Banach spaces-based mixed finite element methods for the coupled Navier–Stokes and Poisson–Nernst–Planck equations*

Claudio I. Correa† Gabriel N. Gatica† Esteban Henríquez†
Ricardo Ruiz-Baier‡ Manuel Solano†

Abstract

In this paper we extend the Banach spaces-based fully mixed approach recently developed for the coupled Stokes and Poisson–Nernst–Planck equations, to cover the coupled Navier–Stokes and Poisson–Nernst–Planck equations. In addition to the velocity and pressure of the fluid, the velocity gradient and the Bernoulli-type stress tensor are added as further unknowns. Similarly, fully mixed formulations for the Poisson and Nernst–Planck sub-problems are achieved by considering, alongside the electrostatic potential and the concentration of ionized particles, the electric current field and total ionic fluxes as new mixed variables. As a consequence, two saddle-point problems, one of them non-linear, and both involving nonlinear source terms depending on the other unknowns, along with a perturbed saddle-point problem that is in turn further perturbed by a bilinear form depending on the remaining unknowns, constitute the resulting variational formulation of the whole coupled system. Fixed-point strategies are then employed to prove, under smallness assumptions on the data, the well-posedness of the continuous and associated Galerkin schemes, the latter for arbitrary finite element subspaces under suitable stability assumptions. The main theoretical tools utilized include the Babuška–Brezzi and Banach–Nečas–Babuška theories in Banach spaces, an abstract result for perturbed saddle-point problems (also in Banach spaces), and the classical Banach and Brouwer fixed-point theorems. Strang-type lemmas are then applied to establish a priori error estimates. Next, specific finite element subspaces (defined by Raviart–Thomas elements of order $k \geqslant 0$ and piecewise polynomials of degree $\leqslant k$) are shown to satisfy the required hypotheses, and this yields specific convergence rates. Finally, several numerical results are reported, confirming the theoretical findings and illustrating the good performance of the method.

Key words: Poisson–Nernst–Planck, Navier–Stokes, fixed-point theory, finite element methods.

Mathematics subject classifications (2000): 35J66, 65J15, 65N12, 65N15, 65N30, 47J26, 76D07.

1 Introduction

Scope. In this paper we develop a Banach spaces-based formulation yielding a new mixed finite element method for the coupled Navier–Stokes and Poisson–Nernst–Planck equations. This coupled PDE system is a remarkable example of multiphysics models where electrically charged ions interact in a complex

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†CI^{2}MA and Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile, email: clcorrea@udec.cl, ggatica@ci2ma.udec.cl, eshenriquez2016@udec.cl,msolano@ing-mat.udec.cl.

‡School of Mathematics, Monash University, 9 Rainforest Walk, Melbourne, VIC 3800, Australia; and World-Class Research Center "Digital biodesign and personalized healthcare", Sechenov University, Moscow, Russia; and Universidad Adventista de Chile, Casilla 7-D, Chillán, Chile, email: ricardo.ruizbaier@monash.edu.

manner, and at different spatial scales, with the flow behaviour of incompressible viscous fluid. Fluid mixtures of this type are essential in modeling fuel cells, ion channel behavior in cell membranes of biological tissues, electrodialysis and similar mechanisms used in the process of water desalination, and many other instances.

These models for single-phase electrohydrodynamic flows are composed by the coupled system of fluid flow (for example, the Navier–Stokes equations), ion transport (the Nernst–Planck equations with advection) and electrostatics (here a generalized Poisson equation). Obtaining accurate and stable numerical solutions for these complex systems is key to produce reliable simulations. While the computation with high-order methods and other schemes has been studied thoroughly in the literature going back several decades, the rigorous theoretical analysis of finite element and similar methods for the system under consideration here, initiated in the classical work [25], where the authors establish convergence of a finite element method using a projection method *à la* Chorin–Temam. Subsequently, a number of discretization methods have been proposed and their numerical analysis (discrete solvability, stability, convergence) has been conducted, including primal [22, 23, 24], primal-mixed (meaning in our context that the equations of Poisson–Nernst–Planck are written in mixed form but the incompressible flow problem is in classical velocity - pressure formulation) [19, 20], discontinuous Galerkin, and virtual elements [14].

One of our goals is to include conservativity of momentum for each one of the equations involved. A way of doing this is to use fully mixed formulations, that is, solving also for other unknowns of interest such as pseudostress-type tensors, vorticity, fluxes, and so on. Using numerical methods based on fully mixed variational formulations enjoys many advantages. However, in such a case, regularity issues may appear in treating the convective and advective terms as well as in the other coupling mechanisms. Remedies exist, for example augmentation (adding redundant Galerkin residual terms to endow the final formulation with the necessary regularity to control nonlinearities in usual Hilbert spaces). While this approach allows us to treat the convective and advective nonlinearities, it fails in maintaining the key feature of local conservativity (of momentum and mass, for example). Relatively recent efforts have been done in designing an alternative approach, where one looks at the fully mixed forms of the underlying problem without augmenting them. In turn, one requires to work on a more general functional setting, for example on Banach spaces. This is a classic idea going back to the work [3], which has got fresh attention due to the possibility of writing more and more complex nonlinearly coupled multiphysics problems in mixed form. As a non-exhaustive list of contributions taking advantage of the use of Banach frameworks for solving the aforementioned kind of problems, we refer to [2, 4, 6, 7, 8, 10, 11, 17, 18, 21].

Using these arguments, in [13] the authors have recently introduced a Banach spaces-based mixed finite element method for a slightly simpler model: the coupled Stokes and Poisson–Nernst–Planck equations. Even if the underlying model difference is just the presence of the convective term, we note that the structure of a fully mixed form for the Navier–Stokes equations requires a different setup – for example, employing different mixed variables sought in different spaces than those used for Stokes flows in fully mixed form. Moreover, the results in this paper extend further the analysis carried out in [13] by utilizing a different fixed-point strategy.

Outline. The rest of the manuscript is organized as follows. Notations and basic definitions to be utilized throughout the paper are collected in the remainder of this section. Section 2 states the strong form of the coupled problem, in its usual primal form, and also defining the new mixed variables. The fully-mixed continuous formulation is defined in Section 3, and its well-posedness analysis is developed in Section 4. The Galerkin method is introduced and analyzed in Section 5 under suitable assumptions on the finite element subspaces employed. In addition to its unique solvability, a generic error estimate is also provided there. Next, Section 6 specifies finite element subspaces satisfying the required stability properties, states the expected orders of convergence, and describes the discrete conservation properties of the resulting method. Finally, Section 7 showcases a number of numerical examples which serve as computational confirmation of the theoretical convergence rates computed in appropriate weighted norms, as well as of the discrete conservativity, and other tests that exemplify the use of the proposed family of fully mixed methods in the simulation of ionized electrolyte flows.

Preliminary definitions and notational conventions. Throughout the paper, Ω is a bounded Lipschitz-continuous domain of R^n, $n \in \{2,3\}$, with polygonal (resp. polyhedral) boundary Γ in R^2 (resp. R^3), and whose outward normal at $\Gamma := \partial\Omega$ is denoted by $\boldsymbol{\nu}$. Standard notation will be adopted for Lebesgue spaces $\mathrm{L}^t(\Omega)$ and Sobolev spaces $\mathrm{W}^{l,t}(\Omega)$ and $\mathrm{W}^{l,t}_0(\Omega)$, with $l \geqslant 0$ and $t \in [1,+\infty)$, whose norms, either for the scalar and vectorial case, are denoted by $\|\cdot\|_{0,t;\Omega}$ and $\|\cdot\|_{l,t;\Omega}$, respectively. Note that $\mathrm{W}^{0,t}(\Omega) = \mathrm{L}^t(\Omega)$, and if $t = 2$ we write $\mathrm{H}^l(\Omega)$ instead of $\mathrm{W}^{l,2}(\Omega)$, with norm and seminorm denoted by $\|\cdot\|_{l,\Omega}$ and $|\cdot|_{l,\Omega}$, respectively. In addition, letting $t,\, t' \in (1,+\infty)$ conjugate to each other, that is such that $1/t + 1/t' = 1$, we denote by $\mathrm{W}^{1/t',t}(\Gamma)$ the trace space of $\mathrm{W}^{1,t}(\Omega)$, and let $\mathrm{W}^{-1/t',t'}(\Gamma)$ be the dual of $\mathrm{W}^{1/t',t}(\Gamma)$ endowed with the norms $\|\cdot\|_{-1/t',t';\Gamma}$ and $\|\cdot\|_{1/t',t;\Gamma}$, respectively. On the other hand, given any generic scalar functional space M, we let $\mathbf{M}$ and $\mathbb{M}$ be the corresponding vectorial and tensorial counterparts, whereas $\|\cdot\|$ will be employed for the norm of any element or operator whenever there is no confusion about the spaces to which they belong. Furthermore, as usual, $\mathbb{I}$ stands for the identity tensor in $\mathbb{R} := \mathrm{R}^{n\times n}$, and $|\cdot|$ denotes the Euclidean norm in $\mathbf{R} := \mathrm{R}^n$. Also, for any vector field $\mathbf{v} = (v_i)_{i=1,n}$ we set the gradient and divergence operators, respectively, as

$$\nabla\mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,n}, \qquad \mathrm{div}(\mathbf{v}) := \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j} \qquad \text{and} \quad \mathbf{v}\otimes\mathbf{w} := (v_i w_j)_{i,j=1,n}\,.$$

Additionally, for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,n}$, we let $\mathbf{div}(\boldsymbol{\tau})$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the tensor inner product operators, and the deviatoric tensor, respectively, as

$$\boldsymbol{\tau}^{\mathrm{t}} = (\tau_{ji})_{i,j=1,n}, \quad \mathrm{tr}(\boldsymbol{\tau}) = \sum_{i=1}^{n} \boldsymbol{\tau}_{ii}, \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{n} \tau_{ij}\zeta_{ij}\,, \quad \text{and} \quad \boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{n}\mathrm{tr}(\boldsymbol{\tau})\mathbb{I}\,.$$

On the other hand, given $t \in (1,+\infty)$, we also introduce the Banach spaces

$$\mathbf{H}(\mathrm{div}_t;\Omega) := \left\{\boldsymbol{\tau} \in \mathbf{L}^2(\Omega) : \quad \mathrm{div}(\boldsymbol{\tau}) \in \mathrm{L}^t(\Omega)\right\}, \tag{1.1a}$$

$$\mathbb{H}(\mathbf{div}_t;\Omega) := \left\{\boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \quad \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^t(\Omega)\right\}, \tag{1.1b}$$

$$\mathbf{H}^t(\mathrm{div}_t;\Omega) := \left\{\boldsymbol{\tau} \in \mathbf{L}^t(\Omega) : \quad \mathrm{div}(\boldsymbol{\tau}) \in \mathrm{L}^t(\Omega)\right\}, \tag{1.1c}$$

which are endowed with the natural norms defined, respectively, by

$$\|\boldsymbol{\tau}\|_{\mathrm{div}_t;\Omega} := \|\boldsymbol{\tau}\|_{0,\Omega} + \|\mathrm{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbf{H}(\mathrm{div}_t;\Omega)\,, \tag{1.2a}$$

$$\|\boldsymbol{\tau}\|_{\mathbf{div}_t;\Omega} := \|\boldsymbol{\tau}\|_{0,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_t;\Omega)\,, \tag{1.2b}$$

$$\|\boldsymbol{\tau}\|_{t,\mathrm{div}_t;\Omega} := \|\boldsymbol{\tau}\|_{0,t;\Omega} + \|\mathrm{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbf{H}^t(\mathrm{div}_t;\Omega)\,. \tag{1.2c}$$

Then, proceeding as in [16, eq. (1.43), Section 1.3.4] (see also [5, Section 4.1] and [10, Section 3.1]), it is easy to show that for each $t \geqslant \frac{2n}{n+2}$ there holds

$$\langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, v\rangle = \int_\Omega \left\{\boldsymbol{\tau}\cdot\nabla v + v\,\mathrm{div}(\boldsymbol{\tau})\right\} \qquad \forall\,(\boldsymbol{\tau}, v) \in \mathbf{H}(\mathrm{div}_t;\Omega)\times \mathrm{H}^1(\Omega)\,, \tag{1.3a}$$

$$\langle \boldsymbol{\tau}\,\boldsymbol{\nu}, \mathbf{v}\rangle = \int_\Omega \left\{\boldsymbol{\tau} : \nabla\mathbf{v} + \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\tau})\right\} \qquad \forall\,(\boldsymbol{\tau}, \mathbf{v}) \in \mathbb{H}(\mathbf{div}_t;\Omega)\times \mathbf{H}^1(\Omega)\,, \tag{1.3b}$$

where $\langle\cdot,\cdot\rangle$ is the duality pairing between $\mathrm{H}^{-1/2}(\Gamma)$ and $\mathrm{H}^{1/2}(\Gamma)$, as well as between $\mathbf{H}^{-1/2}(\Gamma)$ and $\mathbf{H}^{1/2}(\Gamma)$. Furthermore, given $t,\, t' \in (1,+\infty)$ conjugate to each other, there also holds (cf. [15, Corollary B. 57])

$$\langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, v\rangle_\Gamma = \int_\Omega \left\{\boldsymbol{\tau}\cdot\nabla v + v\,\mathrm{div}(\boldsymbol{\tau})\right\} \qquad \forall\,(\boldsymbol{\tau}, v) \in \mathbf{H}^t(\mathrm{div}_t;\Omega)\times \mathrm{W}^{1,t'}(\Omega)\,, \tag{1.4}$$

where $\langle\cdot,\cdot\rangle_\Gamma$ stands for the duality pairing between $\mathrm{W}^{-1/t,t}(\Gamma)$ and $\mathrm{W}^{1/t,t'}(\Gamma)$.

2 The model problem

We consider the electrohydrostatic model describing the flow of a Newtonian and incompressible fluid occupying the domain Ω, and whose mathematical representation is given by the coupled Navier–Stokes and Poisson–Nernst–Planck equations. Its behavior is determined by the concentrations ξ_1 and ξ_2 of ionized particles, and by the electric current field $\boldsymbol{\varphi}$. More precisely, and regarding firstly the fluid, we look for the velocity $\mathbf{u}$ and the pressure p such that $(\mathbf{u}, p)$ is solution to the Navier–Stokes equations

$$\begin{gathered} -\mu\,\Delta\mathbf{u} + \omega\,(\nabla\mathbf{u})\,\mathbf{u} + \nabla p = -(\xi_1-\xi_2)\,\varepsilon^{-1}\boldsymbol{\varphi} + \mathbf{f} \quad \text{in} \quad \Omega\,, \\ \operatorname{div}(\mathbf{u}) = 0 \quad \text{in} \quad \Omega\,, \quad \mathbf{u} = \mathbf{g} \quad \text{on} \quad \Gamma\,, \quad \int_\Omega p = 0\,, \end{gathered} \tag{2.1}$$

where μ is the constant dynamic viscosity, ω is the fluid density, ε is the dielectric coefficient, also known as the electric conductivity coefficient, $\mathbf{f}$ is a source term, $\mathbf{g}$ is the Dirichlet datum for $\mathbf{u}$ on Γ, and the null mean value of p has been incorporated as a uniqueness condition for this unknown. Note that, due to the incompressibility of the fluid (cf. second equation of (2.1)), $\mathbf{g}$ must satisfy the compatibility condition

$$\int_\Gamma \mathbf{g}\cdot\boldsymbol{\nu} = 0\,. \tag{2.2}$$

Furthermore, $\boldsymbol{\varphi}$, ξ_1 and ξ_2 solve the Poisson–Nernst–Planck equations, given by

$$\begin{gathered} \boldsymbol{\varphi} = \varepsilon\,\nabla\chi \quad \text{in} \quad \Omega\,, \quad -\operatorname{div}(\boldsymbol{\varphi}) = (\xi_1-\xi_2) + f \quad \text{in} \quad \Omega\,, \\ \chi = g \quad \text{on} \quad \Gamma\,, \end{gathered} \tag{2.3}$$

where χ is the electrostatic potential, and for each $i \in \{1,2\}$

$$\begin{gathered} \xi_i - \operatorname{div}\big(\kappa_i(\nabla\xi_i + q_i\,\xi_i\,\varepsilon^{-1}\boldsymbol{\varphi}) - \xi_i\,\mathbf{u}\big) = f_i \quad \text{in} \quad \Omega\,, \\ \xi_i = g_i \quad \text{on} \quad \Gamma\,, \end{gathered} \tag{2.4}$$

where κ_1 and κ_2 are the diffusion coefficients, $q_i := \begin{cases} 1 & \text{if } i=1 \\ -1 & \text{if } i=2 \end{cases}$ is the charge of each particle, f, f_1, and f_2 are external forces, and g, g_1 and g_2 are Dirichlet data for χ, ξ_1 and ξ_2, respectively, on Γ. We end the description of the model by remarking that ε, κ_1, and κ_2 are all assumed to be bounded above and below, which means that there exist positive constants ε_0, ε_1, $\underline{\kappa}$, and $\bar{\kappa}$, such that

$$\varepsilon_0 \leqslant \varepsilon(\mathbf{x}) \leqslant \varepsilon_1 \quad \text{and} \quad \underline{\kappa} \leqslant \kappa_i(\mathbf{x}) \leqslant \bar{\kappa} \quad \text{for almost all } \mathbf{x}\in\Omega\,, \quad \forall\, i \in \{1,2\}\,. \tag{2.5}$$

Since we are interested in employing a fully-mixed variational formulation for the coupled model (2.1) – (2.4), we first adopt the approach from [11] (see also [10]) for the fluid and introduce the velocity gradient and the Bernoulli-type stress tensor as further unknowns, that is

$$\mathbf{t} := \nabla\mathbf{u} \quad \text{in} \quad \Omega \quad \text{and} \quad \boldsymbol{\sigma} := \mu\,\mathbf{t} - \frac{\omega}{2}(\mathbf{u}\otimes\mathbf{u}) - p\,\mathbb{I} \quad \text{in} \quad \Omega\,. \tag{2.6}$$

In this way, noting that $\mathbf{div}(\mathbf{u}\otimes\mathbf{u}) = (\nabla\mathbf{u})\mathbf{u} = \mathbf{t}\mathbf{u}$, which follows from the fact that $\operatorname{div}(\mathbf{u}) = 0$, we find that the first equation of (2.1) can be rewritten as

$$-\mathbf{div}(\boldsymbol{\sigma}) + \frac{\omega}{2}\,\mathbf{t}\mathbf{u} = -(\xi_1-\xi_2)\,\varepsilon^{-1}\boldsymbol{\varphi} + \mathbf{f} \quad \text{in} \quad \Omega\,.$$

Next, taking matrix trace and the deviatoric part of the second equation of (2.6), we find that the latter and the incompressibility condition, which becomes now $\operatorname{tr}(\mathbf{t}) = 0$, are equivalent to the pair

$$\boldsymbol{\sigma}^{\mathrm{d}} = \mu\,\mathbf{t} - \frac{\omega}{2}(\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}} \quad \text{in} \quad \Omega \quad \text{and} \quad p = -\frac{1}{n}\operatorname{tr}\Big(\boldsymbol{\sigma} + \frac{\omega}{2}(\mathbf{u}\otimes\mathbf{u})\Big) \quad \text{in} \quad \Omega\,, \tag{2.7}$$

whence the pressure can be eliminated from the formulation and computed afterwards in terms of $\boldsymbol{\sigma}$ and $\mathbf{u}$ as indicated in the second column of (2.7).

On the other hand, for the Nernst–Planck equations we introduce for each $i \in \{1,2\}$ the total fluxes

$$\boldsymbol{\sigma}_i \,:=\, \kappa_i\,(\nabla \xi_i \,+\, q_i\,\xi_i\,\varepsilon^{-1}\boldsymbol{\varphi}) \,-\, \xi_i\,\mathbf{u} \quad \text{in} \quad \Omega\,,$$

so that the respective transport equation reads now $\xi_i \,-\, \operatorname{div}(\boldsymbol{\sigma}_i) \,=\, f_i \quad \text{in} \quad \Omega$. Consequently, (2.1) – (2.4) can then be rewritten in terms of $\mathbf{t}$, $\boldsymbol{\sigma}$, $\mathbf{u}$, $\boldsymbol{\varphi}$, χ, $\boldsymbol{\sigma}_i$ and ξ_i, $i \in \{1,2\}$, as

$$\begin{gathered}
\mathbf{t} \,=\, \nabla \mathbf{u} \quad \text{in} \quad \Omega\,,\\
\boldsymbol{\sigma}^{\mathrm{d}} \,=\, \mu\,\mathbf{t} \,-\, \tfrac{\omega}{2}\,(\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}} \quad \text{in} \quad \Omega\,, \quad \mathbf{div}(\boldsymbol{\sigma}) \,-\, \tfrac{\omega}{2}\,\mathbf{t}\mathbf{u} \,=\, (\xi_1-\xi_2)\,\varepsilon^{-1}\boldsymbol{\varphi} - \mathbf{f} \quad \text{in} \quad \Omega\,,\\
\mathbf{u} \,=\, \mathbf{g} \quad \text{on} \quad \Gamma\,, \quad \int_\Omega \operatorname{tr}\big(\boldsymbol{\sigma} \,+\, \tfrac{\omega}{2}(\mathbf{u}\otimes\mathbf{u})\big) \,=\, 0\,,\\
\tfrac{1}{\varepsilon}\boldsymbol{\varphi} \,=\, \nabla\chi \quad \text{in} \quad \Omega\,, \quad -\operatorname{div}(\boldsymbol{\varphi}) \,=\, (\xi_1-\xi_2)+f \quad \text{in} \quad \Omega\,,\\
\chi \,=\, g \quad \text{on} \quad \Gamma\,,\\
\tfrac{1}{\kappa_i}\boldsymbol{\sigma}_i \,=\, \nabla\xi_i \,+\, q_i\,\xi_i\,\varepsilon^{-1}\boldsymbol{\varphi} \,-\, \kappa_i^{-1}\,\xi_i\,\mathbf{u} \quad \text{in} \quad \Omega\,,\\
\xi_i \,-\, \operatorname{div}(\boldsymbol{\sigma}_i) \,=\, f_i \quad \text{in} \quad \Omega\,, \quad \xi_i \,=\, g_i \quad \text{on} \quad \Gamma\,, \quad i \in \{1,2\}\,.
\end{gathered} \tag{2.8}$$

And we note that the uniqueness condition for p rewrites equivalently as the null mean value constraint for $\operatorname{tr}\big(\boldsymbol{\sigma} \,+\, \tfrac{\omega}{2}(\mathbf{u}\otimes\mathbf{u})\big)$.

3 The fully-mixed formulation

In this section we derive a Banach spaces-based fully-mixed formulation of (2.8). We use the integration by parts formulae (1.3a) - (1.4) along with the Cauchy–Schwarz and Hölder inequalities. We split the exposition into a preliminary discussion on functional spaces, then present each sub-problem separately, and finally state the variational formulation of the whole coupled system (2.8).

3.1 Preliminaries

We begin by determining suitable spaces where to seek the unknowns by taking a closer look at the terms $\frac{\omega}{2}\,\mathbf{t}\mathbf{u}$, $\frac{\omega}{2}(\mathbf{u}\otimes\mathbf{u})$, $(\xi_1-\xi_2)\,\varepsilon^{-1}\boldsymbol{\varphi}$, $q_i\,\xi_i\,\varepsilon^{-1}\boldsymbol{\varphi}$ and $\kappa_i^{-1}\,\xi_i\,\mathbf{u}$ in the second and sixth rows of (2.8). To be more precise, ignoring the bounded functions ε^{-1}, and κ_i^{-1}, as well as the constant q_i, an immediate application of the Cauchy–Schwarz and Hölder inequalities, yields

$$\left|\int_\Omega (\xi_1-\xi_2)\,\boldsymbol{\varphi}\cdot\mathbf{v}\right| \,\leqslant\, \|\xi_1-\xi_2\|_{0,2\ell;\Omega}\,\|\boldsymbol{\varphi}\|_{0,2j;\Omega}\,\|\mathbf{v}\|_{0,\Omega}\,, \tag{3.1a}$$

$$\left|\int_\Omega (\mathbf{u}\otimes\mathbf{u}):\mathbf{s}\right| \,\leqslant\, \|\mathbf{u}\|_{0,4;\Omega}\,\|\mathbf{u}\|_{0,4;\Omega}\,\|\mathbf{s}\|_{0,\Omega}\,, \tag{3.1b}$$

$$\left|\int_\Omega \mathbf{t}\mathbf{u}\cdot\mathbf{v}\right| \,\leqslant\, \|\mathbf{t}\|_{0,\Omega}\,\|\mathbf{u}\|_{0,4;\Omega}\,\|\mathbf{v}\|_{0,4;\Omega}\,, \tag{3.1c}$$

$$\left|\int_\Omega \xi_i\,\boldsymbol{\varphi}\cdot\boldsymbol{\tau}_i\right| \,\leqslant\, \|\xi_i\|_{0,2\ell;\Omega}\,\|\boldsymbol{\varphi}\|_{0,2j;\Omega}\,\|\boldsymbol{\tau}_i\|_{0,\Omega}\,, \tag{3.1d}$$

$$\left|\int_\Omega \xi_i\,\mathbf{u}\cdot\boldsymbol{\tau}_i\right| \,\leqslant\, \|\xi_i\|_{0,2\ell;\Omega}\,\|\mathbf{u}\|_{0,2j;\Omega}\,\|\boldsymbol{\tau}_i\|_{0,\Omega}\,, \tag{3.1e}$$

where $\ell,\, j \in (1,+\infty)$ are conjugate to each other; and $\mathbf{v}$, $\mathbf{s}$, and $\boldsymbol{\tau}_i$ are test functions associated to $\mathbf{u}$, $\mathbf{t}$, and $\boldsymbol{\sigma}_i$, respectively. In this way, denoting

$$\rho \,:=\, 2\ell\,, \quad \varrho \,:=\, \frac{2\ell}{2\ell-1} \ (\text{conjugate of } \rho)\,, \quad r \,:=\, 2j\,, \quad \text{and} \quad s \,:=\, \frac{2j}{2j-1} \ (\text{conjugate of } r)\,, \tag{3.2}$$

it follows that the above expressions make sense for $\xi_i \in \mathrm{L}^\rho(\Omega)$, $\boldsymbol{\varphi} \in \mathbf{L}^r(\Omega)$, $\mathbf{u}, \mathbf{v} \in \mathbf{L}^4(\Omega)$, $\mathbf{t}, \mathbf{s} \in \mathbb{L}^2(\Omega)$, and $\boldsymbol{\tau}_i \in \mathbf{L}^2(\Omega)$. Since we need that $\mathbf{u} \in \mathbf{L}^4(\Omega)$, we impose that $2j \leqslant 4$. The specific choice of ℓ (and hence of j, ρ, r and the respective conjugates ϱ and s) will be addressed later on. In the meantime we consider generic values in (3.2). Moreover, since $\boldsymbol{\varphi} \in \mathbf{L}^r(\Omega)$, from the first equation in the fourth row of (2.8), we deduce that χ should be initially sought in $\mathrm{W}^{1,r}(\Omega)$.

3.2 The Navier–Stokes equations

The analysis of the mixed formulation for the Navier–Stokes equations is inspired by the work done by [6, Section 2.1]. As they do, we first assume that $\mathbf{g} \in \mathbf{H}^{1/2}(\Gamma)$. Then, by a direct application of (1.3b) with $t \geqslant \frac{2n}{n+2}$ and $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_t;\Omega)$, we test the first equation of (2.8) obtaining

$$\int_\Omega \boldsymbol{\tau} : \mathbf{t} + \int_\Omega \mathbf{u} \cdot \mathbf{div}(\boldsymbol{\tau}) \;=\; \langle \boldsymbol{\tau}\,\boldsymbol{\nu}, \mathbf{g}\rangle_\Gamma \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_t;\Omega)\,. \tag{3.3}$$

It is easy to notice that, thanks to Cauchy–Schwarz's inequality and the free trace property of $\mathbf{t}$, the first term of (3.3) makes sense for $\mathbf{t} \in \mathbb{L}^2_{\mathrm{tr}}(\Omega)$, where

$$\mathbb{L}^2_{\mathrm{tr}}(\Omega) \;:=\; \Big\{ \mathbf{s} \in \mathbb{L}^2(\Omega) : \quad \mathrm{tr}(\mathbf{s}) \;=\; 0 \Big\}\,.$$

In turn, knowing that $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^t(\Omega)$, and using Hölder's inequality, we deduce from the second term of (3.3) that, we look for $\mathbf{u} \in \mathbf{L}^{t'}(\Omega)$, where t' is the conjugate of t. On the other hand, testing the first equation of the second row of (2.8) against tensors in $\mathbb{L}^2(\Omega)$, and recalling the orthogonal splitting $\mathbb{L}^2(\Omega) \;=\; \mathbb{L}^2_{\mathrm{tr}}(\Omega) \oplus \mathrm{R}\,\mathbb{I}$, we get

$$-\int_\Omega \boldsymbol{\sigma} : \mathbf{s} + \mu \int_\Omega \mathbf{t} : \mathbf{s} - \frac{\omega}{2} \int_\Omega (\mathbf{u} \otimes \mathbf{u}) : \mathbf{s} \;=\; 0 \qquad \forall\, \mathbf{s} \in \mathbb{L}^2_{\mathrm{tr}}(\Omega)\,, \tag{3.4}$$

from where, by the Cauchy–Schwarz and Hölder inequalities, we deduce that the third term makes sense for $\mathbf{u} \in \mathbf{L}^4(\Omega)$ setting $t' = 4$ and therefore $t = 4/3$. Furthermore, aiming to use the same space of $\boldsymbol{\tau}$, then we seek $\boldsymbol{\sigma} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega)$ as well. On the other hand, as we know that $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{L}^{4/3}(\Omega)$, we test the second equation of the second row of (2.8) against vector functions in $\mathbf{L}^4(\Omega)$, which yields

$$-\int_\Omega \mathbf{div}(\boldsymbol{\sigma}) \cdot \mathbf{v} + \frac{\omega}{2} \int_\Omega \mathbf{t}\mathbf{u} \cdot \mathbf{v} \;=\; \int_\Omega (\xi_2 - \xi_1)\,\varepsilon^{-1} \boldsymbol{\varphi} \cdot \mathbf{v} + \int_\Omega \mathbf{f} \cdot \mathbf{v} \qquad \forall\, \mathbf{v} \in \mathbf{L}^4(\Omega)\,. \tag{3.5}$$

Notice from the above deduction and the already established spaces for $\mathbf{t}$, $\mathbf{u}$ and $\mathbf{v}$, that the first, second and fourth terms of (3.5) are well-defined, the latter if the datum $\mathbf{f}$ belongs to $\mathbf{L}^{4/3}(\Omega)$, which is henceforth assumed. As for the third, which will depend on where to look $\xi := (\xi_1\,,\ \xi_1)$ and $\boldsymbol{\varphi}$, we will refer to it later. We now consider the decomposition

$$\mathbb{H}(\mathbf{div}_{4/3};\Omega) \;=\; \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \oplus \mathrm{R}\,\mathbb{I}\,, \tag{3.6}$$

where

$$\mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \;:=\; \Big\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega) : \quad \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) \;=\; 0 \Big\}\,, \tag{3.7}$$

implying that $\boldsymbol{\sigma}$ can be uniquely decomposed (also using the second equation of the third row of (2.8)), as $\boldsymbol{\sigma} \;=\; \boldsymbol{\sigma}_0 + c_0\mathbb{I}$, where

$$\boldsymbol{\sigma}_0 \;\in\; \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \quad \text{and} \quad c_0 \;:=\; \frac{1}{n|\Omega|} \int_\Omega \mathrm{tr}(\boldsymbol{\sigma}) \;=\; -\frac{\omega}{2\,n\,|\Omega|} \int_\Omega \mathrm{tr}(\mathbf{u} \otimes \mathbf{u})\,. \tag{3.8}$$

Thus, similarly to the case of the pressure, the constant c_0 can be computed once the velocity is known, and hence it only remains to obtain $\boldsymbol{\sigma}_0$. In this regard, we notice that (3.4) and (3.5) do not change if $\boldsymbol{\sigma}$

is replaced by $\boldsymbol{\sigma}_0$. In turn, as $\mathbf{t}$ is sought in $\mathbb{L}^2_{\mathrm{tr}}(\Omega)$, and using the compatibility condition (2.2), we realize that testing (3.3) against $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega)$ is equivalent to doing it against $\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$. Therefore, taking into account the above discussion, and introducing the notations

$$\vec{\mathbf{u}} = (\mathbf{u},\mathbf{t}), \quad \vec{\mathbf{v}} = (\mathbf{v},\mathbf{s}), \quad \vec{\mathbf{w}} = (\mathbf{w},\boldsymbol{\vartheta}) \in \mathbf{H} := \mathbf{L}^4(\Omega) \times \mathbb{L}^2_{\mathrm{tr}}(\Omega), \quad \text{and} \quad \mathbf{Q} := \mathbb{H}_0(\mathbf{div}_{4/3};\Omega),$$

we redenote from now on $\boldsymbol{\sigma}_0$ as simply $\boldsymbol{\sigma} \in \mathbf{Q}$. Then, from the expressions (3.3), (3.4) and (3.5), we state the following mixed formulation for the Navier–Stokes equations: Find $(\vec{\mathbf{u}},\boldsymbol{\sigma}) \in \mathbf{H} \times \mathbf{Q}$ such that

$$\begin{aligned} \mathbf{a}(\vec{\mathbf{u}},\vec{\mathbf{v}}) + \mathbf{c}(\mathbf{u};\vec{\mathbf{u}},\vec{\mathbf{v}}) + \mathbf{b}(\vec{\mathbf{v}},\boldsymbol{\sigma}) &= \mathbf{F}_{\boldsymbol{\xi},\boldsymbol{\varphi}}(\vec{\mathbf{v}}) \qquad \forall\,\vec{\mathbf{v}} \in \mathbf{H}, \\ \mathbf{b}(\vec{\mathbf{u}},\boldsymbol{\tau}) &= \mathbf{G}(\boldsymbol{\tau}) \qquad \forall\,\boldsymbol{\tau} \in \mathbf{Q}, \end{aligned} \tag{3.9}$$

where, given $\mathbf{z} \in \mathbf{L}^4(\Omega)$, the bilinear forms $\mathbf{a} : \mathbf{H} \times \mathbf{H} \to \mathrm{R}$, $\mathbf{b} : \mathbf{H} \times \mathbf{Q} \to \mathrm{R}$, and $\mathbf{c}(\mathbf{z};\cdot,\cdot) : \mathbf{H} \times \mathbf{H} \to \mathrm{R}$, are defined as

$$\mathbf{a}(\vec{\mathbf{w}},\mathbf{v}) := \mu \int_\Omega \boldsymbol{\vartheta} : \mathbf{s} \qquad \forall\,\vec{\mathbf{w}},\,\vec{\mathbf{v}} \in \mathbf{H}, \tag{3.10}$$

$$\mathbf{b}(\vec{\mathbf{v}},\boldsymbol{\tau}) := -\int_\Omega \boldsymbol{\tau} : \mathbf{s} - \int_\Omega \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\tau}) \qquad \forall\,(\vec{\mathbf{v}},\boldsymbol{\tau}) \in \mathbf{H} \times \mathbf{Q}, \tag{3.11}$$

and

$$\mathbf{c}(\mathbf{z};\vec{\mathbf{w}},\vec{\mathbf{v}}) := \frac{\omega}{2}\left\{\int_\Omega \boldsymbol{\vartheta}\,\mathbf{z} \cdot \mathbf{v} - \int_\Omega (\mathbf{w} \otimes \mathbf{z}) : \mathbf{s}\right\} \qquad \forall\,\vec{\mathbf{w}},\vec{\mathbf{v}} \in \mathbf{H}, \tag{3.12}$$

whereas, given $\boldsymbol{\eta} := (\eta_1,\,\eta_2)$ and $\boldsymbol{\phi}$ in the same spaces where $\boldsymbol{\xi}$ and $\boldsymbol{\varphi}$ will be sought respectively, the linear functionals $\mathbf{F}_{\boldsymbol{\eta},\boldsymbol{\phi}} : \mathbf{H} \to \mathrm{R}$ and $\mathbf{G} : \mathbf{Q} \to \mathrm{R}$ are given by

$$\mathbf{F}_{\boldsymbol{\eta},\boldsymbol{\phi}}(\vec{\mathbf{v}}) := \int_\Omega (\eta_2 - \eta_1)\,\varepsilon^{-1}\,\boldsymbol{\phi} \cdot \mathbf{v} + \int_\Omega \mathbf{f} \cdot \mathbf{v} \quad \forall\,\vec{\mathbf{v}} \in \mathbf{H}, \tag{3.13}$$

and

$$\mathbf{G}(\boldsymbol{\tau}) := -\langle \boldsymbol{\tau\nu},\,\mathbf{g}\rangle \quad \forall\,\vec{\mathbf{v}} \in \mathbf{Q}. \tag{3.14}$$

In turn, it is easy to see that $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}(\mathbf{z},\cdot,\cdot)$, and $\mathbf{G}$ are bounded. In fact, using the norms

$$\|\vec{\mathbf{v}}\|_{\mathbf{H}} := \|\mathbf{v}\|_{0,4;\Omega} + \|\mathbf{s}\|_{0,\Omega} \quad \forall\,\vec{\mathbf{v}} := (\mathbf{v},\mathbf{s}) \in \mathbf{H}, \quad \|\boldsymbol{\tau}\|_{\mathbf{Q}} := \|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega} \quad \forall\,\boldsymbol{\tau} \in \mathbf{Q},$$

applying the Cauchy–Schwarz and Hölder inequalities, and using (1.3b) with $\mathbf{v}_{\mathbf{g}} \in \mathbf{H}^1(\Omega)$ such that $\mathbf{v}_{\mathbf{g}}|_\Gamma = \mathbf{g}$ and $\|\mathbf{v}_{\mathbf{g}}\|_{1,\Omega} = \|\mathbf{g}\|_{1/2,\Gamma}$, along with the continuous injection $\mathbf{i}_4 : \mathbf{H}^1(\Omega) \to \mathbf{L}^4(\Omega)$, we find that there exist positive constants, denoted and given as

$$\|\mathbf{a}\| := \mu, \qquad \|\mathbf{b}\| := 1, \qquad \|\mathbf{c}\| := \frac{\omega}{2}, \qquad \|\mathbf{G}\| := (1 + \|\mathbf{i}_4\|)\,\|\mathbf{g}\|_{1/2,\Gamma},$$
$$\text{and} \quad \|\mathbf{F}\| := \max\left\{\varepsilon_0^{-1}\,|\Omega|^{1/4}, 1\right\},$$

such that

$$\begin{aligned} |\mathbf{a}(\vec{\mathbf{w}},\vec{\mathbf{v}})| &\leqslant \|\mathbf{a}\|\,\|\vec{\mathbf{w}}\|_{\mathbf{H}}\,\|\vec{\mathbf{v}}\|_{\mathbf{H}} \qquad \forall\,\vec{\mathbf{w}},\,\vec{\mathbf{v}} \in \mathbf{H}, \\ |\mathbf{b}(\vec{\mathbf{v}},\boldsymbol{\tau})| &\leqslant \|\mathbf{b}\|\,\|\vec{\mathbf{v}}\|_{\mathbf{H}}\,\|\boldsymbol{\tau}\|_{\mathbf{Q}} \qquad \forall\,(\vec{\mathbf{v}},\boldsymbol{\tau}) \in \mathbf{H} \times \mathbf{Q}, \\ |\mathbf{c}(\mathbf{z};\vec{\mathbf{w}},\vec{\mathbf{v}})| &\leqslant \|\mathbf{c}\|\,\|\mathbf{z}\|_{0,4;\Omega}\,\|\vec{\mathbf{w}}\|_{\mathbf{H}}\,\|\vec{\mathbf{v}}\|_{\mathbf{H}} \qquad \forall\,\mathbf{z} \in \mathbf{L}^4(\Omega), \quad \forall\,\vec{\mathbf{w}},\vec{\mathbf{v}} \in \mathbf{H}, \\ |\mathbf{G}(\boldsymbol{\tau})| &\leqslant \|\mathbf{G}\|\,\|\boldsymbol{\tau}\|_{\mathbf{Q}} \qquad \forall\,\boldsymbol{\tau} \in \mathbf{Q}, \end{aligned} \tag{3.15}$$

and

$$|\mathbf{F}_{\boldsymbol{\eta},\boldsymbol{\phi}}(\mathbf{v})| \leqslant \|\mathbf{F}\|\left\{\|\eta_1 - \eta_2\|_{0,\rho;\Omega}\,\|\boldsymbol{\phi}\|_{0,r;\Omega} + \|\mathbf{f}\|_{0,4/3;\Omega}\right\}\|\mathbf{v}\|_{0,4;\Omega} \qquad \forall\,\mathbf{v} \in \mathbf{L}^4(\Omega).$$

Furthermore, simple algebraic calculations show that

$$\mathbf{c}(\mathbf{z};\vec{\mathbf{v}},\vec{\mathbf{v}}) = 0 \qquad \forall\,\mathbf{z} \in \mathbf{L}^4(\Omega), \quad \forall\,\vec{\mathbf{v}} \in \mathbf{H}. \tag{3.16}$$

3.3 The electrostatic potential equations

The derivation of the mixed formulation for the electrostatic potential equations (fourth and fifth rows of (2.8)) has been presented in [13, Section 3.3]. It reads: Find $(\boldsymbol{\varphi}, \chi) \in \mathrm{X}_2 \times \mathrm{M}_1$ such that

$$\begin{aligned} a(\boldsymbol{\varphi}, \boldsymbol{\psi}) + b_1(\boldsymbol{\psi}, \chi) &= \mathrm{F}(\boldsymbol{\psi}) \qquad \forall\, \boldsymbol{\psi} \in \mathrm{X}_1\,, \\ b_2(\boldsymbol{\varphi}, \lambda) &= \mathrm{G}_{\boldsymbol{\xi}}(\lambda) \qquad \forall\, \lambda \in \mathrm{M}_2\,, \end{aligned} \tag{3.17}$$

where

$$\mathrm{X}_2 := \mathbf{H}^r(\mathrm{div}_r; \Omega)\,, \quad \mathrm{M}_1 := \mathrm{L}^r(\Omega)\,, \quad \mathrm{X}_1 := \mathbf{H}^s(\mathrm{div}_s; \Omega)\,, \quad \mathrm{M}_2 := \mathrm{L}^s(\Omega)\,,$$

and the bilinear forms $a : \mathrm{X}_2 \times \mathrm{X}_1 \to \mathrm{R}$ and $b_i : \mathrm{X}_i \times \mathrm{M}_i \to \mathrm{R}$, $i \in \{1, 2\}$, and the functional $\mathrm{F} : \mathrm{X}_1 \to \mathrm{R}$, are defined, respectively, as

$$a(\boldsymbol{\phi}, \boldsymbol{\psi}) := \int_\Omega \frac{1}{\varepsilon}\, \boldsymbol{\phi} \cdot \boldsymbol{\psi} \qquad \forall\, (\boldsymbol{\phi}, \boldsymbol{\psi}) \in \mathrm{X}_2 \times \mathrm{X}_1\,, \tag{3.18}$$

$$b_i(\boldsymbol{\psi}, \lambda) := \int_\Omega \lambda\, \mathrm{div}(\boldsymbol{\psi}) \qquad \forall\, (\boldsymbol{\psi}, \lambda) \in \mathrm{X}_i \times \mathrm{M}_i\,, \tag{3.19}$$

$$\mathrm{F}(\boldsymbol{\psi}) := \langle \boldsymbol{\psi} \cdot \boldsymbol{\nu}, g \rangle_\Gamma \qquad \forall\, \boldsymbol{\psi} \in \mathrm{X}_1\,, \tag{3.20}$$

whereas, given $\boldsymbol{\eta} := (\eta_1, \eta_2) \in \mathbf{L}^\rho(\Omega)$, the functional $\mathrm{G}_{\boldsymbol{\eta}} : \mathrm{M}_2 \to \mathrm{R}$ is defined by

$$\mathrm{G}_{\boldsymbol{\eta}}(\lambda) := -\int_\Omega \lambda\, (\eta_1 - \eta_2) - \int_\Omega f\, \lambda \qquad \forall\, \lambda \in \mathrm{M}_2\,. \tag{3.21}$$

Note from (3.1a) - (3.1e) that η_1 and η_2 must belong to $\mathrm{L}^\rho(\Omega)$. Also, in order for the first term on the right-hand side of (3.21) to make sense, we require that $\rho \geqslant r$.

For the boundedness of a, b_i, $i \in \{1, 2\}$, F, and $\mathrm{G}_{\boldsymbol{\eta}}$, we recall that the norm of X_1 and X_2 are defined by (1.2c) with $t = s$ and $t = r$, respectively, whereas those of M_1 and M_2 are given by $\|\cdot\|_{0,r;\Omega}$ and $\|\cdot\|_{0,s;\Omega}$, respectively. Then, employing again the Cauchy–Schwarz and Hölder inequalities, bounding ε^{-1} according to (2.5), and using that $\|\cdot\|_{0,r;\Omega} \leqslant |\Omega|^{(\rho-r)/\rho r} \|\cdot\|_{0,\rho;\Omega}$, which follows from the fact that $\rho \geqslant r$, we find that there exist positive constants

$$\|a\| := \frac{1}{\varepsilon_0}\,, \qquad \|b_1\| = \|b_2\| := 1\,, \quad \text{and} \quad \|\mathrm{G}\| := \max\left\{1, |\Omega|^{(\rho-r)/\rho r}\right\}\,,$$

such that

$$\begin{aligned} |a(\boldsymbol{\phi}, \boldsymbol{\psi})| &\leqslant \|a\|\, \|\boldsymbol{\phi}\|_{\mathrm{X}_2}\, \|\boldsymbol{\psi}\|_{\mathrm{X}_1} \qquad \forall\, (\boldsymbol{\phi}, \boldsymbol{\psi}) \in \mathrm{X}_2 \times \mathrm{X}_1\,, \\ |b_i(\boldsymbol{\psi}, \lambda)| &\leqslant \|b_i\|\, \|\boldsymbol{\psi}\|_{\mathrm{X}_i}\, \|\lambda\|_{\mathrm{M}_i} \qquad \forall\, (\boldsymbol{\psi}, \lambda) \in \mathrm{X}_i \times \mathrm{M}_1\,, \quad \forall\, i \in \{1, 2\}\,, \quad \text{and} \\ |\mathrm{G}_{\boldsymbol{\eta}}(\lambda)| &\leqslant \|\mathrm{G}\| \left\{ \|\eta_1 - \eta_2\|_{0,\rho;\Omega} + \|f\|_{0,r;\Omega} \right\} \|\lambda\|_{0,s;\Omega} \qquad \forall\, \lambda \in \mathrm{M}_2\,. \end{aligned} \tag{3.22}$$

Regarding the boundedness of F, we need to apply [15, Lemma A.36], which, along with the surjectivity of the trace operator mapping $\mathrm{W}^{1,r}(\Omega)$ onto $\mathrm{W}^{1/s,r}(\Gamma)$, yields the existence of a fixed constant $C_r > 0$, such that for the given $g \in \mathrm{W}^{1/s,r}(\Gamma)$, there exists $v_g \in \mathrm{W}^{1,r}(\Omega)$ satisfying $v_g|_\Gamma = g$ and $\|v_g\|_{1,r;\Omega} \leqslant C_r\, \|g\|_{1/s,r;\Gamma}$. In this way, employing now (1.4), we obtain

$$|\mathrm{F}(\boldsymbol{\psi})| \leqslant \|\mathrm{F}\|\, \|\boldsymbol{\psi}\|_{\mathrm{X}_1} \quad \forall\, \boldsymbol{\psi} \in \mathrm{X}_1\,, \quad \text{with} \quad \|\mathrm{F}\| := C_r\, \|g\|_{1/s,r;\Gamma}\,.$$

We stress that the above derivation is analogous to the one for the boundedness of $\mathbf{G}$ (cf. (3.14)). However, note that, though similar, two different integration by parts formulae, namely (1.3b) and (1.4), are employed, and that the final estimates yielding $\|\mathbf{g}\|_{1/2,\Gamma}$ and $\|g\|_{1/s,r;\Gamma}$ are obtained by an equality and an inequality, respectively.

3.4 The ionized particles concentration equations

The following mixed variational formulation for the ionized particles concentration equations has been proposed in [13, Section 3.4]: Find $(\boldsymbol{\sigma}_i, \xi_i) \in \mathrm{H}_i \times \mathrm{Q}_i$ such that

$$\begin{array}{rcll} a_i(\boldsymbol{\sigma}_i, \boldsymbol{\tau}_i) + c_i(\boldsymbol{\tau}_i, \xi_i) \,-\, c_{\boldsymbol{\varphi},\mathbf{u}}(\boldsymbol{\tau}_i, \xi_i) & = & \mathrm{F}_i(\boldsymbol{\tau}_i) & \forall\, \boldsymbol{\tau}_i \in \mathrm{H}_i\,, \\ c_i(\boldsymbol{\sigma}_i, \eta_i) \,-\, d_i(\xi_i, \eta_i) & = & \mathrm{G}_i(\eta_i) & \forall\, \eta_i \in \mathrm{Q}_i\,, \end{array} \tag{3.23}$$

where

$$\mathrm{H}_i \,:=\, \mathbf{H}(\mathbf{div}_\varrho;\Omega)\,, \qquad \mathrm{Q}_i \,:=\, \mathrm{L}^\rho(\Omega)\,, \tag{3.24}$$

and the bilinear forms $a_i : \mathrm{H}_i \times \mathrm{H}_i \to \mathrm{R}$, $c_i : \mathrm{H}_i \times \mathrm{Q}_i \to \mathrm{R}$, and $d_i : \mathrm{Q}_i \times \mathrm{Q}_i \to \mathrm{R}$, and the functionals $\mathrm{F}_i : \mathrm{H}_i \to R$ and $\mathrm{G}_i : \mathrm{Q}_i \to \mathrm{R}$, are defined, respectively, as

$$a_i(\boldsymbol{\zeta}_i, \boldsymbol{\tau}_i) \,:=\, \int_\Omega \frac{1}{\kappa_i}\, \boldsymbol{\zeta}_i \cdot \boldsymbol{\tau}_i \qquad \forall\, (\boldsymbol{\zeta}_i, \boldsymbol{\tau}_i) \in \mathrm{H}_i \times \mathrm{H}_i\,, \tag{3.25a}$$

$$c_i(\boldsymbol{\tau}_i, \eta_i) \,:=\, \int_\Omega \eta_i\, \mathrm{div}(\boldsymbol{\tau}_i) \qquad \forall\, (\boldsymbol{\tau}_i, \eta_i) \in \mathrm{H}_i \times \mathrm{Q}_i\,, \tag{3.25b}$$

$$d_i(\vartheta, \eta_i) \,:=\, \int_\Omega \vartheta_i\, \eta_i \qquad \forall\, (\vartheta_i, \eta_i) \in \mathrm{Q}_i \times \mathrm{Q}_i\,, \tag{3.25c}$$

$$\mathrm{F}_i(\boldsymbol{\tau}_i) \,:=\, \langle \boldsymbol{\tau}_i \cdot \boldsymbol{\nu}, g_i \rangle \qquad \forall\, \boldsymbol{\tau}_i \in \mathrm{H}_i\,, \tag{3.25d}$$

$$\mathrm{G}_i(\eta_i) \,:=\, -\int_\Omega f_i\, \eta_i \qquad \forall\, \eta_i \in \mathrm{Q}_i\,, \tag{3.25e}$$

whereas, given $(\boldsymbol{\phi}, \mathbf{v}) \in \mathrm{X}_2 \times \mathbf{L}^4(\Omega)$, the bilinear form $c_{\boldsymbol{\phi},\mathbf{v}} : \mathrm{H}_i \times \mathrm{Q}_i \to \mathrm{R}$ is set as

$$c_{\boldsymbol{\phi},\mathbf{v}}(\boldsymbol{\tau}_i, \eta_i) \,:=\, \int_\Omega \Big\{ q_i\, \eta_i\, \varepsilon^{-1}\, \boldsymbol{\phi} \,-\, \kappa_i^{-1}\, \eta_i\, \mathbf{v} \Big\} \cdot \boldsymbol{\tau}_i \qquad \forall\, (\boldsymbol{\tau}_i, \eta_i) \in \mathrm{H}_i \times \mathrm{Q}_i\,.$$

It is concluded that a_i, c_i, d_i, F_i, G_i and $c_{\boldsymbol{\phi},\mathbf{v}}$ are all bounded with the norm defined by (1.2a) with $t \,=\, \varrho$ for H_i, and certainly the norm $\|\cdot\|_{0,\rho;\Omega}$ for Q_i. Indeed, applying the Cauchy–Schwarz and Hölder inequalities, bounding both ε^{-1} and κ^{-1} according to (2.5), noting that $\|\cdot\|_{0,\Omega} \,\leqslant\, |\Omega|^{(\rho-2)/2\rho}\|\cdot\|_{0,\rho;\Omega}$, invoking the identity (1.3a) and the continuous injection $i_\rho : \mathrm{H}^1(\Omega) \to \mathrm{L}^\rho(\Omega)$, similarly as for the boundedness of $\mathbf{G}$ (cf. (3.14)), and utilizing (3.1d) and (3.1e), we find that there exist positive constants

$$\|a_i\| \,:=\, \frac{1}{\underline{\kappa}}\,, \quad \|c_i\| \,:=\, 1\,, \quad \|d_i\| \,:=\, |\Omega|^{(\rho-2)/\rho}\,, \quad \|\mathrm{F}_i\| \,:=\, (1 + \|i_\rho\|)\, \|g_i\|_{1/2,\Gamma}\,,$$
$$\|\mathrm{G}_i\| \,:=\, \|f_i\|_{0,\varrho;\Omega}\,, \quad \text{and} \quad \|c\| \,:=\max\left\{\varepsilon_0^{-1}, \underline{\kappa}^{-1}\right\}\,,$$

such that

$$\begin{array}{l} |a_i(\boldsymbol{\zeta}_i, \boldsymbol{\tau}_i)| \,\leqslant\, \|a_i\|\, \|\boldsymbol{\zeta}_i\|_{\mathrm{H}_i}\, \|\boldsymbol{\tau}_i\|_{\mathrm{H}_i} \qquad \forall\, (\boldsymbol{\zeta}_i, \boldsymbol{\tau}_i) \in \mathrm{H}_i \times \mathrm{H}_i\,, \\ |c_i(\boldsymbol{\tau}_i, \eta_i)| \,\leqslant\, \|c_i\|\, \|\boldsymbol{\tau}_i\|_{\mathrm{H}_i}\, \|\eta_i\|_{\mathrm{Q}_i} \qquad \forall\, (\boldsymbol{\tau}_i, \eta_i) \in \mathrm{H}_i \times \mathrm{Q}_i\,, \\ |d_i(\vartheta_i, \eta_i)| \,\leqslant\, \|d_i\|\, \|\vartheta\|_{\mathrm{Q}_i}\, \|\eta_i\|_{\mathrm{Q}_i} \qquad \forall\, (\vartheta, \eta_i) \in \mathrm{Q}_i \times \mathrm{Q}_i\,, \\ |\mathrm{F}_i(\boldsymbol{\tau}_i)| \,\leqslant\, \|\mathrm{F}_i\|\, \|\boldsymbol{\tau}_i\|_{\mathrm{H}_i} \qquad \forall\, \boldsymbol{\tau}_i \in \mathrm{H}_i\,, \\ |\mathrm{G}_i(\eta_i)| \,\leqslant\, \|\mathrm{G}_i\|\, \|\eta_i\|_{\mathrm{Q}_i} \qquad \forall\, \eta_i \in \mathrm{Q}_i \quad \text{and} \\ |c_{\boldsymbol{\phi},\mathbf{v}}(\boldsymbol{\tau}_i, \eta_i)| \,\leqslant\, \|c\| \Big\{ \|\boldsymbol{\phi}\|_{0,r;\Omega} \,+\, \|\mathbf{v}\|_{0,r;\Omega} \Big\}\, \|\eta_i\|_{0,\rho;\Omega}\, \|\boldsymbol{\tau}_i\|_{0,\Omega} \qquad \forall\, (\boldsymbol{\tau}_i, \eta_i) \in \mathrm{H}_i \times \mathrm{Q}_i\,. \end{array}$$

In the rest of the paper will be used indistinctly either $\|\boldsymbol{\eta}\|_{\mathrm{Q}_1 \times \mathrm{Q}_2}$ or $\|\boldsymbol{\eta}\|_{0,\rho;\Omega}$, where

$$\|\boldsymbol{\eta}\|_{0,\rho;\Omega} \,:=\, \|\eta_1\|_{0,\rho;\Omega} \,+\, \|\eta_2\|_{0,\rho;\Omega} \qquad \forall\, \boldsymbol{\eta} \,:=\, (\eta_1, \eta_2) \in \mathrm{Q}_1 \times \mathrm{Q}_2\,.$$

3.5 The whole coupled formulation

Summarizing the discussion from the previous sections, and putting together (3.9), (3.17), and (3.23), we find that, under the assumptions that $\mathbf{f} \in \mathbf{L}^{4/3}(\Omega)$, $\mathbf{g} \in \mathbf{H}^{1/2}(\Gamma)$, $f \in \mathrm{L}^r(\Omega)$, $g \in \mathrm{W}^{1/s,r}(\Gamma)$, $f_i \in \mathrm{L}^\varrho(\Omega)$, $g_i \in \mathrm{H}^{1/2}(\Gamma)$, and $\rho \geqslant r$, the variational formulation of (2.8) reduces to: Find $(\vec{\mathbf{u}}, \boldsymbol{\sigma}) \in \mathbf{H} \times \mathbf{Q}$, $(\boldsymbol{\varphi}, \chi) \in \mathrm{X}_2 \times \mathrm{M}_1$, and $(\boldsymbol{\sigma}_i, \xi_i) \in \mathrm{H}_i \times \mathrm{Q}_i$, $i \in \{1, 2\}$, such that

$$\begin{aligned}
\mathbf{a}(\vec{\mathbf{u}}, \vec{\mathbf{v}}) + \mathbf{c}(\mathbf{u}; \vec{\mathbf{u}}, \vec{\mathbf{v}}) + \mathbf{b}(\vec{\mathbf{v}}, \boldsymbol{\sigma}) &= \mathbf{F}_{\boldsymbol{\xi},\boldsymbol{\varphi}}(\vec{\mathbf{v}}) && \forall\, \vec{\mathbf{v}} \in \mathbf{H}\,,\\
\mathbf{b}(\vec{\mathbf{u}}, \boldsymbol{\tau}) &= \mathbf{G}(\boldsymbol{\tau}) && \forall\, \boldsymbol{\tau} \in \mathbf{Q}\,,\\
a(\boldsymbol{\varphi}, \boldsymbol{\psi}) + b_1(\boldsymbol{\psi}, \chi) &= \mathrm{F}(\boldsymbol{\psi}) && \forall\, \boldsymbol{\psi} \in X_1\,,\\
b_2(\boldsymbol{\varphi}, \lambda) &= \mathrm{G}_{\boldsymbol{\xi}}(\lambda) && \forall\, \lambda \in M_2\,,\\
a_i(\boldsymbol{\sigma}_i, \boldsymbol{\tau}_i) + c_i(\boldsymbol{\tau}_i, \xi_i) - c_{\boldsymbol{\varphi},\mathbf{u}}(\boldsymbol{\tau}_i, \xi_i) &= \mathrm{F}_i(\boldsymbol{\tau}_i) && \forall\, \boldsymbol{\tau}_i \in \mathrm{H}_i\,,\\
c_i(\boldsymbol{\sigma}_i, \eta_i) - d_i(\xi_i, \eta_i) &= \mathrm{G}_i(\eta_i) && \forall\, \eta_i \in \mathrm{Q}_i\,.
\end{aligned} \tag{3.26}$$

We stress here that the feasible ranges for the indexes ℓ, j, ρ, ϱ, r, and s, are specified below in (4.8).

4 The continuous solvability analysis

In this section we proceed similarly to how it was done in [10] and [17] (see also [2, 5, 6, 13, 18], and some of the references therein) and adopt a fixed-point strategy to analyze the solvability of (3.26)

4.1 The fixed-point approach

We begin by rewriting (3.26) as an equivalent fixed-point equation, for which we first introduce the operator $\mathbf{S} : \mathbf{L}^4(\Omega) \times (\mathrm{Q}_1 \times \mathrm{Q}_2) \times \mathrm{X}_2 \to \mathbf{L}^4(\Omega)$ defined by

$$\mathbf{S}(\mathbf{z}, \boldsymbol{\eta}, \boldsymbol{\phi}) = \mathbf{u} \qquad \forall\, (\mathbf{z}, \boldsymbol{\eta}, \boldsymbol{\phi}) \in \mathbf{L}^4(\Omega) \times (\mathrm{Q}_1 \times \mathrm{Q}_2) \times \mathrm{X}_2\,,$$

where $(\vec{\mathbf{u}}, \boldsymbol{\sigma}) = ((\mathbf{u}, \mathbf{t}), \boldsymbol{\sigma}) \in \mathbf{H} \times \mathbf{Q}$ is the unique solution (conditions for its existence are to be derived below) of the problem (3.9) (equivalently, the first and second rows of (3.26)) when $\mathbf{c}(\mathbf{u}, \cdot, \cdot)$ and $\mathbf{F}_{\boldsymbol{\xi},\boldsymbol{\varphi}}$ are replaced by $\mathbf{c}(\mathbf{z}, \cdot, \cdot)$ and $\mathbf{F}_{\boldsymbol{\eta},\boldsymbol{\phi}}$, respectively, that is

$$\begin{aligned}
\mathbf{a}(\vec{\mathbf{u}}, \vec{\mathbf{v}}) + \mathbf{c}(\mathbf{z}; \vec{\mathbf{u}}, \vec{\mathbf{v}}) + \mathbf{b}(\vec{\mathbf{v}}, \boldsymbol{\sigma}) &= \mathbf{F}_{\boldsymbol{\eta},\boldsymbol{\phi}}(\vec{\mathbf{v}}) && \forall\, \vec{\mathbf{v}} \in \mathbf{H}\,,\\
\mathbf{b}(\vec{\mathbf{u}}, \boldsymbol{\tau}) &= \mathbf{G}(\boldsymbol{\tau}) && \forall\, \boldsymbol{\tau} \in \mathbf{Q}\,.
\end{aligned} \tag{4.1}$$

In turn, we also introduce the operator $\bar{\mathrm{T}} : \mathrm{Q}_1 \times \mathrm{Q}_2 \to \mathrm{X}_2$ defined as

$$\bar{\mathrm{T}}(\boldsymbol{\eta}) := \boldsymbol{\varphi} \qquad \forall\, \boldsymbol{\eta} \in \mathrm{Q}_1 \times \mathrm{Q}_2\,,$$

where $(\boldsymbol{\varphi}, \chi) \in \mathrm{X}_2 \times \mathrm{M}_1$ is the unique solution (to be confirmed below) of problem (3.17) (equivalently, the third and fourth rows of (3.26)) with $\boldsymbol{\eta}$ instead of $\boldsymbol{\xi}$

$$\begin{aligned}
a(\boldsymbol{\varphi}, \boldsymbol{\psi}) + b_1(\boldsymbol{\psi}, \chi) &= \mathrm{F}(\boldsymbol{\psi}) && \forall\, \boldsymbol{\psi} \in X_1\,,\\
b_2(\boldsymbol{\varphi}, \lambda) &= \mathrm{G}_{\boldsymbol{\eta}}(\lambda) && \forall\, \lambda \in M_2\,.
\end{aligned} \tag{4.2}$$

Similarly, for each $i \in \{1, 2\}$, we define the operator $\widetilde{\mathrm{T}}_i : \mathrm{X}_2 \times \mathbf{L}^4(\Omega) \to \mathrm{Q}_i$ as

$$\widetilde{\mathrm{T}}_i(\boldsymbol{\phi}, \mathbf{v}) := \xi_i \qquad \forall\, (\boldsymbol{\phi}, \mathbf{v}) \in \mathrm{X}_2 \times \mathbf{L}^4(\Omega)\,,$$

where $(\boldsymbol{\sigma}_i, \xi_i) \in \mathrm{H}_i \times \mathrm{Q}_i$ is the unique solution (to be confirmed below) of problem (3.23) (equivalently, the fifth and sixth rows of (3.26)) with $(\boldsymbol{\phi}, \mathbf{v})$ instead $(\boldsymbol{\varphi}, \mathbf{u})$, that is

$$\begin{aligned}
a_i(\boldsymbol{\sigma}_i, \boldsymbol{\tau}_i) + c_i(\boldsymbol{\tau}_i, \xi_i) - c_{\boldsymbol{\phi},\mathbf{v}}(\boldsymbol{\tau}_i, \xi_i) &= \mathrm{F}_i(\boldsymbol{\tau}_i) && \forall\, \boldsymbol{\tau}_i \in \mathrm{H}_i\,,\\
c_i(\boldsymbol{\sigma}_i, \eta_i) - d_i(\xi_i, \eta_i) &= \mathrm{G}_i(\eta_i) && \forall\, \eta_i \in \mathrm{Q}_i\,,
\end{aligned} \tag{4.3}$$

so that we can define the operator $\widetilde{\mathrm{T}} : \mathrm{X}_2 \times \mathbf{L}^4(\Omega) \to (\mathrm{Q}_1 \times \mathrm{Q}_2)$ as

$$\widetilde{\mathrm{T}}(\boldsymbol{\phi}, \mathbf{v}) \,:=\, \Big(\widetilde{\mathrm{T}}_1(\boldsymbol{\phi}, \mathbf{v}), \widetilde{\mathrm{T}}_2(\boldsymbol{\phi}, \mathbf{v})\Big) \,=\, (\xi_1, \xi_2) \,=:\, \boldsymbol{\xi} \qquad \forall\, (\boldsymbol{\phi}, \mathbf{v}) \in \mathrm{X}_2 \times \mathbf{L}^4(\Omega)\,. \tag{4.4}$$

Finally, defining the operator $\mathbf{T} : \mathrm{X}_2 \times \mathbf{L}^4(\Omega) \to \mathrm{X}_2 \times \mathbf{L}^4(\Omega)$ as

$$\mathbf{T}(\boldsymbol{\phi}, \mathbf{z}) \,:=\, \Big(\bar{\mathrm{T}}(\widetilde{\mathrm{T}}(\boldsymbol{\phi}, \mathbf{z})), \mathbf{S}\big(\mathbf{z}, \widetilde{\mathrm{T}}(\boldsymbol{\phi}, \mathbf{z}), \bar{\mathrm{T}}(\widetilde{\mathrm{T}}(\boldsymbol{\phi}, \mathbf{z}))\big)\Big)\,, \tag{4.5}$$

we observe that solving (3.26) is equivalent to seeking a fixed point of $\mathbf{T}$, that is: Find $(\boldsymbol{\varphi}, \mathbf{u}) \in \mathrm{X}_2 \times \mathbf{L}^4(\Omega)$ such that

$$\mathbf{T}(\boldsymbol{\varphi}, \mathbf{u}) \,=\, (\boldsymbol{\varphi}, \mathbf{u})\,.$$

4.2 Well-posedness of the uncoupled problems

In this section we show that the problems (4.1), (4.2) and (4.3) are well-posed; and therefore the respective operators $\mathbf{S}$, $\bar{\mathrm{T}}$, and $\widetilde{\mathrm{T}}_i$ are well-defined. To that end, we will employ Babuška–Brezzi theory in Banach spaces for the general case (cf. [3, Theorem 2.1, Corollary 2.1, Section 2.1], and for a particular one [15, Theorem 2.34]), as well as a recently established result for perturbed saddle point formulations in Banach spaces (cf. [12, Theorem 3.4]) along with the Banach–Nečas–Babuška Theorem (also known as the generalized Lax–Milgram Lemma) (cf. [15, Theorem 2.6]).

To prove that, given an arbitrary $(\mathbf{z}, \boldsymbol{\eta}, \boldsymbol{\phi}) \in \mathbf{L}^4(\Omega) \times (\mathrm{Q}_1 \times \mathrm{Q}_2) \times \mathrm{X}_2$, (4.1) is well-posed, equivalently that $\mathbf{S}$ is well-defined, we cite the work done in [6, Section 3.2.1] and the references therein. It has to be emphasized that $\boldsymbol{\alpha}$ will denote the $\mathbf{V}-$ellipticity constant of $\mathbf{a}$, $\boldsymbol{\beta}$ is the constant of the inf-sup condition of $\mathbf{b}$ and $\mathbf{i}_4$ denotes the continuous injection of $\mathbf{H}^1(\Omega)$ into $\mathbf{L}^4(\Omega)$ (for more details see [6, Section 3.2.1]). In turn, they proved the following lemma.

Lemma 4.1. *For each $(\mathbf{z}, \boldsymbol{\eta}, \boldsymbol{\phi}) \in \mathbf{L}^4(\Omega) \times (\mathrm{Q}_1 \times \mathrm{Q}_2) \times \mathrm{X}_2$ there exists a unique $(\vec{\mathbf{u}}, \boldsymbol{\sigma}) \,=\, ((\mathbf{u}, \mathbf{t}), \boldsymbol{\sigma}) \in \mathbf{H} \times \mathbf{Q}$ solution of (4.1), and hence one can define $\mathbf{S}(\mathbf{z}, \boldsymbol{\eta}, \boldsymbol{\phi}) \,:=\, \mathbf{u} \in \mathbf{L}^4(\Omega)$. Moreover, there exists a positive constant $C_{\mathbf{S}}$, depending only on $|\Omega|$, $\|\mathbf{i}_4\|$, μ, ω, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, such that*

$$\|\mathbf{S}(\mathbf{z}, \boldsymbol{\eta}, \boldsymbol{\phi})\|_{0,4;\Omega} \,=\, \|\mathbf{u}\|_{0,4;\Omega} \,\leqslant\, \|\vec{\mathbf{u}}\|_{\mathbf{H}} \leqslant C_{\mathbf{S}} \Big\{ \|\boldsymbol{\eta}\|_{0,\rho;\Omega} \|\boldsymbol{\phi}\|_{0,r;\Omega} + \|\mathbf{f}\|_{0,4/3;\Omega} + (1 + \|\mathbf{z}\|_{0,4;\Omega}) \|\mathbf{g}\|_{1/2,\Gamma} \Big\}. \tag{4.6}$$

Proof. The proof is analogous to that of [6, Lemma 3.1]. □

Furthermore, proceeding similarly to the derivation of (4.6) (see [6, Lemma 3.1]), we get

$$\|\boldsymbol{\sigma}\|_{\mathbf{Q}} = \|\boldsymbol{\sigma}\|_{\mathbf{div}_{4/3};\Omega} \leqslant \bar{C}_{\mathbf{S}} (1 + \|\mathbf{z}\|_{0,4;\Omega}) \Big\{ \|\boldsymbol{\eta}\|_{0,\rho;\Omega} \|\boldsymbol{\phi}\|_{0,r;\Omega} + \|\mathbf{f}\|_{0,4/3;\Omega} + (1 + \|\mathbf{z}\|_{0,4;\Omega}) \|\mathbf{g}\|_{1/2,\Gamma} \Big\}, \tag{4.7}$$

where $\bar{C}_{\mathbf{S}}$ is a positive constant depending, as well, on $|\Omega|$, $\mathbf{i_4}$, μ, ω, $\boldsymbol{\alpha}$, and $\boldsymbol{\beta}$.

In order to prove that, given an arbitrary $\boldsymbol{\eta} \in \mathrm{Q}_1 \times \mathrm{Q}_2$, problem (4.2) is well-posed (and, equivalently, that $\bar{\mathrm{T}}$ is well-defined), we take inspiration from the work done in [13, Section 4.2.2] and the references therein. It should be noted that throughout the analysis performed in [13, Section 4.2.2] for the well-definedness of $\bar{\mathrm{T}}$, suitable ranges were specified for the index of each space (cf. [13, Lemma 4.4]), in particular for ℓ and, consequently, for j, r, s, ρ, and ϱ. In our case, we have that $2j \,\leqslant\, 4$. Therefore, these ranges do not change, and the appropriate ranges needed for the analysis will be as follows

$$\begin{cases} \ell \in [2, +\infty)\,,\ j \in (1, 2]\,, \quad \rho \in [4, +\infty)\,,\ \varrho \in (1, 4/3]\,, \quad r \in (2, 4]\,,\ s \in [4/3, 2) \quad \text{if } n = 2\,, \\ \qquad\quad \ell = 3\,,\ j = 3/2\,, \quad \rho = 6\,,\ \varrho = 6/5\,, \quad r = 3\,,\ s = 3/2 \quad \text{if } n = 3\,. \end{cases} \tag{4.8}$$

On the other hand, as a consequence of [13, Lemmas 4.3 and 4.4] and the boundedness stated in (3.22), we are able to conclude that the operator $\bar{\mathrm{T}}$ is well-defined. More precisely, we denote by $\bar{\alpha}$ and $\bar{\beta}_i$ the inf-sup constants for the bilinear forms a and b_i, $i \in \{1, 2\}$, respectively (cf. [13, Lemmas 4.3 and 4.4, respectively]), and state the following result from [13, Theorem 4.5].

Theorem 4.2. *For each $\boldsymbol{\eta} \in \mathrm{Q}_1 \times \mathrm{Q}_2$ there exists a unique $(\boldsymbol{\varphi}, \chi) \in \mathrm{X}_2 \times \mathrm{M}_1$ solution to (4.2), and hence one can define $\bar{\mathrm{T}}(\boldsymbol{\eta}) := \boldsymbol{\varphi} \in \mathrm{X}_2$. Moreover, there exists a positive constant $C_{\bar{\mathrm{T}}}$, depending only on, ε_0, C_r, $|\Omega|$, $\bar{\alpha}$, and $\bar{\beta}_2$, such that*

$$\|\bar{\mathrm{T}}(\boldsymbol{\eta})\|_{\mathrm{X}_2} := \|\boldsymbol{\varphi}\|_{\mathrm{X}_2} \leqslant C_{\bar{\mathrm{T}}}\Big\{\|g\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega} + \|\boldsymbol{\eta}\|_{0,\rho;\Omega}\Big\}. \tag{4.9}$$

Employing [3, Corollary 2.1, Section 2.1, eq. (2.16)] we observe that the a priori bound for the χ component of the unique solution to (4.2) reduces to

$$\|\chi\|_{\mathrm{M}_1} \leqslant \frac{1}{\bar{\beta}_1}\left(1 + \frac{\|a\|}{\bar{\alpha}}\right)\|\mathrm{F}\|_{\mathrm{X}_1'} + \frac{\|a\|}{\bar{\beta}_1\bar{\beta}_2}\left(1 + \frac{\|a\|}{\bar{\alpha}}\right)\|\mathrm{G}_{\boldsymbol{\eta}}\|_{M_2'}. \tag{4.10}$$

As in (4.9), the same inequality is obtained for (4.10), but with a different constant, in particular depending additionally on $\bar{\beta}_1$. Therefore, as before, we still denote the largest of them by $C_{\bar{\mathrm{T}}}$, and simply say that the right-hand side of (4.9) constitutes the a priori estimate for both $\boldsymbol{\varphi}$ and χ.

Finally, in order to prove that, given an arbitrary $(\boldsymbol{\phi}, \mathbf{v}) \in \mathrm{X}_2 \times \mathbf{L}^4(\Omega)$, (4.3) is well-posed for each $i \in \{1, 2\}$, we observe first that the operator $\widetilde{\mathrm{T}}$ is defined in the same way as in [13, Section 4.2.3].

Therefore we introduce the bilinear forms $\mathbf{A}$, $\mathbf{A}_{\boldsymbol{\phi},\mathbf{v}} : (\mathrm{H}_i \times \mathrm{Q}_i) \times (\mathrm{H}_i \times \mathrm{Q}_i) \to \mathrm{R}$ given by

$$\mathbf{A}\left((\boldsymbol{\zeta}_i, \vartheta_i), (\boldsymbol{\tau}_i, \eta_i)\right) := a_i(\boldsymbol{\zeta}_i, \boldsymbol{\tau}_i) + c_i(\boldsymbol{\tau}_i, \vartheta_i) + c_i(\boldsymbol{\zeta}_i, \eta_i) - d_i(\vartheta, \eta_i), \tag{4.11a}$$

$$\mathbf{A}_{\boldsymbol{\phi},\mathbf{v}}\left((\boldsymbol{\zeta}_i, \vartheta_i), (\boldsymbol{\tau}_i, \eta_i)\right) := \mathbf{A}\left((\boldsymbol{\zeta}_i, \vartheta_i), (\boldsymbol{\tau}_i, \eta_i)\right) - c_{\boldsymbol{\phi},\mathbf{v}}(\boldsymbol{\tau}_i, \vartheta_i), \tag{4.11b}$$

for all $(\boldsymbol{\zeta}_i, \vartheta_i)$, $(\boldsymbol{\tau}_i, \eta_i) \in \mathrm{H}_i \times \mathrm{Q}_i$, so that (4.3) can be re-stated as: Find $(\boldsymbol{\sigma}_i, \xi_i) \in \mathrm{H}_i \times \mathrm{Q}_i$ such that

$$\mathbf{A}_{\boldsymbol{\phi},\mathbf{v}}\left((\boldsymbol{\sigma}_i, \xi_i), (\boldsymbol{\tau}_i, \eta_i)\right) = \mathrm{F}_i(\boldsymbol{\tau}_i) + \mathrm{G}_i(\eta_i) \qquad \forall\, (\boldsymbol{\tau}_i, \eta_i) \in \mathrm{H}_i \times \mathrm{Q}_i.$$

Thus, the proof reduces to first showing that the bilinear forms that are part of $\mathbf{A}$ satisfy the hypotheses of [12, Theorem 3.4] and then combine the consequence of this result with the effect of the extra term given by $c_{\boldsymbol{\phi},\mathbf{v}}(\cdot,\cdot)$, to conclude that $\mathbf{A}_{\boldsymbol{\phi},\mathbf{v}}$ satisfies a global inf-sup condition. Indeed, it is clear from (3.25a), (3.25c) and the upper bound of κ_i (cf. (2.5)) that a_i and d_i are symmetric and positive semi-definite, which proves the assumption i) of [12, Theorem 3.4]. Next, taking into account the definitions of c_i (cf. (3.25b)) and the spaces H_i and Q_i (cf. (3.24)), and using that $\mathrm{L}^\rho(\Omega)$ is isomorphic to its dual $\mathrm{L}^\varrho(\Omega)$, we easily find that the null space V_i of the operator induced by c_i becomes

$$\mathrm{V}_i := \Big\{\boldsymbol{\tau}_i \in \mathrm{H}_i : \quad \mathrm{div}(\boldsymbol{\tau}_i) = 0\Big\}, \tag{4.12}$$

and thus

$$a_i(\boldsymbol{\tau}_i, \boldsymbol{\tau}_i) \geqslant \frac{1}{\bar{\kappa}}\|\boldsymbol{\tau}_i\|^2_{0,\Omega} = \frac{1}{\bar{\kappa}}\|\boldsymbol{\tau}_i\|^2_{\mathrm{div}_\varrho;\Omega} \qquad \forall\, \boldsymbol{\tau}_i \in \mathrm{V}_i, \tag{4.13}$$

from which the hypothesis ii) of [12, Theorem 3.4], i.e., the continuous inf-sup condition a_i, is clearly satisfied with constant $\widetilde{\alpha} := \bar{\kappa}^{-1}$.

From what has been developed in [13, Section 4.2.3], we are in position to establish that, for each $i \in \{1, 2\}$, (4.3) is well-posed, which means, equivalently, that $\widetilde{\mathrm{T}}_i$ is well-defined. Indeed, recalling that $\widetilde{\boldsymbol{\alpha}}_{\mathbf{A}} > 0$ is the inf-sup constant of $\mathbf{A}$ (for more details, see [13, eq. (74), Section 4.2.3]), we proceed to state the following result [13, Theorem 4.6].

Theorem 4.3. *For each $i \in \{1, 2\}$, and for each $(\boldsymbol{\phi}, \mathbf{v}) \in \mathrm{X}_2 \times \mathbf{L}^4(\Omega)$, such that there holds*

$$\|\boldsymbol{\phi}\|_{0,r;\Omega} + \|\mathbf{v}\|_{0,r;\Omega} \leqslant \frac{\widetilde{\boldsymbol{\alpha}}_{\mathbf{A}}}{2\,\|c\|}, \tag{4.14}$$

there exists a unique $(\boldsymbol{\phi}_i, \xi_i) \in \mathrm{H}_i \times \mathrm{Q}_i$ solution to (4.3), and hence one can define $\widetilde{\mathrm{T}}_i(\boldsymbol{\phi}, \mathbf{v}) := \xi_i \in \mathrm{Q}_i$. Moreover, there exists a positive constant $C_{\widetilde{\mathrm{T}}}$, depending only on $\|i_\rho\|$ and $\widetilde{\boldsymbol{\alpha}}_{\mathbf{A}}$, such that

$$\|\widetilde{\mathrm{T}}_i(\boldsymbol{\phi}, \mathbf{v})\|_{\mathrm{Q}_i} = \|\xi_i\|_{\mathrm{Q}_i} \leqslant \|(\boldsymbol{\sigma}_i, \xi_i)\|_{\mathrm{H}_i \times \mathrm{Q}_i} \leqslant C_{\widetilde{\mathrm{T}}}\Big\{\|g_i\|_{1/2,\Gamma} + \|f_i\|_{0,\varrho;\Omega}\Big\}. \tag{4.15}$$

We end this section by observing from the definition of $\widetilde{\mathrm{T}}$ (cf. (4.4)) and the priori estimates given by (4.15) for each $i \in \{1,2\}$, that

$$\|\widetilde{\mathrm{T}}(\boldsymbol{\phi},\mathbf{v})\|_{\mathrm{Q}_1\times\mathrm{Q}_2} \,:=\, \sum_{i=1}^{2} \|\widetilde{\mathrm{T}}_i(\boldsymbol{\phi},\mathbf{v})\|_{\mathrm{Q}_i} \,\leqslant\, C_{\widetilde{\mathrm{T}}} \sum_{i=1}^{2} \Big\{ \|g_i\|_{1/2,\Gamma} \,+\, \|f_i\|_{0,\varrho;\Omega} \Big\}, \tag{4.16}$$

for each $(\boldsymbol{\phi},\mathbf{v}) \in \mathrm{X}_2 \times \mathbf{L}^4(\Omega)$ satisfying (4.14).

4.3 Solvability analysis of the fixed-point scheme

Knowing that the operators $\mathbf{S}$, $\bar{\mathrm{T}}$, $\widetilde{\mathrm{T}}$ and thus also $\mathbf{T}$ are well defined for small data, we now address the solvability of the fixed-point equation (4.5) applying Banach's Theorem. We first derive sufficient conditions under which $\mathbf{T}$ maps the following closed ball (with radius to be specified later on) of $\mathrm{X}_2\times\mathbf{L}^4(\Omega)$ into itself

$$\mathrm{W}(\delta) \,:=\, \Big\{ (\boldsymbol{\phi},\mathbf{z}) \in \mathrm{X}_2 \times \mathbf{L}^4(\Omega) : \quad \|(\boldsymbol{\phi},\mathbf{z})\| \,:=\, \|\boldsymbol{\phi}\|_{\mathrm{X}_2} \,+\, \|\mathbf{z}\|_{0,4;\Omega} \,\leqslant\, \delta \Big\}. \tag{4.17}$$

Then, given $(\boldsymbol{\phi},\mathbf{z}) \in \mathrm{W}(\delta)$, we have from the definition of $\mathbf{T}$ (cf. (4.5)) and the a priori estimate for $\widetilde{\mathrm{T}}$ (cf. (4.16)) that, under the assumption (cf. (4.14))

$$\|\boldsymbol{\phi}\|_{\mathrm{X}_2} + \|\mathbf{z}\|_{0,4,\Omega} \,\leqslant\, \frac{\widetilde{\boldsymbol{\alpha}}_{\mathbf{A}}}{2\,\|c\|}\,,$$

which suggests to define $\delta \,:=\, \frac{\widetilde{\boldsymbol{\alpha}}_{\mathbf{A}}}{2\,\|c\|}$, followed by an application of the a priori estimates for $\mathbf{S}$ (cf. (4.6)) $\bar{\mathrm{T}}$ (cf. (4.9)) and $\widetilde{\mathrm{T}}$ (cf. (4.16)), we deduce

$$\|\mathbf{T}(\boldsymbol{\phi},\mathbf{z})\|_{\mathrm{X}_2\times\mathbf{L}^4(\Omega)} \,\leqslant\, C_{\mathbf{T}} \left\{ \Big(1 \,+\, \Lambda_{g_i,f_i}\Big) \Big(\|g\|_{1/s,r;\Gamma} \,+\, \|f\|_{0,r;\Omega} \,+\, \Lambda_{g_i,f_i} \Big) \,+\, \|\mathbf{f}\|_{0,4/3;\Omega} \,+\, \|\mathbf{g}\|_{1/2,\Gamma} \right\},$$

where $C_{\mathbf{T}}$ is a positive constant depending only on $C_{\mathbf{S}}$, $C_{\bar{\mathrm{T}}}$, $C_{\widetilde{\mathrm{T}}}$, and $(1+\delta)$, and we also define

$$\Lambda_{g_i,f_i} \,:=\, \sum_{i=1}^{2} \Big\{ \|g_i\|_{1/2,\Gamma} + \|f_i\|_{0,\varrho;\Omega} \Big\}.$$

Therefore, we have proved the following lemma.

Lemma 4.4. *Assume that the data are sufficiently small so that*

$$C_{\mathbf{T}} \left\{ \Big(1 \,+\, \Lambda_{g_i,f_i}\Big) \Big(\|g\|_{1/s,r;\Gamma} \,+\, \|f\|_{0,r;\Omega} \,+\, \Lambda_{g_i,f_i} \Big) \,+\, \|\mathbf{f}\|_{0,4/3;\Omega} \,+\, \|\mathbf{g}\|_{1/2,\Gamma} \right\} \,\leqslant\, \delta\,. \tag{4.18}$$

Then, $\mathbf{T}(\mathrm{W}(\delta)) \subseteq \mathrm{W}(\delta)$.

We now aim to prove that the operator $\mathbf{T}$ is Lipschitz-continuous, for which it suffices to show that $\mathbf{S}$, $\bar{\mathrm{T}}$, $\widetilde{\mathrm{T}}_i$ $(i = \{1,2\})$ and $\widetilde{\mathrm{T}}$ satisfy suitable continuity properties. We begin by studying $\mathbf{S}$.

Lemma 4.5. *There exists a positive constant* L_S*, depending on* $\boldsymbol{\alpha}$*,* ε*,* $|\Omega|$ *and* $\|\mathbf{c}\|$*, such that*

$$\begin{aligned} &\|\mathbf{S}(\mathbf{z},\boldsymbol{\eta},\boldsymbol{\phi}) - \mathbf{S}(\mathbf{z}_0,\boldsymbol{\eta}_0,\boldsymbol{\phi}_0)\|_{\mathbf{H}} \\ &\quad\leqslant\, \mathrm{L}_S \Big\{ \mathcal{F}(\mathbf{z}_0,\boldsymbol{\eta}_0,\boldsymbol{\phi}_0)\, \|\mathbf{z}-\mathbf{z}_0\|_{0,4;\Omega} \,+\, \|\boldsymbol{\phi}_0\|_{0,r;\Omega}\, \|\boldsymbol{\eta}-\boldsymbol{\eta}_0\|_{0,\rho;\Omega} \,+\, \|\boldsymbol{\eta}\|_{0,\rho;\Omega}\, \|\boldsymbol{\phi}-\boldsymbol{\phi}_0\|_{0,r;\Omega} \Big\}, \end{aligned} \tag{4.19}$$

for all $(\mathbf{z},\boldsymbol{\eta},\boldsymbol{\phi}), (\mathbf{z}_0,\boldsymbol{\eta}_0,\boldsymbol{\phi}_0) \in \mathbf{L}^4(\Omega) \times (\mathrm{Q}_1\times\mathrm{Q}_2) \times \mathrm{X}_2$*, where*

$$\mathcal{F}(\mathbf{z}_0,\boldsymbol{\eta}_0,\boldsymbol{\phi}_0) \,:=\, C_S \Big\{ \|\boldsymbol{\eta}_0\|_{0,\rho;\Omega}\, \|\boldsymbol{\phi}_0\|_{0,r;\Omega} \,+\, \|\mathbf{f}\|_{0,4/3;\Omega} \,+\, (1 \,+\, \|\mathbf{z}_0\|_{0,4;\Omega})\, \|\mathbf{g}\|_{1/2,\Gamma} \Big\}. \tag{4.20}$$

Proof. Given $(\mathbf{z},\boldsymbol{\eta},\boldsymbol{\phi}),(\mathbf{z}_0,\boldsymbol{\eta}_0,\boldsymbol{\phi}_0) \in \mathbf{L}^4(\Omega)\times(\mathrm{Q}_1\times\mathrm{Q}_2)\times\mathrm{X}_2$, we let $\mathbf{S}(\mathbf{z},\boldsymbol{\eta},\boldsymbol{\phi}) := \mathbf{u}\in\mathbf{L}^4(\Omega)$ and $\mathbf{S}(\mathbf{z}_0,\boldsymbol{\eta}_0,\boldsymbol{\phi}_0) := \mathbf{u}_0\in\mathbf{L}^4(\Omega)$, where $\boldsymbol{\eta}_0 := (\eta_{0,1},\eta_{0,2})$; and $(\vec{\mathbf{u}},\boldsymbol{\sigma}) = ((\mathbf{u},\mathbf{t}),\boldsymbol{\sigma})\in\mathbf{H}\times\mathbf{Q}$ and $(\vec{\mathbf{u}}_0,\boldsymbol{\sigma}_0) = ((\mathbf{u}_0,\mathbf{t}_0),\boldsymbol{\sigma}_0)\in\mathbf{H}\times\mathbf{Q}$ are the respective solutions to (4.1). It follows from the second equations of (4.1) that $\vec{\mathbf{u}}-\vec{\mathbf{u}}_0\in\mathbf{V}$ (where $\mathbf{V}$ denotes the kernel of the operator induced by the bilinear form $\mathbf{b}$ [6, cf. (3.11)]), and then $\mathbf{V}$-ellipticity of $\mathbf{a}$ ([6, cf. (3.12)]) gives

$$\boldsymbol{\alpha}\,\|\vec{\mathbf{u}}-\vec{\mathbf{u}}_0\|^2_{\mathbf{H}} \,\leqslant\, \mathbf{a}(\vec{\mathbf{u}}-\vec{\mathbf{u}}_0,\vec{\mathbf{u}}-\vec{\mathbf{u}}_0)\,. \tag{4.21}$$

In turn, applying the first equations of (4.1) to $\vec{\mathbf{v}} = \vec{\mathbf{u}}-\vec{\mathbf{u}}_0$, we obtain

$$\mathbf{a}(\vec{\mathbf{u}},\vec{\mathbf{u}}-\vec{\mathbf{u}}_0) \,+\, \mathbf{c}(\mathbf{z};\vec{\mathbf{u}},\vec{\mathbf{u}}-\vec{\mathbf{u}}_0) \,=\, \mathbf{F}_{\boldsymbol{\eta},\boldsymbol{\phi}}(\vec{\mathbf{u}}-\vec{\mathbf{u}}_0)\,, \tag{4.22a}$$
$$\mathbf{a}(\vec{\mathbf{u}}_0,\vec{\mathbf{u}}-\vec{\mathbf{u}}_0) \,+\, \mathbf{c}(\mathbf{z}_0;\vec{\mathbf{u}}_0,\vec{\mathbf{u}}-\vec{\mathbf{u}}_0) \,=\, \mathbf{F}_{\boldsymbol{\eta}_0,\boldsymbol{\phi}_0}(\vec{\mathbf{u}}-\vec{\mathbf{u}}_0)\,, \tag{4.22b}$$

so that, subtracting (4.22b) from (4.22a), and using, thanks to the bilinearity of $\mathbf{c}(\mathbf{z};\cdot,\cdot)$ and (3.16), that

$$\mathbf{c}(\mathbf{z};\vec{\mathbf{u}},\vec{\mathbf{u}}-\vec{\mathbf{u}}_0) \,=\, \mathbf{c}(\mathbf{z};\vec{\mathbf{u}}-\vec{\mathbf{u}}_0,\vec{\mathbf{u}}-\vec{\mathbf{u}}_0) + \mathbf{c}(\mathbf{z};\vec{\mathbf{u}}_0,\vec{\mathbf{u}}-\vec{\mathbf{u}}_0) \,=\, \mathbf{c}(\mathbf{z};\vec{\mathbf{u}}_0,\vec{\mathbf{u}}-\vec{\mathbf{u}}_0)\,,$$

we find

$$\mathbf{a}(\vec{\mathbf{u}}-\vec{\mathbf{u}}_0,\vec{\mathbf{u}}-\vec{\mathbf{u}}_0) = (\mathbf{F}_{\boldsymbol{\eta},\boldsymbol{\phi}}-\mathbf{F}_{\boldsymbol{\vartheta},\boldsymbol{\psi}})(\vec{\mathbf{u}}-\vec{\mathbf{u}}_0) + \mathbf{c}(\mathbf{z}_0-\mathbf{z};\vec{\mathbf{u}}_0,\vec{\mathbf{u}}-\vec{\mathbf{u}}_0)\,. \tag{4.23}$$

In turn, it is clear from (3.13) that subtracting and adding $\boldsymbol{\phi}_0$ to the factor $\boldsymbol{\phi}$ in the first term, we get

$$\begin{aligned}\big(\mathbf{F}_{\boldsymbol{\eta},\boldsymbol{\phi}}-\mathbf{F}_{\boldsymbol{\eta}_0,\boldsymbol{\phi}_0}\big)(\vec{\mathbf{u}}-\vec{\mathbf{u}}_0) \,&=\, \int_\Omega \varepsilon^{-1}\Big\{\big(\eta_2-\eta_1\big)\,\boldsymbol{\phi} - \big(\eta_{0,2}-\eta_{0,1}\big)\,\boldsymbol{\phi}_0\Big\}\cdot(\mathbf{u}-\mathbf{u}_0)\\ &=\, \int_\Omega \varepsilon^{-1}\Big\{\big(\eta_2-\eta_1\big)\,\big(\boldsymbol{\phi}-\boldsymbol{\phi}_0\big) + \big((\eta_2-\eta_{0,2})-(\eta_1-\eta_{0,1})\big)\,\boldsymbol{\phi}_0\Big\}\cdot(\mathbf{u}-\mathbf{u}_0)\,.\end{aligned}$$

Then, bearing in mind the boundedness of ε by ε_0 and by the fact that $\|\cdot\|_{0,\Omega} \leqslant |\Omega|^{1/4}\|\cdot\|_{0,4;\Omega}$, we obtain

$$\big(\mathbf{F}_{\boldsymbol{\eta},\boldsymbol{\phi}}-\mathbf{F}_{\boldsymbol{\eta}_0,\boldsymbol{\phi}_0}\big)(\vec{\mathbf{u}}-\vec{\mathbf{u}}_0) \leqslant \varepsilon_0^{-1}\,|\Omega|^{1/4}\Big\{\|\boldsymbol{\eta}\|_{0,\rho,\Omega}\,\|\boldsymbol{\phi}-\boldsymbol{\phi}_0\|_{0,r,\Omega} \,+\, \|\boldsymbol{\eta}-\boldsymbol{\eta}_0\|_{0,\rho,\Omega}\,\|\boldsymbol{\phi}_0\|_{0,r,\Omega}\Big\}\|\vec{\mathbf{u}}-\vec{\mathbf{u}}_0\|_{\mathbf{H}}, \tag{4.24}$$

while the boundedness property of $\mathbf{c}$ (cf. (3.15)) results in

$$\mathbf{c}(\mathbf{z}_0-\mathbf{z};\vec{\mathbf{u}}_0,\vec{\mathbf{u}}-\vec{\mathbf{u}}_0) \,\leqslant\, \|\mathbf{c}\|\,\|\mathbf{z}-\mathbf{z}_0\|_{0,4;\Omega}\,\|\vec{\mathbf{u}}_0\|_{\mathbf{H}}\|\vec{\mathbf{u}}-\vec{\mathbf{u}}_0\|_{\mathbf{H}}\,. \tag{4.25}$$

Finally, employing (4.24) and (4.25) in (4.23), by substituting the resulting estimate into (4.21), simplifying by $\|\vec{\mathbf{u}}-\vec{\mathbf{u}}_0\|_{\mathbf{H}}$ and bounding $\|\vec{\mathbf{u}}_0\|_{\mathbf{H}}$ by the upper bound in (4.6), we arrive at the required inequality (4.19) with $\mathrm{L}_{\mathbf{S}} := \boldsymbol{\alpha}^{-1}\max\big\{\varepsilon_0^{-1}|\Omega|^{1/4},\,\|c\|\big\}$. ☐

The next result establishes the continuity of $\bar{\mathrm{T}}$, whose proof can be found in [13, Lemma 4.9].

Lemma 4.6. *There exists a positive constant $\mathrm{L}_{\bar{\mathrm{T}}}$, depending only on $|\Omega|$, $\bar{\alpha}$, $\bar{\beta}_2$, and $\|a\|$, such that*

$$\|\bar{\mathrm{T}}(\boldsymbol{\eta})-\bar{\mathrm{T}}(\boldsymbol{\eta}_0)\|_{\mathrm{X}_2} \,\leqslant\, \mathrm{L}_{\bar{\mathrm{T}}}\,\|\boldsymbol{\eta}-\boldsymbol{\eta}_0\|_{0,\rho;\Omega} \qquad \forall\,\boldsymbol{\eta},\,\boldsymbol{\eta}_0\in\mathrm{Q}_1\times\mathrm{Q}_2\,. \tag{4.26}$$

In turn, the continuity of $\widetilde{\mathrm{T}}$ is provided in [13, Lemma 4.10].

Lemma 4.7. *There exists a positive constant $\mathrm{L}_{\widetilde{\mathrm{T}}}$, depending only on ε_0, $\underline{\kappa}$, $\widetilde{\boldsymbol{\alpha}}_{\mathbf{A}}$, and $C_{\widetilde{\mathrm{T}}}$, such that*

$$\|\widetilde{\mathrm{T}}(\boldsymbol{\phi},\mathbf{v})-\widetilde{\mathrm{T}}(\boldsymbol{\phi}_0,\mathbf{v}_0)\|_{\mathrm{Q}_1\times\mathrm{Q}_2} \,\leqslant\, \mathrm{L}_{\widetilde{\mathrm{T}}}\sum_{i=1}^{2}\Big\{\|g_i\|_{1/2,\Gamma}+\|f_i\|_{0,\varrho,\Omega}\Big\}\,\|(\boldsymbol{\phi},\mathbf{v})-(\boldsymbol{\phi}_0,\mathbf{v}_0)\|_{\mathrm{X}_2\times\mathbf{L}^4(\Omega)} \tag{4.27}$$

for all $(\boldsymbol{\phi},\mathbf{v})$, $(\boldsymbol{\phi}_0,\mathbf{v}_0)\in\mathrm{X}_2\times\mathbf{L}^4(\Omega)$ *satisfying* (4.14).

Having proved Lemmas 4.5, 4.6, and 4.7, we now aim to derive the continuity of the fixed-point operator $\mathbf{T}$. Given $(\boldsymbol{\phi}, \mathbf{z}), (\boldsymbol{\phi}_0, \mathbf{z}_0) \in \mathrm{W}(\delta)$ (cf. (4.17)), from the definition of $\mathbf{T}$ (cf. (4.5)) we have that

$$\begin{aligned} \|\mathbf{T}(\boldsymbol{\phi}, \mathbf{z}) - \mathbf{T}(\boldsymbol{\phi}_0, \mathbf{z}_0)\|_{\mathrm{X}_2 \times \mathbf{L}^4(\Omega)} \,=\, & \|\bar{\mathrm{T}}\big(\widetilde{\mathrm{T}}(\boldsymbol{\phi}, \mathbf{z})\big) \,-\, \bar{\mathrm{T}}\big(\widetilde{\mathrm{T}}(\boldsymbol{\phi}_0, \mathbf{z}_0)\big)\|_{\mathrm{X}_2} \\ & + \|\mathbf{S}\big(\mathbf{z}, \widetilde{\mathrm{T}}(\boldsymbol{\phi}, \mathbf{z}), \bar{\mathrm{T}}(\widetilde{\mathrm{T}}(\boldsymbol{\phi}, \mathbf{z}))\big) \,-\, \mathbf{S}\big(\mathbf{z}_0, \widetilde{\mathrm{T}}(\boldsymbol{\phi}_0, \mathbf{z}_0), \bar{\mathrm{T}}(\widetilde{\mathrm{T}}(\boldsymbol{\phi}_0, \mathbf{z}_0))\big)\|_{0,4;\Omega} \,. \end{aligned} \tag{4.28}$$

Then, applying the continuity of $\bar{\mathrm{T}}$ (cf. Lemma 4.6, (4.26)) and $\widetilde{\mathrm{T}}$ (cf. Lemma 4.7, (4.27)), we get

$$\|\bar{\mathrm{T}}\big(\widetilde{\mathrm{T}}(\boldsymbol{\phi}, \mathbf{z})\big) \,-\, \bar{\mathrm{T}}\big(\widetilde{\mathrm{T}}(\boldsymbol{\phi}_0, \mathbf{z}_0)\big)\|_{\mathrm{X}_2} \,\leqslant\, \mathrm{L}_0 \sum_{i=1}^{2} \big\{\|g_i\|_{1/2,\Gamma} \,+\, \|f_i\|_{0,\varrho;\Omega}\big\}\, \|(\boldsymbol{\phi}, \mathbf{z}) \,-\, (\boldsymbol{\phi}_0, \mathbf{z}_0)\|_{\mathrm{X}_2 \times \mathbf{L}^4(\Omega)} \,, \tag{4.29}$$

where L_0 is a positive constant depending only on $\mathrm{L}_{\bar{\mathrm{T}}}$ and $\mathrm{L}_{\widetilde{\mathrm{T}}}$. On the other hand, to bound the second term of (4.28), we apply the continuity of $\mathbf{S}$ (cf. Lemma 4.5, (4.19)), in particular, setting $\boldsymbol{\eta} = \widetilde{\mathrm{T}}(\boldsymbol{\phi}, \mathbf{z})$, $\boldsymbol{\eta}_0 = \widetilde{\mathrm{T}}(\boldsymbol{\phi}_0, \mathbf{z}_0)$, $\boldsymbol{\phi} = \bar{\mathrm{T}}(\widetilde{\mathrm{T}}(\boldsymbol{\phi}, \mathbf{z}))$, and $\boldsymbol{\phi}_0 = \bar{\mathrm{T}}(\widetilde{\mathrm{T}}(\boldsymbol{\phi}_0, \mathbf{z}_0))$ in (4.19), followed by the continuity of $\bar{\mathrm{T}}$ (cf. Lemma 4.6, (4.26)) and $\widetilde{\mathrm{T}}$ (cf. Lemma 4.7, (4.27)), we deduce

$$\begin{aligned} & \|\mathbf{S}\big(\mathbf{z}, \widetilde{\mathrm{T}}(\boldsymbol{\phi}, \mathbf{z}), \bar{\mathrm{T}}(\widetilde{\mathrm{T}}(\boldsymbol{\phi}, \mathbf{z}))\big) \,-\, \mathbf{S}\big(\mathbf{z}_0, \widetilde{\mathrm{T}}(\boldsymbol{\phi}_0, \mathbf{z}_0), \bar{\mathrm{T}}(\widetilde{\mathrm{T}}(\boldsymbol{\phi}_0, \mathbf{z}_0))\big)\|_{0,4;\Omega} \\ & \leqslant C_1 \Big\{ \mathcal{F}_{\mathbf{T}}\, \|\mathbf{z} \,-\, \mathbf{z}_0\|_{0,4;\Omega} \,+\, \Lambda_{g_i,f_i} \Big(\|g\|_{1/s,r;\Gamma} \,+\, \|f\|_{0,r;\Omega} \,+\, \Lambda_{g_i,f_i} \Big)\, \|(\boldsymbol{\phi}, \mathbf{z}) \,-\, (\boldsymbol{\phi}_0, \mathbf{z}_0)\|_{\mathrm{X}_2 \times \mathbf{L}^4(\Omega)} \Big\} \,, \end{aligned} \tag{4.30}$$

where C_1 is a positive constant depending only on $C_{\bar{\mathrm{T}}}$, $C_{\widetilde{\mathrm{T}}}$, $\mathrm{L}_{\mathbf{S}}$, $\mathrm{L}_{\bar{\mathrm{T}}}$, and $\mathrm{L}_{\widetilde{\mathrm{T}}}$, and also where

$$\mathcal{F}_{\mathbf{T}} \,:=\, \mathcal{F}(\mathbf{z}_0, \widetilde{\mathrm{T}}(\boldsymbol{\phi}_0, \mathbf{z}_0), \bar{\mathrm{T}}(\widetilde{\mathrm{T}}(\boldsymbol{\phi}_0, \mathbf{z}_0)))\,.$$

In turn, applying the a priori estimates of $\bar{\mathrm{T}}, \widetilde{\mathrm{T}}$ (cf. (4.9),(4.16)), and using that $\|\mathbf{z}_0\|_{0,4;\Omega} \leqslant \delta$, we get

$$\begin{aligned} \mathcal{F}_{\mathbf{T}} \,&=\, \mathcal{F}(\mathbf{z}_0, \widetilde{\mathrm{T}}(\boldsymbol{\phi}_0, \mathbf{z}_0), \bar{\mathrm{T}}(\widetilde{\mathrm{T}}(\boldsymbol{\phi}_0, \mathbf{z}_0))) \\ & \leqslant \Big\{ \|\widetilde{\mathrm{T}}(\boldsymbol{\phi}_0, \mathbf{z}_0)\|_{0,\rho;\Omega}\, \|\bar{\mathrm{T}}(\widetilde{\mathrm{T}}(\boldsymbol{\phi}_0, \mathbf{z}_0))\|_{0,r;\Omega} \,+\, \|\mathbf{f}\|_{0,4/3;\Omega} \,+\, (1 \,+\, \|\mathbf{z}_0\|_{0,4;\Omega})\, \|\mathbf{g}\|_{1/2,\Gamma} \Big\} \\ & \leqslant C_{\mathcal{F}} \Big\{ \Lambda_{g_i,f_i} \Big(\|g\|_{1/s,r;\Gamma} \,+\, \|f\|_{0,r;\Omega} \,+\, \Lambda_{g_i,f_i} \Big) \,+\, \|\mathbf{f}\|_{0,4/3;\Omega} \,+\, \|\mathbf{g}\|_{1/2,\Gamma} \Big\} \,, \end{aligned} \tag{4.31}$$

where $C_{\mathcal{F}} > 0$ is a constant depending only on $C_{\bar{\mathrm{T}}}$, $C_{\widetilde{\mathrm{T}}}$, and δ. Then, replacing the estimate of (4.31) into (4.30), we deduce the existence of a positive constant C_2, depending only on C_1 and $C_{\mathcal{F}}$, such that

$$\begin{aligned} & \|\mathbf{S}\big(\mathbf{z}, \widetilde{\mathrm{T}}(\boldsymbol{\phi}, \mathbf{z}), \bar{\mathrm{T}}(\widetilde{\mathrm{T}}(\boldsymbol{\phi}, \mathbf{z}))\big) \,-\, \mathbf{S}\big(\mathbf{z}_0, \widetilde{\mathrm{T}}(\boldsymbol{\phi}_0, \mathbf{z}_0), \bar{\mathrm{T}}(\widetilde{\mathrm{T}}(\boldsymbol{\phi}_0, \mathbf{z}_0))\big)\|_{0,4;\Omega} \\ & \leqslant C_2 \Big\{ \Lambda_{g_i,f_i} \Big(\|g\|_{1/s,r;\Gamma} \,+\, \|f\|_{0,r;\Omega} \,+\, \Lambda_{g_i,f_i} \Big) \,+\, \|\mathbf{f}\|_{0,4/3;\Omega} \,+\, \|\mathbf{g}\|_{1/2,\Gamma} \Big\}\, \|(\boldsymbol{\phi}, \mathbf{z}) \,-\, (\boldsymbol{\phi}_0, \mathbf{z}_0)\| \,. \end{aligned} \tag{4.32}$$

Finally, from what has been deduced in (4.29) and (4.32), by a straightforward application into (4.28), we arrive at

$$\begin{aligned} & \|\mathbf{T}(\boldsymbol{\phi}, \mathbf{z}) \,-\, \mathbf{T}(\boldsymbol{\phi}_0, \mathbf{z}_0)\|_{\mathrm{X}_2 \times \mathbf{L}^4(\Omega)} \\ & \leqslant \mathrm{L}_{\mathbf{T}} \Big\{ \Lambda_{g_i,f_i} \Big(\|g\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega} + \Lambda_{g_i,f_i} + 1 \Big) + \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{g}\|_{1/2,\Gamma} \Big\} \|(\boldsymbol{\phi}, \mathbf{z}) - (\boldsymbol{\phi}_0, \mathbf{z}_0)\| \,, \end{aligned} \tag{4.33}$$

where $\mathrm{L}_{\mathbf{T}}$ is a positive constant depending only on $C_{\bar{\mathrm{T}}}$, $C_{\widetilde{\mathrm{T}}}$, $\mathrm{L}_{\mathbf{S}}$, $\mathrm{L}_{\bar{\mathrm{T}}}$, $\mathrm{L}_{\widetilde{\mathrm{T}}}$, and δ. Consequently, we are in a position to establish the main result of this section.

Theorem 4.8. *In addition to the hypothesis* (4.18) *of Lemma* 4.4*, assume that*

$$\mathrm{L}_{\mathbf{T}} \Big\{ \Lambda_{g_i,f_i} \Big(\|g\|_{1/s,r;\Gamma} \,+\, \|f\|_{0,r;\Omega} \,+\, \Lambda_{g_i,f_i} \,+\, 1 \Big) \,+\, \|\mathbf{f}\|_{0,4/3;\Omega} \,+\, \|\mathbf{g}\|_{1/2,\Gamma} \Big\} \,<\, 1\,. \tag{4.34}$$

Then, the operator $\mathbf{T}$ *has a unique fixed point* $(\boldsymbol{\varphi}, \mathbf{u}) \in \mathrm{W}(\delta)$*. Equivalently, the coupled problem* (3.26) *has a unique solution* $(\vec{\mathbf{u}}, \boldsymbol{\sigma}) \in \mathbf{H} \times \mathbf{Q}$*,* $(\boldsymbol{\varphi}, \chi) \in \mathrm{X}_2 \times \mathrm{M}_1$*, and* $(\boldsymbol{\sigma}_i, \xi_i) \in \mathrm{H}_i \times \mathrm{Q}_i$*,* $i \in \{1, 2\}$*, with* $(\boldsymbol{\varphi}, \mathbf{u}) \in \mathrm{W}(\delta)$*. Moreover, there hold the following a priori estimates*

$$
\begin{aligned}
\|(\vec{\mathbf{u}}, \boldsymbol{\sigma})\|_{\mathbf{H}\times\mathbf{Q}} &\leqslant C_{\vec{\mathbf{u}},\boldsymbol{\sigma}} \Big\{ \|\boldsymbol{\xi}\|_{0,\rho;\Omega} \|\boldsymbol{\varphi}\|_{0,r;\Omega} + \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{g}\|_{1/2,\Gamma} \Big\}, \\
\|(\boldsymbol{\varphi}, \chi)\|_{\mathrm{X}_2\times\mathrm{M}_1} &\leqslant C_{\bar{\mathrm{T}}} \Big\{ \|g\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega} + \|\boldsymbol{\xi}\|_{0,\rho;\Omega} \Big\}, \quad \textit{and} \\
\|(\boldsymbol{\sigma}_i, \xi_i)\|_{\mathrm{H}_i\times\mathrm{Q}_i} &\leqslant C_{\tilde{\mathrm{T}}} \Big\{ \|g_i\|_{1/2,\Gamma} + \|f_i\|_{0,\varrho;\Omega} \Big\} \quad i \in \{1,2\},
\end{aligned}
\tag{4.35}
$$

where $C_{\vec{\mathbf{u}},\boldsymbol{\sigma}}$ *is a positive constant depending only on* $C_{\mathbf{S}}$ *and* δ*.*

Proof. We first recall that the assumptions of Lemma 4.4 guarantee that $\mathbf{T}$ maps $\mathrm{W}(\delta)$ into itself. Then, bearing in mind the Lipschitz-continuity of $\mathbf{T} : \mathrm{W}(\delta) \to \mathrm{W}(\delta)$ (cf. (4.33)) and the assumption (4.34), a straightforward application of the classical Banach Theorem yields the existence of a unique fixed point $(\boldsymbol{\varphi}, \mathbf{u}) \in \mathrm{W}(\delta)$ of this operator, and hence a unique solution of (3.26). Finally, recalling that $\|\mathbf{u}\|_{0,4;\Omega} \leqslant \delta$, it is easy to see that the a priori estimates provided by (4.6) (cf. Lemma 4.1), (4.9) (cf. Theorem 4.2), and (4.15) (cf. Theorem 4.3) yield (4.35) and finish the proof. □

5 The Galerkin scheme

In this section we introduce the Galerkin scheme of the fully mixed variational formulation (3.26), analyze its solvability by applying a discrete version of the fixed-point approach adopted in Section 4.1, and subsequently derive its a priori error estimate.

5.1 Preliminaries

We first let $\mathbf{H}_h^{\mathbf{u}}$, $\mathbb{H}_h^{\mathbf{t}}$, $\mathbb{H}_h^{\boldsymbol{\sigma}}$, $\mathrm{X}_{i,h}$, $\mathrm{M}_{i,h}$, $\mathrm{H}_{i,h}$, and $\mathrm{Q}_{i,h}$, $i \in \{1,2\}$, be a arbitrary finite element subspaces of the spaces $\mathbf{L}^4(\Omega)$, $\mathbb{L}^2_{\mathrm{tr}}(\Omega)$, $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$, X_i, M_i, H_i, and Q_i, $i \in \{1,2\}$, respectively. Hereafter, h denotes both the sub-index of each subspace and the size of a regular triangulation $\mathcal{T}_h$ of $\bar{\Omega}$ made up of triangles K (when $n = 2$) or tetrahedra K (when $n = 3$) of diameter h_K, so that $h := \max\{h_K : \quad K \in \mathcal{T}_h\}$. The explicit finite element subspaces satisfying the stability assumptions that will be introduced throughout the following analysis will be defined later in Section 6. Then, defining the spaces

$$
\mathbf{H}_h := \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\mathbf{t}}, \quad \mathbf{Q}_h := \mathbb{H}_h^{\boldsymbol{\sigma}} \cap \mathbb{H}_0(\mathbf{div}_{4/3},;\Omega),
$$

and denoting $\vec{\mathbf{u}}_h := (\mathbf{u}_h, \mathbf{t}_h)$, $\vec{\mathbf{v}}_h := (\mathbf{v}_h, \mathbf{s}_h) \in \mathbf{H}_h$, the Galerkin scheme associated with (3.26) reads: Find $(\vec{\mathbf{u}}_h, \boldsymbol{\sigma}_h) \in \mathbf{H}_h \times \mathbf{Q}_h$, $(\boldsymbol{\varphi}_h, \chi_h) \in \mathrm{X}_{2,h} \times \mathrm{M}_{1,h}$, and $(\boldsymbol{\sigma}_{i,h}, \xi_{i,h}) \in \mathrm{H}_{i,h} \times \mathrm{Q}_{i,h}$, $i \in \{1,2\}$, such that

$$
\begin{aligned}
\mathbf{a}(\vec{\mathbf{u}}_h, \vec{\mathbf{v}}_h) + \mathbf{c}(\mathbf{u}_h; \vec{\mathbf{u}}_h, \vec{\mathbf{v}}_h) + \mathbf{b}(\vec{\mathbf{v}}_h, \boldsymbol{\sigma}_h) &= \mathbf{F}_{\boldsymbol{\xi}_h,\boldsymbol{\varphi}_h}(\vec{\mathbf{v}}_h) && \forall\, \vec{\mathbf{v}}_h \in \mathbf{H}_h, \\
\mathbf{b}(\vec{\mathbf{u}}_h, \boldsymbol{\tau}_h) &= \mathbf{G}(\boldsymbol{\tau}_h) && \forall\, \boldsymbol{\tau}_h \in \mathbf{Q}_h, \\
a(\boldsymbol{\varphi}_h, \boldsymbol{\psi}_h) + b_1(\boldsymbol{\psi}_h, \chi_h) &= \mathrm{F}(\boldsymbol{\psi}_h) && \forall\, \boldsymbol{\psi}_h \in \mathrm{X}_{1,h}, \\
b_2(\boldsymbol{\varphi}_h, \lambda_h) &= \mathrm{G}_{\boldsymbol{\xi}_h}(\lambda_h) && \forall\, \lambda_h \in \mathrm{M}_{2,h}, \\
a_i(\boldsymbol{\sigma}_{i,h}, \boldsymbol{\tau}_{i,h}) + c_i(\boldsymbol{\tau}_{i,h}, \xi_{i,h}) - c_{\boldsymbol{\varphi}_h,\mathbf{u}_h}(\boldsymbol{\tau}_{i,h}, \xi_{i,h}) &= \mathrm{F}_i(\boldsymbol{\tau}_{i,h}) && \forall\, \boldsymbol{\tau}_{i,h} \in \mathrm{H}_{i,h}, \\
c_i(\boldsymbol{\sigma}_{i,h}, \eta_{i,h}) - d_i(\xi_{i,h}, \eta_{i,h}) &= \mathrm{G}_i(\eta_{i,h}) && \forall\, \eta_{i,h} \in \mathrm{Q}_{i,h}.
\end{aligned}
\tag{5.1}
$$

At this point we explicit a couple of identities contained in the above discrete formulation, which yield later on (cf. Section 6.4) the conservation properties of our Galerkin solution. Indeed, bearing in mind from Sections 3.2, 3.3, and 3.4 the definitions of the bilinear forms and functionals involved, we notice that when taking $\vec{\mathbf{v}}_h := (\mathbf{v}_h, \mathbf{0}) \in \mathbf{H}_h$, the first equation of (5.1) becomes

$$
\int_\Omega \mathbf{v}_h \cdot \Big(\mathbf{div}(\boldsymbol{\sigma}_h) - (\xi_{1,h} - \xi_{2,h})\, \varepsilon^{-1} \boldsymbol{\varphi}_h - \frac{\omega}{2}\, \mathbf{t}_h\, \mathbf{u}_h + \mathbf{f} \Big) = 0 \qquad \forall\, \mathbf{v}_h \in \mathbf{H}_h^{\mathbf{u}}. \tag{5.2}
$$

In turn, the fourth and sixth equations of (5.1) reduce, respectively, to

$$\int_\Omega \lambda_h \left(\mathrm{div}(\boldsymbol{\varphi}_h) + (\xi_{1,h} - \xi_{2,h}) + f\right) \;=\; 0 \qquad \forall\, \lambda_h \in \mathrm{M}_{2,h}\,, \tag{5.3}$$

and

$$\int_\Omega \eta_{i,h} \left(\mathrm{div}(\boldsymbol{\sigma}_{i,h}) - \xi_{i,h} + f_i\right) \;=\; 0 \qquad \forall\, \eta_{i,h} \in \mathrm{Q}_{i,h}\,. \tag{5.4}$$

Certainly, the respective continuous versions of (5.2), (5.3), and (5.4) are obtained from (3.26) by proceeding analogously.

Next, we adopt the discrete version of the strategy used in Section 4.1 to analyze the solvability of (5.1). Accordingly, we introduce the operator $\mathbf{S}_h : \mathbf{H}_h^{\mathbf{u}} \times (\mathrm{Q}_{1,h} \times \mathrm{Q}_{2,h}) \times \mathrm{X}_{2,h} \to \mathbf{H}_h^{\mathbf{u}}$ defined by

$$\mathbf{S}_h(\mathbf{z}_h, \boldsymbol{\eta}_h, \boldsymbol{\phi}_h) \;=\; \mathbf{u}_h \qquad \forall\, (\mathbf{z}_h, \boldsymbol{\eta}_h, \boldsymbol{\phi}_h) \in \mathbf{H}_h^{\mathbf{u}} \times (\mathrm{Q}_{1,h} \times \mathrm{Q}_{2,h}) \times \mathrm{X}_{2,h}$$

where $(\vec{\mathbf{u}}_h, \boldsymbol{\sigma}_h) \;=\; ((\mathbf{u}_h, \mathbf{t}_h), \boldsymbol{\sigma}_h) \in \mathbf{H}_h \times \mathbf{Q}_h$ is the unique solution (to be derived below under what conditions it does exists) of the first and second rows of (5.1) when $\mathbf{c}(\mathbf{u}_h, \cdot, \cdot)$ and $\mathbf{F}_{\boldsymbol{\xi}_h, \boldsymbol{\varphi}_h}$ are replaced by $\mathbf{c}(\mathbf{z}_h, \cdot, \cdot)$ and $\mathbf{F}_{\boldsymbol{\eta}_h, \boldsymbol{\phi}_h}$, respectively, that is

$$\begin{aligned} \mathbf{a}(\vec{\mathbf{u}}_h, \vec{\mathbf{v}}_h) + \mathbf{c}(\mathbf{z}_h; \vec{\mathbf{u}}_h, \vec{\mathbf{v}}_h) + \mathbf{b}(\vec{\mathbf{v}}_h, \boldsymbol{\sigma}_h) &= \mathbf{F}_{\boldsymbol{\eta}_h, \boldsymbol{\phi}_h}(\vec{\mathbf{v}}_h) \qquad \forall\, \vec{\mathbf{v}}_h \in \mathbf{H}_h\,, \\ \mathbf{b}(\vec{\mathbf{u}}_h, \boldsymbol{\tau}_h) &= \mathbf{G}(\boldsymbol{\tau}_h) \qquad \forall\, \boldsymbol{\tau}_h \in \mathbf{Q}_h\,. \end{aligned} \tag{5.5}$$

In turn, we also introduce the operator $\bar{\mathrm{T}}_h : \mathrm{Q}_{1,h} \times \mathrm{Q}_{2,h} \to \mathrm{X}_{2,h}$ defined as

$$\bar{\mathrm{T}}_h(\boldsymbol{\eta}_h) \;:=\; \boldsymbol{\varphi}_h \qquad \forall\, \boldsymbol{\eta}_h \in \mathrm{Q}_1 \times \mathrm{Q}_2\,,$$

where $(\boldsymbol{\varphi}_h, \chi_h) \in \mathrm{X}_{2,h} \times \mathrm{M}_{1,h}$ is the unique solution (to be confirmed below) of the third and fourth rows of (5.1) with $\boldsymbol{\eta}_h$ instead of $\boldsymbol{\xi}_h$

$$\begin{aligned} a(\boldsymbol{\varphi}_h, \boldsymbol{\psi}_h) + b_1(\boldsymbol{\psi}_h, \chi_h) &= \mathrm{F}(\boldsymbol{\psi}_h) \qquad \forall\, \boldsymbol{\psi}_h \in \mathrm{X}_{1,h}\,, \\ b_2(\boldsymbol{\varphi}_h, \lambda_h) &= \mathrm{G}_{\boldsymbol{\eta}_h}(\lambda_h) \qquad \forall\, \lambda_h \in \mathrm{M}_{2,h}\,. \end{aligned} \tag{5.6}$$

Similarly, for each $i \in \{1, 2\}$, we define the operator $\widetilde{\mathrm{T}}_{i,h} : \mathrm{X}_{2,h} \times \mathbf{H}_h^{\mathbf{u}} \to \mathrm{Q}_{i,h}$ as

$$\widetilde{\mathrm{T}}_{i,h}(\boldsymbol{\phi}_h, \mathbf{v}_h) \;:=\; \xi_{i,h} \qquad \forall\, (\boldsymbol{\phi}_h, \mathbf{v}_h) \in \mathrm{X}_{2,h} \times \mathbf{H}_h^{\mathbf{u}}\,,$$

where $(\boldsymbol{\sigma}_{i,h}, \xi_{i,h}) \in \mathrm{H}_{i,h} \times \mathrm{Q}_{i,h}$ is the unique solution (to be confirmed below) of the fifth and sixth rows of (5.1) with $(\boldsymbol{\phi}_h, \mathbf{v}_h)$ instead $(\boldsymbol{\varphi}_h, \mathbf{u}_h)$, that is

$$\begin{aligned} a_i(\boldsymbol{\sigma}_{i,h}, \boldsymbol{\tau}_{i,h}) + c_i(\boldsymbol{\tau}_{i,h}, \xi_{i,h}) - c_{\boldsymbol{\phi}_h, \mathbf{v}_h}(\boldsymbol{\tau}_{i,h}, \xi_{i,h}) &= \mathrm{F}_i(\boldsymbol{\tau}_{i,h}) \qquad \forall\, \boldsymbol{\tau}_{i,h} \in \mathrm{H}_{i,h}\,, \\ c_i(\boldsymbol{\sigma}_{i,h}, \eta_{i,h}) - d_i(\xi_{i,h}, \eta_{i,h}) &= \mathrm{G}_i(\eta_{i,h}) \qquad \forall\, \eta_{i,h} \in \mathrm{Q}_{i,h}\,, \end{aligned} \tag{5.7}$$

so that we can define the operator $\widetilde{\mathrm{T}}_h : \mathrm{X}_{2,h} \times \mathbf{H}_h^{\mathbf{u}} \to (\mathrm{Q}_{1,h} \times \mathrm{Q}_{2,h})$ as

$$\widetilde{\mathrm{T}}_h(\boldsymbol{\phi}_h, \mathbf{v}_h) \;:=\; \left(\widetilde{\mathrm{T}}_{1,h}(\boldsymbol{\phi}_h, \mathbf{v}_h), \widetilde{\mathrm{T}}_{2,h}(\boldsymbol{\phi}_h, \mathbf{v}_h)\right) \;=\; (\xi_{1,h}, \xi_{2,h}) \;=:\; \boldsymbol{\xi}_h \qquad \forall\, (\boldsymbol{\phi}_h, \mathbf{v}_h) \in \mathrm{X}_{2,h} \times \mathbf{H}_h^{\mathbf{u}}\,. \tag{5.8}$$

Finally, we define the discrete analogue of $\mathbf{T}$ (cf. (4.5)), that is $\mathbf{T}_h : \mathrm{X}_{2,h} \times \mathbf{H}_h^{\mathbf{u}} \to \mathrm{X}_{2,h} \times \mathbf{H}_h^{\mathbf{u}}$ as

$$\mathbf{T}_h(\boldsymbol{\phi}_h, \mathbf{z}_h) \;:=\; \left(\bar{\mathrm{T}}_h(\widetilde{\mathrm{T}}_h(\boldsymbol{\phi}_h, \mathbf{z}_h)), \mathbf{S}_h\big(\mathbf{z}_h, \widetilde{\mathrm{T}}_h(\boldsymbol{\phi}_h, \mathbf{z}_h), \bar{\mathrm{T}}_h(\widetilde{\mathrm{T}}_h(\boldsymbol{\phi}_h, \mathbf{z}_h))\big)\right) \quad \forall\, (\boldsymbol{\phi}_h, \mathbf{z}_h) \in \mathrm{X}_{2,h} \times \mathbf{H}_h^{\mathbf{u}}\,,$$

so that solving (5.1) is equivalent to seeking a fixed point of $\mathbf{T}_h$: Find $(\boldsymbol{\varphi}_h, \mathbf{u}_h) \in \mathrm{X}_{2,h} \times \mathbf{H}_h^{\mathbf{u}}$ such that

$$\mathbf{T}_h(\boldsymbol{\varphi}_h, \mathbf{u}_h) \;=\; (\boldsymbol{\varphi}_h, \mathbf{u}_h)\,. \tag{5.9}$$

5.2 Discrete solvability analysis

In this section we proceed analogously to Section 4.2 and 4.3 and establish the well-posedness of the discrete system (5.1) by studying the solvability of the equivalent fixed-point equation (5.9). In this regard, we emphasize in advance that, the respective analysis being very similar to that developed in previous sections, we limit ourselves here to collecting the main results and providing selected details of their proofs.

Accordingly, we first prove that the discrete operators $\mathbf{S}_h$, $\bar{\mathrm{T}}_h$, and $\widetilde{\mathrm{T}}_{i,h}$, $i \in \{1,2\}$, and hence $\widetilde{\mathrm{T}}_h$ and $\mathbf{T}_h$, are all well-defined, which reduces, equivalently, to showing that problems (5.5), (5.6), and (5.7) are well-posed. For this purpose, we now apply the discrete version of [3, Theorem 2.1, Corollary 2.1, Section 2.1], [15, Theorem 2.34], and [12, Theorem 3.4], which are given by [3, Corollary 2.2, Section 2.2], [15, Proposition 2.42], and [12, Theorem 3.5], respectively. More specifically, following a similar approach from, e.g. [6, Section 4.2] and [13, Section 5.2], our analysis is based on suitable hypotheses that must be satisfied by the finite element subspaces used in (5.1), which are divided according to the requirements of the associated decoupled problems. Explicit examples of discrete spaces verifying these hypotheses will be specified later in Section 6.

According to the above, and to address first the well-definedness of $\mathbf{S}_h$, we assume that

(H.1) there exists a positive constant $\boldsymbol{\beta}_{\mathtt{d}}$, independent of h, such that

$$\sup_{\substack{\vec{\mathbf{v}}_h \in \mathbf{H}_h \\ \vec{\mathbf{v}}_h \neq \mathbf{0}}} \frac{\mathbf{b}(\vec{\mathbf{v}}_h, \boldsymbol{\tau}_h)}{\|\vec{\mathbf{v}}_h\|_{\mathbf{H}}} \geqslant \boldsymbol{\beta}_{\mathtt{d}} \, \|\boldsymbol{\tau}_h\|_{\mathbf{Q}} \quad \forall \, \boldsymbol{\tau}_h \in \mathbf{Q}_h \,.$$

In addition, we let $\mathbf{V}_h$ be the discrete kernel of the bilinear form $\mathbf{b}$, that is

$$\mathbf{V}_h \, := \, \Big\{ \vec{\mathbf{v}}_h \in \mathbf{H}_h : \quad \mathbf{b}(\vec{\mathbf{v}}_h, \boldsymbol{\tau}_h) \, = \, 0 \quad \forall \, \boldsymbol{\tau}_h \in \mathbf{Q}_h \Big\} \,,$$

and suppose

(H.2) there exists a positive constant $C_{\mathtt{d}}$, independent of h, such that

$$\|\mathbf{s}_h\|_{0,\Omega} \, \geqslant \, C_{\mathtt{d}} \, \|\mathbf{v}_h\|_{0,4;\Omega} \quad \forall \, \vec{\mathbf{v}}_h \, := \, (\mathbf{v}_h, \mathbf{s}_h) \in \mathbf{V}_h \,.$$

Then, given $\mathbf{z}_h \in \mathbf{H}_h^{\mathbf{u}}$, it follows from the bilinear form $\mathcal{A}_{z_h} : \mathbf{H}_h \times \mathbf{H}_h \to \mathrm{R}$ defined by (cf. [6, eq. (3.9)], (3.18))

$$\mathcal{A}_{\mathbf{z}_h}(\vec{\mathbf{w}}_h, \vec{\mathbf{v}}_h) \, := \, \mathbf{a}(\vec{\mathbf{w}}_h, \vec{\mathbf{v}}_h) \, + \, \mathbf{c}(\mathbf{z}_h; \mathbf{w}_h, \mathbf{v}_h) \quad \forall \, \vec{\mathbf{w}}_h \,, \vec{\mathbf{v}}_h \in \mathbf{H}_h \,,$$

the identity (3.16), and the assumption **(H.2)**, that

$$\mathcal{A}_{\mathbf{z}_h}(\vec{\mathbf{v}}_h, \vec{\mathbf{v}}_h) \, = \, \mathbf{a}(\vec{\mathbf{v}}_h, \vec{\mathbf{v}}_h) \, = \, \mu \|\mathbf{s}_h\|_{0,\Omega}^2 \, \geqslant \, \frac{\mu}{2} \, C_{\mathtt{d}}^2 \, \|\mathbf{v}_h\|_{0,4;\Omega}^2 + \frac{\mu}{2} \, \|\mathbf{s}_h\|_{0,\Omega}^2 \quad \forall \, \vec{\mathbf{v}}_h \, := \, (\mathbf{v}_h, \mathbf{s}_h) \in \mathbf{V}_h \,,$$

which proves the $\mathbf{V}_h$-ellipticity of $\mathcal{A}_{\mathbf{z}_h}$ with constant $\boldsymbol{\alpha}_{\mathtt{d}} \, := \, \dfrac{\mu}{2} \, \min\{C_{\mathtt{d}}, 1\}$. Thus, the discrete analogue of Lemma 4.1 is as follows.

Lemma 5.1. *For each* $(\mathbf{z}_h, \boldsymbol{\eta}_h, \boldsymbol{\phi}_h) \in \mathbf{H}_h^{\mathbf{u}} \times (\mathrm{Q}_{1,h} \times \mathrm{Q}_{2,h}) \times \mathrm{X}_{2,h}$*, there exists a unique* $(\vec{\mathbf{u}}_h, \boldsymbol{\sigma}_h) := ((\mathbf{u}_h, \mathbf{t}_h), \boldsymbol{\sigma}_h) \in \mathbf{H}_h \times \mathrm{Q}_h$ *solution to (5.5), and hence one can define* $\mathbf{S}_h(\mathbf{z}_h, \boldsymbol{\eta}_h, \boldsymbol{\phi}_h) := \mathbf{u}_h \in \mathbf{H}_h^{\mathbf{u}}$*. Moreover, there exists a positive constant* $C_{\mathbf{S},\mathtt{d}}$*, depending only on* $|\Omega|$, $\|\mathbf{i}_4\|$, μ, $\boldsymbol{\alpha}_{\mathtt{d}}$*, and* $\boldsymbol{\beta}_{\mathtt{d}}$*, such that*

$$\begin{aligned} \|\mathbf{S}_h(\mathbf{z}_h, \boldsymbol{\eta}_h, \boldsymbol{\phi}_h)\|_{0,4;\Omega} \, &= \, \|\mathbf{u}_h\|_{0,4;\Omega} \, \leqslant \, \|\vec{\mathbf{u}}_h\|_{\mathbf{H}} \\ &\leqslant \, C_{\mathbf{S},\mathtt{d}} \Big\{ \|\eta_h\|_{0,\rho;\Omega} \, \|\boldsymbol{\phi}\|_{0,r;\Omega} \, + \, \|\mathbf{f}\|_{0,4/3;\Omega} \, + \, (1 \, + \, \|\mathbf{z}_h\|_{0,4;\Omega}) \|\mathbf{g}\|_{1/2,\Gamma} \Big\} \,. \end{aligned}$$

Proof. The proof is analogous to that of [6, Lemma 4.1]. □

Note here that the discrete analogue of (4.7) reads

$$\|\boldsymbol{\sigma}_h\|_{\mathbf{Q}} \,=\, \|\boldsymbol{\sigma}_h\|_{\mathbf{div}_{4/3};\Omega} \leqslant \bar{C}_{\mathbf{S},\mathtt{d}}\,(1+\|\mathbf{z}_h\|_{0,4;\Omega})\Big\{\|\boldsymbol{\eta}_h\|_{0,\rho;\Omega}\|\boldsymbol{\phi}_h\|_{0,r;\Omega} \,+\, \|\mathbf{f}\|_{4/3;\Omega} + (1+\|\mathbf{z}_h\|_{0,4;\Omega})\|\mathbf{g}\|_{1/2,\Gamma}\Big\},$$

where $\bar{C}_{\mathbf{S},\mathtt{d}}$ is a positive constant depending as well on $|\Omega|$, $\|\mathbf{i}_4\|$, μ, ω, $\boldsymbol{\alpha}_{\mathtt{d}}$, and $\boldsymbol{\beta}_{\mathtt{d}}$.

In turn, for the well-definedness of $\bar{\mathrm{T}}_h$, we need to introduce the discrete kernels of b_1 and b_2, namely

$$\mathrm{K}_{1,h} \,:=\, \Big\{\boldsymbol{\psi}_h \in \mathrm{X}_{1,h} : \quad b_1(\boldsymbol{\psi}_h,\lambda_h) \,=\, 0 \quad \forall\,\lambda_h \in \mathrm{M}_{1,h}\Big\},$$

$$\mathrm{K}_{2,h} \,:=\, \Big\{\boldsymbol{\psi}_h \in \mathrm{X}_{2,h} : \quad b_2(\boldsymbol{\psi}_h,\lambda_h) \,=\, 0 \quad \forall\,\lambda_h \in \mathrm{M}_{2,h}\Big\},$$

respectively, and adopt the following assumptions:

(H.3) there exists a positive constant $\bar{\alpha}_{\mathtt{d}}$, independent of h, such that

$$\sup_{\substack{\boldsymbol{\psi}_h\in\mathrm{K}_{1,h}\\ \boldsymbol{\psi}_h\neq\mathbf{0}}} \frac{a(\boldsymbol{\phi}_h,\boldsymbol{\psi}_h)}{\|\boldsymbol{\psi}_h\|_{\mathrm{X}_1}} \,\geqslant\, \bar{\alpha}_{\mathtt{d}}\,\|\boldsymbol{\phi}_h\|_{\mathrm{X}_2} \qquad \forall\,\boldsymbol{\phi}_h \in \mathrm{K}_{2,h}\,, \quad \text{and}$$

$$\sup_{\boldsymbol{\phi}_h\in\mathrm{K}_{2,h}} a(\boldsymbol{\phi}_h,\boldsymbol{\psi}_h) \,>\, 0 \qquad \forall\,\boldsymbol{\psi}_h \in \mathrm{K}_{1,h}\,, \quad \boldsymbol{\psi}_h \neq \mathbf{0}\,.$$

(H.4) for each $i \in \{1,2\}$ there exists a positive constant $\bar{\beta}_{i,\mathtt{d}}$ independent of h, such that

$$\sup_{\substack{\boldsymbol{\psi}_h\in\mathrm{X}_{i,h}\\ \boldsymbol{\psi}_h\neq\mathbf{0}}} \frac{b_i(\boldsymbol{\psi}_h,\lambda_h)}{\|\boldsymbol{\psi}_h\|_{\mathrm{X}_i}} \,\geqslant\, \bar{\beta}_{i,\mathtt{d}}\,\|\lambda_h\|_{\mathrm{M}_i} \qquad \forall\,\lambda_h \in \mathrm{M}_{i,h}\,.$$

As a consequence of **(H.3)** and **(H.4)** we provide next the discrete version of Theorem 4.2.

Theorem 5.2. *For each $\boldsymbol{\eta}_h \in \mathrm{Q}_{1,h}\times\mathrm{Q}_{2,h}$ there exists a unique $(\boldsymbol{\varphi}_h,\chi_h) \in \mathrm{X}_{2,h}\times\mathrm{M}_{1,h}$ solution to (5.6), and hence one can define $\bar{\mathrm{T}}_h(\boldsymbol{\eta}_h) := \boldsymbol{\varphi}_h \in \mathrm{X}_{2,h}$. Moreover, there exists a positive constant $C_{\bar{\mathrm{T}},\mathtt{d}}$, depending only on, ε_0, C_r, $|\Omega|$, $\bar{\alpha}_{\mathtt{d}}$, and $\bar{\beta}_{2,\mathtt{d}}$ such that*

$$\|\bar{\mathrm{T}}_h(\boldsymbol{\eta}_h)\|_{\mathrm{X}_2} \,:=\, \|\boldsymbol{\varphi}_h\|_{\mathrm{X}_2} \,\leqslant\, C_{\bar{\mathrm{T}},\mathtt{d}}\Big\{\|g\|_{1/s,r;\Gamma} \,+\, \|f\|_{0,r;\Omega} \,+\, \|\boldsymbol{\eta}_h\|_{0,\rho;\Omega}\Big\}\,. \tag{5.10}$$

Proof. See the proof of [13, Theorem 5.2]. □

Analogous to what was explained for the continuous operator $\bar{\mathrm{T}}$, here we can also assume that, except for a constant $C_{\bar{\mathrm{T}},\mathtt{d}}$ depending additionally on $\bar{\beta}_{1,\mathtt{d}}$, the a priori estimate for χ_h, which is now deduced from [3, Corollary 2.2, eq. (2.25)], is also given by the right-hand side of (5.10).

It remains to prove the well-definedness of $\widetilde{\mathrm{T}}_h := (\widetilde{\mathrm{T}}_{1,h},\widetilde{\mathrm{T}}_{2,h})$, for which we first note that, being a_i and c_i symmetric and positive semi-definite in the whole spaces H_i and Q_i, they certainly maintain their properties in $\mathrm{H}_{i,h}$ and $\mathrm{Q}_{i,h}$, respectively, so that the assumption i) of [12, Theorem 3.5] is clearly satisfied. Next, given $i \in \{1,2\}$, we let $\mathrm{V}_{i,h}$ be the discrete kernel of c_i, that is

$$\mathrm{V}_{i,h} \,:=\, \Big\{\boldsymbol{\tau}_{i,h} \in \mathrm{H}_{i,h} : \quad c_i(\boldsymbol{\tau}_{i,h},\eta_{i,h}) \,=\, 0 \quad \forall\,\eta_{i,h} \in \mathrm{Q}_{i,h}\Big\}, \tag{5.11}$$

and consider the hypotheses

(H.5) for each $i \in \{1,2\}$ there holds $\mathrm{div}(\mathrm{H}_{i,h}) \subseteq \mathrm{Q}_{i,h}$, and

(H.6) there exists a positive constant $\widetilde{\beta}_{\mathtt{d}} > 0$, independent of h, such that

$$\sup_{\substack{\boldsymbol{\tau}_{i,h}\in\mathrm{H}_{i,h}\\ \boldsymbol{\tau}_{i,h}\neq\mathbf{0}}} \frac{c_i(\boldsymbol{\tau}_{i,h},\eta_{i,h})}{\|\boldsymbol{\tau}_{i,h}\|_{\mathrm{H}_i}} \,\geqslant\, \widetilde{\beta}_{\mathtt{d}}\,\|\eta_{i,h}\|_{\mathrm{Q}_i} \qquad \forall\,\eta_{i,h} \in \mathrm{Q}_{i,h}\,.$$

It follows from (5.11), the definition of c_i (cf. (3.25b)), and **(H.5)** that

$$\mathrm{V}_{i,h} \,:=\, \Big\{ \boldsymbol{\tau}_{i,h} : \quad \mathrm{div}(\boldsymbol{\tau}_{i,h}) \,=\, 0 \Big\}\,,$$

from which it is easy to notice that $\mathrm{V}_{i,h}$ is contained in the continuous kernel V_i (cf. (4.12)) of c_i, giving rise to the discrete analogue of (4.13), that is

$$a_i(\boldsymbol{\tau}_{i,h}, \boldsymbol{\tau}_{i,h}) \,\geqslant\, \frac{1}{\bar{\kappa}}\, \|\boldsymbol{\tau}_{i,h}\|^2_{\mathrm{div}_{\varrho};\Omega} \qquad \forall\, \boldsymbol{\tau}_{i,h} \in \mathrm{V}_{i,h} \tag{5.12}$$

Thus, it follows from (5.12) that a_i satisfies the hypothesis ii) of [12, Theorem 3.5] with the constant $\widetilde{\alpha}_{\mathtt{d}} := \bar{\kappa}^{-1}$, whereas **(H.6)** itself constitutes assumption iii). Consequently, a direct application of [12, Theorem 3.5] implies the global discrete inf-sup condition for $\mathbf{A}$ (cf. (4.11a)) with a positive constant $\widetilde{\boldsymbol{\alpha}}_{\mathbf{A},\mathtt{d}}$ depending only on $\|a_i\|$, $\|c_i\|$, $\widetilde{\alpha}_{\mathtt{d}}$, and $\widetilde{\beta}_{\mathtt{d}}$, and thus the same property is shared by $\mathbf{A}_{\boldsymbol{\phi}_h,\mathbf{v}_h}$ for each $(\boldsymbol{\phi}_h, \mathbf{v}_h) \in \mathrm{X}_{2,h} \times \mathbf{H}^{\mathbf{u}}_h$, satisfying the discrete version of (4.14), that is

$$\|\boldsymbol{\phi}_h\|_{0,r;\Omega} \,+\, \|\mathbf{v}_h\|_{0,r;\Omega} \,\leqslant\, \frac{\widetilde{\boldsymbol{\alpha}}_{\mathbf{A},\mathtt{d}}}{2\,\|c\|}\,. \tag{5.13}$$

We are now in position of establishing the well-definedness of $\widetilde{\mathrm{T}}_{i,h}$ for each $i \in \{1, 2\}$, for which we cite the following result from [13, Theorem 5.3].

Theorem 5.3. *Given $i \in \{1, 2\}$ and $(\boldsymbol{\phi}_h, \mathbf{v}_h) \in \mathrm{X}_{2,h} \times \mathbf{H}^{\mathbf{u}}_h$ such that (5.13) holds, there exists a unique $(\boldsymbol{\sigma}_{i,h}, \xi_{i,h}) \in \mathrm{H}_{i,h} \times \mathrm{Q}_{i,h}$ solution to (5.7), and hence one can define $\widetilde{\mathrm{T}}_{i,h}(\boldsymbol{\phi}_h, \mathbf{v}_h) := \widetilde{\xi}_{i,h} \in \mathrm{Q}_{i,h}$. Moreover, there exists a positive constant $C_{\widetilde{\mathrm{T}},\mathtt{d}}$, depending only on $\|\mathrm{i}_\rho\|$ and $\widetilde{\boldsymbol{\alpha}}_{\mathbf{A},\mathtt{d}}$, such that*

$$\|\widetilde{\mathrm{T}}_{i,h}(\boldsymbol{\phi}_h, \mathbf{v}_h)\|_{\mathrm{Q}_i} \,=\, \|\xi_{i,h}\|_{\mathrm{Q}_i} \,\leqslant\, \|(\boldsymbol{\sigma}_{i,h}, \xi_{i,h})\|_{\mathrm{H}_i \times \mathrm{Q}_i} \,\leqslant\, C_{\widetilde{\mathrm{T}},\mathtt{d}} \Big\{ \|g_i\|_{1/2,\Gamma} \,+\, \|f_i\|_{0,\varrho;\Omega} \Big\}\,. \tag{5.14}$$

Analogously to the continuous case, it follows from the definition of $\widetilde{\mathrm{T}}_h$ (cf. (5.8)) and the a priori estimates given by (5.14) for each $i \in \{1, 2\}$, that

$$\|\widetilde{\mathrm{T}}_h(\boldsymbol{\phi}_h, \mathbf{v}_h)\|_{\mathrm{Q}_1 \times \mathrm{Q}_2} \,:=\, \sum_{i=1}^{2} \|\widetilde{\mathrm{T}}_{i,h}(\boldsymbol{\phi}_j, \mathbf{v}_h)\|_{\mathrm{Q}_i} \,\leqslant\, C_{\widetilde{\mathrm{T}},\mathtt{d}} \sum_{i=1}^{2} \Big\{ \|g_i\|_{1/2,\Gamma} \,+\, \|f_i\|_{0,\varrho;\Omega} \Big\}$$

for each $(\boldsymbol{\phi}_h, \mathbf{v}_h) \in \mathrm{X}_{2,h} \times \mathbf{H}^{\mathbf{u}}_h$ satisfying (5.13).

Having established that the discrete operators $\mathbf{S}_h$, $\bar{\mathrm{T}}_h$, $\widetilde{\mathrm{T}}_h$, and hence $\mathbf{T}_h$ (under the constraint imposed by (5.13)), are well defined, we now proceed as in Section 4.3 to address the solvability of the fixed-point equation (5.9). Then, letting $\delta_{\mathtt{d}}$ be an arbitrary radius, we define

$$\mathrm{W}(\delta_{\mathtt{d}}) \,:=\, \Big\{ (\boldsymbol{\phi}_h, \mathbf{z}_h) \in \mathrm{X}_{2,h} \times \mathbf{H}^{\mathbf{u}}_h : \quad \|(\boldsymbol{\phi}_h, \mathbf{z}_h)\| \,:=\, \|\boldsymbol{\phi}\|_{\mathrm{X}_2} \,+\, \|\mathbf{z}\|_{0,4;\Omega} \,\leqslant\, \delta_{\mathtt{d}} \Big\}\,.$$

Reasoning analogously to the derivation of Lemma 4.4 (cf. beginning of Section 4.3), we define $\delta_{\mathtt{d}} := \frac{\widetilde{\boldsymbol{\alpha}}_{\mathbf{A},\mathtt{d}}}{2\,\|c\|}$, and deduce that $\mathbf{T}_h$ maps $\mathrm{W}(\delta_{\mathtt{d}})$ into itself under the discrete version of (4.18), i.e.

$$C_{\mathbf{T},\mathtt{d}} \Big\{ \Big(1 \,+\, \Lambda_{g_i,f_i}\Big) \Big(\|g\|_{1/s,r;\Gamma} \,+\, \|f\|_{0,r;\Omega} \,+\, \Lambda_{g_i,f_i} \Big) \,+\, \|\mathbf{f}\|_{0,4/3;\Omega} \,+\, \|\mathbf{g}\|_{1/2,\Gamma} \Big\} \,\leqslant\, \delta_{\mathtt{d}}\,, \tag{5.15}$$

where $C_{\mathbf{T},\mathtt{d}}$ is a positive constant depending only on $C_{\mathbf{S}}$, $C_{\bar{\mathrm{T}}}$, $C_{\widetilde{\mathrm{T}}}$, and $(1 \,+\, \delta_{\mathtt{d}})$.

On the other hand, employing arguments analogous to those used in the proofs of Lemmas 4.1, 4.2, and 4.3, we can prove the continuity properties of $\mathbf{S}_h$, $\bar{\mathrm{T}}_h$, and $\widetilde{\mathrm{T}}_h$, that is the discrete version of (4.19), (4.26), and (4.27), which are exactly as the latter, but with constants denoted $\mathrm{L}_{\mathbf{S},\mathtt{d}}$, $\mathrm{L}_{\bar{\mathrm{T}},\mathtt{d}}$, and $\mathrm{L}_{\widetilde{\mathrm{T}},\mathtt{d}}$. Therefore, following a procedure analogous to the one that gave rise to (4.33), we deduce that, there exists a positive constant $\mathrm{L}_{\mathbf{T},\mathtt{d}}$ which is obtained similarly to $\mathrm{L}_{\mathbf{T}}$, but instead of depending on $C_{\mathbf{S}}$, $C_{\bar{\mathrm{T}}}$, $\mathrm{L}_{\mathbf{S}}$, $\mathrm{L}_{\bar{\mathrm{T}}}$, $\mathrm{L}_{\widetilde{\mathrm{T}}}$, and δ it depends on $C_{\mathbf{S},\mathtt{d}}$, $C_{\bar{\mathrm{T}},\mathtt{d}}$, $\mathrm{L}_{\mathbf{S},\mathtt{d}}$, $\mathrm{L}_{\bar{\mathrm{T}},\mathtt{d}}$, $\mathrm{L}_{\widetilde{\mathrm{T}},\mathtt{d}}$, and $\delta_{\mathtt{d}}$ such that

$$\begin{aligned}\|\mathbf{T}(\boldsymbol{\phi}_h,\mathbf{z}_h) - \mathbf{T}(\boldsymbol{\phi}_0,\mathbf{z}_0)\|_{\mathrm{X}_2\times\mathbf{L}^4(\Omega)} \leqslant \mathrm{L}_{\mathbf{T},\mathtt{d}}\Big\{\Lambda_{g_i,f_i}\Big(\|g\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega} + \Lambda_{g_i,f_i} + 1\Big) \\ + \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{g}\|_{1/2,\Gamma}\Big\}\,\|(\boldsymbol{\phi}_h,\mathbf{z}_h) - (\boldsymbol{\phi}_{h,0},\mathbf{z}_{h,0})\|\,,\end{aligned} \tag{5.16}$$

for all $(\boldsymbol{\phi}_h,\mathbf{z}_h),(\boldsymbol{\phi}_{h,0},\mathbf{z}_{h,0}) \in \mathrm{W}(\delta_{\mathtt{d}})$.

Consequently, we can now establish the main result of this section.

Theorem 5.4. *Assume that the data are sufficiently small so that (5.15) holds. Then, the operator $\mathbf{T}_h$ has a fixed point $(\boldsymbol{\varphi}_h,\mathbf{u}_h) \in \mathrm{W}(\delta_d)$. Equivalently, the coupled problem (5.1) has a solution $(\vec{\mathbf{u}}_h,\boldsymbol{\sigma}_h) \in \mathbf{H}_h\times\mathbf{Q}_h$, $(\boldsymbol{\varphi}_h,\chi_h) \in \mathrm{X}_{2,h}\times\mathrm{M}_{1,h}$, and $(\boldsymbol{\sigma}_{i,h},\xi_{i,h}) \in \mathrm{H}_{i,h}\times\mathrm{Q}_{i,h}$, $i\in\{1,2\}$, with $(\boldsymbol{\varphi}_h,\mathbf{u}_h) \in \mathrm{W}(\delta_{\mathtt{d}})$. Moreover, there hold the following a priori estimates*

$$\begin{aligned}\|(\vec{\mathbf{u}}_h,\boldsymbol{\sigma}_h)\|_{\mathbf{H}\times\mathbf{Q}} &\leqslant C_{\vec{\mathbf{u}},\boldsymbol{\sigma},\mathtt{d}}\Big\{\|\boldsymbol{\xi}_h\|_{0,\rho;\Omega}\,\|\boldsymbol{\varphi}_h\|_{0,r;\Omega} + \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{g}\|_{1/2,\Gamma}\Big\}\,,\\ \|(\boldsymbol{\varphi}_h,\chi_h)\|_{\mathrm{X}_2\times\mathrm{M}_1} &\leqslant C_{\bar{\mathrm{T}},\mathtt{d}}\Big\{\|g\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega} + \|\boldsymbol{\xi}_h\|_{0,\rho;\Omega}\Big\}\,,\quad \textit{and}\\ \|(\sigma_{i,h},\xi_{i,h})\|_{\mathrm{H}_i\times\mathrm{Q}_i} &\leqslant C_{\widetilde{\mathrm{T}},\mathtt{d}}\Big\{\|g_i\|_{1/2,\Gamma} + \|f_i\|_{0,\varrho;\Omega}\Big\}\quad i\in\{1,2\}\,,\end{aligned} \tag{5.17}$$

where $C_{\vec{\mathbf{u}},\boldsymbol{\sigma},\mathtt{d}}$ is a positive constant depending only on $C_{\mathbf{S},\mathtt{d}}$ and $\delta_{\mathtt{d}}$. In addition, under the extra assumption

$$\mathrm{L}_{\mathbf{T},\mathtt{d}}\Big\{\Lambda_{g_i,f_i}\Big(\|g\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega} + \Lambda_{g_i,f_i} + 1\Big) + \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{g}\|_{1/2,\Gamma}\Big\} < 1\,. \tag{5.18}$$

the aforementioned solutions of (5.9) and (5.1) are unique.

Proof. As indicated above, the fact that $\mathbf{T}_h$ maps $\mathrm{W}(\delta_{\mathtt{d}})$ into itself is consequence of (5.15). Then, the continuity of $\mathbf{T}_h$ (cf. (5.16)) and Brouwer's theorem (cf. [9, Theorem 9.9-2]) imply the existence of solution of (5.9). In turn, under the additional hypotheses (5.18), Banach's fixed-point Theorem guarantees the uniqueness of the solution. Additionally, bearing in mind that $\|\mathbf{u}_h\|_{0,4;\Omega} \leqslant \delta_{\mathtt{d}}$, in either case, (4.6), (4.9), (4.15) yield the a priori estimates (5.17) and conclude the proof. □

5.3 A priori error analysis

In this section we consider arbitrary finite element subspaces that satisfy the assumptions specified in Section 5.2, and establish the Céa estimate for the Galerkin error

$$\|(\vec{\mathbf{u}},\boldsymbol{\sigma}) - (\vec{\mathbf{u}}_h,\boldsymbol{\sigma}_h)\|_{\mathbf{H}\times\mathbf{Q}} + \|(\boldsymbol{\varphi},\chi) - (\boldsymbol{\varphi}_h,\chi_h)\|_{\mathrm{X}_2\times\mathrm{M}_1} + \sum_{i=1}^{2}\|(\boldsymbol{\sigma}_i,\xi_i) - (\boldsymbol{\sigma}_{i,h},\xi_{i,h})\|_{\mathrm{H}_i\times\mathrm{Q}_i}\,,$$

where $\big((\vec{\mathbf{u}},\boldsymbol{\sigma}),(\boldsymbol{\varphi},\chi),(\boldsymbol{\sigma}_i,\xi_i)\big) \in \big(\mathbf{H}\times\mathbf{Q}\big)\times\big(\mathrm{X}_2\times\mathrm{M}_1\big)\times\big(\mathrm{H}_i\times\mathrm{Q}_i\big)$, $i\in\big\{1,2\big\}$, is the unique solution of (3.26), and $\big((\vec{\mathbf{u}}_h,\boldsymbol{\sigma}_h),(\boldsymbol{\varphi}_h,\chi_h),(\boldsymbol{\sigma}_{i,h},\xi_{i,h})\big) \in \big(\mathbf{H}_h\times\mathbf{Q}_h\big)\times\big(\mathrm{X}_{2,h}\times\mathrm{M}_{1,h}\big)\times\big(\mathrm{H}_{i,h}\times\mathrm{Q}_{i,h}\big)$, $i\in\big\{1,2\big\}$, is a solution of (5.1). We proceed as in previous related work (see, e.g. [6]) by applying suitable Strang-type estimates to the pairs of associated continuous and discrete schemes arising from (3.26) and (5.1) after splitting them according to the three decoupled equations. Throughout the remainder of this section, given a subspace Z_h of an arbitrary Banach space $\big(Z,\|\cdot\|_Z\big)$, we set

$$\mathrm{dist}\big(z,Z_h\big) := \inf_{z_h\in Z_h}\|z - z_h\|_Z \qquad \forall\, z\in Z\,.$$

We begin the analysis by considering the first two rows of (3.26) and (5.1), so that, employing the estimates provided by [6, eq. (4.27), Section 4.3], we deduce the existence of a positive constant $\widehat{C}_1$, depending only on $\boldsymbol{\alpha}_{\mathtt{d}}$, $\boldsymbol{\beta}_{\mathtt{d}}$, $\|\mathbf{a}\|$, $\|\mathbf{b}\|$, $\|\mathbf{c}\|$, δ, and $\delta_{\mathtt{d}}$, such that

$$\begin{aligned} \|(\vec{\mathbf{u}},\boldsymbol{\sigma}) - (\vec{\mathbf{u}}_h,\boldsymbol{\sigma}_h)\|_{\mathbf{H}\times\mathbf{Q}} \,\leqslant\, \widehat{C}_1 \Big\{ & \mathrm{dist}\big(\vec{\mathbf{u}},\mathbf{H}_h\big) \,+\, \mathrm{dist}\big(\boldsymbol{\sigma},\mathbf{Q}_h\big) \\ & + \|\mathbf{F}_{\boldsymbol{\xi},\boldsymbol{\varphi}} \,-\, \mathbf{F}_{\boldsymbol{\xi}_h,\boldsymbol{\varphi}_h}\|_{\mathbf{H}_h'} \,+\, \|\mathbf{c}(\mathbf{u};\vec{\mathbf{u}},\cdot) \,-\, \mathbf{c}(\mathbf{u}_h;\vec{\mathbf{u}},\cdot)\|_{\mathbf{H}_h'} \Big\}. \end{aligned} \tag{5.19}$$

Thus, proceeding as in (4.24) and using the boundedness of $\mathbf{c}$ (cf. (3.15)), we easily obtain

$$\|\mathbf{F}_{\boldsymbol{\xi},\boldsymbol{\varphi}} - \mathbf{F}_{\boldsymbol{\xi}_h,\boldsymbol{\varphi}_h}\|_{\mathbf{H}_h'} \leqslant \varepsilon_0^{-1}\,|\Omega|^{1/4}\Big\{\|\boldsymbol{\xi}\|_{0,\rho,\Omega}\,\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{0,r,\Omega} \,+\, \|\boldsymbol{\xi}-\boldsymbol{\xi}_h\|_{0,\rho,\Omega}\,\|\boldsymbol{\varphi}_h\|_{0,r,\Omega}\Big\}\,,$$

and

$$\|\mathbf{c}(\mathbf{u};\vec{\mathbf{u}},\cdot) - \mathbf{c}(\mathbf{u}_h;\vec{\mathbf{u}},\cdot)\|_{\mathbf{H}_h'} \,\leqslant\, \|\mathbf{c}\|\,\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\,\|\vec{\mathbf{u}}\|_{\mathbf{H}}\,,$$

which, replaced back into (5.19), yields

$$\begin{aligned} \|(\vec{\mathbf{u}},\boldsymbol{\sigma}) \,-\, (\vec{\mathbf{u}}_h,\boldsymbol{\sigma}_h)\|_{\mathbf{H}\times\mathbf{Q}} \,\leqslant\, & \widehat{C}_1 \Big\{\mathrm{dist}\big(\vec{\mathbf{u}},\mathbf{H}_h\big) \,+\, \mathrm{dist}\big(\boldsymbol{\sigma},\mathbf{Q}_h\big)\Big\} \\ & + \widehat{C}_2 \Big\{\|\boldsymbol{\xi}\|_{0,\rho,\Omega}\,\|\boldsymbol{\varphi} \,-\, \boldsymbol{\varphi}_h\|_{0,r,\Omega} \,+\, \|\boldsymbol{\xi} \,-\, \boldsymbol{\xi}_h\|_{0,\rho,\Omega}\,\|\boldsymbol{\varphi}_h\|_{0,r,\Omega} \,+\, \|\mathbf{u} \,-\, \mathbf{u}_h\|_{0,4;\Omega}\,\|\vec{\mathbf{u}}\|_{\mathbf{H}}\Big\}\,. \end{aligned} \tag{5.20}$$

where $\widehat{C}_2 \,:=\, \widehat{C}_1 \max\big\{\varepsilon_0^{-1}\,|\Omega|^{1/4}, \|c\|\big\}$. Now, using the estimates obtained in [13, eq. (145), Section 5.3] for the third and fourth rows of (3.26) and (5.1), we find that

$$\|(\boldsymbol{\varphi},\chi) \,-\, (\boldsymbol{\varphi}_h,\chi_h)\|_{\mathrm{X}_2\times\mathrm{M}_1} \,\leqslant\, c_{\bar{\mathrm{T}}}\Big\{\mathrm{dist}(\boldsymbol{\varphi},\mathrm{X}_{2,h}) \,+\, \mathrm{dist}(\chi,\mathrm{M}_{1,h}) \,+\, \|\boldsymbol{\xi} \,-\, \boldsymbol{\xi}_h\|_{0,\rho;\Omega}\Big\}\,, \tag{5.21}$$

with a positive constant $c_{\bar{\mathrm{T}}}$ independent of h, and depending in particular on $|\Omega|$, ρ, and r. On the other hand, using the estimates obtained in [13, eq. (147), Section 5.3] for the fifth and sixth rows of (3.26) and (3.26), we get

$$\begin{aligned} \sum_{i=1}^{2}\|(\boldsymbol{\sigma}_i,\xi_i) - (\boldsymbol{\sigma}_{i,h},\xi_{i,h})\|_{\mathrm{H}_i\times\mathrm{Q}_i} \,\leqslant\, & c_{\widetilde{\mathrm{T}}}\Big\{\sum_{i=1}^{2}\big(\mathrm{dist}(\boldsymbol{\sigma}_i,\mathrm{H}_{i,h}) \,+\, \mathrm{dist}(\xi_i,\mathrm{Q}_{i,h})\big) \\ & + \big(\|\boldsymbol{\varphi}\|_{0,r;\Omega} \,+\, \|\mathbf{u}\|_{0,r;\Omega}\big)\,\|\boldsymbol{\xi}-\boldsymbol{\xi}_h\|_{0,\rho;\Omega} + \|\boldsymbol{\xi}_h\|_{0,\rho;\Omega}\,\big(\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{0,r;\Omega} \,+\, \|\mathbf{u}-\mathbf{u}_h\|_{0,r;\Omega}\big)\Big\}\,, \end{aligned} \tag{5.22}$$

with a positive constant $c_{\widetilde{\mathrm{T}}}$ independent of h, and depending in particular on $\|c\|$. For the remainder of the analysis we introduce the partial error

$$\mathtt{E} \,:=\, \|(\vec{\mathbf{u}},\boldsymbol{\sigma}) - (\vec{\mathbf{u}}_h,\boldsymbol{\sigma}_h)\|_{\mathbf{H}\times\mathbf{Q}} \,+\, \sum_{i=1}^{2}\|(\boldsymbol{\sigma}_i,\xi_i) - (\boldsymbol{\sigma}_{i,h},\xi_{i,h})\|_{\mathrm{H}_i\times\mathrm{Q}_i}\,,$$

and appropriately combine estimates (5.20), (5.21), and (5.22). In particular, using the right-hand side of (5.21) to bound $\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{0,r;\Omega}$ in (5.20) and (5.22), by adding up the resulting inequalities, performing some algebraic manipulations, and then using the a priori bounds for $\|\boldsymbol{\varphi}\|_{0,r;\Omega}$, $\|\boldsymbol{\varphi}_h\|_{0,r;\Omega}$, $\|\boldsymbol{\xi}\|_{0,\rho;\Omega}$, $\|\boldsymbol{\xi}_h\|_{0,\rho;\Omega}$, and $\|\mathbf{u}\|_{0,4;\Omega}$ provided by Theorems 4.8 and 5.4, we deduce the existence of a positive constant C_e, depending on $\widehat{C}_1$, $\widehat{C}_2$, $c_{\bar{\mathrm{T}}}$, $c_{\widetilde{\mathrm{T}}}$, δ, $\delta_{\mathtt{d}}$, C_S, $C_{\bar{\mathrm{T}}}$, $C_{\widetilde{\mathrm{T}}}$, $C_{\bar{\mathrm{T}},\mathtt{d}}$, and $C_{\widetilde{\mathrm{T}},\mathtt{d}}$, and hence independent of h, such that

$$\begin{aligned} \mathtt{E} \,\leqslant\, & C_e\Big\{\mathrm{dist}\big((\vec{\mathbf{u}},\boldsymbol{\sigma}),\mathbf{H}_h\times\mathbf{Q}_h\big) + \mathrm{dist}\big((\boldsymbol{\varphi},\chi),\mathrm{X}_{2,h}\times\mathrm{M}_{1,h}\big) + \sum_{i=1}^{2}\mathrm{dist}\big((\boldsymbol{\sigma}_i,\xi_i),\mathrm{H}_{i,h}\times\mathrm{Q}_{i,h}\big)\Big\} \\ & + C_e\Big\{\|\mathbf{g}\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,4/3,\Omega} + \|g\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega} + \sum_{i=1}^{2}\big(\|g_i\|_{1/2,\Gamma} + \|f_i\|_{0,\varrho,\Omega}\big)\Big\}\,\mathtt{E}\,. \end{aligned} \tag{5.23}$$

Consequently, we are in a position to establish the Céa estimate.

Theorem 5.5. *In addition to the hypotheses of Theorems 4.8 and 5.4, assume that*

$$C_e \left\{ \|\mathbf{g}\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,4/3,\Omega} + \|g\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega} + \sum_{i=1}^{2} \big(\|g_i\|_{1/2,\Gamma} + \|f_i\|_{0,\varrho,\Omega} \big) \right\} \leqslant \frac{1}{2}. \tag{5.24}$$

Then, there exists a positive constant C, independent of h, such that

$$\begin{aligned} &\|(\vec{\mathbf{u}}, \boldsymbol{\sigma}) - (\vec{\mathbf{u}}_h, \boldsymbol{\sigma}_h)\|_{\mathbf{H}\times\mathbf{Q}} + \|(\boldsymbol{\varphi}, \chi) - (\boldsymbol{\varphi}_h, \chi_h)\|_{\mathrm{X}_2\times\mathrm{M}_1} + \sum_{i=1}^{2} \|(\boldsymbol{\sigma}_i, \xi_i) - (\boldsymbol{\sigma}_{i,h}, \xi_{i,h})\|_{\mathrm{H}_i\times\mathrm{Q}_i} \\ &\quad \leqslant C \left\{ \operatorname{dist}\big((\boldsymbol{\sigma}, \mathbf{u}), \mathbf{H}_h \times \mathbf{Q}_h\big) + \operatorname{dist}\big((\boldsymbol{\varphi}, \chi), \mathrm{X}_{2,h} \times \mathrm{M}_{1,h}\big) + \sum_{i=1}^{2} \operatorname{dist}\big((\boldsymbol{\sigma}_i, \xi_i), \mathrm{H}_{i,h} \times \mathrm{Q}_{i,h}\big) \right\}. \end{aligned} \tag{5.25}$$

Proof. Under the assumption (5.24), the a priori estimate for $\mathtt{E}$ follows from (5.23), which together with (5.21), yields (5.25) and ends the proof. □

We end this section with the a priori estimate for $\|p - p_h\|_{0,\Omega}$ where p_h is the discrete pressure suggested by the postprocessing formula given by the second identity in (2.7), which, according to (3.8), becomes

$$p_h \,=\, -\frac{1}{n}\operatorname{tr}\Big(\boldsymbol{\sigma}_h \,+\, c_h\,\mathbb{I} \,+\, \frac{\omega}{2}(\mathbf{u}_h \otimes \mathbf{u}_h)\Big), \quad \text{with} \quad c_h \,:=\, -\frac{\omega}{2\,n\,|\Omega|}\int_\Omega \operatorname{tr}(\mathbf{u}_h \otimes \mathbf{u}_h)\,. \tag{5.26}$$

Then, applying the Cauchy–Schwarz inequality, performing some algebraic manipulations, and employing the a priori bounds for $\|\mathbf{u}\|_{0,4;\Omega}$ and $\|\mathbf{u}_h\|_{0,4;\Omega}$, we deduce the existence of a positive constant C, depending on data, but independent of h, such that

$$\|p \,-\, p_h\|_{0,\Omega} \,\leqslant\, C\left\{ \|\boldsymbol{\sigma} \,-\, \boldsymbol{\sigma}_h\|_{0,\Omega} \,+\, \|\mathbf{u} \,-\, \mathbf{u}_h\|_{0,4;\Omega} \right\}.$$

6 Specific finite element subspaces

We now define finite element subspaces satisfying the hypotheses **(H.1)** - **(H.6)** from Section 5.2, and provide the rates of convergences for the Galerkin scheme (5.1).

6.1 Preliminaries

In the following we use the notation introduced at the beginning of Section 5.1. Thus, given an integer $k \geqslant 0$, for each $K \in \mathcal{T}_h$ we let $\mathrm{P}_k(K)$, $\mathbf{P}_k(K)$, and $\mathbb{P}_k(K)$ be the scalar, vector, and tensor versions, respectively, of the space of polynomials of degree $\leqslant k$ defined on K. Similarly, letting $\mathbf{x}$ be a generic vector in R^n, $\mathbf{RT}_k(K) := \mathbf{P}(K) + \mathrm{P}_k(K)\mathbf{x}$ and $\mathbb{RT}_k(K)$ stand for the local Raviart–Thomas space of order k defined on K and its associated tensor counterpart. Additionally, we let $\mathrm{P}_k(\mathcal{T}_h)$, $\mathbf{P}_k(\mathcal{T}_h)$, $\mathbb{P}_k(\mathcal{T}_h)$, $\mathbf{RT}_k(\mathcal{T}_h)$ and $\mathbb{RT}_k(\mathcal{T}_h)$ be the global versions of $\mathrm{P}_k(K)$, $\mathbf{P}_k(K)$, $\mathbb{P}_k(K)$, $\mathbf{RT}_k(K)$ and $\mathbb{RT}_k(K)$, respectively, that is

$$\begin{aligned} \mathrm{P}_k(\mathcal{T}_h) \,&:=\, \Big\{ v_h \in \mathrm{L}^2(\Omega) : \quad v_h|_K \in \mathrm{P}_k(K) \quad \forall\, K \in \mathcal{T}_h \Big\}\,, \\ \mathbf{P}_k(\mathcal{T}_h) \,&:=\, \Big\{ \mathbf{v}_h \in \mathbf{L}^2(\Omega) : \quad \mathbf{v}_h|_K \in \mathbf{P}_k(K) \quad \forall\, K \in \mathcal{T}_h \Big\}\,, \\ \mathbb{P}_k(\mathcal{T}_h) \,&:=\, \Big\{ \mathbf{s}_h \in \mathbb{L}^2(\Omega) : \quad \mathbf{s}_h|_K \in \mathbb{P}_k(K) \quad \forall\, K \in \mathcal{T}_h \Big\}\,, \\ \mathbf{RT}_k(\mathcal{T}_h) \,&:=\, \Big\{ \mathbf{q}_h \in \mathbf{H}(\mathrm{div};\Omega) : \quad \mathbf{q}_h|_K \in \mathbf{RT}_k(K) \quad \forall\, K \in \mathcal{T}_h \Big\}\,, \\ \mathbb{RT}_k(\mathcal{T}_h) \,&:=\, \Big\{ \boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div};\Omega) : \quad \boldsymbol{\tau}_h|_K \in \mathbb{RT}_k(K) \quad \forall\, K \in \mathcal{T}_h \Big\}\,, \end{aligned}$$

We notice here that for each $t \in (1,+\infty)$ there hold the inclusions $\mathrm{P}_k(\mathcal{T}_h) \subseteq \mathrm{L}^t(\Omega)$, $\mathbf{P}_k(\mathcal{T}_h) \subseteq \mathbf{L}^t(\Omega)$, $\mathbb{P}_k \subseteq \mathbb{L}^t(\Omega)$, $\mathbf{RT}_k(\Omega) \subseteq \mathbf{H}(\mathrm{div}_t;\Omega)$, $\mathbf{RT}_k(\Omega) \subseteq \mathbf{H}^t(\mathrm{div}_t;\Omega)$, and $\mathbb{RT}_k(\mathcal{T}_h) \subseteq \mathbb{H}(\mathbf{div}_t;\Omega)$, which are employed below to introduce our specific finite element subspaces. Indeed, we now set

$$\begin{gathered}\mathbf{H}_h^{\mathbf{u}} := \mathbf{P}_k(\mathcal{T}_h)\,, \quad \mathbb{H}_h^{\mathbf{t}} := \mathbb{L}^2_{\mathrm{tr}}(\Omega) \cap \mathbb{P}_k(\mathcal{T}_h)\,, \quad \mathbf{H}_h := \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\mathbf{t}}\,, \quad \mathbb{H}_h^{\boldsymbol{\sigma}} := \mathbb{RT}_k(\mathcal{T}_h)\,, \\ \mathbf{Q}_h := \mathbb{H}_h^{\boldsymbol{\sigma}} \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,, \quad \mathrm{H}_{i,h} := \mathbf{RT}_k(\mathcal{T}_h)\,, \quad \mathrm{Q}_{i,h} := \mathrm{P}_k(\mathcal{T}_h) \\ \mathrm{X}_{2,h} := \mathbf{RT}_k(\mathcal{T}_h)\,, \quad \mathrm{M}_{1,h} := \mathrm{P}_k(\mathcal{T}_h)\,, \quad \mathrm{X}_{1,h} := \mathbf{RT}_k(\mathcal{T}_h)\,, \quad \text{and} \quad \mathrm{M}_{2,h} := \mathrm{P}_k(\mathcal{T}_h)\,.\end{gathered} \tag{6.1}$$

6.2 Verification of the hypotheses (H.1) - (H.6)

We begin by observing that the hypotheses **(H.1)** and **(H.2)** are exactly the same as [6, **(H.1)** and **(H.2)**], particularly is proved in [6, Lemma 5.1]. In turn, we emphasize that **(H.3)** corresponds exactly to [6, **(H.5)**], and hence we omit most details and refer to [6, Section 5.2, Lemma 5.2]. Finally, it is clear from (6.1) that **(H.5)** is trivially satisfied (cf. [16, Lemma 3.6, part (i)]), whereas **(H.6)** was proved precisely by [17, Lemma 4.5].

6.3 The rates of convergence

Here we present the rates of convergence of the Galerkin scheme (5.1) with the specific finite element subspaces introduced in Section 6.1, for which the respective approximation properties were previously collected. In fact, it follows easily from [15, Proposition 1.135] and its vector and tensorial versions, along with interpolation estimates of Sobolev spaces, that those of $\mathbf{H}_h^{\mathbf{u}}$, $\mathbb{H}_h^{\mathbf{t}}$, $\mathrm{Q}_{i,h}$, and $\mathrm{M}_{1,h}$ are given as follows

$(\mathbf{AP}_h^{\mathbf{u}})$ there exists a positive constant C, independent of h, such that for each $l \in [0, k+1]$, and for each $\mathbf{v} \in \mathbf{W}^{l,4}(\Omega)$, there holds

$$\mathrm{dist}(\mathbf{v}, \mathbf{H}_h^{\mathbf{u}}) := \inf_{\mathbf{v}_h \in \mathbf{H}_h^{\mathbf{u}}} \|\mathbf{v} - \mathbf{v}_h\|_{0,4;\Omega} \leqslant C\, h^l\, \|\mathbf{v}\|_{l,4;\Omega}\,,$$

$(\mathbf{AP}_h^{\mathbf{t}})$ there exists a positive constant C, independent of h, such that for each $l \in [0, k+1]$, and for each $\mathbf{s} \in \mathbb{H}^l(\Omega) \cap \mathbb{L}^2_{\mathrm{tr}}(\Omega)$, there holds

$$\mathrm{dist}(\mathbf{s}, \mathbb{H}_h^{\mathbf{t}}) := \inf_{\mathbf{s}_h \in \mathbb{H}_h^{\mathbf{t}}} \|\mathbf{s} - \mathbf{s}_h\|_{0,\Omega} \leqslant C\, h^l\, \|\mathbf{s}\|_{l,\Omega}\,.$$

$\left(\mathbf{AP}_h^{\xi_i}\right)$ there exists a positive constant C, independent of h, such that for each $l \in [0, k+1]$, and for each $\eta_i \in \mathrm{W}^{l,\rho}(\Omega)$, there holds

$$\mathrm{dist}(\eta_i, \mathrm{Q}_{i,h}) := \inf_{\eta_{i,h} \in \mathrm{Q}_{i,h}} \|\eta_i - \eta_{i,h}\|_{0,\rho;\Omega} \leqslant C\, h^l\, \|\eta_i\|_{l,\rho;\Omega}\,,$$

$(\mathbf{AP}_h^{\chi})$ there exists a positive constant C, independent of h, such that for each $l \in [0, k+1]$, and for each $\lambda \in \mathrm{W}^{l,r}(\Omega)$, there holds

$$\mathrm{dist}(\lambda, \mathrm{M}_{1,h}) := \inf_{\lambda_h \in \mathrm{M}_{1,h}} \|\lambda - \lambda_h\|_{0,r;\Omega} \leqslant C\, h^l\, \|\lambda\|_{l,r;\Omega}\,.$$

Furthermore, from [17, eq. (4.6), Section 4.1] and its tensor version, which, as the foregoing ones, are derived classically by using the Deny–Lions Lemma and the corresponding scaling estimates (cf. [15, Lemmas B.67 and 1.101]), we state below the approximation properties of $\mathbf{Q}_h$ and $\mathrm{H}_{i,h}$

$(\mathbf{AP}_h^{\boldsymbol{\sigma}})$ there exists a positive constant C, independent of h, such that for each $l \in [1, k+1]$, and for each $\boldsymbol{\tau} \in \mathbb{H}^l(\Omega) \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$ with $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{W}^{l,4/3}(\Omega)$, there holds

$$\mathrm{dist}(\boldsymbol{\tau}, \mathbf{Q}_h) := \inf_{\boldsymbol{\tau}_h \in \mathbf{Q}_h} \|\boldsymbol{\tau} - \boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega} \leqslant C\, h^l \left\{ \|\boldsymbol{\tau}\|_{l,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{l,4/3;\Omega} \right\},$$

$\left(\mathbf{AP}_h^{\boldsymbol{\sigma}_i}\right)$ there exists a positive constant C, independent of h, such that for each $l \in [1, k+1]$, and for each $\boldsymbol{\tau}_i \in \mathbb{H}^l(\Omega)$ with $\operatorname{div}(\boldsymbol{\tau}_i) \in \mathbf{W}^{l,\varrho}(\Omega)$, there holds

$$\operatorname{dist}(\boldsymbol{\tau}_i, \mathrm{H}_{i,h}) := \inf_{\boldsymbol{\tau}_{i,h} \in \mathrm{H}_{i,h}} \|\boldsymbol{\tau}_i - \boldsymbol{\tau}_{i,h}\|_{\operatorname{div}_\varrho;\Omega} \leqslant C\, h^l \left\{ \|\boldsymbol{\tau}_i\|_{l,\Omega} + \|\operatorname{div}(\boldsymbol{\tau}_i)\|_{l,\varrho;\Omega} \right\}.$$

Finally, that of $\mathrm{X}_{2,h}$, which we recall from [17, Section 4.5 $(\mathbf{AP}_h^{\mathbf{u}})$], becomes

$\left(\mathbf{AP}_h^{\boldsymbol{\varphi}}\right)$ there exists a positive constant C, independent of h, such that for each $l \in [1, k+1]$, and for each $\boldsymbol{\phi} \in \mathbf{W}^{l,r}(\Omega)$ with $\operatorname{div}(\boldsymbol{\phi}) \in \mathrm{W}^{l,r}(\Omega)$, there holds

$$\operatorname{dist}(\boldsymbol{\phi}, \mathrm{X}_2) := \inf_{\boldsymbol{\phi} \in \mathrm{X}_{2,h}} \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{r,\operatorname{div}_r;\Omega} \leqslant C\, h^l \left\{ \|\boldsymbol{\phi}\|_{l,r;\Omega} + \|\operatorname{div}(\boldsymbol{\phi})\|_{l,r;\Omega} \right\}.$$

The rates of convergence of (5.1) are now provided by the following theorem.

Theorem 6.1. *Let* $\big((\vec{\mathbf{u}}, \boldsymbol{\sigma}), (\boldsymbol{\varphi}, \xi), (\boldsymbol{\sigma}_i, \xi_i)\big) \in (\mathbf{H} \times \mathbf{Q}) \times (\mathrm{X}_2 \times \mathrm{M}_1) \times (\mathrm{H}_i \times \mathrm{Q}_i)$, $i \in \{1,2\}$ *be the unique solution of* (3.26) *with* $(\boldsymbol{\varphi}, \mathbf{u}) \in \mathrm{W}(\delta)$, *and let* $\big((\vec{\mathbf{u}}_h, \boldsymbol{\sigma}_h), (\boldsymbol{\varphi}_h, \xi_h), (\boldsymbol{\sigma}_{i,h}, \xi_{i,h})\big) \in (\mathbf{H}_h \times \mathbf{Q}_h) \times (\mathrm{X}_{2,h} \times \mathrm{M}_{1,h}) \times (\mathrm{H}_{i,h} \times \mathrm{Q}_{i,h})$, $i \in \{1,2\}$ *be a solution of* (5.1) *with* $(\boldsymbol{\varphi}_h, \mathbf{u}_h) \in \mathrm{W}(\delta_{\mathrm{d}})$, *which is guaranteed by Theorems* 4.8 *and* 5.4, *respectively. In turn, let* p *and* p_h *be given by* (2.7) *and* (5.26), *respectively. Assume the hypotheses of Theorem* 5.5, *and that there exists* $l \in [1, k+1]$ *such that* $\mathbf{u} \in \mathbf{W}^{l,4}(\Omega)$, $\mathbf{t} \in \mathbb{H}^l(\Omega) \cap \mathbb{L}^2_{\mathrm{tr}}(\Omega)$, $\boldsymbol{\sigma} \in \mathbb{H}^l(\Omega) \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{W}^{l,4/3}(\Omega)$, $\boldsymbol{\varphi} \in \mathbf{W}^{l,r}(\Omega)$, $\operatorname{div}(\boldsymbol{\varphi}) \in \mathrm{W}^{l,r}(\Omega)$, $\chi \in \mathrm{W}^{l,r}(\Omega)$, $\boldsymbol{\sigma}_i \in \mathbb{H}^l(\Omega)$, $\operatorname{div}(\boldsymbol{\sigma}_i) \in \mathbf{W}^{l,\varrho}(\Omega)$, *and* $\xi_i \in \mathrm{W}^{l,\varrho}(\Omega)$, $i \in \{1,2\}$. *Then, there exists a positive constant* C, *independent of* h, *such that*

$$\begin{aligned}
&\|(\vec{\mathbf{u}}, \boldsymbol{\sigma}) - (\vec{\mathbf{u}}_h, \boldsymbol{\sigma}_h)\|_{\mathbf{H}\times\mathbf{Q}} + \|p - p_h\|_{0,\Omega} + \|(\boldsymbol{\varphi}, \chi) - (\boldsymbol{\varphi}_h, \chi_h)\|_{\mathrm{X}_2\times\mathrm{M}_1} + \sum_{i=1}^{2} \|(\boldsymbol{\sigma}_i, \xi_i) - (\boldsymbol{\sigma}_{i,h}, \xi_{i,h})\|_{\mathrm{H}_i\times\mathrm{Q}_i} \\
&\quad \leqslant C\, h^l \Big\{ \|\mathbf{u}\|_{l,4;\Omega} + \|\mathbf{t}\|_{l,\Omega} + \|\boldsymbol{\sigma}\|_{l,\Omega} + \|\mathbf{div}(\boldsymbol{\sigma})\|_{l,4/3;\Omega} + \|\boldsymbol{\varphi}\|_{l,r;\Omega} + \|\operatorname{div}(\boldsymbol{\varphi})\|_{l,r;\Omega} \\
&\qquad + \|\xi\|_{l,r;\Omega} + \sum_{i=1}^{2} \big(\|\boldsymbol{\sigma}_i\|_{l,\Omega} + \|\operatorname{div}(\boldsymbol{\sigma}_i)\|_{l,\varrho;\Omega} + \|\xi_i\|_{l,\rho;\Omega} \big) \Big\}.
\end{aligned}$$

Proof. It follows straightforwardly from Theorem 5.5, (5.26), and the above approximation properties. □

6.4 Conservation properties

We first let $\mathcal{P}_h^k : \mathrm{L}^1(\Omega) \to \mathrm{P}_k(\mathcal{T}_h)$ be the projector defined, for each $v \in \mathrm{L}^1(\Omega)$, as the unique element $\mathcal{P}_h^k(v) \in \mathrm{P}_k(\mathcal{T}_h)$ such that

$$\int_\Omega \mathcal{P}_h^k(v)\, q_h = \int_\Omega v\, q_h \qquad \forall\, q_h \in \mathrm{P}_k(\mathcal{T}_h), \tag{6.2}$$

and let $\boldsymbol{\mathcal{P}}_h^k : \mathbf{L}^1(\Omega) \to \mathbf{P}_k(\mathcal{T}_h)$ be its corresponding vector version. Then, according to the definitions of the specific finite element subspaces $\mathbf{H}_h^{\mathbf{u}}$, $\mathrm{M}_{2,h}$ and $\mathrm{Q}_{i,h}$, $i \in \{1,2\}$, provided in (6.1), we readily deduce from (5.2), (5.3), and (5.4), that there hold

$$\boldsymbol{\mathcal{P}}_h^k\Big(\mathbf{div}(\boldsymbol{\sigma}_h) - (\xi_{1,h} - \xi_{2,h})\,\varepsilon^{-1}\boldsymbol{\varphi}_h - \frac{\omega}{2}\,\mathbf{t}_h\,\mathbf{u}_h + \mathbf{f}\Big) = \mathbf{0} \quad \text{in} \quad \Omega, \tag{6.3}$$

$$\mathcal{P}_h^k\big(\operatorname{div}(\boldsymbol{\varphi}_h) + (\xi_{1,h} - \xi_{2,h}) + f\big) = 0 \quad \text{in} \quad \Omega, \tag{6.4}$$

and

$$\mathcal{P}_h^k\big(\operatorname{div}(\boldsymbol{\sigma}_{i,h}) - \xi_{i,h} + f_i\big) = 0 \quad \text{in} \quad \Omega, \tag{6.5}$$

respectively, which constitute the discrete conservation of momentum properties of (5.1).

7 Numerical results

The computational tests in this section have been realized using the finite element library `FEniCS` [1]. The nonlinear algebraic systems are solved with Newton's method with a residual tolerance of 10^{-6}. The linear systems are solved with the direct method MUMPS. The zero-mean condition for the trace of the $\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$ component $\boldsymbol{\sigma}_0$ of the original Bernoulli-type stress tensor $\boldsymbol{\sigma}$, is enforced using a real Lagrange multiplier. Recall that in Section 3.2, $\boldsymbol{\sigma}_0$ was simply redenoted $\boldsymbol{\sigma}$.

7.1 Verification of convergence

We choose the arbitrary model parameters $\mu = \varepsilon = 0.1$, $\omega = 0.5$, $\kappa_1 = 0.01$, $\kappa_2 = 0.2$, and, letting $\mathbf{x} := (x, y)$ (resp. $\mathbf{x} := (x, y, z)$) be a generic vector of R^2 (resp. R^3), define the following manufactured exact solutions to (2.8) in 2D and 3D, respectively

$$\text{On } \Omega = (0,1)^2\text{:}\quad \begin{cases} \mathbf{u}(\mathbf{x}) = \begin{pmatrix} \cos(\pi x)\sin(\pi y) \\ -\sin(\pi x)\cos(\pi y) \end{pmatrix}, \quad p(\mathbf{x}) = x^4 - y^4, \\ \xi_1(\mathbf{x}) = \exp(-xy), \quad \xi_2(\mathbf{x}) = \cos^2(xy), \quad \chi(\mathbf{x}) = \sin(x)\cos(y), \end{cases}$$

$$\text{On } \Omega = (0,1)^3\text{:}\quad \begin{cases} \mathbf{u}(\mathbf{x}) = \begin{pmatrix} \sin(\pi x)\cos(\pi y)\cos(\pi z) \\ -2\cos(\pi x)\sin(\pi y)\cos(\pi z) \\ \cos(\pi x)\cos(\pi y)\sin(\pi z) \end{pmatrix}, \quad p(\mathbf{x}) = x^4 - \frac{1}{2}(y^4 + z^4), \\ \xi_1(\mathbf{x}) = \exp(-xy + z), \quad \xi_2(\mathbf{x}) = \cos^2(xyz), \quad \chi(\mathbf{x}) = \sin(x)\cos(y)\sin(z), \end{cases}$$

and mixed variables

$$\mathbf{t} = \nabla\mathbf{u}, \quad \boldsymbol{\sigma} = \mu\nabla\mathbf{u} - \frac{\omega}{2}(\mathbf{u}\otimes\mathbf{u}) - p\mathbb{I}, \quad \boldsymbol{\varphi} = \varepsilon\nabla\chi, \quad \boldsymbol{\sigma}_i = \kappa_i(\nabla\xi_i + q_i\xi_i\varepsilon^{-1}\boldsymbol{\varphi}) - \xi_i\mathbf{u}\,.$$

With these smooth fields we construct forcing/source terms and non-homogeneous Dirichlet boundary conditions $\mathbf{f}, \mathbf{g}, f_i, g_i$. For the 3D case we take the Banach exponents $r = 3$, $s = 3/2$, $\rho = 6$, $\varrho = 6/5$, while for the 2D computations we use $r = \rho = 4$, $s = \varrho = 4/3$. The problem is numerically solved on a sequence of $\mathtt{n}_k^{\max}$ successively refined regular meshes. Errors in the norms from Theorem 6.1 are separated in the contribution from each unknown. The error history is portrayed in Figure 7.1, where in the 2D case we also run the convergence tests for the second-order scheme (using $k = 1$). It is noted that, irrespective of the spatial dimension or the polynomial degree, the method converges optimally. Furthermore, Figure 7.2 shows approximate solutions for primary and mixed variables, all fields sufficiently well captured.

We also study the conservation features of the method explained in Section 6.4, for which the following numbers

$$\mathtt{mom}_h := \big\|\boldsymbol{\mathcal{P}}_h^k\big(\mathbf{div}(\boldsymbol{\sigma}_h) - (\xi_{1,h} - \xi_{2,h})\,\varepsilon^{-1}\boldsymbol{\varphi}_h - \frac{\omega}{2}\,\mathbf{t}_h\,\mathbf{u}_h + \mathbf{f}\big)\big\|_{\ell^\infty},$$

$$\mathtt{pot}_h := \|\mathcal{P}_h^k\big(\mathrm{div}(\boldsymbol{\varphi}_h) + (\xi_{1,h} - \xi_{2,h}) + f\big)\|_{\ell^\infty}, \quad \mathtt{tra}_{i,h} := \|\mathcal{P}_h^k\big(\mathrm{div}(\boldsymbol{\sigma}_{i,h}) - \xi_{i,h} + f_i\big)\|_{\ell^\infty},$$

are computed at each refinement level and tabulated in Table 7.1 together with the total error

$$\mathtt{e} := \|(\vec{\mathbf{u}}, \boldsymbol{\sigma}) - (\vec{\mathbf{u}}_h, \boldsymbol{\sigma}_h)\|_{\mathbf{H}\times\mathbf{Q}} + \|p - p_h\|_{0,\Omega} + \|(\boldsymbol{\varphi}, \chi) - (\boldsymbol{\varphi}_h, \chi_h)\|_{\mathrm{X}_2\times\mathrm{M}_1} + \sum_{i=1}^{2}\|(\boldsymbol{\sigma}_i, \xi_i) - (\boldsymbol{\sigma}_{i,h}, \xi_{i,h})\|_{\mathrm{H}_i\times\mathrm{Q}_i}\,,$$

and its experimental convergence rate $\mathtt{r} = \log(\mathtt{e}/\widehat{\mathtt{e}})[\log(h/\widehat{h})]^{-1}$, where $\mathtt{e}$ and $\widehat{\mathtt{e}}$ denote errors produced on two consecutive meshes of sizes h and $\widehat{h}$, respectively. Note here that the above $\|\cdot\|_{\ell^\infty}$ norms are computed by considering the degrees of freedom defining uniquely the respective scalar or vector piecewise polynomials to which they are applied. We report on the 2D case only (in 3D we obtain analogous results). The expected optimal convergence of the total error, and the announced local conservativity are confirmed. We also see that after the first mesh refinement the number of Newton iterations required for convergence is four.

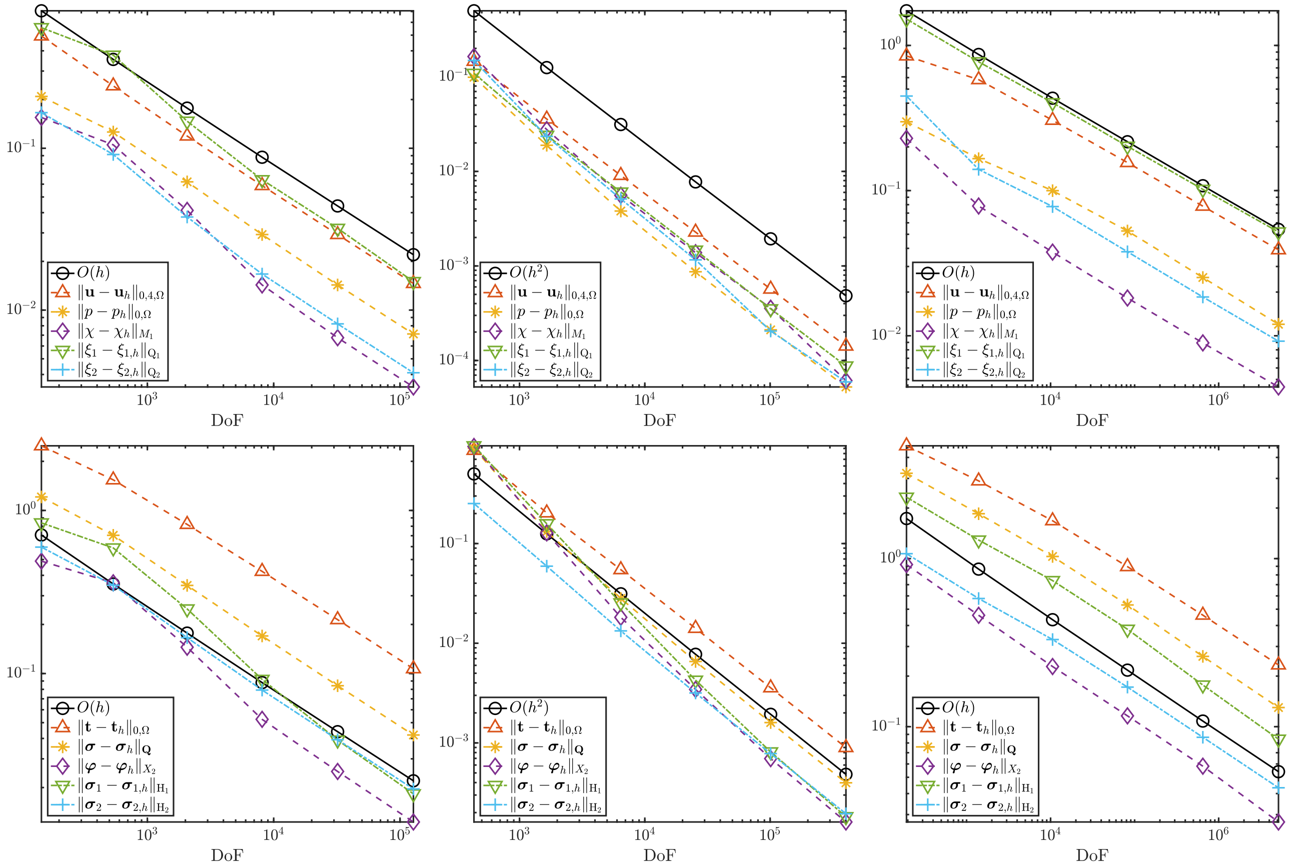

Figure 7.1: Convergence test in 2D and 3D. Error history associated with the fully mixed method for $k=0$ and in 2D (left), for $k=1$ and in 2D (middle), and for $k=0$ and 3D (right). Primary variables (top) and mixed variables (bottom).

DoF	h	e	r	$\mathtt{mom}_h$	$\mathtt{pot}_h$	$\mathtt{tra}_{1,h}$	$\mathtt{tra}_{2,h}$	it
				$k=0$				
145	0.707	7.20e+00	*	1.11e-13	1.18e-16	2.80e-16	1.23e-16	5
537	0.354	4.48e+00	0.69	1.05e-10	5.53e-16	1.02e-15	1.76e-16	4
2'065	0.177	2.13e+00	1.07	3.33e-10	2.65e-15	2.18e-15	3.05e-16	4
8'097	0.088	9.69e-01	1.14	7.30e-12	4.90e-15	6.78e-15	7.53e-16	4
32'065	0.044	4.60e-01	1.08	2.44e-12	1.24e-14	2.21e-14	1.49e-15	4
127'617	0.022	2.24e-01	1.04	3.93e-12	1.27e-13	1.02e-13	2.80e-14	4
				$k=1$				
433	0.707	5.42e+00	*	1.12e-07	4.43e-15	3.28e-15	9.77e-16	5
1'649	0.354	9.17e-01	2.56	5.17e-12	6.73e-15	1.50e-14	1.51e-15	4
6'433	0.177	1.78e-01	2.37	2.58e-12	1.81e-14	2.57e-14	2.33e-15	4
25'409	0.088	3.70e-02	2.26	2.90e-12	3.51e-14	5.34e-14	4.12e-15	4
100'997	0.044	8.28e-03	2.16	2.80e-12	8.73e-14	1.45e-13	1.03e-14	4
402'689	0.022	1.97e-03	2.07	2.54e-12	1.98e-13	2.82e-13	2.34e-14	4

Table 7.1: Convergence test in 2D. Total error, experimental rates of convergence, ℓ^∞-norm of the projected residual of the momentum, potential, and ionic transport equations, and Newton iteration count. Computations with the two lowest-order polynomial degrees.

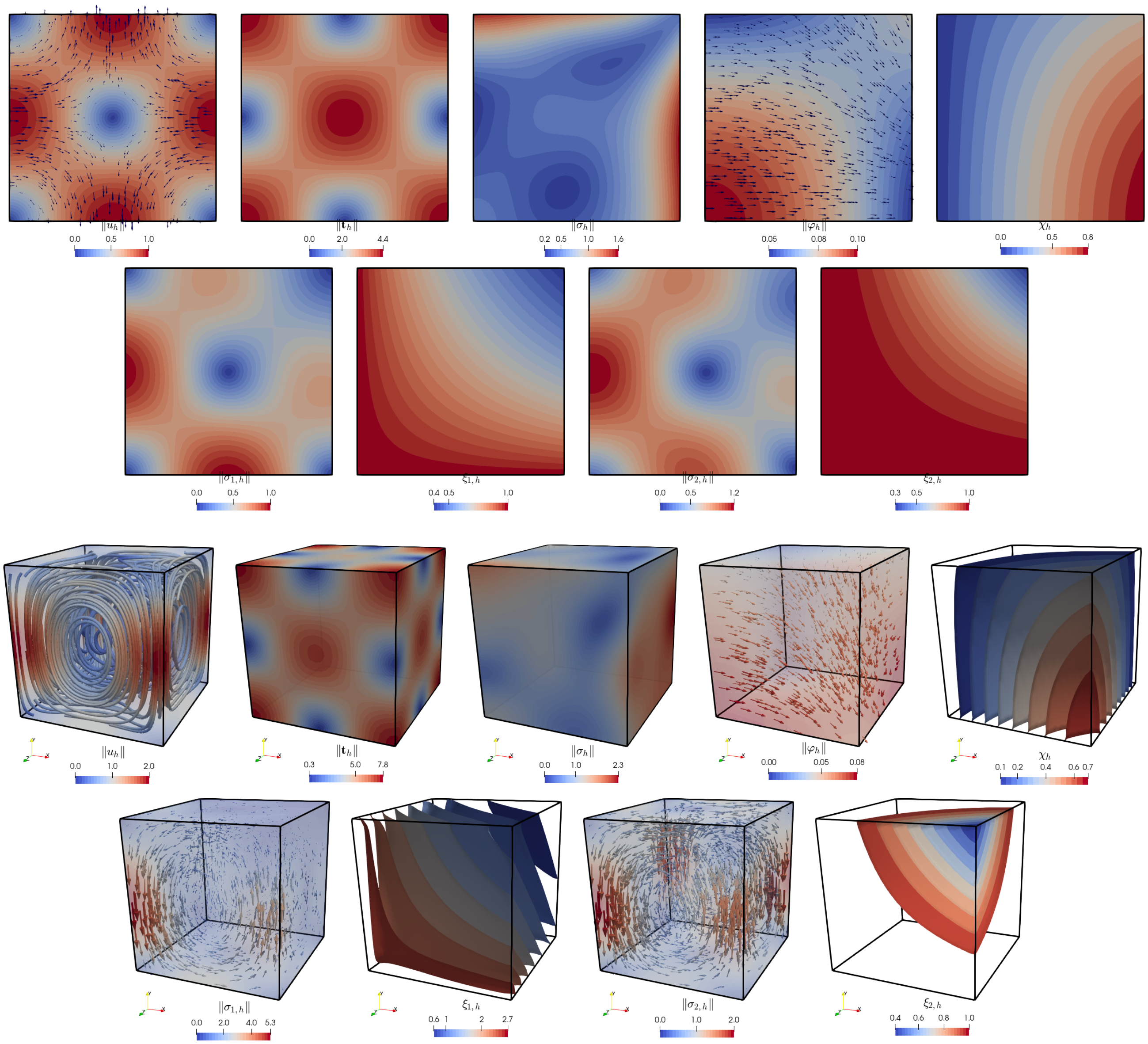

Figure 7.2: Convergence test in 2D and 3D. Approximate velocity, velocity gradient, Bernoulli-type stress tensor, electric field, electrostatic potential, flux of cations, concentration of cations, flux of anions, concentration of anions, computed with the first-order method.

7.2 Ion spreading in a charged enclosure

In order to further validate our numerical methods, inspired by the tests in [23, Section 5.2] we simulate the phenomenon of electrodiffusion of ions in a charged reservoir. We follow the parametrization used there, but we consider only constant coefficients (the referenced paper focuses on concentration-dependent density, viscosity, and diffusivity). Another simplification with respect to [23] is that we only take the canonical momentum $\omega\mathbf{u}$ (that is, without mass diffusion or migration due to the ionic species).

The domain is $\Omega = (0,1) \times (0,2)$, which we discretize into a structured mesh of 10'000 triangles. As illustrated in Figure 7.3, the boundary conditions are as follows: for the fluid flow we impose no-slip $\mathbf{u} = \mathbf{0}$ everywhere on the boundary. For the chemical species we assume that the normal trace of the total fluxes is zero everywhere on the boundary $\boldsymbol{\sigma}_i \cdot \boldsymbol{\nu} = 0$ (that is, the boundary is considered impenetrable for the ionic quantities), which is imposed essentially in the space $\mathrm{H}_{i,h}$. For the electrostatic sub-system we consider two separate sub-boundaries: on the top segment $(y = 2)$ we prescribe a given potential χ_0

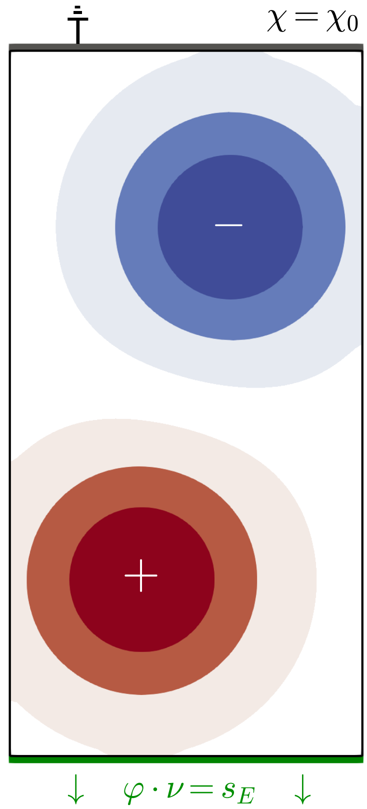

Figure 7.3: Ion spreading in a charged enclosure. Set up of the geometry, boundary conditions for the electrostatic equations, and initial distribution of positively and negatively charged ion particles.

(representing a ground condition, imposed naturally), on the vertical walls of the reservoir we set zero normal trace of the electric field $\boldsymbol{\varphi} \cdot \boldsymbol{\nu} = 0$, and the bottom segment is regarded as a positively charged surface $\boldsymbol{\varphi} \cdot \boldsymbol{\nu} = s_E$ (the two last conditions are imposed essentially).

The mixing/spreading process is intrinsically time-dependent and so we include in the formulation the following modified versions of the fully-discrete momentum and ion conservation equations

$$\frac{1}{\Delta t}\int_\Omega \mathbf{u}_h^{m+1} \cdot \mathbf{v}_h + \mathbf{a}(\vec{\mathbf{u}}_h^{m+1}, \vec{\mathbf{v}}_h) + \mathbf{c}(\mathbf{u}_h^{m+1}; \vec{\mathbf{u}}_h^{m+1}, \vec{\mathbf{v}}_h) + \mathbf{b}(\vec{\mathbf{v}}_h, \boldsymbol{\sigma}_h) \;=\; \frac{1}{\Delta t}\int_\Omega \mathbf{u}_h^{m} \cdot \mathbf{v}_h + \mathbf{F}_{\boldsymbol{\xi}_h, \boldsymbol{\varphi}_h}(\vec{\mathbf{v}}_h)\,,$$

$$c_i(\boldsymbol{\sigma}_{i,h}^{m+1}, \eta_{i,h}) \,-\, d_i\left(\frac{1}{\Delta t}\xi_{i,h}^{m+1}, \eta_{i,h}\right) \;=\; \mathrm{G}_i(\eta_{i,h}) \,-\, d_i\left(\frac{1}{\Delta t}\xi_{i,h}^{m}, \eta_{i,h}\right),$$

for all $\vec{\mathbf{v}}_h \in \mathbf{H}_h$, and for all $\eta_{i,h} \in \mathrm{Q}_{i,h}$, respectively, where the superscripts $m, m+1$ denote approximations at time instants t^m, t^{m+1} using backward Euler's method. For this we take a constant time step $\Delta t = 0.01$ and conduct the simulations until the final time $t = 2.5$. The initial velocity is zero and the initial concentrations of positively and negatively charged particles are as follows

$$\xi_{i,0}(\mathbf{x}) = \frac{\hat{\xi}_0}{2\pi R^2}\exp\left\{-\frac{(x - \frac{1}{2} + \frac{q_i}{8})^2 + (y - 1 + \frac{q_i}{2})^2}{2R^2}\right\},$$

respectively (see also the sketch in Figure 7.3). The model parameters are as follows

$$\omega = 1, \quad \epsilon = 0.5, \quad \mu = 0.08, \quad \kappa_1 = \kappa_2 = 0.01, \quad s_E = 1, \quad \chi_0 = 0, \quad \hat{\xi}_0 = 3, \quad R = \frac{1}{4}.$$

The numerical solutions are displayed in Figure 7.4, where we plot snapshots at five time instants of the net charge (computed as the difference between positively and negatively charged ion species) and the line integral convolution (similar to streamlines) of the fluid velocity. Exactly as in [23, Figure 6], in our case we observe that the flow patterns that occur thanks to the interaction of difference of potential and charges (different on the top and bottom boundaries) permit spreading into the reservoir, and the net charge figures show the expected decay due to dissipation.

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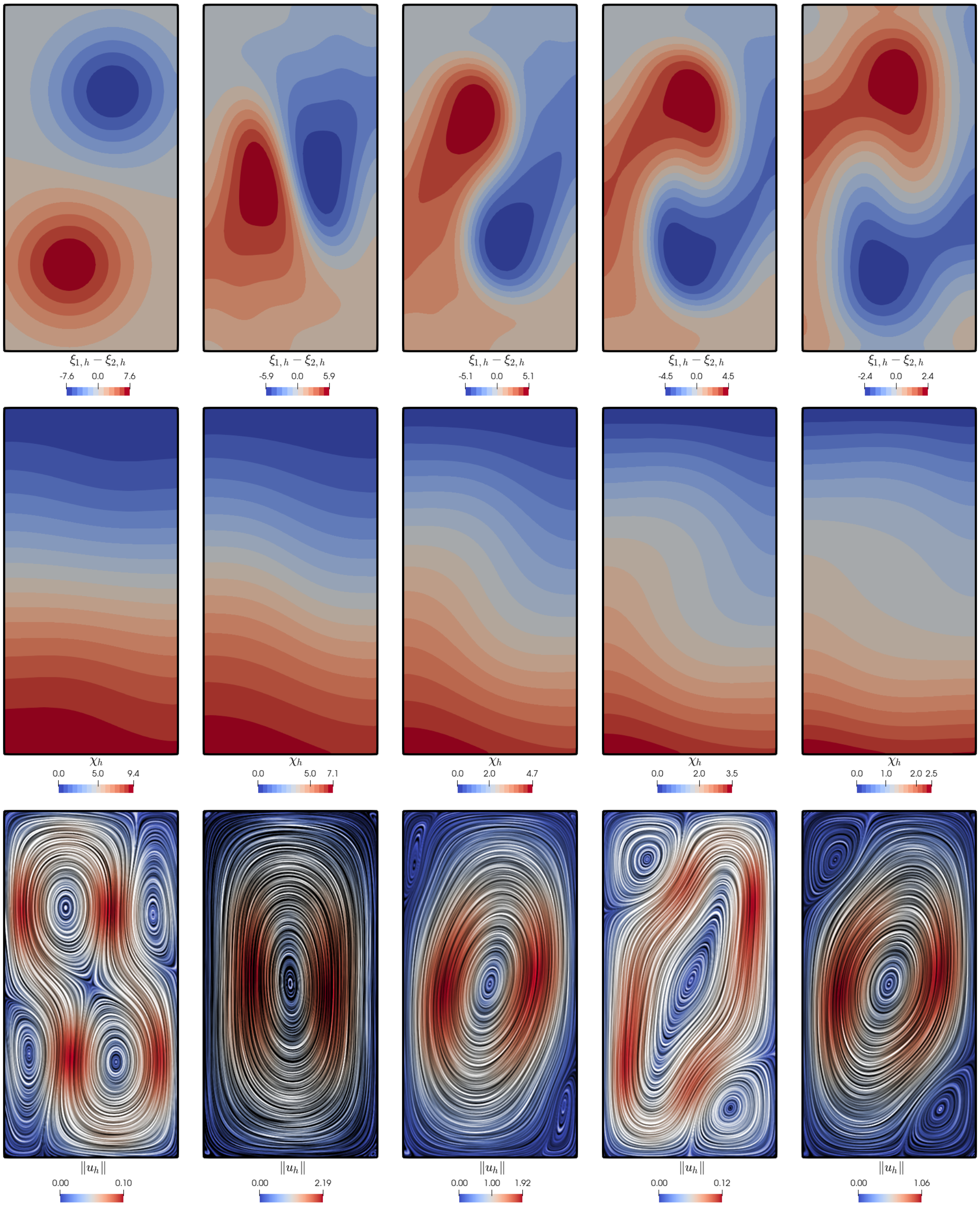

Figure 7.4: Ion spreading in a charged enclosure. Net charge (top), electric potential (middle), and velocity line integral convolution (bottom) at times $t = 0, 0.5, 0.75, 1, 2.5$ (from left to right columns).

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