = 1$ (right).

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