

Unconditionally stable and energy conserving discretization of the dynamic von Kármán equations

Carsten Carstensen* Neela Nataraj† Ricardo Ruiz-Baier‡ Aamir Yousuf§

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Abstract

A fully discrete approximation of the dynamic von Kármán equations combines nonconforming Morley finite element methods for spatial discretization with an energy conserving *modified* unconditionally stable Newmark second-order time-stepping scheme. Brouwer’s fixed-point theorem establishes existence of a solution to the fully discrete scheme and further uniqueness and stability estimates follow for small loads. Optimal order a priori error estimates in the piecewise energy norm with quadratic convergence in time are derived for the fully discrete scheme. The results of the numerical experiments validate the theoretical error bounds.

Key words. von Kármán system, stability, finite element methods, error estimates.

1 Introduction

1.1 Mathematical Model

The von Kármán equations provide a fundamental mathematical framework for describing the nonlinear bending behavior of thin elastic plates. These equations, originally developed in the context of classical elasticity, account for large deflections while retaining the assumption of small strains, making them particularly useful in modeling geometrically nonlinear deformations. This paper analyzes a discretization scheme for the dynamic von Kármán system

$$u_{tt} + \Delta^2 u - [u, v] = f(\mathbf{x}, t) \text{ at } (\mathbf{x}, t) \in \Omega \times (0, T], \quad (1.1a)$$

$$\Delta^2 v + \frac{1}{2}[u, u] = 0 \text{ at } (\mathbf{x}, t) \in \Omega \times [0, T] \quad (1.1b)$$

with initial and clamped boundary conditions

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad u_t(\mathbf{x}, 0) = u_1(\mathbf{x}) \text{ in } \Omega; \quad u = \frac{\partial u}{\partial \mathbf{n}} = 0, v = \frac{\partial v}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega \times (0, T]. \quad (1.1c)$$

Here and throughout the paper, $[\bullet, \bullet]$ is the Monge–Ampère form operator or von Kármán bracket,

$$[u, v] := \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial y^2} \frac{\partial^2 v}{\partial x^2} - 2 \frac{\partial^2 u}{\partial x \partial y} \frac{\partial^2 v}{\partial x \partial y},$$

while Ω is a bounded polygonal Lipschitz domain in $\mathbb{R}^2$ with outward-pointing unit normal $\mathbf{n}$ to the boundary $\partial\Omega$, $\Delta^2 w := \partial^4 w / \partial x^4 + 2\partial^4 w / \partial x^2 \partial y^2 + \partial^4 w / \partial y^4$ denotes the biharmonic operator, $\partial u / \partial \mathbf{n} = \nabla u \cdot \mathbf{n}$ (resp. $\partial v / \partial \mathbf{n} = \nabla v \cdot \mathbf{n}$) is the outer normal derivative of u (resp. v) on $\partial\Omega$, and u_t, u_{tt} denote the first- and second-order derivatives with respect to time, respectively. The system of equations (1.1) describes the transversal displacement u and the Airy-stress function v of a vibrating plate, whose boundary is clamped in transversal direction but free in horizontal direction. For a derivation of the equations from physical principles, we refer to [19] and the references therein.

*Institut für Mathematik, Humboldt-Universität zu Berlin, 10117 Berlin, Germany and Distinguished Visiting Professor, Department of Mathematics, Indian Institute of Technology Bombay, Powai, Mumbai, Maharashtra (cc@math.hu-berlin.de).

†Department of Mathematics, Indian Institute of Technology Bombay, Powai, Mumbai, Maharashtra, India (neela@math.iitb.ac.in).

‡School of Mathematics, Monash University, 9 Rainforest Walk, 3800 Clayton VIC, Australia (ricardo.ruizbaier@monash.edu).

§IITB–Monash Research Academy, Indian Institute of Technology Bombay, Powai, Mumbai, Maharashtra, India (aamir72@iitb.ac.in).

1.2 Literature overview

Several contributions in the literature discuss the discretization of the static von Kármán equations [12, 30, 32, 33]; [4, 20, 28] and the references therein study existence of solutions, regularity, and bifurcation phenomena. The weak solutions $u, v \in H_0^2(\Omega)$ to the static von Kármán equations belong to $H_0^2(\Omega) \cap H^{2+\sigma}(\Omega)$ with the index of elliptic regularity $\sigma > 1/2$ determined by the interior angles of the polygonal boundary $\partial\Omega$ with $\sigma = 1$ for convex domains [8]. In [32] and [33], the authors analyze conforming and nonconforming finite elements for the static von Kármán model and derive the optimal-order error estimates in the energy, H^1 , and L^2 norms. A unified analysis for fourth-order quadratic semilinear problems in [15] applies to Morley, discontinuous Galerkin, C^0 interior penalty, and the weakly over-penalized symmetric interior penalty schemes. A conforming C^1 virtual element method in [30] approximates isolated solutions to the von Kármán equations.

Despite the rich literature on the numerical analysis of the von Kármán static model, not many results are available for its challenging time-dependent counterpart. The underlying biharmonic wave equation has been the subject of several numerical studies: [3] considers the C^1 -conforming Bogner–Fox–Schmit elements in space coupled with Galerkin and collocation methods in time and [35] analyzes lowest-order nonstandard finite element methods (Morley, discontinuous Galerkin, and C^0 interior penalty) combined with explicit and implicit time-stepping schemes. One reference for the dynamic von Kármán equation is [7] with a semidiscrete (in time) energy conserving finite difference schemes. The dynamic von Kármán equation (1.1) over rectangular domains in [40] employs Chebyshev and Legendre-collocation methods for the spatial discretization and the implicit Newmark- β scheme combined with a nonlinear fixed-point algorithm in each time step. Bilbao et al. [5–7] have developed a line of finite-difference schemes for the dynamic von Kármán plate, from closed-form energy-conserving but conditionally stable schemes.

1.3 Contributions

This paper presents a rigorous analysis and implementation of a *novel* fully discrete scheme for the dynamic von Kármán equations with clamped boundary conditions. A nonconforming Morley finite element method for the spatial discretization and a *modified* Newmark second-order time discretization scheme in this article lead to an *unconditionally stable* scheme. The main contributions of this work are as follows.

- An implicit modified Crank–Nicolson scheme leads to an unconditionally stable quadratic scheme with optimal convergence rates for the initial discretization.
- A classical implicit time-integration approach for plate dynamics is the Newmark constant-average acceleration method [34] that is unconditionally stable for linear problems. *It is observed that the Newmark constant-average acceleration method loses its unconditional stability in the nonlinear problem* [2, Page no. 9]. Since the Courant–Friedrichs–Lewy (CFL) condition $k = \mathcal{O}(h^2)$ [2, eqn. (36)] appears too restrictive, we propose a *new* scheme which preserves the *energy conservation* property at discrete level and exploits the advantage of unconditional stability of implicit schemes.
- The smallness assumption on the data u_0, u_1 , and the time step ensures the well-posedness of the fully discrete scheme. Optimal-order a priori error estimates in the energy norm are obtained for the fully discrete solution under additional smallness assumptions on f : We obtain quadratic convergence in time and optimal convergence rates in space.
- Numerical experiments validate the theoretical analysis and confirm the predicted spatial and temporal convergence rates in Example 1 of Section 6. Example 2 in Section 6 demonstrates that the von Kármán thin plate model is effective. The findings indicate that as the plate thickness decreases, the two-dimensional simulations closely approximate the results from three-dimensional modeling.

1.4 Outline of the paper

The remaining parts of this section introduce overall notation. Section 2 presents the weak formulation of the problem, describes the Morley finite element space and the fully discrete scheme, and states the main results of the paper. Proofs follow in Sections 4–5. Section 3 introduces a companion operator, a Ritz projection, and other auxiliary results. Section 4 establishes the well-posedness and a priori error estimates for the initial time steps of the fully discrete scheme, while Section 5 addresses the remaining time levels. Finally, Section 6 presents the Newton–Raphson procedure together with the results of numerical experiments.

1.5 Notation

Standard notations on Lebesgue and Sobolev spaces [22] apply throughout the paper. We denote the L^2 scalar product by $(\bullet, \bullet)_{L^2(\Omega)}$ and its induced norm by $\|\bullet\|$; the same notation applies to the vector and tensor-valued functions. Throughout this paper, $\mathcal{T}$ denotes a shape-regular triangulation of the bounded polygonal Lipschitz domain Ω into triangles, $H^m(\mathcal{T})$ denotes the Hilbert space $\prod_{K \in \mathcal{T}} H^m(K)$, and $P_r(\mathcal{T}) \subset L^2(\Omega)$ is the space of piecewise polynomials of degree at most r . The notation $\|\bullet\|$ denotes the global seminorm $|\bullet|_{H^2(\Omega)}$ and $\|\bullet\|_{\text{pw}}$ is the piecewise seminorm $|\bullet|_{H^2(\mathcal{T})}$ for $\mathcal{T}$ -piecewise H^2 -functions. The operators D^2 and D_{pw}^2 denote the global and piecewise Hessian. The notation $a \lesssim b$ abbreviates $a \leq Cb$ with a generic constant C , that is independent of the discretization parameters. The weighted arithmetic-geometric mean inequality $ab \leq \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2$ for positive real numbers a, b, ϵ applies throughout.

Let $(X, \|\bullet\|_X)$ be a Hilbert space and $0 \leq a < b \leq T$, the Bochner space $L^p(a, b; X)$ consists of all strongly measurable functions $\varphi : (a, b) \rightarrow X$ [22] with finite norm

$$\|g\|_{L^p(a,b;X)} := \left(\int_a^b \|g(t)\|_X^p dt \right)^{1/p} \text{ for } 1 \leq p < \infty \quad \text{and else} \quad \|g\|_{L^\infty(a,b;X)} := \text{ess sup}_{a \leq t \leq b} \|g(t)\|_X.$$

For $m \in \mathbb{N}$, the Sobolev-Bochner space $H^m(a, b; X)$ consists of all functions $g \in L^2(a, b; X)$ such that $\frac{d^k g}{dt^k} \in L^2(a, b; X)$ for $k = 0, 1, \dots, m$ and is equipped with the norm

$$\|g\|_{H^m(a,b;X)} := \left(\sum_{k=0}^m \left\| \frac{d^k g}{dt^k} \right\|_{L^2(a,b;X)}^2 \right)^{1/2} = \left(\sum_{k=0}^m \int_a^b \left\| \frac{d^k g(t)}{dt^k} \right\|_X^2 dt \right)^{1/2}.$$

The space $C^k([a, b]; X)$ denote all k -times continuously differentiable functions $g : [a, b] \rightarrow X$ with

$$\|g\|_{C^k([a,b];X)} := \sum_{0 \leq i \leq k} \max_{a \leq t \leq b} \left\| \frac{\partial^i g}{\partial t^i} \right\|_X < \infty.$$

2 Main results

This section presents the weak formulation of (1.1), summarizes its well-posedness results from the literature, and derives energy conservation identities in Subsection 2.1. Subsection 2.2 discusses the spatial and temporal discretization framework. Subsection 2.3 motivates an unconditionally stable modified Newmark fully discrete scheme (2.10) from the energy conservation identity (2.3) at the discrete level and Subsection 2.4 proposes this scheme to approximate the solution at higher time steps. Subsection 2.5 introduces the initial approximations at the first two time steps using a modified Crank–Nicolson scheme. The stability and error estimates for the fully-discrete scheme are stated in Subsection 2.6.

2.1 Weak formulation

Let $a(\bullet, \bullet) : H_0^2(\Omega) \times H_0^2(\Omega) \rightarrow \mathbb{R}$ denote the energy scalar product in $H_0^2(\Omega)$ defined by

$$a(\varphi, \psi) := \int_{\Omega} D^2 \varphi : D^2 \psi \, d\mathbf{x} \text{ for all } \varphi, \psi \in H_0^2(\Omega).$$

Let $b(\bullet, \bullet, \bullet) : H_0^2(\Omega) \times H_0^2(\Omega) \times H_0^2(\Omega) \rightarrow \mathbb{R}$ be the symmetric trilinear form given by

$$b(\varphi, \phi, \psi) := - \int_{\Omega} [\varphi, \phi] \psi \, d\mathbf{x} \text{ for all } \varphi, \phi, \psi \in H_0^2(\Omega).$$

The global Sobolev embedding $H_0^2(\Omega) \hookrightarrow L^\infty(\Omega)$ [1] and the Friedrichs inequality reveal $\|\varphi\|_{L^\infty(\Omega)} \leq C_S \|\varphi\|$ for all $\varphi \in H_0^2(\Omega)$ and $C_S > 0$. Hölder and Cauchy–Schwarz inequalities lead to

$$|b(\varphi, \chi, \psi)| \leq \|\psi\|_{L^\infty(\Omega)} \int_{\Omega} |[\varphi, \chi]| \, d\mathbf{x} \leq C_S \|\psi\| \int_{\Omega} |[\varphi, \chi]| \, d\mathbf{x} \leq C_S \|\varphi\| \|\chi\| \|\psi\|. \quad (2.1)$$

The weak formulation of (1.1) seeks $(u, v) \in (L^\infty(0, T; H_0^2(\Omega)))^2$ with $u_t, u_{tt} \in L^\infty(0, T; L^2(\Omega))$ such that, at a.e

$$0 < t < T,$$

$$(u_{tt}, \varphi)_{L^2(\Omega)} + a(u, \varphi) + b(u, \varphi, v) = (f, \varphi)_{L^2(\Omega)} \quad \text{for all } \varphi \in H_0^2(\Omega), \quad (2.2a)$$

$$a(v, \psi) - \frac{1}{2}b(u, u, \psi) = 0 \quad \text{for all } \psi \in H_0^2(\Omega), \quad (2.2b)$$

$$u(0) = u_0 \text{ and } u_t(0) = u_1. \quad (2.2c)$$

The dependence of functions on t will be skipped whenever there is no risk of confusion, e.g., $u := u(t)$ and $v = v(t)$ in (2.2).

Existence, uniqueness, and regularity results

The well-posedness of the dynamic von Kármán equations depends on the regularity of the given data and the smoothness properties of the domain. In Table 2.1, we present the main existence and uniqueness results from the literature when Ω is a smooth bounded domain in $\mathbb{R}^2$. These results highlight the assumptions on given data and the corresponding regularity of solution. Moreover, under the assumptions $u_0 \in H_0^2(\Omega)$, $u_1 \in L^2(\Omega)$, and $f \in L^2(0, T; L^2(\Omega))$ (see [9] in Table 2.1 below), the Faedo–Galerkin technique (cf. [24, Eqn. 4.3]) leads to the following *energy conservation* identity

$$\|u_t\|^2 + \|D^2 u\|^2 + \|D^2 v\|^2 = \|u_1\|^2 + \|D^2 u_0\|^2 + \|D^2 v_0\|^2 + 2 \int_0^t (f, u_t)_{L^2(\Omega)} dt \quad \text{at any } t \in [0, T]. \quad (2.3)$$

Consider (2.2b) at $t = 0$ and test it against v_0 . Then, from (2.1) we obtain the inequality $|v_0|_{H^2(\Omega)} \leq C_S \|u_0\|^2$. This,

Table 2.1: Summary of existence, uniqueness, and regularity results for (1.1) ($\Omega \subset \mathbb{R}^2$ is smooth and bounded)

Ref.	Details
[39]	Assumptions: $u_0 \in H_D^{5+\delta}(\Omega) := \{\varphi \in H^{5+\delta} \cap H^4(\Omega) \cap H_0^2(\Omega) : \Delta^2 \varphi = 0 \text{ on } \partial\Omega\}$, $u_1 \in H^{3+\delta}(\Omega) \cap H_0^2(\Omega)$ for $0 < \delta < \frac{1}{2}$; $f = 0$ Result: Local unique solution on maximal interval $I^* = [T_-, T_+)$ containing zero Regularity: $u \in C^0(I^*, H_D^{5+\delta}(\Omega)) \cap C^1(I^*, H^{3+\delta}(\Omega) \cap H_0^2(\Omega)) \cap C^2(I^*, H^{1+\delta}(\Omega) \cap H_0^1(\Omega))$ (Classical solution)
[29]	Assumptions: Case 1: $u_0 \in H^4(\Omega) \cap H_0^2(\Omega)$, $u_1 \in H_0^2(\Omega)$, $f = 0$ Case 2: $u_0 \in H_D^{5+\delta}(\Omega)$, $u_1 \in H^{3+\delta}(\Omega) \cap H_0^2(\Omega)$ for $0 < \delta < \frac{1}{2}$, $f = 0$ Result: Global existence and uniqueness for both cases Regularity: Case 1: $u \in C^0(\mathbb{R}, H^4(\Omega) \cap H_0^2(\Omega)) \cap C^1(\mathbb{R}, H^2(\Omega)) \cap C^2(\mathbb{R}, L^2(\Omega))$ Case 2: $u \in C^0(\mathbb{R}, H_D^{5+\delta}(\Omega)) \cap C^1(\mathbb{R}, H^{3+\delta}(\Omega)) \cap C^2(\mathbb{R}, H^{1+\delta}(\Omega))$ (Classical solution)
[9]	Assumptions: $u_0 \in H_0^2(\Omega)$, $u_1 \in L^2(\Omega)$, $f \in L^2(0, T; L^2(\Omega))$ Result: Existence and uniqueness of weak solution on $[0, T]$ Regularity: $u \in L^\infty(0, T; H_0^2(\Omega))$, $u_t \in L^\infty(0, T; L^2(\Omega))$, $u_{tt} \in L^\infty(0, T; H^{-2}(\Omega))$, $v \in L^\infty(0, T; H_0^2(\Omega))$

Cauchy-Schwarz and Young's inequalities reveal $2(f, u_t)_{L^2(\Omega)} \leq 2\|f\|\|u_t\| \leq T\|f\|^2 + \|u_t\|^2/T$ and lead to

$$\begin{aligned} \|u_t\|^2 + |u|_{H^2(\Omega)}^2 + |v|_{H^2(\Omega)}^2 &\leq \|u_0\|^2(1 + C_S^2 \|u_0\|^2) + \|u_1\|^2 + T\|f\|_{L^2(0, T; L^2(\Omega))}^2 + \frac{1}{T} \int_0^t \|u_t\|^2 dt \\ &=: M'_1(u_0, u_1, f) + \frac{1}{T} \int_0^t \|u_t\|^2 dt \end{aligned}$$

with $\int_0^t \|f\|^2 dt \leq \|f\|_{L^2(0, T; L^2(\Omega))}^2$ in the first inequality. Gronwall's inequality [17, Lemma 2.1] leads to stability of the continuous solution

$$\|u_t\|^2 + |u|_{H^2(\Omega)}^2 + |v|_{H^2(\Omega)}^2 \leq \left(1 + \frac{1}{T} \int_0^t \exp\left(\frac{t-s}{T}\right) ds\right) M'_1(u_0, u_1, f) \leq e M'_1(u_0, u_1, f) =: \check{M}_1(u_0, u_1, f) \quad (2.4)$$

with the Euler number $e = 2.71$.

2.2 Preliminaries

For a triangle $K \in \mathcal{T}$ with diameter h_K , area $|K|$, and the outward unit normal n_K along ∂K , let $h = h_{\max} := \max_{K \in \mathcal{T}} h_K$. Let $\mathcal{V}(\Omega)$ (resp. $\mathcal{V}(\partial\Omega)$) denote the set of all interior (resp. boundary) vertices of $\mathcal{T}$ and let $\mathcal{V} = \mathcal{V}(\Omega) \cup \mathcal{V}(\partial\Omega)$. Let $\mathcal{E}(\Omega)$ (resp. $\mathcal{E}(\partial\Omega)$) denote the set of all interior (resp. boundary) edges of $\mathcal{T}$ and let $\mathcal{E} = \mathcal{E}(\Omega) \cup \mathcal{E}(\partial\Omega)$. The nonconforming Morley FE space $M(\mathcal{T})$ [14] reads (see Figure 2.1)

$$\begin{aligned} M'(\mathcal{T}) &:= \{ \varphi_M \in P_2(\mathcal{T}) : \varphi_M \text{ is continuous at interior vertices} \\ &\quad \text{and its normal derivatives are continuous} \\ &\quad \text{at the midpoints of interior edges} \}, \\ M(\mathcal{T}) &:= \{ \varphi_M \in M'(\mathcal{T}) : \varphi_M \text{ vanishes at the vertices of } \partial\Omega \\ &\quad \text{and its normal derivatives vanish} \\ &\quad \text{at the midpoints of boundary edges of } \partial\Omega \}. \end{aligned}$$

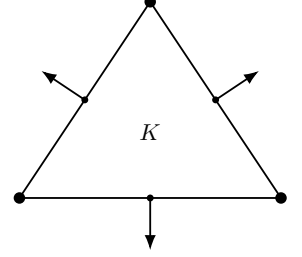


Figure 2.1: Morley finite element.

Define a discrete bilinear form $a_{pw}(\bullet, \bullet) : (H_0^2(\Omega) + M(\mathcal{T})) \times (H_0^2(\Omega) + M(\mathcal{T})) \rightarrow \mathbb{R}$ by

$$a_{pw}(\varphi, \psi) := \int_{\Omega} D_{pw}^2 \varphi : D_{pw}^2 \psi \, dx \text{ for all } \varphi, \psi \in H_0^2(\Omega) + M(\mathcal{T}).$$

This is a semi-scalar product and induces a semi-norm on $H^2(\mathcal{T})$ given by

$$\|\varphi\|_{pw}^2 := \sum_{K \in \mathcal{T}} |\varphi|_{H^2(K)}^2 = a_{pw}(\varphi, \varphi) \quad \text{for all } \varphi \in H^2(\mathcal{T}). \quad (2.5)$$

It is known that $a_{pw}(\bullet, \bullet)$ is a scalar product on $H_0^2(\Omega) + M(\mathcal{T})$ [13, 23] and $(H_0^2(\Omega) + M(\mathcal{T}), a_{pw}(\bullet, \bullet))$ is a Hilbert space with the induced norm $\|\bullet\|_{pw}$.

The piecewise trilinear form $b_{pw}(\bullet, \bullet, \bullet) : (H_0^2(\Omega) + M(\mathcal{T})) \times (H_0^2(\Omega) + M(\mathcal{T})) \times (H_0^2(\Omega) + M(\mathcal{T})) \rightarrow \mathbb{R}$ is defined by

$$b_{pw}(\varphi, \chi, \psi) := - \sum_{K \in \mathcal{T}} \int_K [\varphi, \chi] \psi \, dx =: - \int_{\Omega} [\varphi, \chi]_{pw} \psi \, dx \quad \text{for all } \varphi, \chi, \psi \in H_0^2(\Omega) + M(\mathcal{T}).$$

The trilinear form b_{pw} is merely symmetric with respect to the first two arguments, that is, $b_{pw}(\varphi, \chi, \psi) = b_{pw}(\chi, \varphi, \psi)$. The definition of $b_{pw}(\bullet, \bullet, \bullet)$ from the last displayed expression, a Hölder's inequality, and a discrete Sobolev embedding

$$\|\psi\|_{L^\infty(\Omega)} \leq C_{ds} \|\psi\|_{pw} \quad (2.6)$$

(cf. [11, Lemma 4.7] and [12, Lemma 2.5] for proofs) reveal

$$|b_{pw}(\varphi, \chi, \psi)| \leq C_{ds} \|\varphi\|_{pw} \|\chi\|_{pw} \|\psi\|_{pw}. \quad (2.7)$$

The time discretization on a uniform partition $0 = t_0 < t_1 < t_2 < \dots < t_N = T$ of the interval $[0, T]$ into $N \in \mathbb{N}$ congruent intervals with $t_n = nk$, focuses on a uniform time step $k = T/N$. The following abbreviations apply to any function $\varphi(\mathbf{x}, t)$:

$$\varphi^n := \varphi(\mathbf{x}, t_n) = \varphi(t_n), \quad \varphi^{n+1/2} := \frac{1}{2} (\varphi^{n+1} + \varphi^n), \quad \varphi^{n,1/4} := \frac{1}{4} (\varphi^{n+1} + 2\varphi^n + \varphi^{n-1}), \quad (2.8a)$$

$$\bar{\partial}_t \varphi^{n+1/2} := \frac{\varphi^{n+1} - \varphi^n}{k}, \quad \bar{\partial}_t^2 \varphi^n := \frac{\varphi^{n+1} - 2\varphi^n + \varphi^{n-1}}{k^2}, \quad \delta_t \varphi^n := \frac{\varphi^{n+1} - \varphi^{n-1}}{2k}. \quad (2.8b)$$

The identities below result from (2.8a)-(2.8b) and are straightforward.

$$k \bar{\partial}_t^2 \varphi^n = \bar{\partial}_t \varphi^{n+1/2} - \bar{\partial}_t \varphi^{n-1/2}, \quad 2\varphi^{n,1/4} = \varphi^{n+1/2} + \varphi^{n-1/2}, \quad (2.9a)$$

$$\varphi^{n+1} - \varphi^{n-1} = 2(\varphi^{n+1/2} - \varphi^{n-1/2}) = k(\bar{\partial}_t \varphi^{n+1/2} + \bar{\partial}_t \varphi^{n-1/2}) = 2k \delta_t \varphi^n. \quad (2.9b)$$

2.3 Motivation

The algorithmic energy momentum method by Simo and Tarnow's [37,38] first established that midpoint or trapezoidal time discretizations of nonlinear elastodynamics fail to conserve energy exactly and could be numerically unstable in a related but different application to shells. They introduced a stress evaluation built from a discrete gradient that restores exact conservation of energy and momentum. Bilbao et al. [5, 7] later adapted this discrete gradient framework to the von Kármán bracket and derived closed form energy conserving finite difference schemes that are *stable conditionally*, later generalized to the full system with in plane inertia. While [6] establish unconditional stability in general, their treatment of the plate case falls back to a conditionally stable variant; to the best of our knowledge *no* fully-discrete scheme with rigorously proven unconditional stability is available for the nonlinear, coupled dynamic von Kármán system on a nonconforming spatial discretization with particular attention to the choice for the initial approximations.

Suppose that U^n denotes the approximation in a semi-discrete problem at a time $t_n = nk$ and consider $U(t)$ as a piecewise linear spline in a uniform time discretization with time-step size $k > 0$. Let $U^{n+1/2} := (U^{n+1} + U^n)/2$ be the approximation at the midpoint $t_{n+1/2} = (n + 1/2)k$ and $\bar{\partial}_t U^{n+1/2} := (U^{n+1} - U^n)/k$ be the approximation of the velocity in the time interval (t_{n-1}, t_n) . The discretization of the time derivative u_t at time t_n by a central difference quotient $k^{-2}(U^{n+1} - 2U^n + U^{n-1}) = k^{-1}(U^{n+1/2} - U^{n-1/2})$ is the most natural choice and thereafter we need an approximation of $\Delta^2 u$ at time t_n by an average of values of U^{n-1}, U^n, U^{n+1} to allow energy conservation. It is known from [35] that $(U^{n+1/2} + U^{n-1/2})/2$ allows for the following arguments. The resulting discrete problem for the wave equation and $f \equiv 0$ reads

$$k^{-1}(\bar{\partial}_t U^{n+1/2} - \bar{\partial}_t U^{n-1/2}, \varphi)_{L^2(\Omega)} + \frac{1}{2}a_{\text{pw}}(U^{n+1/2} + U^{n-1/2}, \varphi) = 0.$$

The test with $\varphi = k(\bar{\partial}_t U^{n+1/2} + \bar{\partial}_t U^{n-1/2}) = U^{n+1} - U^{n-1} = 2(U^{n+1/2} - U^{n-1/2})$ leads to telescoping terms

$$\|\bar{\partial}_t U^{n+1/2}\|^2 - \|\bar{\partial}_t U^{n-1/2}\|^2 + \||U^{n+1/2}\|_{\text{pw}}^2 - \||U^{n-1/2}\|_{\text{pw}}^2 = 0$$

that (up to initial values for $U^0, U^1, \bar{\partial}_t U^{1/2}$) provides energy conservation (for the wave equation) viz.

$$E'_{n+1/2} := \|\bar{\partial}_t U^{n+1/2}\|^2 + \||U^{n+1/2}\|_{\text{pw}}^2 = E'_{n-1/2} \text{ for } n \in \mathbb{N}.$$

Adopt the notation $V^n, V^{n+1/2}, V^{n+1}$ for piecewise linear approximations for the variable $V(t)$ that approximates v in (1.1b). This preliminary consideration eventually reveals a technique for energy conservation and its repeated application favours the subsequent scheme

$$k^{-1}(\bar{\partial}_t U^{n+1/2} - \bar{\partial}_t U^{n-1/2}, \varphi)_{L^2(\Omega)} + \frac{1}{2}a_{\text{pw}}(U^{n+1/2} + U^{n-1/2}, \varphi) + \frac{1}{4}b_{\text{pw}}(U^{n+1/2} + U^{n-1/2}, \varphi, V^{n+1/2} + V^{n-1/2}) = 0.$$

The test with $\varphi = k(\bar{\partial}_t U^{n+1/2} + \bar{\partial}_t U^{n-1/2}) = U^{n+1} - U^{n-1} = 2(U^{n+1/2} - U^{n-1/2})$ and the above calculations lead to

$$2E'_{n-1/2} - 2E'_{n+1/2} = b_{\text{pw}}(U^{n+1/2}, U^{n+1/2}, V^{n+1/2} + V^{n-1/2}) - b_{\text{pw}}(U^{n-1/2}, U^{n-1/2}, V^{n+1/2} + V^{n-1/2})$$

provided the trilinear form b_{pw} is symmetric in the first two components. The second discrete equation may be evaluated in an averaged sense with

$$b_{\text{pw}}(U^{n+1/2}, U^{n+1/2}, \psi) = 2a_{\text{pw}}(V^{n+1/2}, \psi).$$

For the test function $\psi = V^{n+1/2} + V^{n-1/2}$, this leads to

$$E'_{n-1/2} - E'_{n+1/2} = a_{\text{pw}}(V^{n+1/2} - V^{n-1/2}, V^{n+1/2} + V^{n-1/2}) = \||V^{n+1/2}\|_{\text{pw}}^2 - \||V^{n-1/2}\|_{\text{pw}}^2$$

and so to energy conservation in the sense of

$$E_{n+1/2} := E'_{n+1/2} + \||V^{n+1/2}\|_{\text{pw}}^2 = E_{n-1/2}.$$

This novel motivation for the semi-discrete scheme is examined for the fully-discrete scheme below in this paper.

2.4 Fully-discrete scheme

This subsection presents the fully discrete scheme for the von Kármán equations under the assumption that discrete solutions at the first two time steps are available. Initial discretization computes the discrete solutions at the first two time steps in Subsection 2.5 below. We state the main well-posedness and error estimate results for the scheme here, while the detailed proofs follow in Section 5.

Let (u_M^n, v_M^n) denote the Morley approximation of $(u(t), v(t))$ at time t_n for $n = 0, 1, \dots, N$. Assume that (u_M^0, v_M^0) and $(u_M^1, v_M^1) \in \mathbf{M}(\mathcal{T}) := \mathbf{M}(\mathcal{T}) \times \mathbf{M}(\mathcal{T})$ be approximations of the continuous solution $(u(t), v(t))$ at times t_0 and t_1 , respectively. For $n = 1, 2, \dots, N - 1$, the fully discrete scheme motivated in Subsection 2.3 seeks the solution $(u_M^{n+1}, v_M^{n+1}) \in \mathbf{M}(\mathcal{T})$ to

$$(\bar{\partial}_t^2 u_M^n, \varphi_M)_{L^2(\Omega)} + a_{\text{pw}}(u_M^{n,1/4}, \varphi_M) + b_{\text{pw}}(u_M^{n,1/4}, \varphi_M, v_M^{n,1/4}) = (f^{n,1/4}, \varphi_M)_{L^2(\Omega)} \quad \text{for all } \varphi_M \in \mathbf{M}(\mathcal{T}), \quad (2.10a)$$

$$a_{\text{pw}}(v_M^{n+1/2}, \psi_M) - \frac{1}{2} b_{\text{pw}}(u_M^{n+1/2}, u_M^{n+1/2}, \psi_M) = 0 \quad \text{for all } \psi_M \in \mathbf{M}(\mathcal{T}). \quad (2.10b)$$

With reference to (2.10a), a Newmark scheme for u_{tt} with $t > t_1$ and the approximation $b_{\text{pw}}(u_M^{n,1/4}, \varphi, v_M^{n,1/4})$ for the non-linear term imply a quadratic convergence in time. To see this, we look back at the key identity (2.3) and Subsection 2.3 for the nonlinear term that ensures the *energy conservation* of (2.2). For its discrete counterpart, in the proof of stability, the test function is chosen as $2k\delta_t u_M^n$ in (2.10a) and the linear terms telescope perfectly as $\|\bar{\partial}_t u_M^{n+1/2}\|^2 - \|\bar{\partial}_t u_M^{n-1/2}\|^2 + \|u_M^{n+1/2}\|_{\text{pw}}^2 - \|u_M^{n-1/2}\|_{\text{pw}}^2$. From the design of (2.10b), the non-linear term reduces to

$$\begin{aligned} b_{\text{pw}}(u_M^{n,1/4}, 2k\delta_t u_M^n, v_M^{n,1/4}) &= b_{\text{pw}}(u_M^{n+1/2}, u_M^{n+1/2}, v_M^{n,1/4}) - b_{\text{pw}}(u_M^{n-1/2}, u_M^{n-1/2}, v_M^{n,1/4}) \\ &= \|v_M^{n+1/2}\|_{\text{pw}}^2 - \|v_M^{n-1/2}\|_{\text{pw}}^2. \end{aligned} \quad (2.11)$$

The second equality in (2.11) holds precisely because $v_M^{n+1/2}$ satisfies (2.10b): (2.10b) is therefore not an independent approximation of a separate Partial Differential Equation (PDE), but the exact discrete condition on v forced by the requirement that the nonlinear energy balance closes. The sum of (2.11) over time steps then leads to a global discrete energy inequality with no restriction on Δt . Beyond accuracy, the chosen weighted time-averaging strategy is motivated by stability: The discrete energy balance for the nonlinear von Kármán system ensures unconditional stability, unlike classical implicit Newmark schemes which are only conditionally stable in the nonlinear setting (see, e.g., [2, eqn. (36)]). Theorem 2.4(a) provides the details of *unconditional stability of this discrete scheme*. The design is further guided by the requirement that the same quarter-average structure must simultaneously deliver a discrete energy law (see Subsection 2.3 for details.)

2.5 Initial discretization

We propose approximations for the solution (u, v) at the initial time discretization points t_0 and t_1 . This is crucial for the design of the fully discrete formulation in Subsection 2.3, since the multi-step scheme (2.10) requires not only the solution at these two initial steps, but its stability and error estimates also depend critically on these approximations. The results on well-posedness and error control are stated in Theorem 2.1 and Theorem 2.3 respectively, while the proofs follow in Section 4.

For $u_0 \in H_0^2(\Omega)$, we propose (u_M^0, v_M^0) as an approximation of $(u(t), v(t))$ at $t = t_0$ defined by

$$a_{\text{pw}}(u_M^0, \varphi_M) = a(u_0, J\varphi_M), \quad a_{\text{pw}}(v_M^0, \psi_M) = \frac{1}{2} b_{\text{pw}}(u_M^0, u_M^0, \psi_M) \quad \text{for all } (\varphi_M, \psi_M) \in \mathbf{M}(\mathcal{T}). \quad (2.12)$$

Next, for $u_1 \in L^2(\Omega)$, the approximation (u_M^1, v_M^1) of the continuous solution $(u(t), v(t))$ at $t = t_1$ is then derived from

$$2k^{-1}(\bar{\partial}_t u_M^{1/2} - u_1, \varphi_M)_{L^2(\Omega)} + a_{\text{pw}}(u_M^{1/2}, \varphi_M) + b_{\text{pw}}(u_M^{1/2}, \varphi_M, v_M^{1/2}) = (f^{1/2}, \varphi_M)_{L^2(\Omega)}, \quad (2.13a)$$

$$a_{\text{pw}}(v_M^1, \psi_M) - \frac{1}{2} b_{\text{pw}}(u_M^1, u_M^1, \psi_M) = 0. \quad (2.13b)$$

The key identity (2.3) for the nonlinear term ensures the *energy conservation* of (2.2). To arrive at discrete counterpart of (2.3) in the proof of stability, the test function will be $\bar{\partial}_t u_M^{1/2}$. For the discrete scheme in (2.13), elementary manipulations lead to

$$b_{\text{pw}}(u_M^{1/2}, \bar{\partial}_t u_M^{1/2}, v_M^{1/2}) = \frac{1}{2k} (b_{\text{pw}}(u_M^1, u_M^1, v_M^{1/2}) - b_{\text{pw}}(u_M^0, u_M^0, v_M^{1/2})) = \frac{1}{2k} (\|v_M^1\|_{\text{pw}}^2 - \|v_M^0\|_{\text{pw}}^2). \quad (2.14)$$

The above identity facilitates the proof of stability of the discrete scheme (see Theorem 2.1(c) below, for details).

2.6 Key results

In this subsection, we present the main results on stability and error estimates for the fully discrete scheme (2.10)-(2.13). The proofs are given in Sections 4-5.

The first theorem provides sufficient conditions under which (2.12) and (2.13) are well-posed. The proof is provided in Subsection 4.1 and utilizes the expression

$$\check{M}_2(u_0, u_1, f, k) := \|J\|^2 \|u_0\|^2 \left(1 + \frac{1}{4} C_{\text{ds}}^2 \|J\|^2 \|u_0\|^2\right) + 4\|u_1\|^2 + k^2 \|f\|_{C([0, t_1]; L^2(\Omega))}^2$$

with the operator norm $\|J\| := \|J\|_{\mathcal{L}(M(\mathcal{T}), H_0^2(\Omega))}$ for a bounded linear companion operator of Subsection 3.1 below. Recall C_{ds} from (2.6).

Theorem 2.1 (well-posedness). *Suppose that the initial data satisfy $u_0 \in H_0^2(\Omega)$, $u_1 \in L^2(\Omega)$, and the source $f \in C([0, t_1]; L^2(\Omega))$. Then (a)-(d) hold viz.*

- (a) (2.12) is well-posed with $\|u_M^0\|_{\text{pw}} \leq \|J\| \|u_0\|$ and $\|v_M^0\|_{\text{pw}} \leq \frac{1}{2} C_{\text{ds}} \|J\|^2 \|u_0\|^2$,
- (b) there exists a solution (u_M^1, v_M^1) to (2.13),
- (c) (u_M^1, v_M^1) satisfies $\|u_M^1\|_{\text{pw}} + \|v_M^1\|_{\text{pw}} \leq \sqrt{2} (\check{M}_2(u_0, u_1, f, k))^{1/2}$,
- (d) the solution (u_M^1, v_M^1) to (2.13) is unique under smallness assumptions on u_0, u_1 , and the temporal discretization parameter k , precisely $M_2(u_0, u_1, f, k) := C_{\text{ds}}^2 (\|J\|^2 \|u_0\|^2 + \check{M}_2(u_0, u_1, f, k)) \leq 1$.

Remark 2.2 (well-posedness of (2.12)). *Theorem 2.1(a) does not require smallness data assumptions. This is because (2.12) is linear and its equations decouple. Hence the unique solvability of its first equation follows from the Lax–Milgram lemma, and the solution is used to uniquely solve the second equation.*

The next result provides the discretization error bounds for (2.12)-(2.13). The proof is postponed to Subsection 4.2.

Theorem 2.3 (initial error bounds). *Suppose that the solution u to (2.2) belongs to $H^2(0, t_1; H_0^2(\Omega) \cap H^{2+\sigma}(\Omega))$ and $u_{\text{int}} \in L^\infty(0, t_1; L^2(\Omega))$. Let $\sigma \in (1/2, 1]$ be an index of elliptic regularity for the biharmonic problem [8]. Then*

$$\|\bar{\partial}_t(u^{1/2} - u_M^{1/2})\| + \|u^{1/2} - u_M^{1/2}\|_{\text{pw}} + \|v^{1/2} - v_M^{1/2}\|_{\text{pw}} = \mathcal{O}(h^\sigma + k^2).$$

The next theorem provides sufficient conditions under which the scheme (2.10) is well-posed. This result utilizes the expression

$$\check{M}_3(u_0, u_1, f, k) := e^2 (\|J\|^2 \|u_0\|^2 (1 + \frac{1}{4} C_{\text{ds}}^2 \|J\|^2 \|u_0\|^2) + 2\|u_1\|^2 + \frac{k^2}{2} \|f\|_{C([0, t_1]; L^2(\Omega))}^2) + Tk \sum_{n=1}^m \|f^{n, 1/4}\|^2$$

and the proof is presented in Subsection 5.1.

Theorem 2.4 (well-posedness). *Suppose that the initial data satisfy the smoothness assumptions $u_0 \in H_0^2(\Omega)$, $u_1 \in L^2(\Omega)$, and the source $f \in C([0, T]; L^2(\Omega))$. Then*

- (a) the fully discrete scheme (2.10) is unconditionally stable, in the sense that if (u_M^{m+1}, v_M^{m+1}) solves (2.10) for any $1 \leq m \leq N-1$, then $\|u_M^{m+1/2}\|_{\text{pw}} + \|v_M^{m+1/2}\|_{\text{pw}} \leq \sqrt{2} (\check{M}_3(u_0, u_1, f, k))^{1/2}$,
- (b) for any $n = 1, 2, \dots, N-1$, there exists a solution (u_M^{n+1}, v_M^{n+1}) to (2.10),
- (c) If $M_3(u_0, u_1, f, k) := C_{\text{ds}}^2 \check{M}_3(u_0, u_1, f, k) < 1$ then the solution (u_M^{n+1}, v_M^{n+1}) to (2.10) is unique.

Next, we present a theorem that provides the error bounds for the discretization (2.10). It utilizes the expression $M_1(u_0, u_1, f) := \|J\| C_{\text{ds}} (\check{M}_1(u_0, u_1, f))^{1/2}$ with $\check{M}_1(u_0, u_1, f)$ from (2.4). The proof is provided in Subsection 5.2.

Theorem 2.5 (energy error bounds). *Let $u_0 \in H_0^2(\Omega)$, $u_1 \in L^2(\Omega)$, and $f \in H^1(0, T; L^2(\Omega))$ be such that $M_1(u_0, u_1, f) \leq 1/2$. Suppose that $u, u_t, v, v_t \in H^1(0, T; H_0^2(\Omega) \cap H^{2+\sigma}(\Omega))$ and $u \in H^4(0, T; L^2(\Omega))$. Then, for any $1 \leq m \leq n$ with $1 \leq n \leq N-1$, we have*

$$\|\bar{\partial}_t(u^{m+1/2} - u_M^{m+1/2})\| + \|u^{m+1/2} - u_M^{m+1/2}\|_{\text{pw}} + \|v^{m+1/2} - v_M^{m+1/2}\|_{\text{pw}} = \mathcal{O}(h^\sigma + k^2).$$

3 Auxiliary results

The fourth-order terms and the nonlinearity in the coupled system create several technical difficulties, and these are further enhanced by the complexities of second-order time discretization. In this section, we present some technical results that prepare us for the proof of the main results presented later on.

3.1 Companion operator

In contrast to other averaging operators [10], the conforming companion operator (see, e.g., [14, 16, 23]) preserves the integral mean of the function and its Hessian, and it also satisfies an enhanced best-approximation property. In this article, companion operator serves two main purposes: (a) it allows a nonstandard Ritz projection in Subsection 3.2, below. This projection possesses optimal approximation features and plays a central role in the error analysis, (b) during the derivation of the error bounds, the test functions in the continuous weak formulation are chosen from the range of this operator. This produces cleaner estimates and removes the need for boundary-term computations that arise when the PDE is tested with discrete functions. The next lemma gives the construction of this operator and utilizes the C^1 -conforming Hsieh–Clough–Tocher (HCT($\mathcal{T}$)) element [18].

Lemma 3.1 (companion operator – [14, Lemma 3.7] and [16, Lemma 5.1]). *There exists a linear mapping $J : M(\mathcal{T}) \rightarrow (HCT(\mathcal{T}) + P_8(\mathcal{T})) \cap H_0^2(\Omega)$ such that any $\varphi_M \in M(\mathcal{T})$ satisfies*

$$\begin{aligned} (a) \quad & J\varphi_M(z) = \varphi_M(z) \quad \text{for } z \in \mathcal{V}, \quad (b) \quad \nabla(J\varphi_M)(z) = |\mathcal{T}(z)|^{-1} \sum_{K \in \mathcal{T}(z)} (\nabla\varphi_M|_K)(z) \quad \text{for } z \in \mathcal{V}(\Omega), \\ (c) \quad & \int_e \frac{\partial J\varphi_M}{\partial \mathbf{n}} ds = \int_e \frac{\partial \varphi_M}{\partial \mathbf{n}} ds \quad \text{for any } e \in \mathcal{E}, \quad (d) \quad \|\varphi_M - J\varphi_M\|_{pw} \leq C_1 \min_{\varphi \in H_0^2(\Omega)} \|\varphi_M - \varphi\|_{pw}, \\ (e) \quad & \|\varphi_M - J\varphi_M\|_{H^s(\mathcal{T})} \leq C_2 h^{2-s} \min_{\varphi \in H_0^2(\Omega)} \|\varphi - \varphi_M\|_{pw} \quad C_2 > 0, \quad \text{and } 0 \leq s \leq 2, \\ (f) \quad & \Pi^0(\varphi_M - J\varphi_M) = 0 \quad \text{and} \quad \Pi^0(D_{pw}^2(\varphi_M - J\varphi_M)) = 0. \end{aligned}$$

3.2 Modified Ritz projection

The *modified* Ritz projection $\mathcal{R}_M : H_0^2(\Omega) \rightarrow M(\mathcal{T})$ is defined as:

$$a_{pw}(\mathcal{R}_M \varphi, \varphi_M) = a(\varphi, J\varphi_M) \quad \text{for all } \varphi_M \in M(\mathcal{T}), \varphi \in H_0^2(\Omega). \quad (3.1)$$

The choice $\varphi_M = \mathcal{R}_M \varphi$ in (3.1), together with the continuity of $a(\bullet, \bullet)$ and J , imply the *stability* of $\mathcal{R}_M$

$$\|\mathcal{R}_M \varphi\|_{pw} \leq \|J\| \|\varphi\| \quad \text{for all } \varphi \in H_0^2(\Omega). \quad (3.2)$$

The next lemma states the approximation properties of (3.1). For a proof, see [31, Appendix A.5–A.6].

Lemma 3.2 (approximation properties for $\mathcal{R}_M$ [13, 31]). *Let $\varphi \in H_0^2(\Omega) \cap H^{2+\sigma}(\Omega)$, and let $\mathcal{R}_M \varphi$ be its Ritz projection defined in (3.1). Then, there exists a constant $C_3 > 0$ such that*

$$\|\varphi - \mathcal{R}_M \varphi\| + h^\sigma \|\varphi - \mathcal{R}_M \varphi\|_{pw} \leq C_3 h^{2\sigma} \|\varphi\|_{H^{2+\sigma}(\Omega)}. \quad (3.3)$$

3.3 Trilinear forms

Lemma 3.3 (properties of trilinear forms). *For all $\psi, \xi \in H_0^2(\Omega)$, $\varphi, \chi \in H_0^2(\Omega) + M(\mathcal{T})$, $\zeta \in H_0^2(\Omega) \cap H^{2+\sigma}(\Omega)$, and $\chi_M, \psi_M \in M(\mathcal{T})$, there exist positive constants C_4, C_5, C_6, C_7 , and C_8 such that*

$$\begin{aligned} (a) \quad & |b(\xi, \psi, J\chi_M)|, |b(\xi, J\chi_M, \psi)| \leq C_4 \|\xi\| \|\psi\| \|\chi_M\|_{pw}, \\ (b) \quad & |b_{pw}(\varphi, J\chi_M, \psi_M)|, |b_{pw}(\varphi, \chi_M, J\psi_M)| \leq C_5 \|\varphi\|_{pw} \|\chi_M\|_{pw} \|\psi_M\|_{pw}, \\ (c) \quad & |b_{pw}(\varphi, \chi, (J - I)\psi_M)|, |b_{pw}(\varphi, (J - I)\psi_M, \chi)| \leq C_6 \|\varphi\|_{pw} \|\chi\|_{pw} \|\psi_M\|_{pw}, \\ (d) \quad & |b_{pw}(\psi, \varphi, (J - I)\chi_M)| \leq C_7 h \|\psi\| \|\varphi\|_{pw} \|\chi_M\|_{pw}, \quad \text{and} \end{aligned}$$

$$(e) \quad |b_{\text{pw}}(\zeta, (J - I)\chi_M, \xi)| \leq C_8 h^\sigma \|\zeta\|_{H^{2+\sigma}(\Omega)} \|\chi_M\|_{\text{pw}} \|\xi\|.$$

Proof. (a) An application of (2.1) and the continuity of J reveals that

$$|b(\xi, \psi, J\chi_M)| \leq C_5 \|\xi\| \|\psi\| \|J\chi_M\| \leq C_4 \|\xi\| \|\psi\| \|\chi_M\|_{\text{pw}}$$

with $C_4 := \|J\|C_5$. The bound for $b(\varphi, J\chi_M, \psi)$ follows analogously.

(b) The boundedness of the discrete trilinear form $b_{\text{pw}}(\bullet, \bullet, \bullet)$ from (2.7) and continuity of J lead to

$$|b_{\text{pw}}(\varphi, J\chi_M, \psi_M)| \leq C_{\text{dS}} \|\varphi\|_{\text{pw}} \|J\chi_M\| \|\psi_M\|_{\text{pw}} \leq C_5 \|\varphi\|_{\text{pw}} \|\chi_M\|_{\text{pw}} \|\psi_M\|_{\text{pw}}$$

with $C_5 := \|J\|C_{\text{dS}}$. The bound for $b_{\text{pw}}(\varphi, \chi_M, J\psi_M)$ follows analogously.

(c) The boundedness of $b_{\text{pw}}(\bullet, \bullet, \bullet)$ from (2.7) and Lemma 3.1(d) allow us to show that

$$|b_{\text{pw}}(\varphi, \chi, (J - I)\psi_M)| \leq C_{\text{dS}} \|\varphi\|_{\text{pw}} \|\chi\|_{\text{pw}} \|(J - I)\psi_M\|_{\text{pw}} \leq C_6 \|\varphi\|_{\text{pw}} \|\chi\|_{\text{pw}} \|\psi_M\|_{\text{pw}}$$

with $C_6 := C_1 C_{\text{dS}}$. The bound for $b_{\text{pw}}(\varphi, (J - I)\psi_M, \chi)$ follows analogously.

(d) The definition of $b_{\text{pw}}(\bullet, \bullet, \bullet)$, a Cauchy-Schwarz inequality, and an inverse estimate $h_K \|(J - I)\chi_M\|_{L^\infty(K)} \leq C_{\text{inv}} \|(J - I)\chi_M\|_{L^2(K)}$ [21, Lemma 12.1] lead to

$$\begin{aligned} |b_{\text{pw}}(\psi, \varphi, (J - I)\chi_M)| &\leq \|\psi\| \|\varphi\|_{\text{pw}} \|(J - I)\chi_M\|_{L^\infty(\Omega)} \\ &\leq C_{\text{inv}} \|\psi\| \|\varphi\|_{\text{pw}} h_{\mathcal{T}}^{-1} \|(J - I)\chi_M\|_{L^2(\Omega)} \leq C_7 h \|\psi\| \|\varphi\|_{\text{pw}} \|\chi_M\|_{\text{pw}}, \end{aligned}$$

with Lemma 3.1(e) in the last step and $C_7 := C_2 C_{\text{inv}}$.

(e) The definition of $b_{\text{pw}}(\bullet, \bullet, \bullet)$, (c), a Cauchy-Schwarz inequality, and $b_{\text{pw}}(\mathcal{R}_M \zeta, (J - I)\chi_M, \Pi_0 \xi) = 0$ (Lemma 3.1(f)) show

$$\begin{aligned} |b_{\text{pw}}(\zeta, (J - I)\chi_M, \xi)| &= |b_{\text{pw}}(\zeta - \mathcal{R}_M \zeta, (J - I)\chi_M, \xi)| + |b_{\text{pw}}(\mathcal{R}_M \zeta, (J - I)\chi_M, \xi)| \\ &\leq C_6 \|\zeta - \mathcal{R}_M \zeta\|_{\text{pw}} \|\chi_M\|_{\text{pw}} \|\xi\| + \|\mathcal{R}_M \zeta\|_{\text{pw}} \|(J - I)\chi_M\|_{\text{pw}} \|\xi - \Pi_0 \xi\|_{L^\infty(\Omega)}. \end{aligned}$$

We apply (3.3) to bound the first term on the right-hand side, (3.2), Lemma 3.1(d), and $\|\xi - \Pi_0 \xi\|_{L^\infty(\Omega)} \leq C_{\Pi_0} h \|\xi\|_{H^2(\Omega)}$ [21, Theorem 18.16] to bound the second term on the right-hand side above. This eventually gives

$$\begin{aligned} |b_{\text{pw}}(\zeta, (J - I)\chi_M, \xi)| &\leq C_3 C_6 h^\sigma \|\zeta\|_{H^{2+\sigma}(\Omega)} \|\chi_M\|_{\text{pw}} \|\xi\| + \|J\| C_1 C_{\Pi_0} h \|\zeta\| \|\chi_M\|_{\text{pw}} \|\xi\|_{H^2(\Omega)} \\ &\leq C_8 h^\sigma \|\zeta\|_{H^{2+\sigma}(\Omega)} \|\chi_M\|_{\text{pw}} \|\xi\|_{H^2(\Omega)} \end{aligned}$$

with $\|\xi\| \leq \|\xi\|_{H^2(\Omega)}$, $\|\zeta\| \leq \|\zeta\|_{H^{2+\sigma}(\Omega)}$ and $C_8 := C_3 C_6 + \|J\| C_1 C_{\Pi_0} |\Omega|^{\frac{1-\sigma}{2}}$ in the last step. $\square$

3.4 Temporal discretization results

Lemma 3.4. *Let $(X, \|\bullet\|_X)$ be a Hilbert space and $\varphi : [0, T] \rightarrow X$. If $\varphi \in C([t_{m-1}, t_{m+1}]; X)$, for some $1 \leq m \leq N - 1$, then*

$$(a) \quad \|\varphi^m\|_X, \|\varphi^{m+1/2}\|_X \leq \|\varphi\|_{C([t_m, t_{m+1}]; X)} \text{ and } (b) \quad \|\varphi^{m, 1/4}\|_X \leq \|\varphi\|_{C([t_{m-1}, t_{m+1}]; X)}.$$

In addition, if $\varphi_t \in L^\infty(t_{m-1}, t_{m+2}; X)$ for some $1 \leq m \leq N - 2$, then

$$(c) \quad \|\bar{\partial}_t \varphi^{m+1/2}\|_X \leq \|\varphi_t\|_{L^\infty(t_m, t_{m+1}; X)}, \quad (d) \quad \|\varphi^{m+1/2} - \varphi^{m-1/2}\|_X \leq \sqrt{k/2} \|\varphi_t\|_{L^2(t_{m-1}, t_{m+1}; X)},$$

$$(e) \quad \|\delta_t \varphi^m\|_X \leq \|\varphi_t\|_{L^\infty(t_{m-1}, t_{m+1}; X)}, \text{ and } (f) \quad \|\varphi^{m+1, 1/4} - \varphi^{m, 1/4}\|_X \leq \sqrt{k/2} \|\varphi_t\|_{L^2(t_{m-1}, t_{m+2}; X)}.$$

Furthermore, if $\varphi \in H^2(0, T; X)$, then (g) $k \sum_{n=1}^m \|\bar{\partial}_t^2 \varphi^n\|_X^2 \leq \frac{4}{3} \|\varphi_{tt}\|_{L^2(0, T; X)}^2$.

Proof. (a) From (2.8a), we have $\varphi^m = \varphi(t_m)$ and hence

$$\|\varphi^m\|_X = \|\varphi(t_m)\|_X \leq \|\varphi\|_{C([t_m, t_{m+1}]; X)}. \quad (3.4)$$

The definition of $\varphi^{m+1/2}$ from (2.8a), triangle inequality, and arguments as above, imply that

$$\|\varphi^{m+1/2}\|_X = \frac{1}{2}\|\varphi^m + \varphi^{m+1}\|_X \leq \frac{1}{2}(\|\varphi^m\|_X + \|\varphi^{m+1}\|_X) \leq \|\varphi\|_{C([t_m, t_{m+1}]; X)}.$$

(b) The proof proceeds analogous to that of (a) by using definition (2.8a) and triangle inequality.

(c) From the definition (2.8b), there holds $\bar{\partial}_t \varphi^{m+1/2} = k^{-1}(\varphi^{m+1} - \varphi^m) = k^{-1} \int_{t_m}^{t_{m+1}} \varphi_t(t) dt$, and hence

$$\|\bar{\partial}_t \varphi^{m+1/2}\|_X = k^{-1} \left\| \int_{t_m}^{t_{m+1}} \varphi_t(t) dt \right\|_X \leq k^{-1} \int_{t_m}^{t_{m+1}} \|\varphi_t(t)\|_X dt \leq \|\varphi_t\|_{L^\infty(t_m, t_{m+1}; X)}.$$

(d) Since $\varphi^{m+1/2} - \varphi^{m-1/2} = \frac{1}{2}(\varphi^{m+1} - \varphi^{m-1})$ holds from (2.8a), a Cauchy-Schwarz inequality shows

$$\|\varphi^{m+1/2} - \varphi^{m-1/2}\|_X = \frac{1}{2} \left\| \int_{t_{m-1}}^{t_{m+1}} \varphi_t dt \right\|_X \leq \frac{1}{2} \int_{t_{m-1}}^{t_{m+1}} \|\varphi_t\|_X dt \leq \sqrt{\frac{k}{2}} \|\varphi_t\|_{L^2(t_{m-1}, t_{m+1}; X)}.$$

(e) Since $\delta_t \varphi^m = \frac{1}{k}(\varphi^{m+1/2} - \varphi^{m-1/2})$, analogous arguments to (d) provide

$$\|\delta_t \varphi^m\|_X = \frac{1}{k} \|\varphi^{m+1/2} - \varphi^{m-1/2}\|_X \leq \frac{1}{\sqrt{2k}} \|\varphi_t\|_{L^2(t_{m-1}, t_{m+1}; X)} \leq \|\varphi_t\|_{L^\infty(t_{m-1}, t_{m+1}; X)}.$$

(f) A triangle inequality $\|\varphi^{m+1, 1/4} - \varphi^{m, 1/4}\|_X = \frac{1}{2} \|\varphi^{m+3/2} - \varphi^{m+1/2}\|_X + \|\varphi^{m+1/2} - \varphi^{m-1/2}\|_X$ and (d) lead to

$$\|\varphi^{m+1, 1/4} - \varphi^{m, 1/4}\|_X \leq \sqrt{\frac{k}{8}} (\|\varphi_t\|_{L^2(t_m, t_{m+2}; X)} + \|\varphi_t\|_{L^2(t_{m-1}, t_{m+1}; X)}) \leq \sqrt{\frac{k}{2}} \|\varphi_t\|_{L^2(t_{m-1}, t_{m+2}; X)}.$$

(g) From Taylor's theorem (with integral remainder), we have

$$\varphi^{n+1} = \varphi^n + k\varphi_t^n + \int_{t_n}^{t_{n+1}} (t_{n+1} - t)\varphi_{tt} dt \text{ and } \varphi^{n-1} = \varphi^n - k\varphi_t^n + \int_{t_{n-1}}^{t_n} (t_{n-1} - t)\varphi_{tt} dt.$$

These two identities and (2.8b) allow us to assert that

$$\begin{aligned} k^2 \|\bar{\partial}_t^2 \varphi^n\|_X &= \|\varphi^{n+1} - 2\varphi^n + \varphi^{n-1}\|_X = \left\| \int_{t_n}^{t_{n+1}} (t_{n+1} - t)\varphi_{tt} dt + \int_{t_{n-1}}^{t_n} (t_{n-1} - t)\varphi_{tt} dt \right\|_X \\ &\leq \int_{t_n}^{t_{n+1}} |t_{n+1} - t| \|\varphi_{tt}\|_X dt + \int_{t_{n-1}}^{t_n} |t_{n-1} - t| \|\varphi_{tt}\|_X dt. \end{aligned}$$

An appeal to a Cauchy-Schwarz inequality and $\int_{t_n}^{t_{n+1}} (t_{n+1} - t)^2 dt = \frac{k^3}{3} = \int_{t_{n-1}}^{t_n} (t_{n-1} - t)^2 dt$ and elementary manipulations provide

$$k \sum_{n=1}^m \|\bar{\partial}_t^2 \varphi^n\|_X^2 \leq \frac{2}{3} \left(\sum_{n=1}^m \int_{t_n}^{t_{n+1}} \|\varphi_{tt}\|_X^2 dt + \sum_{n=1}^m \int_{t_{n-1}}^{t_n} \|\varphi_{tt}\|_X^2 dt \right) \leq \frac{4}{3} \|\varphi_{tt}\|_{L^2(0, T; X)}^2.$$

This concludes the proof. $\square$

Lemma 3.5. Let $\xi = \varphi - \mathcal{R}_M \varphi$. If (a) $\varphi \in C([t_m, t_{m+1}]; H_0^2(\Omega) \cap H^{2+\sigma}(\Omega))$ for some $0 \leq m \leq N-1$, then $\|\xi^m\|_{pw}, \|\xi^{m+1/2}\|_{pw} \leq C_3 h^\sigma \|\varphi\|_{C([t_m, t_{m+1}]; H^{2+\sigma}(\Omega))}$, (b) $\varphi \in C([t_{m-1}, t_{m+1}]; H_0^2(\Omega) \cap H^{2+\sigma}(\Omega))$ for some $1 \leq m \leq N-1$, then $\|\xi^{m, 1/4}\|_{pw} \leq C_3 h^\sigma \|\varphi\|_{C([t_{m-1}, t_{m+1}]; H^{2+\sigma}(\Omega))}$.

Furthermore, $\varphi \in H^1(t_{m-1}, t_{m+2}; H_0^2(\Omega) \cap H^{2+\sigma}(\Omega))$ for some $1 \leq m \leq N-2$ satisfies

(c) $\|k\delta_t \xi^m\|_{pw} \leq C_3 \sqrt{k/2} h^\sigma \|\varphi_t\|_{L^2(t_{m-1}, t_{m+1}; H^{2+\sigma}(\Omega))}$, (d) $\|\xi^{m+1, 1/4} - \xi^{m, 1/4}\|_{pw} \leq C_3 \sqrt{k/2} h^\sigma \|\varphi_t\|_{L^2(t_{m-1}, t_{m+2}; H^{2+\sigma}(\Omega))}$.

Proof. (a) Lemma 3.2 and arguments similar to those used in (3.4) reveals

$$\|\xi^m\|_{pw} = \|\varphi^m - \mathcal{R}_M \varphi^m\|_{pw} \leq C_3 h^\sigma \|\varphi^m\|_{H^{2+\sigma}(\Omega)} \leq C_3 h^\sigma \|\varphi\|_{C([t_m, t_{m+1}]; H^{2+\sigma}(\Omega))}. \quad (3.5)$$

Definition (2.8a) and triangle inequality followed by application of (3.5) twice, shows that

$$\|\xi^{m+1/2}\|_{\text{pw}} = \frac{1}{2} \|\xi^m + \xi^{m+1}\|_{\text{pw}} \leq \frac{1}{2} (\|\xi^m\|_{\text{pw}} + \|\xi^{m+1}\|_{\text{pw}}) \leq C_3 h^\sigma \|\varphi\|_{C([t_m, t_{m+1}]; H^{2+\sigma}(\Omega))}.$$

(b) The proof follows analogous to (a).

(c) Owing to similar arguments employed in Lemma 3.4(d), we can assert that

$$\|k\delta_t \xi^m\|_{\text{pw}} = \|\xi^{m+1/2} - \xi^{m-1/2}\|_{\text{pw}} \leq \sqrt{\frac{k}{2}} \left(\int_{t_{m-1}}^{t_{m+1}} \|\xi_t\|_{H^2(\Omega)}^2 dt \right)^{1/2}.$$

The inequality $\left(\int_{t_{m-1}}^{t_{m+1}} \|\xi_t\|_{H^2(\Omega)}^2 dt \right)^{1/2} \leq C_3 h^\sigma \|\xi_t\|_{L^2(t_{m-1}, t_{m+1}; H^2(\Omega))}$ follows from the definition of $\xi(t)$ and (3.3).

(d) The definition (2.9a) and triangle inequality leads to

$$\|\xi^{m+1,1/4} - \xi^{m,1/4}\|_{\text{pw}} \leq \frac{1}{2} \|\xi^{m+3/2} - \xi^{m+1/2}\|_{\text{pw}} + \frac{1}{2} \|\xi^{m+1/2} - \xi^{m-1/2}\|_{\text{pw}}.$$

As a consequence of (c) we conclude the proof. $\square$

Lemma 3.6. (discrete Gronwall [25, Lemma 4.1]) *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be three non-negative sequences, with $\{c_n\}$ monotone, that satisfy $a_m + b_m \leq c_m + \mu \sum_{n=0}^{m-1} a_n$, $\mu > 0$, $a_0 + b_0 \leq c_0$. Then for $m \geq 0$, it holds that $a_m + b_m \leq c_m e^{m\mu}$.*

Lemma 3.7. (truncation error [25, Theorem 4.4]) *Let $\varphi \in C^2([0, T]; L^2(\Omega))$ satisfy*

$$(a) \quad \varphi_{ttt} \in L^\infty(0, t_1; L^2(\Omega)), \text{ then } \|2k^{-1}(\bar{\partial}_t \varphi^{1/2} - \varphi_t(0)) - \varphi_{tt}^{1/2}\| \leq k \|\varphi_{ttt}\|_{L^\infty(0, t_1; L^2(\Omega))},$$

$$(b) \quad \varphi \in H^4(0, T; L^2(\Omega)), \text{ then } k \sum_{n=1}^{N-1} \|\bar{\partial}_t^2 \varphi^n - \varphi_{tt}^{n,1/4}\|^2 \leq C(\varphi_{ttt}) k^4.$$

The constant $C(\varphi_{ttt})$ in (b) depends on $\|\varphi_{ttt}\|_{L^2(0, T; L^2(\Omega))}$.

4 Proofs of results for the initial discretization

This section presents the proofs of the well-posedness result in Theorem 2.1 and the a priori error estimates in Theorem 2.3 for the initialization schemes (2.12)–(2.13).

4.1 Well-posedness of initial discretizations

The proof of Theorem 2.1(a) (resp. Theorem 2.1(b)) below establishes the existence of the discrete solution (u_M^0, v_M^0) (resp. (u_M^1, v_M^1)) to (2.12) (resp. (2.13)). Theorem 2.1(c) establishes stability of the solution to the initialization schemes (2.12)–(2.13) and Theorem 2.1(d) provides the uniqueness. This result is subsequently employed in the error analysis of Theorem 2.3 and serves as the base case for the inductive well-posedness argument for the fully discrete scheme (2.10) in Theorem 2.4.

Proof of Theorem 2.1(a). The proof relies on a Lax-Milgram lemma. For given $u_0 \in H_0^2(\Omega)$, the map $a(u_0, J(\bullet)) : M(\mathcal{T}) \rightarrow \mathbb{R}$ is linear. This follows directly from the fact that J is linear and $a(\bullet, \bullet)$ is bilinear. Also, from continuity of $a(\bullet, \bullet)$ and J we have

$$|a(u_0, J\varphi_M)| \leq \|u_0\| \|J\varphi_M\|_{H^2(\Omega)} \leq \|J\| \|u_0\| \|\varphi_M\|_{\text{pw}}.$$

This implies that $a(u_0, J(\bullet))$ is a bounded linear functional on $M(\mathcal{T})$. This, the continuity and ellipticity of the bilinear form $a_{\text{pw}}(\bullet, \bullet)$ on $M(\mathcal{T})$ lead to the existence of a unique solution u_M^0 to the first equation in (2.12). Similarly, the boundedness of $b_{\text{pw}}(u_M^0, u_M^0, \bullet) : M(\mathcal{T}) \rightarrow \mathbb{R}$ from (2.7), and the ellipticity of $a_{\text{pw}}(\bullet, \bullet)$ ensures the existence of a unique solution v_M^0 to the second equation in (2.12).

The bound $\|u_M^0\|_{\text{pw}} \leq \|J\| \|u_0\|$ follows from $\varphi_M = u_M^0$ in (2.12), the continuity of $a(\bullet, \bullet)$, and J . The choice $\psi_M = v_M^0$ in (2.12), (2.7), and the a priori bound for u_M^0 lead to $\|v_M^0\|_{\text{pw}} \leq \frac{1}{2} C_{\text{dS}} \|J\|^2 \|u_0\|^2$. $\square$

For any $(\varphi_M, \psi_M) \in \mathbf{M}(\mathcal{T})$, recall from (2.13) that the solution (u_M^1, v_M^1) of (2.13) (near $(u(t_1), v(t_1))$) satisfies

$$2k^{-1}(\bar{\partial}_t u_M^{1/2} - u_1, \varphi_M)_{L^2(\Omega)} + a_{\text{pw}}(u_M^{1/2}, \varphi_M) + b_{\text{pw}}(u_M^{1/2}, \varphi_M, v_M^{1/2}) = (f^{1/2}, \varphi_M)_{L^2(\Omega)},$$

$$a_{\text{pw}}(v_M^1, \psi_M) - \frac{1}{2}b_{\text{pw}}(u_M^1, u_M^1, \psi_M) = 0.$$

Unlike the coupled system in (2.12), where the first equation does not involve v_M^0 (allowing the equations to be solved sequentially and the system to remain linear), the proof of the existence result for this nonlinear coupled system requires say, fixed-point theory; we employ a corollary of Brouwer fixed-point theorem.

Theorem 4.1 (Brouwer fixed-point theorem [27, Theorem 5.2.5] [22, Page no. 529]). *Let $\mathbf{H}$ be a finite dimensional Hilbert space with the norm $\|\bullet\|_{\mathbf{H}}$ induced from the inner product $(\bullet, \bullet)_{\mathbf{H}}$. Let $\mathcal{L} : \mathbf{H} \rightarrow \mathbf{H}$ be a continuous mapping such $(\mathcal{L}(\varphi), \varphi)_{\mathbf{H}} \geq 0$ for all $\varphi \in \mathbf{H}$ with $\|\varphi\|_{\mathbf{H}} = R > 0$. Then there exists $\varphi^* \in \mathbf{H}$ with $\|\varphi^*\|_{\mathbf{H}} \leq R$ such that $\mathcal{L}(\varphi^*) = 0$.*

Proof of Theorem 2.1(b). We follow four steps. The first step describes the settings in the context of Brouwer's fixed-point theorem. This is followed by verification of the assumptions in Steps 2 and 3. Step 4 concludes the proof.

Step 1. Setting. Choose $\mathbf{H} = \mathbf{M}(\mathcal{T})$ and for all $\Psi_M = (\psi_{M1}, \psi_{M2})$, $\Phi_M = (\varphi_{M1}, \varphi_{M2}) \in \mathbf{M}(\mathcal{T})$, define the inner product in $\mathbf{H}$ by $(\Psi_M, \Phi_M)_{\text{pw}} := 2(\psi_{M1}, \varphi_{M1})_{L^2(\Omega)} + \frac{k^2}{2}(a_{\text{pw}}(\psi_{M1}, \varphi_{M1}) + a_{\text{pw}}(\psi_{M2}, \varphi_{M2}))$. The norm induced by this inner product is $\|\Phi_M\|_{\text{pw}} = (\Phi_M, \Phi_M)_{\text{pw}}^{1/2}$. The definition of $\|\bullet\|_{\text{pw}}$ leads to

$$\sqrt{2}\|\varphi_{M1}\| \leq \|\Phi_M\|_{\text{pw}}, \quad k\|\varphi_{Mj}\|_{\text{pw}} \leq \sqrt{2}\|\Phi_M\|_{\text{pw}} \quad \text{for } j = 1, 2. \quad (4.1)$$

Next, we introduce the map $\mathcal{G}_{\Psi_M} : \mathbf{M}(\mathcal{T}) \rightarrow \mathbb{R}$ that helps to define $\mathcal{L}$ via a Riesz representation. For a fixed $\Psi_M = (\psi_{M1}, \psi_{M2}) \in \mathbf{M}(\mathcal{T})$ define

$$\mathcal{G}_{\Psi_M}(\Phi_M) := (\Psi_M, \Phi_M)_{\text{pw}} + \frac{k^2}{4}[b_{\text{pw}}(\psi_{M1} + 2u_M^0, \varphi_{M1}, \psi_{M2}) - b_{\text{pw}}(\psi_{M1} + u_M^0, \psi_{M1} + u_M^0, \varphi_{M2})]$$

$$- k^2[(f^{1/2} + 2k^{-1}u_1, \varphi_{M1})_{L^2(\Omega)} - a_{\text{pw}}(u_M^0, \varphi_{M1}) + \frac{1}{2}a_{\text{pw}}(v_M^0, \varphi_{M2})]. \quad (4.2)$$

(The definition of $\mathcal{G}_{\Psi_M}$ is in fact motivated by (2.13) as seen in Step 4). It is easy to verify that $\mathcal{G}_{\Psi_M}(\bullet)$ is linear. We establish its continuity with respect to $\|\bullet\|_{\text{pw}}$. Since (2.7) and (4.1) hold and $k \leq T$, we conclude that

$$\begin{aligned} & \frac{k^2}{4} |b_{\text{pw}}(\psi_{M1} + 2u_M^0, \varphi_{M1}, \psi_{M2}) - b_{\text{pw}}(\psi_{M1} + u_M^0, \psi_{M1} + u_M^0, \varphi_{M2})| \\ & \leq C_{\text{ds}} \frac{k^2}{4} [\|\psi_{M1} + 2u_M^0\|_{\text{pw}} \|\varphi_{M1}\|_{\text{pw}} \|\psi_{M2}\|_{\text{pw}} + \|\psi_{M1} + u_M^0\|_{\text{pw}}^2 \|\varphi_{M2}\|_{\text{pw}}] \\ & \leq C_{\text{ds}} \frac{T}{\sqrt{8}} [\|\psi_{M1} + 2u_M^0\|_{\text{pw}} \|\psi_{M2}\|_{\text{pw}} + \|\psi_{M1} + u_M^0\|_{\text{pw}}^2] \|\Phi_M\|_{\text{pw}}. \end{aligned}$$

Elementary manipulations with (4.1) and $k \leq T$ provide

$$k^2 |(f^{1/2}, \varphi_{M1})_{L^2(\Omega)} + 2k^{-1}(u_1, \varphi_{M1})_{L^2(\Omega)}| \leq \frac{1}{\sqrt{2}}(T^2 \|f^{1/2}\| + 2T \|u_1\|) \|\Phi_M\|_{\text{pw}},$$

$$k^2 |a_{\text{pw}}(u_M^0, \varphi_{M1}) - \frac{1}{2}a_{\text{pw}}(v_M^0, \varphi_{M2})| \leq \sqrt{2}T(\|u_M^0\|_{\text{pw}} + \frac{1}{2}\|v_M^0\|_{\text{pw}}) \|\Phi_M\|_{\text{pw}}.$$

A combination of the last three displayed inequalities and $(\Psi_M, \Phi_M)_{\text{pw}} \leq \|\Psi_M\|_{\text{pw}} \|\Phi_M\|_{\text{pw}}$ in (4.2) reveals that $\mathcal{G}_{\Psi_M}(\bullet)$ is a bounded linear functional in the Hilbert space $(\mathbf{M}(\mathcal{T}), \|\bullet\|_{\text{pw}})$. Its Riesz representation $\mathcal{L}(\Psi_M)$ defines the map $\mathcal{L} : \mathbf{M}(\mathcal{T}) \rightarrow \mathbf{M}(\mathcal{T})$ given by

$$(\mathcal{L}(\Psi_M), \Phi_M)_{\text{pw}} = \mathcal{G}_{\Psi_M}(\Phi_M) \quad \text{for all } \Phi_M \in \mathbf{M}(\mathcal{T}). \quad (4.3)$$

Step 2. $\mathcal{L}$ is continuous. For $\Psi_M = (\psi_{M1}, \psi_{M2})$, $\Psi_M^* = (\psi_{M1}^*, \psi_{M2}^*) \in \mathbf{M}(\mathcal{T})$ and for all $\Phi_M = (\varphi_{M1}, \varphi_{M2}) \in \mathbf{M}(\mathcal{T})$, the relations (4.2) and (4.3) show

$$(\mathcal{L}(\Psi_M) - \mathcal{L}(\Psi_M^*), \Phi_M)_{\text{pw}} = (\Psi_M - \Psi_M^*, \Phi_M)_{\text{pw}} + \frac{k^2}{4}[b_{\text{pw}}(\psi_{M1} + 2u_M^0, \varphi_{M1}, \psi_{M2})$$

$$-b_{\text{pw}}(\psi_{\text{M1}} + u_{\text{M}}^0, \psi_{\text{M1}} + u_{\text{M}}^0, \varphi_{\text{M2}}) - b_{\text{pw}}(\psi_{\text{M1}}^* + 2u_{\text{M}}^0, \varphi_{\text{M1}}, \psi_{\text{M2}}^*) + b_{\text{pw}}(\psi_{\text{M1}}^* + u_{\text{M}}^0, \psi_{\text{M1}}^* + u_{\text{M}}^0, \varphi_{\text{M2}})].$$

Abbreviate $\mathcal{L}(\Psi_{\text{M}}) - \mathcal{L}(\Psi_{\text{M}}^*) = \Theta_{\text{M}} = (\theta_{\text{M1}}, \theta_{\text{M2}})$ and test $\Phi_{\text{M}} = \Theta_{\text{M}}$ in the above expression to infer

$$\begin{aligned} \|\Theta_{\text{M}}\|_{\text{pw}}^2 &= (\Psi_{\text{M}} - \Psi_{\text{M}}^*, \Theta_{\text{M}})_{\text{pw}} + \frac{k^2}{4} [b_{\text{pw}}(\psi_{\text{M1}} + 2u_{\text{M}}^0, \theta_{\text{M1}}, \psi_{\text{M2}}) - b_{\text{pw}}(\psi_{\text{M1}} + u_{\text{M}}^0, \psi_{\text{M1}} + u_{\text{M}}^0, \theta_{\text{M2}}) \\ &\quad - b_{\text{pw}}(\psi_{\text{M1}}^* + 2u_{\text{M}}^0, \theta_{\text{M1}}, \psi_{\text{M2}}^*) + b_{\text{pw}}(\psi_{\text{M1}}^* + u_{\text{M}}^0, \psi_{\text{M1}}^* + u_{\text{M}}^0, \theta_{\text{M2}})]. \end{aligned} \quad (4.4)$$

Applying Cauchy-Schwarz inequality we readily obtain $(\Psi_{\text{M}} - \Psi_{\text{M}}^*, \Theta_{\text{M}})_{\text{pw}} \leq \|\Psi_{\text{M}} - \Psi_{\text{M}}^*\|_{\text{pw}} \|\Theta_{\text{M}}\|_{\text{pw}}$. The linearity of $b_{\text{pw}}(\cdot, \cdot, \cdot)$ in each component and elementary manipulations then show

$$\begin{aligned} &b_{\text{pw}}(\psi_{\text{M1}} + 2u_{\text{M}}^0, \theta_{\text{M1}}, \psi_{\text{M2}}) - b_{\text{pw}}(\psi_{\text{M1}}^* + 2u_{\text{M}}^0, \theta_{\text{M1}}, \psi_{\text{M2}}^*) \\ &= b_{\text{pw}}(\psi_{\text{M1}} - \psi_{\text{M1}}^*, \theta_{\text{M1}}, \psi_{\text{M2}}) + b_{\text{pw}}(\psi_{\text{M1}}^* + 2u_{\text{M}}^0, \theta_{\text{M1}}, \psi_{\text{M2}} - \psi_{\text{M2}}^*), \\ &b_{\text{pw}}(\psi_{\text{M1}}^* + u_{\text{M}}^0, \psi_{\text{M1}}^* + u_{\text{M}}^0, \theta_{\text{M2}}) - b_{\text{pw}}(\psi_{\text{M1}} + u_{\text{M}}^0, \psi_{\text{M1}} + u_{\text{M}}^0, \theta_{\text{M2}}) \\ &= b_{\text{pw}}(\psi_{\text{M1}}^* - \psi_{\text{M1}}, \psi_{\text{M1}}^* + u_{\text{M}}^0, \theta_{\text{M2}}) + b_{\text{pw}}(\psi_{\text{M1}} + u_{\text{M}}^0, \psi_{\text{M1}}^* - \psi_{\text{M1}}, \theta_{\text{M2}}). \end{aligned}$$

The boundedness of $b_{\text{pw}}(\cdot, \cdot, \cdot)$ from (2.7) and (4.1) with the abbreviation $(\psi_{\text{M1}} - \psi_{\text{M1}}^*, \psi_{\text{M2}} - \psi_{\text{M2}}^*) = \Psi_{\text{M}} - \Psi_{\text{M}}^*$, $(\theta_{\text{M1}}, \theta_{\text{M2}}) = \Theta_{\text{M}}$ lead to

$$\begin{aligned} b_{\text{pw}}(\psi_{\text{M1}} - \psi_{\text{M1}}^*, \theta_{\text{M1}}, \psi_{\text{M2}}) &\leq C_{\text{dS}} \|\psi_{\text{M1}} - \psi_{\text{M1}}^*\|_{\text{pw}} \|\theta_{\text{M1}}\|_{\text{pw}} \|\psi_{\text{M2}}\|_{\text{pw}} \\ &\leq 2C_{\text{dS}} k^{-2} \|\Psi_{\text{M}} - \Psi_{\text{M}}^*\|_{\text{pw}} \|\Theta_{\text{M}}\|_{\text{pw}} \|\psi_{\text{M2}}\|_{\text{pw}}. \end{aligned}$$

Analogous arguments reveal

$$\begin{aligned} &b_{\text{pw}}(\psi_{\text{M1}}^* + 2u_{\text{M}}^0, \theta_{\text{M1}}, \psi_{\text{M2}} - \psi_{\text{M2}}^*) + b_{\text{pw}}(\psi_{\text{M1}}^* - \psi_{\text{M1}}, \psi_{\text{M1}}^* + u_{\text{M}}^0, \theta_{\text{M2}}) + b_{\text{pw}}(\psi_{\text{M1}} + u_{\text{M}}^0, \psi_{\text{M1}} - \psi_{\text{M1}}^*, \theta_{\text{M2}}) \\ &\leq 2C_{\text{dS}} k^{-2} (\|\psi_{\text{M1}}^* + 2u_{\text{M}}^0\|_{\text{pw}} + \|\psi_{\text{M1}}^* + u_{\text{M}}^0\|_{\text{pw}} + \|\psi_{\text{M1}} + u_{\text{M}}^0\|_{\text{pw}}) \|\Psi_{\text{M}} - \Psi_{\text{M}}^*\|_{\text{pw}} \|\Theta_{\text{M}}\|_{\text{pw}}. \end{aligned}$$

A combination of all this in (4.4) shows that the map $\mathcal{L}$ is (locally Lipschitz) continuous.

Step 3. $(\mathcal{L}(\Psi_{\text{M}}), \Psi_{\text{M}})_{\text{pw}} \geq 0$. Select $\Psi_{\text{M}} \in \mathbf{M}(\mathcal{T})$ such that $\|\Psi_{\text{M}}\|_{\text{pw}} = R$, where

$$R = k \sqrt{2k^2 \|f\|_{C([0, t_1]; L^2(\Omega))}^2 + 8\|u_1\|^2 + \|J\|^2 \|u_0\|^2 (4 + C_{\text{dS}}^2 \|J\|^2 \|u_0\|^2)}. \quad (4.5)$$

First observe that the linearity, symmetry of $b_{\text{pw}}(\cdot, \cdot, \cdot)$ in its first two components, and (2.12) shows

$$b_{\text{pw}}(\psi_{\text{M1}} + 2u_{\text{M}}^0, \psi_{\text{M1}}, \psi_{\text{M2}}) - b_{\text{pw}}(\psi_{\text{M1}} + u_{\text{M}}^0, \psi_{\text{M1}} + u_{\text{M}}^0, \psi_{\text{M2}}) = -b_{\text{pw}}(u_{\text{M}}^0, u_{\text{M}}^0, \psi_{\text{M2}}) = -2a_{\text{pw}}(v_{\text{M}}^0, \psi_{\text{M2}}).$$

The choice $\Phi_{\text{M}} = \Psi_{\text{M}}$ in (4.3), (4.2), and the last displayed identity lead to

$$(\mathcal{L}(\Psi_{\text{M}}), \Psi_{\text{M}})_{\text{pw}} = \|\Psi_{\text{M}}\|_{\text{pw}}^2 - k^2 (f^{1/2}, \psi_{\text{M1}})_{L^2(\Omega)} - 2k(u_1, \psi_{\text{M1}})_{L^2(\Omega)} + k^2 a_{\text{pw}}(u_{\text{M}}^0, \psi_{\text{M1}}) - k^2 a_{\text{pw}}(v_{\text{M}}^0, \psi_{\text{M2}}).$$

A Cauchy-Schwarz inequality and Young's inequality with $a = k^2 \|f^{1/2} + 2k^{-1}u_1\|$, $b = \|\psi_{\text{M1}}\|$, $\epsilon = 1/2$ followed by a triangle inequality and the bound $\|f^{1/2}\| \leq \|f\|_{C([0, t_1]; L^2(\Omega))}$ from Lemma 3.4(a) reveal

$$k^2 (f^{1/2} + 2k^{-1}u_1, \psi_{\text{M1}})_{L^2(\Omega)} \leq k^2 \|f^{1/2} + 2k^{-1}u_1\| \|\psi_{\text{M1}}\| \leq \frac{k^4}{2} \|f\|_{C([0, t_1]; L^2(\Omega))}^2 + 2k^2 \|u_1\|^2 + \|\psi_{\text{M1}}\|^2.$$

The continuity of bilinear form $a_{\text{pw}}(\cdot, \cdot)$, Young's inequality with $a = \|u_{\text{M}}^0\|_{\text{pw}}$, $b = \|\psi_{\text{M1}}\|_{\text{pw}}$, and $\epsilon = 2$ (resp. $a = \|v_{\text{M}}^0\|_{\text{pw}}$, $b = \|\psi_{\text{M2}}\|_{\text{pw}}$, and $\epsilon = 2$), and the bounds from (a) reveal

$$\begin{aligned} |a_{\text{pw}}(u_{\text{M}}^0, \psi_{\text{M1}})| &\leq \|u_{\text{M}}^0\|_{\text{pw}} \|\psi_{\text{M1}}\|_{\text{pw}} \leq \|u_{\text{M}}^0\|_{\text{pw}}^2 + \frac{1}{4} \|\psi_{\text{M1}}\|_{\text{pw}}^2 \leq \|J\|^2 \|u_0\|^2 + \frac{1}{4} \|\psi_{\text{M1}}\|_{\text{pw}}^2, \\ (\text{resp. } a_{\text{pw}}(v_{\text{M}}^0, \psi_{\text{M2}})) &\leq \|v_{\text{M}}^0\|_{\text{pw}} \|\psi_{\text{M2}}\|_{\text{pw}} \leq \|v_{\text{M}}^0\|_{\text{pw}}^2 + \frac{1}{4} \|\psi_{\text{M2}}\|_{\text{pw}}^2 \leq \frac{1}{4} C_{\text{dS}}^2 \|J\|^4 \|u_0\|_{H^2(\Omega)}^4 + \frac{1}{4} \|\psi_{\text{M2}}\|_{\text{pw}}^2. \end{aligned}$$

A combination of the four last displayed results and the definition of R from (4.5) show

$$(\mathcal{L}(\Psi_{\text{M}}), \Psi_{\text{M}})_{\text{pw}} \geq \|\Psi_{\text{M}}\|_{\text{pw}}^2 - (\|\psi_{\text{M1}}\|^2 + \frac{k^2}{4} \|\psi_{\text{M1}}\|_{\text{pw}}^2 + \frac{k^2}{4} \|\psi_{\text{M2}}\|_{\text{pw}}^2) - \frac{R^2}{4}.$$

Recall that $\Psi_M \in \mathbf{M}(\mathcal{T})$ is chosen such that $\|\Psi_M\|_{pw}^2 = 2\|\psi_{M1}\|^2 + \frac{k^2}{2}\|\psi_{M1}\|_{pw}^2 + \frac{k^2}{2}\|\psi_{M2}\|_{pw}^2 = R^2$. Utilize this in the last displayed inequality to obtain

$$(\mathcal{L}(\Psi_M), \Psi_M)_{pw} \geq R^2 - \frac{R^2}{2} - \frac{R^2}{4} \geq 0.$$

Step 4. Conclusion. Steps 2 and 3 show that $\mathcal{L}$ is continuous and there exists $R > 0$ such that $(\mathcal{L}(\Psi_M), \Psi_M)_{pw} \geq 0$ for all Ψ_M with $\|\Psi_M\|_{pw} = R$. Theorem 4.1 establishes the existence of $\Psi_M^* \in \mathbf{M}(\mathcal{T})$ such that $\|\Psi_M^*\|_{pw} \leq R$ and $\mathcal{L}(\Psi_M^*) = 0$. Define $(u_M^1, v_M^1) := (\psi_{M1}^* + u_M^0, \psi_{M2}^* - v_M^0)$, choose $\Phi_M = k^{-2}(\varphi_M, 0)$ (resp. $\Phi_M = k^{-2}(0, 2\psi_M)$) in (4.3), and utilize $\mathcal{L}(\Psi_M^*) = 0$ to obtain (2.13a) (resp. (2.13b)). $\square$

Proof of Theorem 2.1(c). This part establishes the stability for (u_M^1, v_M^1) by deriving a discrete energy estimate for the scheme (2.13), with the key ingredient being the nonlinear identity (2.14).

Choose $\varphi_M = u_M^1 - u_M^0 = k\bar{\partial}_t u_M^{1/2}$ in (2.13a), utilize $a_{pw}(u_M^{1/2}, u_M^1 - u_M^0) = \frac{1}{2}(\|u_M^1\|_{pw}^2 - \|u_M^0\|_{pw}^2)$ and (2.14) to obtain

$$2\|\bar{\partial}_t u_M^{1/2}\|^2 + \frac{1}{2}(\|u_M^1\|_{pw}^2 - \|u_M^0\|_{pw}^2 + \|v_M^1\|_{pw}^2 - \|v_M^0\|_{pw}^2) = (f^{1/2} + 2k^{-1}u_1, u_M^1 - u_M^0)_{L^2(\Omega)}. \quad (4.6)$$

A Cauchy-Schwarz and a Young's inequality with $a = \|f^{1/2} + 2k^{-1}u_1\|$, $b = \|u_M^1 - u_M^0\|$, and $\epsilon = k^2/2$ reveal

$$(f^{1/2} + 2k^{-1}u_1, u_M^1 - u_M^0)_{L^2(\Omega)} \leq \|f^{1/2} + 2k^{-1}u_1\| \|u_M^1 - u_M^0\| \leq \frac{k^2}{2}\|f\|_{C([0,t_1];L^2(\Omega))}^2 + 2\|u_1\|^2 + \|\bar{\partial}_t u_M^{1/2}\|^2 \quad (4.7)$$

with $\|f^{1/2}\| \leq \|f\|_{C([0,t_1];L^2(\Omega))}$ from Lemma 3.4(a) in the last inequality. A combination of (4.6)-(4.7) and the bounds $\|u_M^0\|_{pw} \leq \|J\| \|u_0\|$ and $\|v_M^0\|_{pw} \leq \frac{1}{2}C_{ds}\|J\|^2 \|u_0\|^2$ from (a) shows

$$2\|\bar{\partial}_t u_M^{1/2}\|^2 + \|u_M^1\|_{pw}^2 + \|v_M^1\|_{pw}^2 \leq \|J\|^2 \|u_0\|^2 \left(1 + \frac{1}{4}C_{ds}^2 \|J\|^2 \|u_0\|^2\right) + 4\|u_1\|^2 + k^2\|f\|_{C([0,t_1];L^2(\Omega))}^2. \quad (4.8)$$

Elementary algebra concludes the proof. $\square$

Proof of Theorem 2.1(d). If possible, let (u_M^1, v_M^1) and $(\hat{u}_M^1, \hat{v}_M^1)$ be two solutions to (2.13) and $e_M = u_M^1 - \hat{u}_M^1$, $\bar{e}_M = v_M^1 - \hat{v}_M^1$. For all $(\varphi_M, \psi_M) \in \mathbf{M}(\mathcal{T})$, (2.13) provides

$$\begin{aligned} \frac{2}{k^2}(e_M, \varphi_M)_{L^2(\Omega)} + \frac{1}{2}a_{pw}(e_M, \varphi_M) + \frac{1}{4}b_{pw}(u_M^1 + u_M^0, \varphi_M, v_M^1 + v_M^0) - \frac{1}{4}b_{pw}(\hat{u}_M^1 + u_M^0, \varphi_M, \hat{v}_M^1 + v_M^0) &= 0, \\ a_{pw}(\bar{e}_M, \psi_M) &= \frac{1}{2}b_{pw}(u_M^1, u_M^1, \psi_M) - \frac{1}{2}b_{pw}(\hat{u}_M^1, \hat{u}_M^1, \psi_M). \end{aligned}$$

The choices of test functions $\varphi_M = 4e_M$ and $\psi_M = 2\bar{e}_M$ in the two equations above, respectively, lead to

$$\frac{8}{k^2}\|e_M\|^2 + 2\|e_M\|_{pw}^2 = -b_{pw}(u_M^1 + u_M^0, e_M, v_M^1 + v_M^0) + b_{pw}(\hat{u}_M^1 + u_M^0, e_M, \hat{v}_M^1 + v_M^0), \quad (4.9a)$$

$$2\|\bar{e}_M\|_{pw}^2 = b_{pw}(u_M^1, u_M^1, \psi_M) - b_{pw}(\hat{u}_M^1, \hat{u}_M^1, \psi_M). \quad (4.9b)$$

Add $\pm b_{pw}(\hat{u}_M^1 + u_M^0, e_M, v_M^1 + v_M^0)$ and utilize trilinearity of $b_{pw}(\bullet, \bullet, \bullet)$ to simplify the right-hand side of (4.9a) as

$$-b_{pw}(u_M^1 + u_M^0, e_M, v_M^1 + v_M^0) + b_{pw}(\hat{u}_M^1 + u_M^0, e_M, \hat{v}_M^1 + v_M^0) = -b_{pw}(e_M, e_M, v_M^1 + v_M^0) - b_{pw}(\hat{u}_M^1 + u_M^0, e_M, \bar{e}_M).$$

Add $\pm b_{pw}(\hat{u}_M^1, u_M^1, \psi_M)$ and utilize symmetry of $b_{pw}(\bullet, \bullet, \bullet)$ to rewrite the right-hand side of (4.9b) as

$$b_{pw}(u_M^1, u_M^1, \bar{e}_M) - b_{pw}(\hat{u}_M^1, \hat{u}_M^1, \bar{e}_M) = b_{pw}(e_M, u_M^1, \bar{e}_M) + b_{pw}(\hat{u}_M^1, e_M, \bar{e}_M) = b_{pw}(u_M^1 + \hat{u}_M^1, e_M, \bar{e}_M).$$

A combination of last four displayed equations and boundedness of $b_{pw}(\bullet, \bullet, \bullet)$ from (2.7) imply that

$$\begin{aligned} \frac{8}{k^2}\|e_M\|^2 + 2\|e_M\|_{pw}^2 + 2\|\bar{e}_M\|_{pw}^2 &= -b_{pw}(e_M, e_M, v_M^1 + v_M^0) + b_{pw}(u_M^1 - u_M^0, e_M, \bar{e}_M) \\ &\leq C_{ds}\|u_M^1 + u_M^0\|_{pw}\|e_M\|_{pw}^2 + C_{ds}\|u_M^1 - u_M^0\|_{pw}\|e_M\|_{pw}\|\bar{e}_M\|_{pw}. \end{aligned}$$

Ignore the first non-negative term on the left-hand side and apply Young's inequality with $a = C_{\text{ds}} \|u_M^1 - u_M^0\|_{\text{pw}} \|e_M\|_{\text{pw}}$, $b = \|\bar{e}_M\|_{\text{pw}}$, and $\epsilon = 1$ for the last term on the right-hand side to derive

$$4\|e_M\|_{\text{pw}}^2 + 4\|\bar{e}_M\|_{\text{pw}}^2 \leq C_{\text{ds}} (4\|v_M^{1/2}\|_{\text{pw}} + C_{\text{ds}} \|u_M^1 - u_M^0\|_{\text{pw}}^2) \|e_M\|_{\text{pw}}^2 + \|\bar{e}_M\|_{\text{pw}}^2. \quad (4.10)$$

Choose $\psi_M = v_M^0$ (resp. $\psi_M = v_M^1$) in (2.13b) and utilize (2.5) followed by (2.7) to obtain

$$\|v_M^0\|_{\text{pw}} \leq \frac{1}{2} C_{\text{ds}} \|u_M^0\|_{\text{pw}}^2 \quad (\text{resp. } \|v_M^1\|_{\text{pw}} \leq \frac{1}{2} C_{\text{ds}} \|u_M^1\|_{\text{pw}}^2).$$

This, a triangle inequality, the bounds for $\|u_M^0\|_{\text{pw}}$ from (a), and that of $\|u_M^1\|_{\text{pw}}$ from (4.8) yield

$$\begin{aligned} 4\|v_M^{1/2}\|_{\text{pw}} + C_{\text{ds}} \|u_M^1 - u_M^0\|_{\text{pw}}^2 &\leq 2(\|v_M^0\|_{\text{pw}} + \|v_M^1\|_{\text{pw}}) + 2C_{\text{ds}} (\|u_M^0\|_{\text{pw}}^2 + \|u_M^1\|_{\text{pw}}^2) \\ &\leq 3C_{\text{ds}} (\|u_M^0\|_{\text{pw}}^2 + \|u_M^1\|_{\text{pw}}^2) \leq 3C_{\text{ds}} (\|J\|^2 \|u_0\|^2 + \check{M}_2(u_0, u_1, f, k)) = 3C_{\text{ds}}^{-1} M_2(u_0, u_1, f, k). \end{aligned} \quad (4.11)$$

Utilize this in the right-hand side of (4.10) to obtain

$$4\|e_M\|_{\text{pw}}^2 + 3\|\bar{e}_M\|_{\text{pw}}^2 \leq 3M_2(u_0, u_1, f, k) \|e_M\|_{\text{pw}}^2. \quad (4.12)$$

For sufficiently small u_0 , u_1 , and k , we assume $M_2(u_0, u_1, f, k) \leq 1$ and then (4.12) reveals $\|e_M\|_{\text{pw}}^2 + 3\|\bar{e}_M\|_{\text{pw}}^2 \leq 0$. $\square$

4.2 A priori error analysis for initial discretization

For any $n = 0, 1, 2, \dots, N$, let us split the errors as follows:

$$u^n - u_M^n = (u^n - \mathcal{R}_M u^n) + (\mathcal{R}_M u^n - u_M^n) := \rho^n + \zeta^n, \quad (4.13a)$$

$$v^n - v_M^n = (v^n - \mathcal{R}_M v^n) + (\mathcal{R}_M v^n - v_M^n) := \chi^n + \theta^n. \quad (4.13b)$$

This, the linearity of $\mathcal{R}_M$, and the definitions in (2.8a)-(2.8b) lead to

$$\begin{aligned} \rho^{n+1/2} &= \frac{1}{2}(\rho^{n+1} + \rho^n), \quad \rho^{n,1/4} = \frac{1}{4}(\rho^{n+1} + 2\rho^n + \rho^{n-1}), \\ \bar{\partial}_t \rho^{n+1/2} &= \frac{1}{k}(\rho^{n+1} - \rho^n), \quad \delta_t \rho^n = \frac{1}{2k}(\rho^{n+1} - \rho^{n-1}), \quad \bar{\partial}_t^2 \rho^n = \frac{1}{k^2}(\rho^{n+1} - 2\rho^n + \rho^{n-1}). \end{aligned}$$

Analog notation replace ρ^n by ζ^n , χ^n , and θ^n .

Proof of Theorem 2.3. We follow *five* steps. Firstly, we derive the error equation. Step 2 bounds the linear terms appearing in the error equation. In Step 3, the nonlinear terms are rearranged. Bounds for these terms are derived in Step 4. In Step 5, the truncation term is estimated. Step 6 concludes the proof.

Step 1. Error equation. For any $\varphi_M \in M(\mathcal{T})$, choose the test function in (2.2a) as $J\varphi_M$ and consider the average of (2.2a) at $t = t_0$ and t_1 . Subtract (2.13a) from the resultant expression to arrive at

$$\begin{aligned} (u_t^{1/2}, J\varphi_M)_{L^2(\Omega)} + a(u^{1/2}, J\varphi_M) - 2k^{-1}(\bar{\partial}_t u_M^{1/2} - u_1, \varphi_M)_{L^2(\Omega)} - a_{\text{pw}}(u_M^{1/2}, \varphi_M) \\ = (f^{1/2}, (J - I)\varphi_M)_{L^2(\Omega)} - \frac{1}{2}(b(u^1, J\varphi_M, v^1) + b(u_0, J\varphi_M, v^0)) + b_{\text{pw}}(u_M^{1/2}, \varphi_M, v_M^{1/2}). \end{aligned} \quad (4.14)$$

The linearity of $b(\bullet, \bullet, \bullet)$ in each component and definitions (2.9a)-(2.9b) with elementary algebra lead to

$$b(u^1, J\varphi_M, v^1) + b(u_0, J\varphi_M, v^0) = 2b(u^{1/2}, J\varphi_M, v^{1/2}) + \frac{k^2}{2}b(\bar{\partial}_t u^{1/2}, J\varphi_M, \bar{\partial}_t v^{1/2}).$$

Then, $a(u^{1/2}, J\varphi_M) = a_{\text{pw}}(\mathcal{R}_M u^{1/2}, \varphi_M)$ from (3.1), (4.13a), and elementary manipulations in (4.14) establish

$$\begin{aligned} 2k^{-1}(\bar{\partial}_t \zeta^{1/2}, \varphi_M)_{L^2(\Omega)} + a_{\text{pw}}(\zeta^{1/2}, \varphi_M) &= (f^{1/2} - u_t^{1/2}, (J - I)\varphi_M)_{L^2(\Omega)} - 2k^{-1}(\bar{\partial}_t \rho^{1/2}, \varphi_M)_{L^2(\Omega)} \\ &+ (2k^{-1}(\bar{\partial}_t u^{1/2} - u_1) - u_t^{1/2}, \varphi_M)_{L^2(\Omega)} + b_{\text{pw}}(u_M^{1/2}, \varphi_M, v_M^{1/2}) - b(u^{1/2}, J\varphi_M, v^{1/2}) - \frac{k^2}{4}b(\bar{\partial}_t u^{1/2}, J\varphi_M, \bar{\partial}_t v^{1/2}). \end{aligned}$$

From (2.12) and (3.1), $u_M^0 = \mathcal{R}_M u_0$. This and (4.13a) imply $\zeta^0 = 0$. The choice $\varphi_M = \zeta^{1/2}$ and $2\zeta^{1/2} = \zeta^1 + \zeta^0 = \zeta^1 - \zeta^0 = k\bar{\partial}_t \zeta^{1/2}$ and (2.5) in the above displayed equation reveals

$$\begin{aligned} \|\bar{\partial}_t \zeta^{1/2}\|^2 + \|\zeta^{1/2}\|_{\text{pw}}^2 &= (f^{1/2} - u_t^{1/2}, (J - I)\zeta^{1/2})_{L^2(\Omega)} + (\bar{\partial}_t u^{1/2} - u_1 - \frac{k}{2}u_t^{1/2} - \bar{\partial}_t \rho^{1/2}, \bar{\partial}_t \zeta^{1/2})_{L^2(\Omega)} \\ &\quad + b_{\text{pw}}(u_M^{1/2}, \zeta^{1/2}, v_M^{1/2}) - b(u^{1/2}, J\zeta^{1/2}, v^{1/2}) - \frac{k^2}{4}b(\bar{\partial}_t u^{1/2}, J\zeta^{1/2}, \bar{\partial}_t v^{1/2}). \end{aligned} \quad (4.15)$$

Choose the test function in (2.2b) as $J\psi_M$ for any $\psi_M \in M(\mathcal{T})$ and consider the average of (2.2b) at $t = t_0$ and t_1 to obtain $a(v^{1/2}, J\psi_M) = \frac{1}{4}(b(u^1, u^1, J\psi_M) + b(u_0, u_0, J\psi_M))$. Subtract this from $a_{\text{pw}}(v_M^{1/2}, \psi_M) = \frac{1}{4}(b_{\text{pw}}(u_M^1, u_M^1, \psi_M) + b_{\text{pw}}(u_M^0, u_M^0, \psi_M))$ for all $\psi_M \in M(\mathcal{T})$ (obtained from a combination of (2.12) and (2.13b)) to show

$$\begin{aligned} a(v^{1/2}, J\psi_M) - a_{\text{pw}}(v_M^{1/2}, \psi_M) \\ = \frac{1}{4}(b(u^1, u^1, J\psi_M) + b(u_0, u_0, J\psi_M) - b_{\text{pw}}(u_M^1, u_M^1, \psi_M) - b_{\text{pw}}(u_M^0, u_M^0, \psi_M)). \end{aligned} \quad (4.16)$$

The definition of $\mathcal{R}_M$ once again shows $a(v^{1/2}, J\psi_M) = a_{\text{pw}}(\mathcal{R}_M v^{1/2}, \psi_M)$. The identity $b_{\text{pw}}(\varphi^1, \varphi^1, \chi) + b_{\text{pw}}(\varphi^0, \varphi^0, \chi) = 4b_{\text{pw}}(\varphi^{1/2}, \varphi^{1/2}, \chi) - 2b_{\text{pw}}(\varphi^0, \varphi^1, \chi)$ follows from the trilinearity of $b_{\text{pw}}(\bullet, \bullet, \bullet)$ and its symmetry in first two components. A combination of these steps for $\psi_M = \theta^{1/2}$ in (4.16) with the definitions $\theta^{1/2}$ from (4.13b) and $\|\bullet\|_{\text{pw}}$ from (2.5), eventually shows that

$$\|\theta^{1/2}\|_{\text{pw}}^2 = b(u^{1/2}, u^{1/2}, J\theta^{1/2}) - \frac{1}{2}b(u_0, u^1, J\theta^{1/2}) - b_{\text{pw}}(u_M^{1/2}, u_M^{1/2}, \theta^{1/2}) + \frac{1}{2}b_{\text{pw}}(u_M^0, u_M^1, \theta^{1/2}). \quad (4.17)$$

A combination of (4.15) and (4.17) with a re-grouping of the nonlinear and truncation term, lead to

$$\begin{aligned} \|\bar{\partial}_t \zeta^{1/2}\|^2 + \|\zeta^{1/2}\|_{\text{pw}}^2 + \|\theta^{1/2}\|_{\text{pw}}^2 \\ = \left[(f^{1/2} - u_t^{1/2}, (J - I)\zeta^{1/2})_{L^2(\Omega)} + (\bar{\partial}_t u^{1/2} - u_1 - \frac{k}{2}u_t^{1/2} - \bar{\partial}_t \rho^{1/2}, \bar{\partial}_t \zeta^{1/2})_{L^2(\Omega)} \right] \\ + \left[b_{\text{pw}}(u_M^{1/2}, \zeta^{1/2}, v_M^{1/2}) - b(u^{1/2}, J\zeta^{1/2}, v^{1/2}) \right] + \left[b(u^{1/2}, u^{1/2}, J\theta^{1/2}) - b_{\text{pw}}(u_M^{1/2}, u_M^{1/2}, \theta^{1/2}) \right] \\ + \frac{1}{2} \left[b_{\text{pw}}(u_M^0, u_M^1, \theta^{1/2}) - b(u_0, u^1, J\theta^{1/2}) \right] - \frac{k^2}{4}b(\bar{\partial}_t u^{1/2}, J\zeta^{1/2}, \bar{\partial}_t v^{1/2}). \end{aligned} \quad (4.18)$$

The first term on the right-hand side is linear contributions, the last term is the nonlinear truncation term, and the remaining terms involve the trilinear terms.

Step 2. Bound for linear terms in (4.18). The three inequalities below come from Cauchy-Schwarz inequality (applied in all the three), the definition of $\bar{\partial}_t \rho^{1/2}$ followed by Lemma 3.1(e), Lemma 3.7(a), and Lemma 3.2, respectively:

$$\begin{aligned} (f^{1/2} - u_t^{1/2}, (J - I)\zeta^{1/2})_{L^2(\Omega)} &\leq \|f^{1/2} - u_t^{1/2}\| \|(J - I)\zeta^{1/2}\| \leq C_2 h^2 \|f^{1/2} - u_t^{1/2}\| \|\zeta^{1/2}\|_{\text{pw}}, \\ k(2k^{-1}(\bar{\partial}_t u^{1/2} - u_1) - u_t^{1/2}, \bar{\partial}_t \zeta^{1/2})_{L^2(\Omega)} &\leq k \|2k^{-1}(\bar{\partial}_t u^{1/2} - u_1) - u_t^{1/2}\| \|\bar{\partial}_t \zeta^{1/2}\| \leq k^2 \|u_{ttt}\|_{L^\infty(0, t_1; L^2(\Omega))} \|\bar{\partial}_t \zeta^{1/2}\|, \\ |(\bar{\partial}_t \rho^{1/2}, \bar{\partial}_t \zeta^{1/2})_{L^2(\Omega)}| &\leq \|\bar{\partial}_t \rho^{1/2}\| \|\bar{\partial}_t \zeta^{1/2}\| = \frac{1}{k} \left\| \int_0^{t_1} \rho_t dt \right\| \|\bar{\partial}_t \zeta^{1/2}\| \leq C_3 h^{2\sigma} \|u_t\|_{L^\infty(0, t_1; H^{2+\sigma}(\Omega))} \|\bar{\partial}_t \zeta^{1/2}\|. \end{aligned}$$

We apply Young's inequality to each of the three inequalities above with the following choices: $a = C_2 h^2 \|f^{1/2} - u_t^{1/2}\|$, $b = \|\zeta^{1/2}\|_{\text{pw}}$, $\epsilon = 8$ (for the first term); $a = k \|u_{ttt}\|_{L^\infty(0, t_1; L^2(\Omega))}$, $b = \|\bar{\partial}_t \zeta^{1/2}\|$, $\epsilon = 1$ (for the second term); $a = C_3 h^{2\sigma} \|u_t\|_{L^\infty(0, t_1; H^{2+\sigma}(\Omega))}$, $b = \|\bar{\partial}_t \zeta^{1/2}\|$, $\epsilon = 2$ (for the third term). This leads to

$$\begin{aligned} (f^{1/2} - u_t^{1/2}, (J - I)\zeta^{1/2})_{L^2(\Omega)} + (\bar{\partial}_t u^{1/2} - u_1 - \frac{k}{2}u_t^{1/2} - \bar{\partial}_t \rho^{1/2}, \bar{\partial}_t \zeta^{1/2})_{L^2(\Omega)} \\ \leq C(f, u_t, u_{tt}, u_{ttt})(h^{4\sigma} + k^4) + \frac{1}{16} \|\zeta^{1/2}\|_{\text{pw}}^2 + \frac{1}{2} \|\bar{\partial}_t \zeta^{1/2}\|^2. \end{aligned} \quad (4.19)$$

In the sequel, the notation C denotes a generic term that depends on $f, u, v, u_t, v_t, u_{tt}, v_{tt}, u_{ttt}$ etc. with the dependence pointed out explicitly, but is independent of the spatial and temporal discretization parameters.

Step 3. Rearrangement of nonlinear terms in (4.18). The nonlinear terms on the right-hand side of (4.18) require a clever manipulation to introduce errors that can be controlled. The trilinearity of $b(\bullet, \bullet, \bullet)$ and $b_{pw}(\bullet, \bullet, \bullet)$, their symmetry with respect to the first and second variables, (4.13a)-(4.13b), and elementary algebra leads to

$$\begin{aligned} & b_{pw}(u_M^{1/2}, \zeta^{1/2}, v_M^{1/2}) - b(u^{1/2}, J\zeta^{1/2}, v^{1/2}) = b_{pw}(u^{1/2}, (I - J)\zeta^{1/2}, v^{1/2}) \\ & - b_{pw}(\zeta^{1/2}, \zeta^{1/2}, v_M^{1/2}) - b_{pw}(u^{1/2}, \zeta^{1/2}, \theta^{1/2}) - b_{pw}(\rho^{1/2}, \zeta^{1/2}, v_M^{1/2}) - b_{pw}(u^{1/2}, \zeta^{1/2}, \chi^{1/2}). \end{aligned}$$

Analogous arguments also imply that

$$\begin{aligned} & b(u^{1/2}, u^{1/2}, J\theta^{1/2}) - b_{pw}(u_M^{1/2}, u_M^{1/2}, \theta^{1/2}) = b_{pw}(u^{1/2}, u^{1/2}, (J - I)\theta^{1/2}) + b_{pw}(u^{1/2}, \zeta^{1/2}, \theta^{1/2}) \\ & + b_{pw}(\zeta^{1/2}, u_M^{1/2}, \theta^{1/2}) + b_{pw}(u^{1/2}, \rho^{1/2}, \theta^{1/2}) + b_{pw}(\rho^{1/2}, u_M^{1/2}, \theta^{1/2}), \\ & \frac{1}{2} [b_{pw}(u_M^0, u_M^1, \theta^{1/2}) - b(u_0, u^1, J\theta^{1/2})] = -\frac{1}{2} b_{pw}(u_0, u^1, (J - I)\theta^{1/2}) - b_{pw}(u_M^0, \zeta^{1/2}, \theta^{1/2}) \\ & - \frac{1}{2} b_{pw}(u_M^0, \rho^1, \theta^{1/2}) - \frac{1}{2} b_{pw}(\rho^0, u^1, \theta^{1/2}) \end{aligned}$$

with $u_M^0 = \mathcal{R}_M u_0$ from (2.12) (which shows $\zeta^0 = 0$ and $\zeta^1 = 2\zeta^{1/2}$ from (4.13a)) in the last identity. Since $b_{pw}(\zeta^{1/2}, u_M^{1/2}, \theta^{1/2}) = b_{pw}(u_M^{1/2}, \zeta^{1/2}, \theta^{1/2})$, two terms appearing on the right-hand side of the above two identities can be combined as $b_{pw}(\zeta^{1/2}, u_M^{1/2}, \theta^{1/2}) - b_{pw}(u_M^0, \zeta^{1/2}, \theta^{1/2}) = \frac{1}{2} b_{pw}(u_M^1 - u_M^0, \zeta^{1/2}, \theta^{1/2})$. Since $b_{pw}(u^{1/2}, \zeta^{1/2}, \theta^{1/2})$ cancels out, the nonlinear terms on the right-hand side of (4.18) rearrange as

$$\begin{aligned} & -b_{pw}(\zeta^{1/2}, \zeta^{1/2}, v_M^{1/2}) + \frac{1}{2} b_{pw}(u_M^1 - u_M^0, \zeta^{1/2}, \theta^{1/2}) + b_{pw}(u^{1/2}, (I - J)\zeta^{1/2}, v^{1/2}) + b_{pw}(u^{1/2}, u^{1/2}, (J - I)\theta^{1/2}) \\ & - \frac{1}{2} b_{pw}(u_0, u^1, (J - I)\theta^{1/2}) - b_{pw}(\rho^{1/2}, \zeta^{1/2}, v_M^{1/2}) - b_{pw}(u^{1/2}, \zeta^{1/2}, \chi^{1/2}) + b_{pw}(u^{1/2}, \rho^{1/2}, \theta^{1/2}) \\ & + b(\rho^{1/2}, u_M^{1/2}, \theta^{1/2}) - \frac{1}{2} b_{pw}(u_M^0, \rho^1, \theta^{1/2}) - \frac{1}{2} b(\rho^0, u^1, \theta^{1/2}) := T_1 + T_2 + \dots + T_{11}. \end{aligned} \quad (4.20)$$

Step 4. Bound for nonlinear terms in (4.20). Applying (2.7) (twice) shows $T_1 + T_2 := -b_{pw}(\zeta^{1/2}, \zeta^{1/2}, v_M^{1/2}) + \frac{1}{2} b_{pw}(u_M^1 - u_M^0, \zeta^{1/2}, \theta^{1/2})$ can be bounded viz.

$$\begin{aligned} T_1 + T_2 & \leq |b_{pw}(\zeta^{1/2}, \zeta^{1/2}, v_M^{1/2})| + \frac{1}{2} |b_{pw}(u_M^1 - u_M^0, \zeta^{1/2}, \theta^{1/2})| \\ & \leq C_{ds} \|\zeta^{1/2}\|_{pw}^2 \|v_M^{1/2}\|_{pw} + \frac{1}{2} C_{ds} \|u_M^1 - u_M^0\|_{pw} \|\zeta^{1/2}\|_{pw} \|\theta^{1/2}\|_{pw} \\ & \leq C_{ds} (\|v_M^{1/2}\|_{pw} + \frac{1}{4} C_{ds} \|u_M^1 - u_M^0\|_{pw}^2) \|\zeta^{1/2}\|_{pw}^2 + \frac{1}{4} \|\theta^{1/2}\|_{pw}^2 \end{aligned}$$

with the Young's inequality (taking $a = \frac{1}{2} C_{ds} \|u_M^1 - u_M^0\|_{pw} \|\zeta^{1/2}\|_{pw}$, $b = \|\theta^{1/2}\|_{pw}$, $\epsilon = 2$) for the second term in the penultimate step. This, (4.11) and the smallness assumption $M_2(u_0, u_1, f, k) \leq 1$ from Theorem 2.1(d) lead to

$$T_1 + T_2 \leq \frac{3}{4} M_2(u_0, u_1, f, k) \|\zeta^{1/2}\|_{pw}^2 + \frac{1}{4} \|\theta^{1/2}\|_{pw}^2 \leq \frac{3}{4} \|\zeta^{1/2}\|_{pw}^2 + \frac{1}{4} \|\theta^{1/2}\|_{pw}^2.$$

Lemma 3.3(e), Lemma 3.4(a), and Young's inequality (skipping details for brevity) lead to

$$T_3 := b_{pw}(u^{1/2}, (I - J)\zeta^{1/2}, v^{1/2}) \leq C_8 h^\sigma \|u\|_{C([0, t_1]; H^{2+\sigma}(\Omega))} \|v\|_{C([0, t_1]; H^2(\Omega))} \|\zeta^{1/2}\|_{pw} \leq C(u, v) h^{2\sigma} + \frac{1}{64} \|\zeta^{1/2}\|_{pw}^2.$$

Analogous arguments with Lemma 3.3(d) reveal

$$\begin{aligned} T_4 + T_5 & := b_{pw}(u^{1/2}, u^{1/2}, (J - I)\theta^{1/2}) - \frac{1}{2} b_{pw}(u_0, u^1, (I - J)\theta^{1/2}) \\ & \leq |b_{pw}(u^{1/2}, u^{1/2}, (J - I)\theta^{1/2})| + \frac{1}{2} |b_{pw}(u_0, u^1, (I - J)\theta^{1/2})| \\ & \leq C_7 h (\|u\|_{C([0, t_1]; H^2(\Omega))}^2 + \frac{1}{2} \|u_0\|_{H^2(\Omega)} \|u\|_{C([0, t_1]; H^2(\Omega))}) \|\theta^{1/2}\|_{pw} \leq C(u) h^2 + \frac{1}{8} \|\theta^{1/2}\|_{pw}^2. \end{aligned}$$

Next, we utilize (2.7), Lemma 3.5(a) followed by Lemma 3.4(a) to obtain

$$\begin{aligned} T_6 + T_7 &:= -b_{\text{pw}}(\rho^{1/2}, \zeta^{1/2}, v_M^{1/2}) - b_{\text{pw}}(u^{1/2}, \zeta^{1/2}, \chi^{1/2}) \leq |b_{\text{pw}}(\rho^{1/2}, \zeta^{1/2}, v_M^{1/2})| + |b_{\text{pw}}(u^{1/2}, \zeta^{1/2}, \chi^{1/2})| \\ &\leq C_{\text{dS}} C_3 h^\sigma (\|u\|_{C([0,t_1]; H^{2+\sigma}(\Omega))} \|v_M^{1/2}\|_{\text{pw}} + \|u\|_{C([0,t_1]; H^2(\Omega))} \|v\|_{C([0,t_1]; H^{2+\sigma}(\Omega))}) \|\zeta^{1/2}\|_{\text{pw}} \leq C(u) h^{2\sigma} + \frac{1}{32} \|\zeta^{1/2}\|_{\text{pw}}^2 \end{aligned}$$

with Young's inequality, the bound $\|v_M^{1/2}\|_{\text{pw}} \leq \frac{3}{4} M_2(u_0, u_1, f, k) \leq \frac{3}{4}$ from (4.11), and the smallness assumption from Theorem 2.1(d) in the last step. Similarly, we infer

$$\begin{aligned} T_8 + T_9 + T_{10} + T_{11} &\leq |b_{\text{pw}}(u^{1/2}, \rho^{1/2}, \theta^{1/2})| + |b(\rho^{1/2}, u_M^{1/2}, \theta^{1/2})| + |\frac{1}{2} b_{\text{pw}}(u_M^0, \rho^1, \theta^{1/2})| + |\frac{1}{2} b(\rho^0, u^1, \theta^{1/2})| \\ &\leq C_{\text{dS}} C_3 \|u\|_{C([0,t_1]; H^{2+\sigma}(\Omega))} h^\sigma (\|u\|_{C([0,t_1]; H^2(\Omega))} + \|u_M^{1/2}\|_{\text{pw}} + \frac{1}{2} \|u_M^0\|_{\text{pw}} + \frac{1}{2} \|u\|_{C([0,t_1]; H^2(\Omega))}) \|\theta^{1/2}\|_{\text{pw}} \\ &\leq C(u) h^{2\sigma} + \frac{1}{2} \|\theta^{1/2}\|_{\text{pw}}^2, \end{aligned}$$

where we have utilized $\|u_M^{1/2}\|_{\text{pw}} \leq \frac{1}{2} (\|u_M^0\|_{\text{pw}} + \|u_M^1\|_{\text{pw}})$ and the bound from (4.11) in the last step.

Step 5. Control of the truncation term. Apply Lemma 3.3(a) and Lemma 3.4(c) to observe

$$\frac{k^2}{4} |b(\bar{\partial}_t u^{1/2}, J \zeta^{1/2}, \bar{\partial}_t v^{1/2})| \leq C_4 \frac{k^2}{4} \|u_t\|_{L^\infty(0,t_1; H^2(\Omega))} \|v_t\|_{L^\infty(0,t_1; H^2(\Omega))} \|\zeta^{1/2}\|_{\text{pw}} \leq C(u_t, v_t) k^4 + \frac{1}{64} \|\zeta^{1/2}\|_{\text{pw}}^2$$

with Young's inequality ($a = \frac{1}{8} C_4 k^2 \|u_t\|_{L^\infty(0,t_1; H^2(\Omega))} \|v_t\|_{L^\infty(0,t_1; H^2(\Omega))}$, $b = \|\zeta^{1/2}\|_{\text{pw}}$, and $\epsilon = 32$) in the last step.

Step 6. Consolidation. A Combination of (4.19), the bounds of the terms in (4.20) from Step 4, Step 5, and (4.18) lead to

$$\frac{1}{2} \|\bar{\partial}_t \zeta^{1/2}\|^2 + \frac{1}{8} \|\zeta^{1/2}\|_{\text{pw}}^2 + \frac{1}{8} \|\theta^{1/2}\|_{\text{pw}}^2 \leq C(u_0, f, u, v, u_t, v_t, u_{tt}, u_{ttt})(h^{2\sigma} + k^4). \quad (4.21)$$

This and triangle inequalities show $\|\bar{\partial}_t(u^{1/2} - u_M^{1/2})\| + \|u^{1/2} - u_M^{1/2}\|_{\text{pw}} + \|v^{1/2} - v_M^{1/2}\|_{\text{pw}} \leq \|\bar{\partial}_t \rho^{1/2}\| + \|\rho^{1/2}\|_{\text{pw}} + \|\chi^{1/2}\|_{\text{pw}} + \|\bar{\partial}_t \zeta^{1/2}\| + \|\zeta^{1/2}\|_{\text{pw}} + \|\theta^{1/2}\|_{\text{pw}}$. Using Lemma 3.2 for the first three terms and (4.21) to the remaining terms, we conclude the proof. $\square$

5 Proofs of results for fully discrete scheme

This section presents the proofs of the well-posedness result in Theorem 2.4 and the a priori error estimates in Theorem 2.5 for the fully discrete scheme (2.10). Subsection 5.1 establishes the existence, uniqueness, and stability of the discrete solution (u_M^{n+1}, v_M^{n+1}) via an inductive argument, where the base case is established in Theorem 2.1. Subsection 5.2 derives the optimal error estimates in the energy norm, achieving optimal convergence in space and quadratic convergence in time, building on the error equation, approximation properties of the Ritz projection, and the truncation error bounds from Lemma 3.7.

5.1 Well-posedness of fully discrete scheme

This subsection presents the proof of Theorem 2.4. We begin by proving Theorem 2.4(a), which establishes that any solution (u_M^{n+1}, v_M^{n+1}) to (2.10), if it exists, satisfies an a priori bound via a discrete energy estimate, with the nonlinear identity (2.11) as the key ingredient. This result reveals that the scheme (2.10) preserves the energy balance property at a discrete level and ensures its unconditional stability.

Proof of Theorem 2.4(a). We establish the a priori stability bound (5.6) for the fully discrete scheme (2.10) via a discrete energy argument, with the nonlinear identity (2.11) and a discrete Gronwall lemma as the key ingredients.

Choose $\varphi_M = u_M^{n+1} - u_M^{n-1} = k(\bar{\partial}_t u_M^{n+1/2} + \bar{\partial}_t u_M^{n-1/2}) = 2(u_M^{n+1/2} - u_M^{n-1/2}) = 2k\delta_t u_M^n$ (from (2.9b)) in (2.10a) to obtain

$$\begin{aligned} &k(\bar{\partial}_t^2 u_M^n, \bar{\partial}_t u_M^{n+1/2} + \bar{\partial}_t u_M^{n-1/2})_{L^2(\Omega)} + 2a_{\text{pw}}(u_M^{n,1/4}, u_M^{n+1/2} - u_M^{n-1/2}) \\ &+ 2kb_{\text{pw}}(u_M^{n,1/4}, \delta_t u_M^n, v_M^{n,1/4}) = k(f^{n,1/4}, \bar{\partial}_t u_M^{n+1/2} + \bar{\partial}_t u_M^{n-1/2})_{L^2(\Omega)}. \end{aligned} \quad (5.1)$$

Utilize $\bar{\partial}_t^2 u_M^n = k^{-1}(\bar{\partial}_t u_M^{n+1/2} - \bar{\partial}_t u_M^{n-1/2})$ from (2.8b) and $u_M^{n,1/4} = \frac{1}{2}(u_M^{n+1/2} + u_M^{n-1/2})$ from (2.9a), linearity, and symmetry of $(\bullet, \bullet)_{L^2(\Omega)}$ and $a_{pw}(\bullet, \bullet)$ to show

$$\begin{aligned} k(\bar{\partial}_t^2 u_M^n, \bar{\partial}_t u_M^{n+1/2} + \bar{\partial}_t u_M^{n-1/2})_{L^2(\Omega)} &= (\bar{\partial}_t u_M^{n+1/2} - \bar{\partial}_t u_M^{n-1/2}, \bar{\partial}_t u_M^{n+1/2} + \bar{\partial}_t u_M^{n-1/2})_{L^2(\Omega)} = \|\bar{\partial}_t u_M^{n+1/2}\|^2 - \|\bar{\partial}_t u_M^{n-1/2}\|^2, \\ 2a_{pw}(u_M^{n,1/4}, u_M^{n+1/2} - u_M^{n-1/2}) &= a_{pw}(u_M^{n+1/2} + u_M^{n-1/2}, u_M^{n+1/2} - u_M^{n-1/2}) = \|u_M^{n+1/2}\|_{pw}^2 - \|u_M^{n-1/2}\|_{pw}^2. \end{aligned}$$

A combination of these identities and (2.11) in (5.1) leads to

$$\begin{aligned} &\|\bar{\partial}_t u_M^{n+1/2}\|^2 - \|\bar{\partial}_t u_M^{n-1/2}\|^2 + \|u_M^{n+1/2}\|_{pw}^2 - \|u_M^{n-1/2}\|_{pw}^2 \\ &+ \|v_M^{n+1/2}\|_{pw}^2 - \|v_M^{n-1/2}\|_{pw}^2 = k(f^{n,1/4}, \bar{\partial}_t u_M^{n+1/2} + \bar{\partial}_t u_M^{n-1/2})_{L^2(\Omega)}. \end{aligned}$$

We sum over $n = 1, 2, \dots, m$ and utilize the telescopic behavior of the left-hand side terms to derive

$$\begin{aligned} \|\bar{\partial}_t u_M^{m+1/2}\|^2 + \|u_M^{m+1/2}\|_{pw}^2 + \|v_M^{m+1/2}\|_{pw}^2 &= \|\bar{\partial}_t u_M^{1/2}\|^2 + \|u_M^{1/2}\|_{pw}^2 + \|v_M^{1/2}\|_{pw}^2 \\ &+ k \sum_{n=1}^m (f^{n,1/4}, \bar{\partial}_t u_M^{n+1/2} + \bar{\partial}_t u_M^{n-1/2})_{L^2(\Omega)} \text{ for all } 1 \leq m \leq N-1. \end{aligned} \quad (5.2)$$

An application of Cauchy-Schwarz and Young's inequalities with $a = \|f^{n,1/4}\|$, $b = \|\bar{\partial}_t u_M^{n+1/2} + \bar{\partial}_t u_M^{n-1/2}\|$, $\epsilon = 2T$ followed by a summation from $n = 1$ to m , and $\|\bar{\partial}_t u_M^{n+1/2} + \bar{\partial}_t u_M^{n-1/2}\|^2 \leq 2\|\bar{\partial}_t u_M^{n+1/2}\|^2 + 2\|\bar{\partial}_t u_M^{n-1/2}\|^2$ from triangle inequality lead to

$$k \sum_{n=1}^m (f^{n,1/4}, \bar{\partial}_t u_M^{n+1/2} + \bar{\partial}_t u_M^{n-1/2})_{L^2(\Omega)} \leq Tk \sum_{n=1}^m \|f^{n,1/4}\|^2 + \frac{1}{2} \|\bar{\partial}_t u_M^{m+1/2}\|^2 + \frac{k}{T} \sum_{n=0}^m \|\bar{\partial}_t u_M^{n+1/2}\|^2 \quad (5.3)$$

with $\frac{k}{T} \leq 1$ in the second term on the right-hand side of (5.3). A combination of (5.2)-(5.3) reads

$$\begin{aligned} &\frac{1}{2} \|\bar{\partial}_t u_M^{m+1/2}\|^2 + \|u_M^{m+1/2}\|_{pw}^2 + \|v_M^{m+1/2}\|_{pw}^2 \\ &\leq \|\bar{\partial}_t u_M^{1/2}\|^2 + \|u_M^{1/2}\|_{pw}^2 + \|v_M^{1/2}\|_{pw}^2 + Tk \sum_{n=1}^m \|f^{n,1/4}\|^2 + \mu \sum_{n=0}^{m-1} \frac{1}{2} \|\bar{\partial}_t u_M^{n+1/2}\|^2 \end{aligned}$$

with $\mu = 2\frac{k}{T}$. Lemma 3.6 with $a_m = \frac{1}{2} \|\bar{\partial}_t u_M^{m+1/2}\|^2$, $b_m = \|u_M^{m+1/2}\|_{pw}^2 + \|v_M^{m+1/2}\|_{pw}^2$, and $c_m = \|\bar{\partial}_t u_M^{1/2}\|^2 + \|u_M^{1/2}\|_{pw}^2 + \|v_M^{1/2}\|_{pw}^2$ (resp. $c_m = \|\bar{\partial}_t u_M^{1/2}\|^2 + \|u_M^{1/2}\|_{pw}^2 + \|v_M^{1/2}\|_{pw}^2 + Tk \sum_{n=1}^m \|f^{n,1/4}\|^2$) if $m = 0$ (resp. if $m \geq 1$), leads to

$$\frac{1}{2} \|\bar{\partial}_t u_M^{m+1/2}\|^2 + \|u_M^{m+1/2}\|_{pw}^2 + \|v_M^{m+1/2}\|_{pw}^2 \leq e^2 (\|\bar{\partial}_t u_M^{1/2}\|^2 + \|u_M^{1/2}\|_{pw}^2 + \|v_M^{1/2}\|_{pw}^2 + Tk \sum_{n=1}^m \|f^{n,1/4}\|^2) \quad (5.4)$$

with $e^{m\mu} = e^{2m\frac{k}{T}} \leq e^2$ from $mk \leq T$ in the last step. Triangle inequalities and (4.8) result in

$$\begin{aligned} \|\bar{\partial}_t u_M^{1/2}\|^2 + \|u_M^{1/2}\|_{pw}^2 + \|v_M^{1/2}\|_{pw}^2 &\leq \|\bar{\partial}_t u_M^{1/2}\|^2 + \frac{1}{2} (\|u_M^1\|_{pw}^2 + \|v_M^1\|_{pw}^2 + \|u_M^0\|_{pw}^2 + \|v_M^0\|_{pw}^2) \\ &\leq \|J\|^2 \|u_0\|^2 (1 + \frac{1}{4} C_{ds}^2 \|J\|^2 \|u_0\|^2) + 2\|u_1\|^2 + \frac{k^2}{2} \|f\|_{C([0,t_1];L^2(\Omega))}^2 \end{aligned} \quad (5.5)$$

with the bounds for $\|u_M^0\|_{pw}$ and $\|v_M^0\|_{pw}$ from Theorem 2.1(a) in the last step. Then (5.5), (5.4), and elementary algebra lead to

$$\frac{1}{2} \|\bar{\partial}_t u_M^{m+1/2}\|^2 + \|u_M^{m+1/2}\|_{pw}^2 + \|v_M^{m+1/2}\|_{pw}^2 \leq \check{M}_3(u_0, u_1, f, k). \quad (5.6)$$

Some elementary manipulations conclude the proof. $\square$

We next prove Theorem 2.4(b), which establishes the existence of a solution (u_M^{n+1}, v_M^{n+1}) to the nonlinear coupled system (2.10) at step t_{n+1} . The proof proceeds by induction, assuming the existence of solutions at levels t_n and t_{n-1} , where the base case is furnished by Theorem 2.1. The existence is established via a corollary of Brouwer's fixed-point theorem (Theorem 4.1), utilizing the a priori bounds from part (a).

Proof of Theorem 2.4(b). The existence follows as in Theorem 2.1(b) and hence we highlight the choices of the inner product and the mapping, which differ because the initial discretization and fully discrete schemes are different.

Choose $\mathbf{H} = \mathbf{M}(\mathcal{T})$ and for all $\Psi_{\mathbf{M}} = (\psi_{M1}, \psi_{M2})$, $\Phi_{\mathbf{M}} = (\varphi_{M1}, \varphi_{M2}) \in \mathbf{M}(\mathcal{T})$, consider the inner product in $\mathbf{H}$ defined as $(\Psi_{\mathbf{M}}, \Phi_{\mathbf{M}})_{\text{pw}} := (\psi_{M1}, \varphi_{M1})_{L^2(\Omega)} + \frac{k^2}{4} (a_{\text{pw}}(\psi_{M1}, \varphi_{M1}) + a_{\text{pw}}(\psi_{M2}, \varphi_{M2}))$. Given $\Psi_{\mathbf{M}} = (\psi_{M1}, \psi_{M2}) \in \mathbf{M}(\mathcal{T})$ and motivated by (2.10), we define $\mathcal{G}_{\Psi_{\mathbf{M}}} : \mathbf{M}(\mathcal{T}) \rightarrow \mathbb{R}$ by

$$\begin{aligned} \mathcal{G}_{\Psi_{\mathbf{M}}}(\Phi_{\mathbf{M}}) &= (\Psi_{\mathbf{M}}, \Phi_{\mathbf{M}})_{\text{pw}} + \frac{k^2}{16} [b_{\text{pw}}(\psi_{M1} + 4u_M^{n-1/2}, \varphi_{M1}, \psi_{M2}) - b_{\text{pw}}(\psi_{M1} + 2u_M^{n-1/2}, \psi_{M1} + 2u_M^{n-1/2}, \varphi_{M2})] \\ &\quad - k^2 [(f^{n,1/4} + 2k^{-1} \bar{\partial}_t u_M^{n-1/2}, \varphi_{M1})_{L^2(\Omega)} - a_{\text{pw}}(u_M^{n-1/2}, \varphi_{M1}) + \frac{1}{2} a_{\text{pw}}(v_M^{n-1/2}, \varphi_{M2})]. \end{aligned} \quad (5.7)$$

Note that $\mathcal{G}_{\Psi_{\mathbf{M}}}(\bullet)$ is linear and bounded with respect to $\|\bullet\|_{\text{pw}}$ and hence by the Riesz representation theorem, we have that $\mathcal{L} : \mathbf{M}(\mathcal{T}) \rightarrow \mathbf{M}(\mathcal{T})$ defined by $(\mathcal{L}(\Psi_{\mathbf{M}}), \Phi_{\mathbf{M}})_{\text{pw}} = \mathcal{G}_{\Psi_{\mathbf{M}}}(\Phi_{\mathbf{M}})$ is well defined and continuous. To apply Theorem 4.1 it remains to prove that there exists some $R > 0$ such that $(\mathcal{L}(\Psi_{\mathbf{M}}), \Psi_{\mathbf{M}})_{\text{pw}} \geq 0$, for all $\Psi_{\mathbf{M}} \in \mathbf{M}(\mathcal{T})$ with $\|\Psi_{\mathbf{M}}\|_{\text{pw}} = R$. For this, we set

$$R^2 := 2k [k^2 \|f\|_{C([0,T];L^2(\Omega))}^2 + 4 \|\bar{\partial}_t u_M^{n-1/2}\|^2 + 2 \|u_M^{n-1/2}\|_{\text{pw}}^2 + 2 \|v_M^{n-1/2}\|_{\text{pw}}^2], \quad (5.8)$$

and choose $\Psi_{\mathbf{M}} \in \mathbf{M}(\mathcal{T})$ such that $\|\Psi_{\mathbf{M}}\|_{\text{pw}} = R$. Selecting $\Phi_{\mathbf{M}} = \Psi_{\mathbf{M}}$ in (5.7) we observe that the linearity, symmetry of $b_{\text{pw}}(\bullet, \bullet)$ in first two components, and (2.10)(b) implies $b_{\text{pw}}(\psi_{M1} + 4u_M^{n-1/2}, \psi_{M1}, \psi_{M2}) - b_{\text{pw}}(\psi_{M1} + 2u_M^{n-1/2}, \psi_{M1} + 2u_M^{n-1/2}, \psi_{M2}) = -4b_{\text{pw}}(u_M^{n-1/2}, u_M^{n-1/2}, \psi_{M2}) = -8a_{\text{pw}}(v_M^{n-1/2}, \psi_{M2})$. Hence

$$\begin{aligned} (\mathcal{L}(\Psi_{\mathbf{M}}), \Psi_{\mathbf{M}})_{\text{pw}} &= \mathcal{G}_{\Psi_{\mathbf{M}}}(\Psi_{\mathbf{M}}) \\ &= \|\Psi_{\mathbf{M}}\|_{\text{pw}}^2 - k^2 [(f^{n,1/4} + 2k^{-1} \bar{\partial}_t u_M^{n-1/2}, \varphi_{M1})_{L^2(\Omega)} - a_{\text{pw}}(u_M^{n-1/2}, \psi_{M1}) + a_{\text{pw}}(v_M^{n-1/2}, \psi_{M2})]. \end{aligned} \quad (5.9)$$

Cauchy-Schwarz and Young's inequalities with $a = k^2 \|f^{n,1/4} + 2k^{-1} \bar{\partial}_t u_M^{n-1/2}\|$, $b = \|\psi_{M1}\|$, $\epsilon = k^2$, and $\|f^{n,1/4}\| \leq \|f\|_{C([0,T];L^2(\Omega))}$ from Lemma 3.4(b) reveal

$$\begin{aligned} k^2 (f^{n,1/4} + 2k^{-1} \bar{\partial}_t u_M^{n-1/2}, \psi_{M1})_{L^2(\Omega)} &\leq k^2 \|f^{n,1/4} + 2k^{-1} \bar{\partial}_t u_M^{n-1/2}\| \|\psi_{M1}\| \\ &\leq \frac{k^4}{2} \|f^{n,1/4} + 2k^{-1} \bar{\partial}_t u_M^{n-1/2}\|^2 + \frac{1}{2} \|\psi_{M1}\|^2 \leq k^4 \|f\|_{C([0,T];L^2(\Omega))}^2 + 4k^2 \|\bar{\partial}_t u_M^{n-1/2}\|^2 + \frac{1}{2} \|\psi_{M1}\|^2. \end{aligned}$$

The continuity of the bilinear form $a_{\text{pw}}(\bullet, \bullet)$ and Young's inequality (applied twice) gives

$$|a_{\text{pw}}(u_M^{n-1/2}, \psi_{M1})| + |a_{\text{pw}}(v_M^{n-1/2}, \psi_{M2})| \leq 2(\|u_M^{n-1/2}\|_{\text{pw}}^2 + \|v_M^{n-1/2}\|_{\text{pw}}^2) + \frac{1}{8} (\|\psi_{M1}\|_{\text{pw}}^2 + \|\psi_{M2}\|_{\text{pw}}^2).$$

Combine the last two displayed inequalities with $\|\Psi_{\mathbf{M}}\|_{\text{pw}}^2 = \|\psi_{M1}\|^2 + \frac{k^2}{4} (\|\psi_{M1}\|_{\text{pw}}^2 + \|\psi_{M2}\|_{\text{pw}}^2) = R^2$ and utilize (5.8) to obtain

$$\begin{aligned} k^2 [(f^{n,1/4} + 2k^{-1} \bar{\partial}_t u_M^{n-1/2}, \varphi_{M1})_{L^2(\Omega)} - a_{\text{pw}}(u_M^{n-1/2}, \psi_{M1}) - a_{\text{pw}}(v_M^{n-1/2}, \psi_{M2})] \\ \leq \frac{1}{2} R^2 + k^2 (k^2 \|f\|_{C([0,T];L^2(\Omega))}^2 + 4 \|\bar{\partial}_t u_M^{n-1/2}\|^2 + 2 \|u_M^{n-1/2}\|_{\text{pw}}^2 + 2 \|v_M^{n-1/2}\|_{\text{pw}}^2) = \frac{3}{4} R^2. \end{aligned}$$

This and (5.9) along with the fact that $\|\Psi_{\mathbf{M}}\|_{\text{pw}} = R$ leads to $(\mathcal{L}(\Psi_{\mathbf{M}}), \Psi_{\mathbf{M}})_{\text{pw}} \geq \frac{R^2}{4} > 0$. Theorem 4.1 shows that there exists $\Psi_{\mathbf{M}}^* = (\psi_{M1}^*, \psi_{M2}^*) \in \mathbf{M}(\mathcal{T})$ such that $\|\Psi_{\mathbf{M}}^*\|_{\text{pw}} \leq R$ and $\mathcal{L}(\Psi_{\mathbf{M}}^*) = 0$. Define $(u_M^{n+1}, v_M^{n+1}) = (\psi_{M1}^* + u_M^{n-1}, \psi_{M2}^* - 2v_M^{n-1})$. This, (5.7), and the definition of $\mathcal{L}$ show that (u_M^{n+1}, v_M^{n+1}) solves the fully-discrete scheme (2.10). $\square$

We conclude this subsection with the proof of Theorem 2.4(c), which establishes the uniqueness of the solution (u_M^{n+1}, v_M^{n+1}) to (2.10) at step t_{n+1} . The proof again proceeds by induction, assuming uniqueness at levels t_n and t_{n-1} , and employs the a priori bounds from part (a) and the smallness assumption of the given data.

Proof of Theorem 2.4(c). If possible, let (u_M^{n+1}, v_M^{n+1}) and $(\hat{u}_M^{n+1}, \hat{v}_M^{n+1})$ be two solutions of (2.10) at t_{n+1} given that (u_M^n, v_M^n) (resp. (u_M^{n-1}, v_M^{n-1})) are unique solutions at t_n (resp. t_{n-1}). Define $e_M := u_M^{n+1} - \hat{u}_M^{n+1}$ and $\bar{e}_M := v_M^{n+1} - \hat{v}_M^{n+1}$. With the definitions from (2.8a) and (2.8b), (2.10) implies

$$\frac{1}{k^2} (e_M, \varphi_M)_{L^2(\Omega)} + \frac{1}{4} a_{\text{pw}}(e_M, \varphi_M) + \frac{1}{16} b_{\text{pw}}(u_M^{n+1} + 2u_M^n + u_M^{n-1}, \varphi_M, v_M^{n+1} + 2v_M^n + v_M^{n-1})$$

$$\begin{aligned}
& -\frac{1}{16}b_{\text{pw}}(\hat{u}_M^{n+1} + 2u_M^n + u_M^{n-1}, \varphi_M, \hat{v}_M^{n+1} + 2v_M^n + v_M^{n-1}) = 0 \text{ for all } \varphi_M \in \mathcal{M}(\mathcal{T}), \\
a_{\text{pw}}(\bar{e}_M, \psi_M) &= \frac{1}{4}b_{\text{pw}}(u_M^{n+1} + u_M^n, u_M^{n+1} + u_M^n, \psi_M) - \frac{1}{4}b_{\text{pw}}(\hat{u}_M^{n+1} + u_M^n, \hat{u}_M^{n+1} + u_M^n, \psi_M) \text{ for all } \psi_M \in \mathcal{M}(\mathcal{T}).
\end{aligned}$$

The test functions $\varphi_M = 16e_M$ and $\psi_M = 4\bar{e}_M$ in the above system of equation lead to

$$\begin{aligned}
16k^{-2}\|e_M\|^2 + 4\|e_M\|_{\text{pw}}^2 &= -b_{\text{pw}}(u_M^{n+1} + 2u_M^n + u_M^{n-1}, e_M, v_M^{n+1} + 2v_M^n + v_M^{n-1}) \\
&\quad + b_{\text{pw}}(\hat{u}_M^{n+1} + 2u_M^n + u_M^{n-1}, e_M, \hat{v}_M^{n+1} + 2v_M^n + v_M^{n-1}), \tag{5.10a}
\end{aligned}$$

$$4\|\bar{e}_M\|_{\text{pw}}^2 = b_{\text{pw}}(u_M^{n+1} + u_M^n, u_M^{n+1} + u_M^n, \bar{e}_M) - b_{\text{pw}}(\hat{u}_M^{n+1} + u_M^n, \hat{u}_M^{n+1} + u_M^n, \bar{e}_M). \tag{5.10b}$$

The linearity of $b_{\text{pw}}(\bullet, \bullet, \bullet)$ in each component and its symmetry in the first two components reveal that the terms on the right-hand sides of (5.10a) and (5.10b) can be combined as $2b_{\text{pw}}(u_M^{n+1/2} - u_M^{n-1/2}, e_M, \bar{e}_M) - 4b_{\text{pw}}(e_M, e_M, v_M^{n,1/4})$. This, an addition of (5.10a) and (5.10b), and the boundedness of $b_{\text{pw}}(\bullet, \bullet, \bullet)$ from (2.7) leads to

$$\begin{aligned}
8k^{-2}\|e_M\|^2 + 2\|e_M\|_{\text{pw}}^2 + 2\|\bar{e}_M\|_{\text{pw}}^2 &= b_{\text{pw}}(u_M^{n+1/2} - u_M^{n-1/2}, e_M, \bar{e}_M) - 2b_{\text{pw}}(e_M, e_M, v_M^{n,1/4}) \\
&\leq C_{\text{ds}}(\|u_M^{n+1/2} - u_M^{n-1/2}\|_{\text{pw}}\|e_M\|_{\text{pw}}\|\bar{e}_M\|_{\text{pw}} + \|v_M^{n+1/2} + v_M^{n-1/2}\|_{\text{pw}}\|e_M\|_{\text{pw}}^2).
\end{aligned}$$

A Young's inequality with $a = \|u_M^{n+1/2} - u_M^{n-1/2}\|_{\text{pw}}\|e_M\|_{\text{pw}}$, $b = \|\bar{e}_M\|_{\text{pw}}$, and $\epsilon = C_{\text{ds}}/2$, and a triangle inequality imply

$$\|u_M^{n+1/2} - u_M^{n-1/2}\|_{\text{pw}}\|e_M\|_{\text{pw}}\|\bar{e}_M\|_{\text{pw}} \leq \frac{C_{\text{ds}}}{2}(\|u_M^{n+1/2}\|_{\text{pw}}^2 + \|u_M^{n-1/2}\|_{\text{pw}}^2)\|e_M\|_{\text{pw}}^2 + C_{\text{ds}}^{-1}\|\bar{e}_M\|_{\text{pw}}^2.$$

Choose $\psi = v_M^{n+1/2}$ in (2.10b) and utilize (2.7) to show $\|v_M^{n+1/2}\|_{\text{pw}} \leq \frac{1}{2}C_{\text{ds}}\|u_M^{n+1/2}\|_{\text{pw}}^2$. From the bound $\|v_M^{n-1/2}\|_{\text{pw}} \leq \frac{1}{2}C_{\text{ds}}\|u_M^{n-1/2}\|_{\text{pw}}^2$ (obtained by replacing n by $n-1$ in the last inequality) and triangle inequality, we have

$$\|v_M^{n+1/2} + v_M^{n-1/2}\|_{\text{pw}} \leq \|v_M^{n+1/2}\|_{\text{pw}} + \|v_M^{n-1/2}\|_{\text{pw}} \leq \frac{1}{2}C_{\text{ds}}(\|u_M^{n+1/2}\|_{\text{pw}}^2 + \|u_M^{n-1/2}\|_{\text{pw}}^2).$$

In turn, a combination of the last three displayed inequalities followed by (5.6) (applied twice) implies

$$8k^{-2}\|e_M\|^2 + 2\|e_M\|_{\text{pw}}^2 + \|\bar{e}_M\|_{\text{pw}}^2 \leq C_{\text{ds}}^2(\|u_M^{n+1/2}\|_{\text{pw}}^2 + \|u_M^{n-1/2}\|_{\text{pw}}^2)\|e_M\|_{\text{pw}}^2 \leq 2M_3(u_0, u_1, f, k)\|e_M\|_{\text{pw}}^2.$$

Under the smallness assumption on u_0, u_1, k from Theorem (2.4)(c), we have $M_3(u_0, u_1, f, k) = \delta$ for some $\delta < 1$. This shows that $8k^{-2}\|e_M\|^2 + 2(1-\delta)\|e_M\|_{\text{pw}}^2 + \|\bar{e}_M\|_{\text{pw}}^2 \leq 0$, which is only possible if $\|e_M\|_{\text{pw}} = \|\bar{e}_M\|_{\text{pw}} = 0$. $\square$

5.2 A priori error analysis

In this subsection, we first present the error equation. This is followed by Lemmas 5.1-5.3 with estimates for the intermediate terms appearing in the error equation. The proof of the error bound in energy-norm for the fully-discrete scheme concludes this subsection with optimal convergence in space and quadratic convergence in time.

Error equation. By the definition of the companion operator, we have $J\varphi_M \in H_0^2(\Omega)$ for all $\varphi_M \in \mathcal{M}(\mathcal{T})$. With this choice of the test function, consider the convex combination of (2.2a) with coefficients $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{1}{4}$ at the three time levels t_{n-1} , t_n , and t_{n+1} , respectively, and subtract (2.10a) from resultant equation to obtain

$$\begin{aligned}
& (u_{tt}^{n,1/4}, J\varphi_M)_{L^2(\Omega)} + a(u^{n,1/4}, J\varphi_M) - (\bar{\partial}_t^2 u_M^n, \varphi_M)_{L^2(\Omega)} - a_{\text{pw}}(u_M^{n,1/4}, \varphi_M) = (f^{n,1/4}, (J-I)\varphi_M)_{L^2(\Omega)} \\
& - \frac{1}{4}[b(u^{n+1}, J\varphi_M, v^{n+1}) + 2b(u^n, J\varphi_M, v^n) + b(u^{n-1}, J\varphi_M, v^{n-1})] + b_{\text{pw}}(u_M^{n,1/4}, \varphi_M, v_M^{n,1/4}). \tag{5.11}
\end{aligned}$$

The linearity of $b(\bullet, \bullet, \bullet)$ in each component and (2.9a)-(2.9b) with elementary algebra leads to the identity

$$\begin{aligned}
& b(u^{n+1}, J\varphi_M, v^{n+1}) + 2b(u^n, J\varphi_M, v^n) + b(u^{n-1}, J\varphi_M, v^{n-1}) \\
& = 4b(u^{n,1/4}, J\varphi_M, v^{n,1/4}) + 2k^2[b(\delta_t u^n, J\varphi_M, \delta_t v^n) + \frac{k^2}{8}b(\bar{\partial}_t^2 u^n, J\varphi_M, \bar{\partial}_t^2 v^n)].
\end{aligned}$$

This, $a(u^{n,1/4}, J\varphi_M) = a_{\text{pw}}(\mathcal{R}_M u^{n,1/4}, \varphi_M)$ from (3.1), (4.13a), and some elementary manipulations in (5.11) with the truncation error $r^n := \bar{\partial}_t^2 u^n - u_{tt}^{n,1/4}$ lead to

$$\begin{aligned} (\bar{\partial}_t^2 \zeta^n, \varphi_M)_{L^2(\Omega)} + a_{\text{pw}}(\zeta^{n,1/4}, \varphi_M) &= (f^{n,1/4} - u_{tt}^{n,1/4}, (J - I)\varphi_M)_{L^2(\Omega)} + (r^n - \bar{\partial}_t^2 \rho^n, \varphi_M)_{L^2(\Omega)} \\ &\quad + b_{\text{pw}}(u_M^{n,1/4}, \varphi_M, v_M^{n,1/4}) - b(u^{n,1/4}, J\varphi_M, v^{n,1/4}) \\ &\quad + \frac{k^2}{2} [b(\delta_t u^n, J\varphi_M, \delta_t v^n) + \frac{k^2}{8} b(\bar{\partial}_t^2 u^n, J\varphi_M, \bar{\partial}_t^2 v^n)]. \end{aligned}$$

The choice $\varphi_M = 2k\delta_t \zeta^n$ in the above equation, (2.9a)-(2.9b), and (2.5) reveal

$$\begin{aligned} \|\bar{\partial}_t \zeta^{n+1/2}\|^2 - \|\bar{\partial}_t \zeta^{n-1/2}\|^2 + \|\zeta^{n+1/2}\|_{\text{pw}}^2 - \|\zeta^{n-1/2}\|_{\text{pw}}^2 &= 2k(f^{n,1/4} - u_{tt}^{n,1/4}, (J - I)\delta_t \zeta^n) + 2k(r^n - \bar{\partial}_t^2 \rho^n, \delta_t \zeta^n) \\ &\quad + 2k[b_{\text{pw}}(u_M^{n,1/4}, \delta_t \zeta^n, v_M^{n,1/4}) - b(u^{n,1/4}, J\delta_t \zeta^n, v^{n,1/4})] + k^3 [b(\delta_t u^n, J\delta_t \zeta^n, \delta_t v^n) + \frac{k^2}{8} b(\bar{\partial}_t^2 u^n, J\delta_t \zeta^n, \bar{\partial}_t^2 v^n)]. \end{aligned} \quad (5.12)$$

Choose the test function in (2.2b) as $J\psi_M$ and consider the difference at $t = t_{n+1}$ and $t = t_{n-1}$. Utilize the linearity of $a(\bullet, \bullet)$ to show $a(v^{n+1} - v^{n-1}, J\psi_M) = \frac{1}{2}b(u^{n+1}, u^{n+1}, J\psi_M) - \frac{1}{2}b(u^{n-1}, u^{n-1}, J\psi_M)$ for all $\psi_M \in M(\mathcal{T})$. Also multiply (2.10b) by 2 and consider its difference at n and $n+1$ steps to obtain $2a_{\text{pw}}(v_M^{n+1/2} - v_M^{n-1/2}, \psi_M) = b_{\text{pw}}(u_M^{n+1/2}, u_M^{n+1/2}, \psi_M) - b_{\text{pw}}(u_M^{n-1/2}, u_M^{n-1/2}, \psi_M)$. Subtract the two equations and utilize (2.9a) to arrive at

$$\begin{aligned} a(v^{n+1} - v^{n-1}, J\psi_M) - a_{\text{pw}}(v_M^{n+1} - v_M^{n-1}, \psi_M) &= \frac{1}{2} (b(u^{n+1}, u^{n+1}, J\psi_M) - b(u^{n-1}, u^{n-1}, J\psi_M)) \\ &\quad - b_{\text{pw}}(u_M^{n+1/2}, u_M^{n+1/2}, \psi_M) + b_{\text{pw}}(u_M^{n-1/2}, u_M^{n-1/2}, \psi_M). \end{aligned} \quad (5.13)$$

The identity $a(v^{n+1} - v^{n-1}, J\psi_M) = a_{\text{pw}}(\mathcal{R}_M(v^{n+1} - v^{n-1}), \psi_M)$ from (3.1) and an application of (4.13b) rewrite the left-hand side of (5.13) as

$$a(v^{n+1} - v^{n-1}, J\psi_M) - a_{\text{pw}}(v_M^{n+1} - v_M^{n-1}, \psi_M) = a_{\text{pw}}(\theta^{n+1} - \theta^{n-1}, \psi_M).$$

The linearity of $b(\bullet, \bullet, \bullet)$ (and $b_{\text{pw}}(\bullet, \bullet, \bullet)$) and (2.9a)-(2.9b) modify the right-hand side of (5.13) as

$$\begin{aligned} \frac{1}{2} (b(u^{n+1}, u^{n+1}, J\psi_M) - b(u^{n-1}, u^{n-1}, J\psi_M)) &= 2kb(u^{n,1/4}, \delta_t u^n, J\psi_M) + \frac{k^3}{2} b(\bar{\partial}_t^2 u^n, \delta_t u^n, J\psi_M), \\ b_{\text{pw}}(u_M^{n+1/2}, u_M^{n+1/2}, \psi_M) - b_{\text{pw}}(u_M^{n-1/2}, u_M^{n-1/2}, \psi_M) &= 2kb_{\text{pw}}(u_M^{n,1/4}, \delta_t u_M^n, \psi_M). \end{aligned}$$

A combination of the last three displayed identities in (5.13) shows

$$a_{\text{pw}}(\theta^{n+1} - \theta^{n-1}, \psi_M) = 2k(b(u^{n,1/4}, \delta_t u^n, J\psi_M) - b_{\text{pw}}(u_M^{n,1/4}, \delta_t u_M^n, \psi_M)) + \frac{k^3}{2} b(\bar{\partial}_t^2 u^n, \delta_t u^n, J\psi_M).$$

The choice $\psi_M = \theta^{n,1/4} = \frac{1}{2}(\theta^{n+1/2} + \theta^{n-1/2})$ and the identity $\theta^{n+1} - \theta^{n-1} = 2(\theta^{n+1/2} - \theta^{n-1/2})$ from (2.8a) then gives

$$\|\theta^{n+1/2}\|_{\text{pw}}^2 - \|\theta^{n-1/2}\|_{\text{pw}}^2 = 2k[b(u^{n,1/4}, \delta_t u^n, J\theta^{n,1/4}) - b_{\text{pw}}(u_M^{n,1/4}, \delta_t u_M^n, \theta^{n,1/4})] + \frac{k^3}{2} b(\bar{\partial}_t^2 u^n, \delta_t u^n, J\theta^{n,1/4}).$$

We then add this and (5.12), sum from $n = 1, 2, \dots, m$, and utilize the telescopic property to arrive at

$$\begin{aligned} \|\bar{\partial}_t \zeta^{m+1/2}\|^2 + \|\zeta^{m+1/2}\|_{\text{pw}}^2 + \|\theta^{m+1/2}\|_{\text{pw}}^2 &= [\|\bar{\partial}_t \zeta^{1/2}\|^2 + \|\zeta^{1/2}\|_{\text{pw}}^2 + \|\theta^{1/2}\|_{\text{pw}}^2] \\ &\quad + 2k \sum_{n=1}^m [(f^{n,1/4} - u_{tt}^{n,1/4}, (J - I)\delta_t \zeta^n) + (r^n - \bar{\partial}_t^2 \rho^n, \delta_t \zeta^n)] \\ &\quad + 2k \sum_{n=1}^m [b_{\text{pw}}(u_M^{n,1/4}, \delta_t \zeta^n, v_M^{n,1/4}) - b(u^{n,1/4}, J\delta_t \zeta^n, v^{n,1/4}) + b(u^{n,1/4}, \delta_t u^n, J\theta^{n,1/4}) - b_{\text{pw}}(u_M^{n,1/4}, \delta_t u_M^n, \theta^{n,1/4})] \\ &\quad + k^3 \sum_{n=1}^m [b(\delta_t u^n, J\delta_t \zeta^n, \delta_t v^n) + \frac{k^2}{8} b(\bar{\partial}_t^2 u^n, J\delta_t \zeta^n, \bar{\partial}_t^2 v^n) + \frac{1}{2} b(\bar{\partial}_t^2 u^n, \delta_t u^n, J\theta^{n,1/4})] \\ &=: T_1 + T_2 + T_3 + T_4 \quad \text{for } 1 \leq m \leq N-1. \end{aligned} \quad (5.14)$$

This is analogous to (4.18); with an (additional) initial error contribution T_1 , linear contributions T_2 , nonlinear contributions T_3 , and truncation terms T_4 . The term T_1 is controlled in (4.21). The estimates for T_2 - T_4 are technical and we present them in the Lemmas 5.1-5.3 below. Combining these arguments gives the proof of Theorem 2.5.

Lemma 5.1 (bound of linear term- T_2). *Let $f \in H^1(0, T; L^2(\Omega))$, $u_{tt} \in L^2(0, T; H^{2+\sigma}(\Omega))$, and $u \in H^4(0, T; L^2(\Omega))$. Then the following bound holds:*

$$\begin{aligned} T_2 &:= 2k \sum_{n=1}^m [(f^{n,1/4} - u_{tt}^{n,1/4}, (J - I)\delta_t \zeta^n)_{L^2(\Omega)} + (r^n - \bar{\partial}_t^2 \rho^n, \delta_t \zeta^n)_{L^2(\Omega)}] \leq C(u_{tt}, u_{ttt}, u_{tttt}, f, f_t)(h^{2\sigma} + k^4) \\ &+ \frac{1}{32} [\|\zeta^{1/2}\|_{pw}^2 + \|\zeta^{m+1/2}\|_{pw}^2] + \frac{7}{8} [\|\bar{\partial}_t \zeta^{1/2}\|^2 + \|\bar{\partial}_t \zeta^{m+1/2}\|^2] + k \sum_{n=0}^{m-1} \|\zeta^{n+1/2}\|_{pw}^2 + \frac{k}{T} \sum_{n=0}^{m-1} \|\bar{\partial}_t \zeta^{n+1/2}\|^2. \end{aligned}$$

Proof. The definition (2.8b) and the identity (2.9b) show $k\delta_t \zeta^n = \zeta^{n+1/2} - \zeta^{n-1/2}$. Then, we obtain

$$\begin{aligned} k \sum_{n=1}^m (f^{n,1/4} - u_{tt}^{n,1/4}, (J - I)\delta_t \zeta^n)_{L^2(\Omega)} &= (f^{m,1/4} - u_{tt}^{m,1/4}, (J - I)\zeta^{m+1/2})_{L^2(\Omega)} + (f^{1,1/4} - u_{tt}^{1,1/4}, (I - J)\zeta^{1/2})_{L^2(\Omega)} \\ &+ \sum_{n=1}^{m-1} ((f^{n+1,1/4} - f^{n,1/4}) - (u_{ttt}^{n+1,1/4} - u_{ttt}^{n,1/4}), (I - J)\zeta^{n+1/2})_{L^2(\Omega)}. \end{aligned} \quad (5.15)$$

A Cauchy-Schwarz inequality, Lemma 3.1(e), and Young's inequality with $a = C_2 h^2 \|f^{m,1/4} - u_{tt}^{m,1/4}\|$, $b = \|\zeta^{m+1/2}\|_{pw}$, and $\epsilon = 32$ lead to

$$\begin{aligned} (f^{m,1/4} - u_{tt}^{m,1/4}, (J - I)\zeta^{m+1/2})_{L^2(\Omega)} &\leq \|f^{m,1/4} - u_{tt}^{m,1/4}\| \|(J - I)\zeta^{m+1/2}\| \\ &\leq 32C_2^2 h^4 (\|f\|_{C(0,T;L^2(\Omega))}^2 + \|u_{tt}\|_{C(0,T;L^2(\Omega))}^2) + \frac{1}{64} \|\zeta^{m+1/2}\|_{pw}^2 \end{aligned} \quad (5.16)$$

with $\|f^{m,1/4} - u_{tt}^{m,1/4}\| \leq 2\|f\|_{C(0,T;L^2(\Omega))}^2 + 2\|u_{tt}\|_{C(0,T;L^2(\Omega))}^2$ in the last step. Analogous arguments show

$$(f^{1,1/4} - u_{tt}^{1,1/4}, (I - J)\zeta^{1/2})_{L^2(\Omega)} \leq 32C_2^2 h^4 (\|f\|_{C(0,T;L^2(\Omega))}^2 + \|u_{tt}\|_{C(0,T;L^2(\Omega))}^2) + \frac{1}{64} \|\zeta^{1/2}\|_{pw}^2. \quad (5.17)$$

The definition (2.9a) reveals

$$(f^{n+1,1/4} - f^{n,1/4}) - (u_{ttt}^{n+1,1/4} - u_{ttt}^{n,1/4}) = \frac{1}{4} \left(\int_{t_n}^{t_{n+2}} (f_t(t) - u_{ttt}(t)) dt + \int_{t_{n-1}}^{t_{n+1}} (f_t(t) - u_{ttt}(t)) dt \right). \quad (5.18)$$

A Cauchy-Schwarz inequality, Lemma 3.1(e), and elementary manipulations lead to

$$\begin{aligned} \left(\int_{t_n}^{t_{n+2}} (f_t(t) - u_{ttt}(t)) dt, (I - J)\zeta^{n+1/2} \right)_{L^2(\Omega)} &\leq \left\| \int_{t_n}^{t_{n+2}} (f_t(t) - u_{ttt}(t)) dt \right\| \|(I - J)\zeta^{n+1/2}\| \\ &\leq C_2 h^2 \left\| \int_{t_n}^{t_{n+2}} (f_t(t) - u_{ttt}(t)) dt \right\| \|\zeta^{n+1/2}\|_{pw} \leq C_2 \sqrt{2k} h^2 \|f_t - u_{ttt}\|_{L^2(t_n, t_{n+2}, L^2(\Omega))} \|\zeta^{n+1/2}\|_{pw} \end{aligned} \quad (5.19)$$

with $\left\| \int_{t_n}^{t_{n+2}} (f_t(t) - u_{ttt}(t)) dt \right\| \leq \int_{t_n}^{t_{n+2}} \|f_t(t) - u_{ttt}(t)\| dt \leq \sqrt{2k} \|f_t - u_{ttt}\|_{L^2(t_n, t_{n+2}, L^2(\Omega))}$ in the last step. A Young's inequality with $a = C_2 h^2 \|f_t - u_{ttt}\|_{L^2(t_n, t_{n+2}, L^2(\Omega))}$, $b = \sqrt{2k} \|\zeta^{n+1/2}\|_{pw}$, and $\epsilon = 1$ on the right-hand side of (5.19), and a triangle inequality result in

$$\left(\int_{t_n}^{t_{n+2}} (f_t(t) - u_{ttt}(t)) dt, (I - J)\zeta^{n+1/2} \right)_{L^2(\Omega)} \leq C_2^2 h^4 (\|f_t\|_{L^2(t_n, t_{n+2}, L^2(\Omega))}^2 + \|u_{ttt}\|_{L^2(t_n, t_{n+2}, L^2(\Omega))}^2) + k \|\zeta^{n+1/2}\|_{pw}^2.$$

Analogous arguments show

$$\left(\int_{t_{n-1}}^{t_{n+1}} (f_t(t) - u_{ttt}(t)) dt, (I - J)\zeta^{n+1/2} \right)_{L^2(\Omega)} \leq C_2^2 h^4 (\|f_t\|_{L^2(t_{n-1}, t_{n+1}, L^2(\Omega))}^2 + \|u_{ttt}\|_{L^2(t_{n-1}, t_{n+1}, L^2(\Omega))}^2) + k \|\zeta^{n+1/2}\|_{pw}^2.$$

A combination of last two inequalities in (5.18) and $\sum_{i=1}^{m-1} \|\varphi\|_{L^2(t_i, t_{i+2}, L^2(\Omega))}^2 \leq 2\|\varphi\|_{L^2(0,T;L^2(\Omega))}^2$ for $\varphi \in \{f, u_{ttt}\}$, reveals

$$\sum_{n=1}^{m-1} ((f^{n+1,1/4} - f^{n,1/4}) - (u_{ttt}^{n+1,1/4} - u_{ttt}^{n,1/4}), (I - J)\zeta^{n+1/2})_{L^2(\Omega)}$$

$$\leq C_2^2 h^4 (\|f_t\|_{L^2(0,T;L^2(\Omega))}^2 + \|u_{ttt}\|_{L^2(0,T;L^2(\Omega))}^2) + \frac{k}{2} \sum_{n=1}^{m-1} \|\zeta^{n+1/2}\|_{\text{pw}}^2.$$

This and a combination of (5.15)-(5.17) with $h^4 \lesssim h^{2\sigma}$ establish

$$\begin{aligned} & 2k \sum_{n=1}^m (f^{n,1/4} - u_{tt}^{n,1/4}, (J - I)\delta_t \zeta^n)_{L^2(\Omega)} \\ & \leq C(u_{tt}, u_{ttt}, f, f_t) h^{2\sigma} + \frac{1}{32} [\|\zeta^{1/2}\|_{\text{pw}}^2 + \|\zeta^{m+1/2}\|_{\text{pw}}^2] + k \sum_{n=1}^{m-1} \|\zeta^{n+1/2}\|_{\text{pw}}^2. \end{aligned} \quad (5.20)$$

Cauchy-Schwarz and Young's inequalities (with $a = \|r^n - \bar{\partial}_t^2 \rho^n\|$, $b = \|\delta_t \zeta^n\|$, and $\epsilon = 4T/7$), a summation over n , a triangle inequality $\|r^n - \bar{\partial}_t^2 \rho^n\|^2 \leq 2\|r^n\|^2 + 2\|\bar{\partial}_t^2 \rho^n\|^2$, the bounds from Lemma 3.7(b) for the truncation error, and Lemma 3.2 result in

$$2k \sum_{n=1}^m (r^n - \bar{\partial}_t^2 \rho^n, \delta_t \zeta^n)_{L^2(\Omega)} \leq \frac{8T}{7} [C(u_{ttt})k^4 + \frac{4}{3}h^{4\sigma}\|u_{tt}\|_{L^2(0,T;H^{2+\sigma}(\Omega))}^2] + \frac{7k}{4T} \sum_{n=1}^m \|\delta_t \zeta^n\|^2.$$

From (2.9b), we have $2\delta_t \zeta^n = \bar{\partial}_t \zeta^{n-1/2} + \bar{\partial}_t \zeta^{n+1/2}$. This and a triangle inequality infer

$$\begin{aligned} \frac{k}{T} \sum_{n=1}^m \|\delta_t \zeta^n\|_{\text{pw}}^2 & \leq \frac{k}{2T} \sum_{n=1}^m [\|\bar{\partial}_t \zeta^{n-1/2}\|_{\text{pw}}^2 + \|\bar{\partial}_t \zeta^{n+1/2}\|_{\text{pw}}^2] \leq \frac{k}{2T} [\|\bar{\partial}_t \zeta^{1/2}\|_{\text{pw}}^2 + \|\bar{\partial}_t \zeta^{m+1/2}\|_{\text{pw}}^2] \\ & + \frac{k}{T} \sum_{n=1}^{m-1} \|\bar{\partial}_t \zeta^{n+1/2}\|_{\text{pw}}^2 \leq \frac{1}{2} [\|\bar{\partial}_t \zeta^{1/2}\|_{\text{pw}}^2 + \|\bar{\partial}_t \zeta^{m+1/2}\|_{\text{pw}}^2] + \frac{k}{T} \sum_{n=1}^{m-1} \|\bar{\partial}_t \zeta^{n+1/2}\|_{\text{pw}}^2 \end{aligned}$$

with $k \leq T$ for the first two terms on the right-hand side. The last two displayed inequalities and $h^{4\sigma} \leq h^{2\sigma}|\Omega|^\sigma$ reveal

$$2k \sum_{n=1}^m (r^n - \bar{\partial}_t^2 \rho^n, \delta_t \zeta^n)_{L^2(\Omega)} \leq C(u_{tt}, u_{ttt})(k^4 + h^{2\sigma}) + \frac{7}{8} [\|\bar{\partial}_t \zeta^{1/2}\|^2 + \|\bar{\partial}_t \zeta^{m+1/2}\|^2] + \frac{7k}{4} \sum_{n=1}^{m-1} \|\bar{\partial}_t \zeta^{n+1/2}\|^2.$$

This in combination with (5.20) and the addition of non-negative terms $k\|\zeta^{1/2}\|_{\text{pw}}^2 + \frac{7k}{4}\|\bar{\partial}_t \zeta^{1/2}\|^2$ to the right-hand side conclude the proof. $\square$

The next lemma provides bounds for the nonlinear contributions T_3 , representing the main difficulty. The estimates rely on the stability of continuous and discrete trilinear forms, approximation properties of the Ritz projection (cf. Lemma 3.2) and of the companion operator (cf. Lemma 3.1(d, e)).

Lemma 5.2 (Bound for nonlinear term T_3). *Let $u_0 \in H_0^2(\Omega)$, $u_1 \in L^2(\Omega)$, and $f \in L^2(0, T, L^2(\Omega))$ be such that $M_1(u_0, u_1, f) \leq 1/2$. Assume that $u, v, \in H^1(0, T; H_0^2(\Omega) \cap H^{2+\sigma}(\Omega))$, $u_t, v_t \in L^\infty(0, T; H^2(\Omega))$. Then*

$$\begin{aligned} T_3 &:= 2k \sum_{n=1}^m [b_{\text{pw}}(u_{\text{M}}^{n,1/4}, \delta_t \zeta^n, v_{\text{M}}^{n,1/4}) - b(u^{n,1/4}, J\delta_t \zeta^n, v^{n,1/4}) + b(u^{n,1/4}, \delta_t u^n, J\theta^{n,1/4}) - b_{\text{pw}}(u_{\text{M}}^{n,1/4}, \delta_t u_{\text{M}}^n, \theta^{n,1/4})] \\ &\leq C(u, u_t, v, v_t) h^{2\sigma} + \frac{17}{32} \|\zeta^{1/2}\|_{\text{pw}}^2 + \frac{9}{16} \|\theta^{1/2}\|_{\text{pw}}^2 + \frac{25}{32} \|\zeta^{m+1/2}\|_{\text{pw}}^2 + \frac{13}{16} \|\theta^{m+1/2}\|_{\text{pw}}^2 \\ &\quad + k(C_5 [\|u_t\|_{L^\infty(0,T;H^2(\Omega))} + \|v_t\|_{L^\infty(0,T;H^2(\Omega))}] + 3) \sum_{n=0}^{m-1} \|\zeta^{n+1/2}\|_{\text{pw}}^2 \\ &\quad + k(C_5 \|u_t\|_{L^\infty(0,T;H^2(\Omega))} + \frac{9}{8T}) \sum_{n=0}^{m-1} \|\theta^{n+1/2}\|_{\text{pw}}^2 \text{ holds for any } m \text{ with } 1 \leq m \leq N-1. \end{aligned}$$

Proof. Definitions (2.8a)-(2.8b) and linearity of $b_{\text{pw}}(\bullet, \bullet, \bullet)$ in each component imply the relation

$$\begin{aligned} & b_{\text{pw}}(u_{\text{M}}^{n,1/4}, \delta_t \zeta^n, v_{\text{M}}^{n,1/4}) - b(u^{n,1/4}, J\delta_t \zeta^n, v^{n,1/4}) = b_{\text{pw}}(\zeta^{n,1/4}, \delta_t \zeta^n, \theta^{n,1/4}) \\ & - b_{\text{pw}}(\zeta^{n,1/4}, \delta_t \zeta^n, \mathcal{R}_{\text{M}} v^{n,1/4}) + b_{\text{pw}}(\rho^{n,1/4}, \delta_t \zeta^n, \theta^{n,1/4}) - b_{\text{pw}}(\rho^{n,1/4}, \delta_t \zeta^n, \mathcal{R}_{\text{M}} v^{n,1/4}) \end{aligned}$$

$$-b_{\text{pw}}(u^{n,1/4}, \delta_t \zeta^n, \theta^{n,1/4}) - b_{\text{pw}}(u^{n,1/4}, \delta_t \zeta^n, \chi^{n,1/4}) + b_{\text{pw}}(u^{n,1/4}, (I - J)\delta_t \zeta^n, v^{n,1/4}).$$

Analogous arguments with $\delta_t(\mathcal{R}_M u^n) = \mathcal{R}_M \delta_t u^n$ from linearity of $\mathcal{R}_M$ lead to

$$\begin{aligned} & b(u^{n,1/4}, \delta_t u^n, J\theta^{n,1/4}) - b_{\text{pw}}(u_M^{n,1/4}, \delta_t u_M^n, \theta^{n,1/4}) = -b_{\text{pw}}(\zeta^{n,1/4}, \delta_t \zeta^n, \theta^{n,1/4}) \\ & + b_{\text{pw}}(\zeta^{n,1/4}, \mathcal{R}_M \delta_t u^n, \theta^{n,1/4}) - b_{\text{pw}}(\rho^{n,1/4}, \delta_t \zeta^n, \theta^{n,1/4}) + b_{\text{pw}}(\rho^{n,1/4}, \mathcal{R}_M \delta_t u^n, \theta^{n,1/4}) \\ & + b_{\text{pw}}(u^{n,1/4}, \delta_t \zeta^n, \theta^{n,1/4}) + b_{\text{pw}}(u^{n,1/4}, \delta_t \rho^n, \theta^{n,1/4}) + b_{\text{pw}}(u^{n,1/4}, \delta_t u^n, (J - I)\theta^{n,1/4}). \end{aligned}$$

Then, noticing that $b_{\text{pw}}(\zeta^{n,1/4}, \delta_t \zeta^n, \theta^{n,1/4})$, $b_{\text{pw}}(\rho^{n,1/4}, \delta_t \zeta^n, \theta^{n,1/4})$, and $b_{\text{pw}}(u^{n,1/4}, \delta_t \zeta^n, \theta^{n,1/4})$ cancel out, we arrive at

$$\begin{aligned} T_3 = 2k \sum_{n=1}^m & \left[-b_{\text{pw}}(\zeta^{n,1/4}, \delta_t \zeta^n, \mathcal{R}_M v^{n,1/4}) + b_{\text{pw}}(\zeta^{n,1/4}, \mathcal{R}_M \delta_t u^n, \theta^{n,1/4}) - b_{\text{pw}}(\rho^{n,1/4}, \delta_t \zeta^n, \mathcal{R}_M v^{n,1/4}) \right. \\ & + b_{\text{pw}}(\rho^{n,1/4}, \mathcal{R}_M \delta_t u^n, \theta^{n,1/4}) - b_{\text{pw}}(u^{n,1/4}, \delta_t \zeta^n, \chi^{n,1/4}) + b_{\text{pw}}(u^{n,1/4}, \delta_t \rho^n, \theta^{n,1/4}) \\ & \left. + b_{\text{pw}}(u^{n,1/4}, (I - J)\delta_t \zeta^n, v^{n,1/4}) + b_{\text{pw}}(u^{n,1/4}, \delta_t u^n, (J - I)\theta^{n,1/4}) \right] =: \sum_{i=1}^8 S_i. \end{aligned} \quad (5.21)$$

The rest of the proof focuses on bounding the terms $S_1, S_2, \dots, S_8$.

Bound for S_1 . Using $2\zeta^{n,1/4} = \zeta^{n+1/2} + \zeta^{n-1/2}$, $k\delta_t \zeta^n = \zeta^{n+1/2} - \zeta^{n-1/2}$ (obtained from (2.8a)-(2.8b)) followed by linearity of $b_{\text{pw}}(\bullet, \bullet, \bullet)$ in each component and symmetry in first two components show

$$-2kb_{\text{pw}}(\zeta^{n,1/4}, \delta_t \zeta^n, \mathcal{R}_M v^{n,1/4}) = -b_{\text{pw}}(\zeta^{n+1/2}, \zeta^{n+1/2}, \mathcal{R}_M v^{n,1/4}) + b_{\text{pw}}(\zeta^{n-1/2}, \zeta^{n-1/2}, \mathcal{R}_M v^{n,1/4}).$$

Utilize $\mathcal{R}_M v^{n,1/4} = \mathcal{R}_M v^{n+1/2} - \frac{k}{2}\mathcal{R}_M \delta_t v^n$ and $\mathcal{R}_M v^{n,1/4} = \mathcal{R}_M v^{n-1/2} + \frac{k}{2}\mathcal{R}_M \delta_t v^n$ in the third components of first and second term, respectively, to infer

$$\begin{aligned} -2kb_{\text{pw}}(\zeta^{n,1/4}, \delta_t \zeta^n, \mathcal{R}_M v^{n,1/4}) & = -b_{\text{pw}}(\zeta^{n+1/2}, \zeta^{n+1/2}, \mathcal{R}_M v^{n+1/2}) + b_{\text{pw}}(\zeta^{n-1/2}, \zeta^{n-1/2}, \mathcal{R}_M v^{n-1/2}) \\ & + \frac{k}{2}(b_{\text{pw}}(\zeta^{n+1/2}, \zeta^{n+1/2}, \mathcal{R}_M \delta_t v^n) + b_{\text{pw}}(\zeta^{n-1/2}, \zeta^{n-1/2}, \mathcal{R}_M \delta_t v^n)). \end{aligned}$$

A summation over $n = 1, 2, \dots, m$ (for $1 \leq m \leq N - 1$), the definition $S_1 := -2k \sum_{n=1}^m b_{\text{pw}}(\zeta^{n,1/4}, \delta_t \zeta^n, \mathcal{R}_M v^{n,1/4})$, and a telescopic sum property in the first two terms on the right-hand side show that

$$\begin{aligned} S_1 & = -b_{\text{pw}}(\zeta^{m+1/2}, \zeta^{m+1/2}, \mathcal{R}_M v^{m+1/2}) + b_{\text{pw}}(\zeta^{1/2}, \zeta^{1/2}, \mathcal{R}_M v^{1/2}) \\ & + \frac{k}{2} \sum_{n=1}^m (b_{\text{pw}}(\zeta^{n+1/2}, \zeta^{n+1/2}, \mathcal{R}_M \delta_t v^n) + b_{\text{pw}}(\zeta^{n-1/2}, \zeta^{n-1/2}, \mathcal{R}_M \delta_t v^n)). \end{aligned}$$

Utilize $\mathcal{R}_M v^{m+1/2} = \mathcal{R}_M v^{m,1/4} + \frac{k}{2}\mathcal{R}_M \delta_t v^m$ and linearity of $b_{\text{pw}}(\bullet, \bullet, \bullet)$ in third component to obtain

$$\begin{aligned} & -b_{\text{pw}}(\zeta^{m+1/2}, \zeta^{m+1/2}, \mathcal{R}_M v^{m+1/2}) + \frac{k}{2} \sum_{n=1}^m b_{\text{pw}}(\zeta^{n+1/2}, \zeta^{n+1/2}, \mathcal{R}_M \delta_t v^n) \\ & = -b_{\text{pw}}(\zeta^{m+1/2}, \zeta^{m+1/2}, \mathcal{R}_M v^{m,1/4}) + \frac{k}{2} \sum_{n=1}^{m-1} b_{\text{pw}}(\zeta^{n+1/2}, \zeta^{n+1/2}, \mathcal{R}_M \delta_t v^n). \end{aligned}$$

A combination of last two displayed identities reveal

$$\begin{aligned} S_1 & = -b_{\text{pw}}(\zeta^{m+1/2}, \zeta^{m+1/2}, \mathcal{R}_M v^{m,1/4}) + b_{\text{pw}}(\zeta^{1/2}, \zeta^{1/2}, \mathcal{R}_M v^{1/2}) \\ & + \frac{k}{2} \left[\sum_{n=1}^{m-1} b_{\text{pw}}(\zeta^{n+1/2}, \zeta^{n+1/2}, \mathcal{R}_M \delta_t v^n) + \sum_{n=1}^m b_{\text{pw}}(\zeta^{n-1/2}, \zeta^{n-1/2}, \mathcal{R}_M \delta_t v^n) \right]. \end{aligned} \quad (5.22)$$

Then, from (2.7), (3.2), $C_5 = \|J\|_{C_{\text{ds}}}$, $\|v^{m,1/4}\|_{\text{pw}} \leq \frac{1}{4}(\|v^{m-1}\|_{\text{pw}} + 2\|v^m\|_{\text{pw}} + \|v^{m+1}\|_{\text{pw}})$ from (2.8a), a triangle inequality, (2.4) with $t = t_{m-1}, t_m$, and t_{m+1} , we obtain

$$|b_{\text{pw}}(\zeta^{m+1/2}, \zeta^{m+1/2}, \mathcal{R}_M v^{m,1/4})| \leq C_{\text{ds}} \|\zeta^{m+1/2}\|_{\text{pw}}^2 \|\mathcal{R}_M v^{m,1/4}\|_{\text{pw}}$$

$$\leq C_5 \|\zeta^{m+1/2}\|_{\text{pw}}^2 \|\nu^{m,1/4}\|_{\text{pw}} \leq M_1(u_0, u_1, f) \|\zeta^{m+1/2}\|_{\text{pw}}^2 \leq \frac{1}{2} \|\zeta^{m+1/2}\|_{\text{pw}}^2.$$

The last inequality follows from the smallness data assumption $M_1(u_0, u_1, f) \leq 1/2$. Similar arguments lead to

$$b_{\text{pw}}(\zeta^{1/2}, \zeta^{1/2}, \mathcal{R}_M \nu^{1/2}) \leq \frac{1}{2} \|\zeta^{1/2}\|_{\text{pw}}^2.$$

Recall $C_5 := C_{\text{ds}} \|J\|$. The inequalities (2.7), (3.2), and the bound

$$\|\delta_t \nu^n\|_{\text{pw}} \leq \|\delta_t \nu^n\|_{H^2(\Omega)} \leq \|\nu_t\|_{L^\infty(0,T;H^2(\Omega))} \quad (5.23)$$

from Lemma 3.4(e) reveal

$$\begin{aligned} b_{\text{pw}}(\zeta^{n+1/2}, \zeta^{n+1/2}, \mathcal{R}_M \delta_t \nu^n) &\leq C_5 \|\zeta^{n+1/2}\|_{\text{pw}}^2 \|\delta_t \nu^n\|_{\text{pw}} \leq C_5 \|\nu_t\|_{L^\infty(0,T;H^2(\Omega))} \|\zeta^{n+1/2}\|_{\text{pw}}^2, \\ b_{\text{pw}}(\zeta^{n-1/2}, \zeta^{n-1/2}, \mathcal{R}_M \delta_t \nu^n) &\leq C_5 \|\zeta^{n-1/2}\|_{\text{pw}}^2 \|\delta_t \nu^n\|_{\text{pw}} \leq C_5 \|\nu_t\|_{L^\infty(0,T;H^2(\Omega))} \|\zeta^{n-1/2}\|_{\text{pw}}^2. \end{aligned}$$

A combination of all this with an addition of the term $C_5 k \|\nu_t\|_{L^\infty(0,T;H^2(\Omega))} \|\zeta^{1/2}\|_{\text{pw}}^2$ on the right-hand side in (5.22) leads to

$$S_1 \leq \frac{1}{2} (\|\zeta^{1/2}\|_{\text{pw}}^2 + \|\zeta^{m+1/2}\|_{\text{pw}}^2) + C_5 k \|\nu_t\|_{L^\infty(0,T;H^2(\Omega))} \sum_{n=0}^{m-1} \|\zeta^{n+1/2}\|_{\text{pw}}^2. \quad \square$$

Bound for S_2 . Recall $2\zeta^{n,1/4} = \zeta^{n+1/2} + \zeta^{n-1/2}$ and $2\theta^{n,1/4} = \theta^{n+1/2} + \theta^{n-1/2}$ from (2.8a). The linearity of $b_{\text{pw}}(\bullet, \bullet, \bullet)$ in first and third components, (2.7), and (3.2) lead to

$$\begin{aligned} S_2 := 2k \sum_{n=1}^m b_{\text{pw}}(\zeta^{n,1/4}, \mathcal{R}_M \delta_t u^n, \theta^{n,1/4}) &\leq \frac{1}{2} C_5 k \sum_{n=1}^m \|\delta_t u^n\|_{\text{pw}} [\|\zeta^{n+1/2}\|_{\text{pw}} \|\theta^{n+1/2}\|_{\text{pw}} \\ &\quad + \|\zeta^{n+1/2}\|_{\text{pw}} \|\theta^{n-1/2}\|_{\text{pw}} + \|\zeta^{n-1/2}\|_{\text{pw}} \|\theta^{n+1/2}\|_{\text{pw}} + \|\zeta^{n-1/2}\|_{\text{pw}} \|\theta^{n-1/2}\|_{\text{pw}}]. \end{aligned} \quad (5.24)$$

We now control the terms on the right-hand side of (5.24). A Young's inequality with $a = \sqrt{C_5 \|\delta_t u^n\|_{\text{pw}}} \|\zeta^{n+1/2}\|_{\text{pw}}$, $b = \sqrt{C_5 \|\delta_t u^n\|_{\text{pw}}} \|\theta^{n+1/2}\|_{\text{pw}}$, and $\epsilon = 1$ (resp. $a = C_5 \|\delta_t u^n\|_{\text{pw}} \|\zeta^{n+1/2}\|_{\text{pw}}$, $b = \|\theta^{n+1/2}\|_{\text{pw}}$, and $\epsilon = 2$) for $1 \leq n \leq m-1$ (resp. for $n = m$) reveals

$$\frac{1}{2} C_5 k \|\delta_t u^n\|_{\text{pw}} \|\zeta^{n+1/2}\|_{\text{pw}} \|\theta^{n+1/2}\|_{\text{pw}} \leq \begin{cases} \frac{1}{4} C_5 k \|\delta_t u^n\|_{\text{pw}} [\|\zeta^{n+1/2}\|_{\text{pw}}^2 + \|\theta^{n+1/2}\|_{\text{pw}}^2] & \text{if } 1 \leq n \leq m-1, \\ \frac{1}{2} C_5 k^2 \|\delta_t u^m\|_{\text{pw}}^2 \|\zeta^{m+1/2}\|_{\text{pw}}^2 + \frac{1}{4} \|\theta^{m+1/2}\|_{\text{pw}}^2 & \text{if } n = m. \end{cases} \quad (5.25)$$

Then, (2.8b), a triangle inequality, (2.4), and the smallness data assumption $M_1(u_0, u_1, f) \leq 1/2$ show

$$\frac{1}{2} C_5^2 k^2 \|\delta_t u^m\|_{\text{pw}}^2 \leq \frac{1}{4} C_5^2 (\|u^{m+1}\|_{\text{pw}}^2 + \|u^{m-1}\|_{\text{pw}}^2) \leq \frac{1}{2} (M_1(u_0, u_1, f))^2 \leq 1/8. \quad (5.26)$$

This, the bound similar to (5.23) (with ν replaced by u) in (5.25), and a summation from $n = 1$ to $n = m$ reveal

$$\begin{aligned} &\frac{1}{2} C_5 k \sum_{n=1}^m \|\delta_t u^n\|_{\text{pw}} \|\zeta^{n+1/2}\|_{\text{pw}} \|\theta^{n+1/2}\|_{\text{pw}} \\ &\leq \frac{1}{4} C_5 k \|u_t\|_{L^\infty(0,T;H^2(\Omega))} \sum_{n=0}^{m-1} [\|\zeta^{n+1/2}\|_{\text{pw}}^2 + \|\theta^{n+1/2}\|_{\text{pw}}^2] + \frac{1}{8} \|\zeta^{m+1/2}\|_{\text{pw}}^2 + \frac{1}{8} \|\theta^{m+1/2}\|_{\text{pw}}^2 \end{aligned} \quad (5.27)$$

with the addition of $\frac{1}{4} C_5 k \|u_t\|_{L^\infty(0,T;H^2(\Omega))} [\|\zeta^{1/2}\|_{\text{pw}}^2 + \|\theta^{1/2}\|_{\text{pw}}^2]$ on the right-hand side of the last inequality.

Next we bound the second term on the right-hand side of (5.24). Young's inequality with $a = \|\zeta^{n+1/2}\|_{\text{pw}}$, $b = \|\theta^{n-1/2}\|_{\text{pw}}$, and $\epsilon = 1$ followed by (5.23) (with ν replaced by u) for $1 \leq n \leq m-1$ and (5.26) for $n = m$ reveals

$$\frac{1}{2} C_5 k \sum_{n=1}^m \|\delta_t u^n\|_{\text{pw}} \|\zeta^{n+1/2}\|_{\text{pw}} \|\theta^{n-1/2}\|_{\text{pw}} \leq \frac{1}{4} C_5 k \sum_{n=1}^m \|\delta_t u^n\|_{\text{pw}} [\|\zeta^{n+1/2}\|_{\text{pw}}^2 + \|\theta^{n-1/2}\|_{\text{pw}}^2]$$

$$\leq \frac{1}{8} \|\zeta^{m+1/2}\|_{\text{pw}}^2 + \frac{1}{4} C_5 k \|u_t\|_{L^\infty(0,T;H^2(\Omega))} \sum_{n=0}^{m-1} \left[\|\zeta^{n+1/2}\|_{\text{pw}}^2 + \|\theta^{n+1/2}\|_{\text{pw}}^2 \right]$$

with the addition of a non-negative term $\frac{1}{4} C_5 k \|u_t\|_{L^\infty(0,T;H^2(\Omega))} \|\zeta^{1/2}\|^2$ on the right-hand side.

Interchanging the roles of ζ^n and θ^n , we bound the third term on the right-hand side of (5.24) as

$$\begin{aligned} & \frac{1}{2} C_5 k \sum_{n=1}^m \|\delta_t u^n\|_{\text{pw}} \|\zeta^{n-1/2}\|_{\text{pw}} \|\theta^{n+1/2}\|_{\text{pw}} \\ & \leq \frac{1}{8} \|\theta^{m+1/2}\|_{\text{pw}}^2 + \frac{1}{4} C_5 k \|u_t\|_{L^\infty(0,T;H^2(\Omega))} \sum_{n=0}^{m-1} \left[\|\zeta^{n+1/2}\|_{\text{pw}}^2 + \|\theta^{n+1/2}\|_{\text{pw}}^2 \right]. \end{aligned}$$

Analogous arguments lead to the bounds of the last term on the right-hand side of (5.24) as

$$\frac{1}{2} C_5 k \sum_{n=1}^m \|\delta_t u^n\|_{\text{pw}} \|\zeta^{n-1/2}\|_{\text{pw}} \|\theta^{n-1/2}\|_{\text{pw}} \leq \frac{1}{4} C_5 k \|u_t\|_{L^\infty(0,T;H^2(\Omega))} \sum_{n=0}^{m-1} \left[\|\zeta^{n+1/2}\|_{\text{pw}}^2 + \|\theta^{n+1/2}\|_{\text{pw}}^2 \right].$$

Putting together the last three displayed inequalities and (5.27) in (5.24) results in

$$S_2 \leq \frac{1}{4} \left[\|\zeta^{m+1/2}\|_{\text{pw}}^2 + \|\theta^{m+1/2}\|_{\text{pw}}^2 \right] + C_5 k \|u_t\|_{L^\infty(0,T;H^2(\Omega))} \sum_{n=0}^{m-1} \left[\|\zeta^{n+1/2}\|_{\text{pw}}^2 + \|\theta^{n+1/2}\|_{\text{pw}}^2 \right]. \quad \square$$

Bound for S_3 . The identity $k\delta_t \zeta^n = \zeta^{n+1/2} - \zeta^{n-1/2}$ from (2.9b) shows

$$-k b_{\text{pw}}(\rho^{n,1/4}, \delta_t \zeta^n, \mathcal{R}_M v^{n,1/4}) = -b_{\text{pw}}(\rho^{n,1/4}, \zeta^{n+1/2} - \zeta^{n-1/2}, \mathcal{R}_M v^{n,1/4}).$$

This and elementary manipulations motivate a split for S_3 in

$$\begin{aligned} \frac{1}{2} S_3 &:= -k \sum_{n=1}^m b_{\text{pw}}(\rho^{n,1/4}, \delta_t \zeta^n, \mathcal{R}_M v^{n,1/4}) = -b_{\text{pw}}(\rho^{m,1/4}, \zeta^{m+1/2}, \mathcal{R}_M v^{m,1/4}) + b_{\text{pw}}(\rho^{1,1/4}, \zeta^{1/2}, \mathcal{R}_M v^{1,1/4}) \\ &+ \sum_{n=1}^{m-1} b_{\text{pw}}(\rho^{n+1,1/4} - \rho^{n,1/4}, \zeta^{n+1/2}, \mathcal{R}_M v^{n+1,1/4}) + \sum_{n=1}^{m-1} b_{\text{pw}}(\rho^{n,1/4}, \zeta^{n+1/2}, \mathcal{R}_M (v^{n+1,1/4} - v^{n,1/4})). \end{aligned} \quad (5.28)$$

The boundedness from (2.7), (3.2), Lemma 3.5(b), and Lemma 3.4(b) imply

$$|b_{\text{pw}}(\rho^{m,1/4}, \zeta^{m+1/2}, \mathcal{R}_M v^{m,1/4})| \leq C_3 C_5 h^\sigma \|u\|_{C([0,T];H^{2+\sigma}(\Omega))} \|v\|_{C([0,T];H^2(\Omega))} \|\zeta^{m+1/2}\|_{\text{pw}}.$$

A Young's inequality with $a = C_3 C_5 h^\sigma \|u\|_{C([0,T];H^{2+\sigma}(\Omega))} \|v\|_{C([0,T];H^2(\Omega))}$, $b = \|\zeta^{m+1/2}\|_{\text{pw}}$, and $\epsilon = 96$ controls the first term on the right-hand side of (5.28) viz.

$$|b_{\text{pw}}(\rho^{m,1/4}, \zeta^{m+1/2}, \mathcal{R}_M v^{m,1/4})| \leq C(u, v) h^{2\sigma} + \frac{1}{192} \|\zeta^{m+1/2}\|_{\text{pw}}^2. \quad (5.29)$$

Analogous arguments with $m = 1$ establish

$$b_{\text{pw}}(\rho^{1,1/4}, \zeta^{1/2}, \mathcal{R}_M v^{1,1/4}) \leq C(u, v) h^{2\sigma} + \frac{1}{192} \|\zeta^{1/2}\|_{\text{pw}}^2. \quad (5.30)$$

Exactly similar arguments with Lemma 3.5(d) and Lemma 3.4(f) lead to

$$\begin{aligned} b_{\text{pw}}(\rho^{n+1,1/4} - \rho^{n,1/4}, \zeta^{n+1/2}, \mathcal{R}_M v^{n+1,1/4}) &\leq C_3 C_5 \sqrt{k/2} h^\sigma \|u_t\|_{L^2(t_{n-1}, t_{n+2}; H^{2+\sigma}(\Omega))} \|v\|_{C([0,T];H^2(\Omega))} \|\zeta^{n+1/2}\|_{\text{pw}}, \\ b_{\text{pw}}(\rho^{n,1/4}, \zeta^{n+1/2}, \mathcal{R}_M (v^{n+1,1/4} - v^{n,1/4})) &\leq C_3 C_5 \sqrt{k/2} h^\sigma \|u\|_{C([0,T];H^{2+\sigma}(\Omega))} \|v_t\|_{L^2(t_{n-1}, t_{n+2}; H^2(\Omega))} \|\zeta^{n+1/2}\|_{\text{pw}}. \end{aligned}$$

A Young's inequality with $b = \sqrt{k/2} \|\zeta^{n+1/2}\|_{\text{pw}}$, $\epsilon = 1$, $a = C_3 C_5 h^\sigma \|u_t\|_{L^2(t_{n-1}, t_{n+2}; H^{2+\sigma}(\Omega))} \|v\|_{C([0,T];H^2(\Omega))}$ (resp. $a = C_3 C_5 h^\sigma \|u\|_{C([0,T];H^{2+\sigma}(\Omega))} \|v_t\|_{L^2(t_{n-1}, t_{n+2}; H^2(\Omega))}$), and the sum for $n = 1$ to m , and some re-arrangements reveal

$$\sum_{n=1}^{m-1} b_{\text{pw}}(\rho^{n+1,1/4}, \zeta^{n+1/2}, \mathcal{R}_M (v^{n+1,1/4} - v^{n,1/4})) \leq C(u, v_t) h^{2\sigma} + \frac{k}{4} \|\zeta^{n+1/2}\|_{\text{pw}}^2$$

$$\left(\text{resp. } \sum_{n=1}^{m-1} b_{\text{pw}}(\rho^{n+1,1/4} - \rho^{n,1/4}, \zeta^{n+1/2}, \mathcal{R}_M v^{n+1,1/4}) \leq C(u, v) h^{2\sigma} + \frac{k}{4} \|\zeta^{n+1/2}\|_{\text{pw}}^2 \right).$$

A combination of (5.29)-(5.30) and the last two displayed inequalities in (5.28) lead to

$$S_3 \leq C(u, v, u_t, v_t) h^{2\sigma} + \frac{1}{96} [\|\zeta^{1/2}\|_{\text{pw}}^2 + \|\zeta^{m+1/2}\|_{\text{pw}}^2] + k \sum_{n=0}^{m-1} \|\zeta^{n+1/2}\|_{\text{pw}}^2$$

with the addition of non-negative term $k \|\zeta^{1/2}\|_{\text{pw}}^2$ on the right-hand side. $\square$

Bound for S_4 . The inequalities in (2.7), (3.2), and Lemma 3.5(b) followed by the definition $k\delta_t u^n = u^{n+1/2} - u^{n-1/2}$, and the bound from Lemma 3.4(d) reveal

$$k b_{\text{pw}}(\rho^{n,1/4}, \mathcal{R}_M \delta_t u^n, \theta^{n,1/4}) \leq C_3 C_5 \sqrt{k/2} h^\sigma \|u\|_{C([0,T]; H^{2+\sigma}(\Omega))} \|u_t\|_{L^2(t_{n-1}, t_{n+1}; H^2(\Omega))} \|\theta^{n,1/4}\|_{\text{pw}}.$$

The identity (2.9a), a triangle inequality, and some elementary manipulations lead to

$$\begin{aligned} \frac{k}{T} \sum_{n=1}^m \|\theta^{n,1/4}\|_{\text{pw}}^2 &\leq \frac{k}{2T} \sum_{n=1}^m [\|\theta^{n-1/2}\|_{\text{pw}}^2 + \|\theta^{n+1/2}\|_{\text{pw}}^2] \\ &\leq \frac{k}{2T} [\|\theta^{1/2}\|_{\text{pw}}^2 + \|\theta^{m+1/2}\|_{\text{pw}}^2] + \frac{k}{T} \sum_{n=1}^{m-1} \|\theta^{n+1/2}\|_{\text{pw}}^2 \\ &\leq \frac{1}{2} [\|\theta^{1/2}\|_{\text{pw}}^2 + \|\theta^{m+1/2}\|_{\text{pw}}^2] + \frac{k}{T} \sum_{n=1}^{m-1} \|\theta^{n+1/2}\|_{\text{pw}}^2 \end{aligned} \quad (5.31)$$

with $k \leq T$ for the first two terms on the right-hand side of last inequality. This, a Young's inequality with $a = C_3 C_5 h^\sigma \|u\|_{C([0,T]; H^{2+\sigma}(\Omega))} \|u_t\|_{L^2(t_{n-1}, t_{n+1}; H^2(\Omega))}$, $b = \sqrt{k/2} \|\theta^{n,1/4}\|_{\text{pw}}$, and $\epsilon = 4/3T$ lead to

$$S_4 := 2k \sum_{n=1}^{m-1} b_{\text{pw}}(\rho^{n,1/4}, \mathcal{R}_M \delta_t u^n, \theta^{n,1/4}) \leq C(u, u_t) h^{2\sigma} + \frac{3}{16} [\|\theta^{1/2}\|_{\text{pw}}^2 + \|\theta^{m+1/2}\|_{\text{pw}}^2] + \frac{3k}{8T} \sum_{n=0}^{m-1} \|\theta^{n+1/2}\|_{\text{pw}}^2$$

with an addition of non-negative term $\frac{3k}{8T} \|\theta^{1/2}\|_{\text{pw}}^2$ on the right-hand side. $\square$

Bound for S_5 . The details of the bound for $S_5 := -2k \sum_{n=1}^m b_{\text{pw}}(u^{n,1/4}, \delta_t \zeta^n, \chi^{n,1/4})$ follows verbatim from the bound for S_3 in with $u^{n,1/4}$ replaced by $\rho^{n,1/4}$ and $\chi^{n,1/4}$ by $\mathcal{R}_M v^{n,1/4}$. Hence the details are skipped and the final bound is

$$S_5 \leq C(u, v, u_t, v_t) h^{2\sigma} + \frac{1}{96} [\|\zeta^{1/2}\|_{\text{pw}}^2 + \|\zeta^{m+1/2}\|_{\text{pw}}^2] + k \sum_{n=0}^{n-1} \|\zeta^{n+1/2}\|_{\text{pw}}^2. \quad \square$$

Bound for S_6 . The boundedness from (2.7), Lemma 3.4(b), $k\delta_t \rho^n = \rho^{n+1/2} - \rho^{n-1/2}$ from (2.9b), and Lemma 3.5(c) lead to

$$k b_{\text{pw}}(u^{n,1/4}, \delta_t \rho^n, \theta^{n,1/4}) \leq C_{\text{ds}} C_3 \sqrt{k/2} h^\sigma \|u\|_{C([0,T]; H^2(\Omega))} \|u_t\|_{L^2(t_{n-1}, t_{n+2}; H^{2+\sigma}(\Omega))} \|\theta^{n,1/4}\|_{\text{pw}}.$$

This and analogous arguments as for the bound of S_4 lead to

$$S_6 \leq C(u, u_t) h^{2\sigma} + \frac{3}{16} [\|\theta^{1/2}\|_{\text{pw}}^2 + \|\theta^{m+1/2}\|_{\text{pw}}^2] + \frac{3k}{8T} \sum_{n=0}^{m-1} \|\theta^{n+1/2}\|_{\text{pw}}^2. \quad \square$$

Bound for S_7 . The identity $k\delta_t \zeta^n = \zeta^{n+1/2} - \zeta^{n-1/2}$ from (2.9b) and elementary manipulations analogous to S_3 reveal

$$\begin{aligned} \frac{1}{2} S_7 &:= k \sum_{n=1}^m b_{\text{pw}}(u^{n,1/4}, (I - J) \delta_t \zeta^n, v^{n,1/4}) = b_{\text{pw}}(u^{m,1/4}, (I - J) \zeta^{m+1/2}, v^{m,1/4}) \\ &\quad + b_{\text{pw}}(u^{1,1/4}, (J - I) \zeta^{1/2}, v^{1,1/4}) + \sum_{n=1}^{m-1} b_{\text{pw}}(u^{n+1,1/4} - u^{n,1/4}, (J - I) \zeta^{n+1/2}, v^{n+1,1/4}) \end{aligned}$$

$$+ \sum_{n=1}^{m-1} b_{\text{pw}}(u^{n,1/4}, (J - I)\zeta^{n+1/2}, v^{n+1,1/4} - v^{n,1/4}). \quad (5.32)$$

Lemma 3.3(e) and a Young's inequality with $a = C_8 h^\sigma \|u^{m,1/4}\|_{H^{2+\sigma}(\Omega)} \|v^{m,1/4}\|_{H^2(\Omega)}$, $b = \|\zeta^{m+1/2}\|_{\text{pw}}$, and $\epsilon = 96$ provide

$$\begin{aligned} b_{\text{pw}}(u^{m,1/4}, (I - J)\zeta^{m+1/2}, v^{m,1/4}) &\leq C_8 h^\sigma \|u^{m,1/4}\|_{H^{2+\sigma}(\Omega)} \|\zeta^{m+1/2}\|_{\text{pw}} \|v^{m,1/4}\|_{H^2(\Omega)} \\ &\leq C(u, v) h^{2\sigma} + \frac{1}{192} \|\zeta^{m+1/2}\|_{\text{pw}}^2. \end{aligned} \quad (5.33)$$

Analogous arguments with $m = 1$ show

$$b_{\text{pw}}(u^{1,1/4}, (I - J)\zeta^{1/2}, v^{1,1/4}) \leq C(u, v) h^{2\sigma} + \frac{1}{192} \|\zeta^{1/2}\|_{\text{pw}}^2. \quad (5.34)$$

Utilize Lemma 3.3(e), Lemma 3.4(b, f), and a Young's inequality with $a = C_8 h^\sigma \|u_t\|_{L^2(t_{n-1}, t_{n+2}; H^{2+\sigma}(\Omega))} \|v\|_{C(0, T; H^2(\Omega))}$ (resp. $a = C_8 h^\sigma \|u\|_{C(0, T; H^{2+\sigma}(\Omega))} \|v_t\|_{L^2(t_{n-1}, t_{n+2}; H^2(\Omega))}$), $b = \sqrt{k/2} \|\zeta^{n+1/2}\|_{\text{pw}}$, $\epsilon = 1$, followed by some elementary manipulations to bound the third (resp. fourth) term on the right-hand side of (5.32) as

$$\begin{aligned} \sum_{n=1}^{m-1} b_{\text{pw}}(u^{n+1,1/4} - u^{n,1/4}, (J - I)\zeta^{n+1/2}, v^{n+1,1/4}) &\leq C(u_t, v) h^{2\sigma} + \frac{k}{4} \sum_{n=1}^{m-1} \|\zeta^{n+1/2}\|_{\text{pw}}^2 \\ \left(\text{resp. } \sum_{n=1}^{m-1} b_{\text{pw}}(u^{n,1/4}, (J - I)\zeta^{n+1/2}, v^{n+1,1/4} - v^{n,1/4}) \right) &\leq C(u, v_t) h^{2\sigma} + \frac{k}{4} \sum_{n=1}^{m-1} \|\zeta^{n+1/2}\|_{\text{pw}}^2. \end{aligned}$$

Combining this estimate with (5.33)-(5.34) in (5.32) leads to

$$S_7 \leq C(u, v, u_t, v_t) h^{2\sigma} + \frac{1}{96} [\|\zeta^{1/2}\|_{\text{pw}}^2 + \|\zeta^{m+1/2}\|_{\text{pw}}^2] + k \sum_{n=0}^{m-1} \|\zeta^{n+1/2}\|_{\text{pw}}^2. \quad \square$$

Bound for S_8 . Apply Lemma 3.3(d), $k\delta_t u^n = u^{n+1/2} - u^{n-1/2}$ from (2.9b), and Lemma 3.4(b, d) to obtain

$$\begin{aligned} k b_{\text{pw}}(\delta_t u^n, u^{n,1/4}, (J - I)\theta^{n,1/4}) &\leq C_7 h^\sigma \|k\delta_t u^n\|_{\text{pw}} \|u^{n,1/4}\|_{\text{pw}} \|\theta^{n,1/4}\|_{\text{pw}} \\ &\leq C_7 \sqrt{k/2} h^\sigma \|u_t\|_{L^2(t_{n-1}, t_{n+1}; H^2(\Omega))} \|u\|_{C([0, T]; H^2(\Omega))} \|\theta^{n,1/4}\|_{\text{pw}}. \end{aligned}$$

This, a Young's inequality with $a = C_7 h^\sigma \|u_t\|_{L^2(t_{n-1}, t_{n+1}; H^2(\Omega))} \|u\|_{C([0, T]; H^2(\Omega))}$, $b = \sqrt{k/2} \|\theta^{n,1/4}\|_{\text{pw}}$, and $\epsilon = 4/3T$ followed by some elementary manipulations and (5.31) lead to

$$S_8 := 2k \sum_{n=1}^m b_{\text{pw}}(\delta_t u^n, u^{n,1/4}, (J - I)\theta^{n,1/4}) \leq C(u, u_t) h^{2\sigma} + \frac{3}{16} [\|\theta^{1/2}\|_{\text{pw}}^2 + \|\theta^{m+1/2}\|_{\text{pw}}^2] + \frac{3k}{8T} \sum_{n=0}^{m-1} \|\theta^{n+1/2}\|_{\text{pw}}^2$$

with an addition of the non-negative term $\frac{3k}{8T} \|\theta^{1/2}\|_{\text{pw}}^2$ on the right-hand side. A combination of (5.14), (5.21), and the bounds for $S_1, S_2, \dots, S_8$ from above conclude the proof. $\square$

The subsequent lemma controls the truncation error for the nonlinear term T_4 (in error equation (5.14)).

Lemma 5.3 (Truncation error for the nonlinear term T_4). *If $u_t, v_t \in H^1(0, T; H^2(\Omega))$, then for any m with $1 \leq m \leq N - 1$, the following bound holds:*

$$\begin{aligned} T_4 &:= k^3 \sum_{n=1}^m \left[b(\delta_t u^n, J\delta_t \zeta^n, \delta_t v^n) + \frac{k^2}{8} b(\bar{\delta}_t^2 u^n, J\delta_t \zeta^n, \bar{\delta}_t^2 v^n) + \frac{1}{2} b(\bar{\delta}_t^2 u^n, \delta_t u^n, J\theta^{n,1/4}) \right] \\ &\leq C(u_t, u_{tt}, v_t, v_{tt}) k^4 + \frac{1}{16} [\|\zeta^{1/2}\|_{\text{pw}}^2 + \|\theta^{1/2}\|_{\text{pw}}^2 + \|\zeta^{m+1/2}\|_{\text{pw}}^2 + \|\theta^{m+1/2}\|_{\text{pw}}^2] \\ &\quad + k \left(1 + \frac{1}{16T} \right) \sum_{n=0}^{m-1} \|\zeta^{n+1/2}\|_{\text{pw}}^2 + \frac{k}{8T} \sum_{n=0}^{m-1} \|\theta^{n+1/2}\|_{\text{pw}}^2. \end{aligned}$$

Proof. The identity $k\delta_t\zeta^n = \zeta^{n+1/2} - \zeta^{n-1/2}$ from (2.9b) and some elementary algebra lead to

$$\begin{aligned} k \sum_{n=1}^m b(\delta_t u^n, J\delta_t \zeta^n, \delta_t v^n) &= b(\delta_t u^m, J\zeta^{m+1/2}, \delta_t v^m) - b(\delta_t u^1, J\zeta^{1/2}, \delta_t v^1) \\ &\quad - \sum_{n=1}^{m-1} b(\delta_t u^{n+1} - \delta_t u^n, J\zeta^{n+1/2}, \delta_t v^{n+1}) - \sum_{n=1}^{m-1} b(\delta_t u^n, J\zeta^{n+1/2}, \delta_t v^{n+1} - \delta_t v^n). \end{aligned} \quad (5.35)$$

We establish bounds for the terms on the right-hand side of (5.35). Lemma 3.3(a), (5.23) (and then with v replaced by u) reveal

$$\begin{aligned} b(\delta_t u^m, J\zeta^{m+1/2}, \delta_t v^m) &\leq C_4 \|\delta_t u^m\|_{pw} \|\zeta^{m+1/2}\|_{pw} \|\delta_t v^m\|_{pw} \\ &\leq C_4 \|u_t\|_{L^\infty(0,T;H^2(\Omega))} \|v_t\|_{L^\infty(0,T;H^2(\Omega))} \|\zeta^{m+1/2}\|_{pw}. \end{aligned}$$

A Young's inequality with $a = C_4 \|u_t\|_{L^\infty(0,T;H^2(\Omega))} \|v_t\|_{L^\infty(0,T;H^2(\Omega))}$, $b = \|\zeta^{m+1/2}\|_{pw}$, and $\epsilon = 16k^2$ provides

$$b(\delta_t u^m, J\zeta^{m+1/2}, \delta_t v^m) \leq C(u_t, v_t)k^2 + \frac{k^{-2}}{32} \|\zeta^{m+1/2}\|_{pw}^2. \quad (5.36)$$

Analogous arguments show

$$|b(\delta_t u^1, J\zeta^{1/2}, \delta_t v^1)| \leq C(u_t, v_t)k^2 + \frac{k^{-2}}{32} \|\zeta^{1/2}\|_{pw}^2. \quad (5.37)$$

Utilize Lemma 3.3(a), the identity $\delta_t u^{n+1} - \delta_t u^n = \frac{k}{2}(\bar{\partial}_t^2 u^{n+1} + \bar{\partial}_t^2 u^n)$ (resp. $\delta_t v^{n+1} - \delta_t v^n = \frac{k}{2}(\bar{\partial}_t^2 v^{n+1} + \bar{\partial}_t^2 v^n)$) obtained from (2.8b), and (5.23) (for v and u) to obtain

$$\begin{aligned} |b(\delta_t u^{n+1} - \delta_t u^n, J\zeta^{n+1/2}, \delta_t v^{n+1})| &\leq \frac{C_4}{2} k \|\bar{\partial}_t^2 u^{n+1} + \bar{\partial}_t^2 u^n\|_{pw} \|v_t\|_{L^\infty(0,T;H^2(\Omega))} \|\zeta^{n+1/2}\|_{pw}, \\ (\text{resp. } |b(\delta_t u^n, J\zeta^{n+1/2}, \delta_t v^{n+1} - \delta_t v^n)| &\leq \frac{C_4}{2} k \|u_t\|_{L^\infty(0,T;H^2(\Omega))} \|\bar{\partial}_t^2 v^{n+1} + \bar{\partial}_t^2 v^n\|_{pw} \|\zeta^{n+1/2}\|_{pw}). \end{aligned}$$

A Young's inequality with $a = \frac{C_4}{2} k \|\bar{\partial}_t^2 u^{n+1} + \bar{\partial}_t^2 u^n\|_{pw} \|v_t\|_{L^\infty(0,T;H^2(\Omega))}$, $b = \|\zeta^{n+1/2}\|_{pw}$, and $\epsilon = k$ (resp. $a = \frac{C_4}{2} k \|u_t\|_{L^\infty(0,T;H^2(\Omega))} \|\bar{\partial}_t^2 v^{n+1} + \bar{\partial}_t^2 v^n\|_{pw}$, $b = \|\zeta^{n+1/2}\|_{pw}$ and $\epsilon = k$) and the triangle inequality lead to

$$\begin{aligned} |b(\delta_t u^{n+1} - \delta_t u^n, J\zeta^{n+1/2}, \delta_t v^{n+1})| &\leq \frac{C_4^2}{4} k^3 (\|\bar{\partial}_t^2 u^{n+1}\|_{pw}^2 + \|\bar{\partial}_t^2 u^n\|_{pw}^2) \|v_t\|_{L^\infty(0,T;H^2(\Omega))}^2 + \frac{1}{2k} \|\zeta^{n+1/2}\|_{pw}^2, \\ |b(\delta_t u^n, J\zeta^{n+1/2}, \delta_t v^{n+1} - \delta_t v^n)| &\leq \frac{C_4^2}{4} k^3 \|u_t\|_{L^\infty(0,T;H^2(\Omega))}^2 (\|\bar{\partial}_t^2 v^{n+1}\|_{pw}^2 + \|\bar{\partial}_t^2 v^n\|_{pw}^2) + \frac{1}{2k} \|\zeta^{n+1/2}\|_{pw}^2. \end{aligned}$$

A combination of these two inequalities and (5.36)-(5.37) in (5.35), the estimate $\sum_{n=1}^{m-1} (\|\bar{\partial}_t^2 \varphi^{n+1}\|_{pw}^2 + \|\bar{\partial}_t^2 \varphi^n\|_{pw}^2) \leq 2 \sum_{n=1}^m \|\bar{\partial}_t^2 \varphi^n\|_{H^2(\Omega)}$, and Lemma 3.4(g) (applied twice) reveals

$$k^3 \sum_{n=1}^m b(\delta_t u^n, J\delta_t \zeta^n, \delta_t v^n) \leq C(u_t, u_{tt}, v_t, v_{tt})k^4 + \frac{1}{32} [\|\zeta^{1/2}\|_{pw}^2 + \|\zeta^{m+1/2}\|_{pw}^2] + k \sum_{n=1}^{m-1} \|\zeta^{n+1/2}\|_{pw}^2. \quad (5.38)$$

Utilizing Lemma 3.3(a), the inequality $\|k\bar{\partial}_t^2 v^n\|_{H^2(\Omega)} \leq \|\bar{\partial}_t v^{n+1/2}\|_{H^2(\Omega)} + \|\bar{\partial}_t v^{n-1/2}\|_{H^2(\Omega)} \leq 2\|v_t\|_{L^\infty(0,T;H^2(\Omega))}$ (obtained from triangle inequality and Lemma 3.4(c), and Young's inequality with $a = 2C_4 \|\bar{\partial}_t^2 u^n\|_{H^2(\Omega)} \|v_t\|_{L^\infty(0,T;H^2(\Omega))}$, $b = \|\delta_t \zeta^n\|_{pw}$, and $\epsilon = 4Tk$, gives

$$\begin{aligned} kb(\bar{\partial}_t^2 u^n, J\delta_t \zeta^n, \bar{\partial}_t^2 v^n) &\leq C_4 \|\bar{\partial}_t^2 u^n\|_{pw} \|\delta_t \zeta^n\|_{pw} \|k\bar{\partial}_t^2 v^n\|_{pw} \leq 2C_4 \|\bar{\partial}_t^2 u^n\|_{H^2(\Omega)} \|v_t\|_{L^\infty(0,T;H^2(\Omega))} \|\delta_t \zeta^n\|_{pw} \\ &\leq 8C_4^2 Tk \|\bar{\partial}_t^2 u^n\|_{H^2(\Omega)}^2 \|v_t\|_{L^\infty(0,T;H^2(\Omega))}^2 + \frac{1}{8Tk} \|\delta_t \zeta^n\|_{pw}^2. \end{aligned}$$

A summation from $n = 1$ to m and Lemma 3.4(g) shows

$$\frac{k^5}{8} \sum_{n=1}^m b(\bar{\partial}_t^2 u^n, J\delta_t \zeta^n, \bar{\partial}_t^2 v^n) \leq C(u_{tt}, v_t)k^4 + \frac{k^3}{64T} \sum_{n=1}^m \|\delta_t \zeta^n\|_{pw}^2.$$

The identity $k\delta_t \zeta^n = \zeta^{n+1/2} - \zeta^{n-1/2}$ from (2.9b) followed by the triangle inequality leads to

$$\begin{aligned} \frac{k^3}{4T} \sum_{n=1}^m \|\delta_t \zeta^n\|_{pw}^2 &\leq \frac{k}{2T} \sum_{n=1}^m [\|\zeta^{n-1/2}\|_{pw}^2 + \|\zeta^{n+1/2}\|_{pw}^2] \leq \frac{k}{2T} [\|\zeta^{1/2}\|_{pw}^2 + \|\zeta^{m+1/2}\|_{pw}^2] \\ &\quad + \frac{k}{T} \sum_{n=1}^{m-1} \|\zeta^{n+1/2}\|_{pw}^2 \leq \frac{1}{2} [\|\zeta^{1/2}\|_{pw}^2 + \|\zeta^{m+1/2}\|_{pw}^2] + \frac{k}{T} \sum_{n=1}^{m-1} \|\zeta^{n+1/2}\|_{pw}^2 \end{aligned}$$

with $k \leq T$ for the first two terms on the right-hand side. A combination of the last two inequalities reveal

$$\frac{k^5}{8} \sum_{n=1}^m b(\bar{\partial}_t^2 u^n, J\delta_t \zeta^n, \bar{\partial}_t^2 v^n) \leq C(u_{tt}, v_t)k^4 + \frac{1}{32} [\|\zeta^{1/2}\|_{pw}^2 + \|\zeta^{m+1/2}\|_{pw}^2] + \frac{k}{16T} \sum_{n=1}^{m-1} \|\zeta^{n+1/2}\|_{pw}^2.$$

Lemma 3.3(a) and Lemma 3.4(e), a Young's inequality with $a = C_4 \|\bar{\partial}_t^2 u^n\|_{H^2(\Omega)} \|u_t\|_{L^\infty(0,T;H^2(\Omega))}$, $b = \|\theta^{n,1/4}\|_{pw}$, and $\epsilon = 2Tk^2$ reveal

$$b(\bar{\partial}_t^2 u^n, \delta_t u^n, J\theta^{n,1/4}) \leq C_4^2 T k^2 \|\bar{\partial}_t^2 u^n\|_{H^2(\Omega)}^2 \|u_t\|_{L^\infty(0,T;H^2(\Omega))}^2 + \frac{1}{4Tk^2} \|\theta^{n,1/4}\|_{pw}^2.$$

This, together with the estimate from Lemma 3.4(g) and (5.31) lead to

$$\frac{k^3}{2} \sum_{n=1}^m b(\bar{\partial}_t^2 u^n, \delta_t u^n, J\theta^{n,1/4}) \leq C(u_t, u_{tt})k^4 + \frac{1}{16} [\|\theta^{1/2}\|_{pw}^2 + \|\theta^{m+1/2}\|_{pw}^2] + \frac{k}{8T} \sum_{n=1}^{m-1} \|\theta^{n+1/2}\|_{pw}^2. \quad (5.39)$$

The proof follows by (5.38)-(5.39), adding $k(1 + \frac{1}{16T}) \|\zeta^{1/2}\|_{pw}^2 + \frac{k}{8T} \|\theta^{1/2}\|_{pw}^2$ on the right-hand side. $\square$

We are in position to present the proof of Theorem 2.5, which utilizes the bounds for T_1 - T_4 from (4.21) and Lemmas 5.1-5.3 into the error equation (5.14) and concludes via a discrete Gronwall inequality.

Proof of Theorem 2.5. The bounds for T_2 , T_3 , and T_4 from Lemmas 5.1-5.3 in (5.14) lead to

$$\begin{aligned} \frac{1}{8} [\|\bar{\partial}_t \zeta^{m+1/2}\|^2 + \|\zeta^{m+1/2}\|_{pw}^2 + \|\theta^{m+1/2}\|_{pw}^2] &\leq C(u, u_t, u_{tt}, u_{ttt}, u_{tttt}, v, v_t, v_{tt}, f, f_t)(h^{2\sigma} + k^4) \\ &\quad + \frac{15}{8} \|\bar{\partial}_t \zeta^{1/2}\|^2 + \frac{13}{8} \|\zeta^{1/2}\|_{pw}^2 + \frac{13}{14} \|\theta^{1/2}\|_{pw}^2 + \frac{\mu^* k}{T} \sum_{n=0}^m \frac{1}{8} [\|\bar{\partial}_t \zeta^{n+1/2}\|^2 + \|\zeta^{n+1/2}\|_{pw}^2 + \|\theta^{n+1/2}\|_{pw}^2] \end{aligned}$$

with the constant

$$\mu^* = \max \left\{ 8, 8T(C_5 [\|u_t\|_{L^\infty(0,T;H^2(\Omega))} + \|v_t\|_{L^\infty(0,T;H^2(\Omega))}] + 5 + \frac{1}{16T}), 8TC_5 \|u_t\|_{L^\infty(0,T;H^2(\Omega))} + 10 \right\}.$$

Next, we apply Lemma 3.6 with the following setting: $a_m = \frac{1}{8} [\|\bar{\partial}_t \zeta^{m+1/2}\|^2 + \|\zeta^{m+1/2}\|_{pw}^2 + \|\theta^{m+1/2}\|_{pw}^2]$, $b_m = 0$, $c_m = C(u, u_t, u_{tt}, u_{ttt}, u_{tttt}, v, v_t, v_{tt}, f, f_t)(h^{2\sigma} + k^4) + \frac{15}{8} \|\bar{\partial}_t \zeta^{1/2}\|^2 + \frac{43}{32} \|\zeta^{1/2}\|_{pw}^2 + \frac{13}{14} \|\theta^{1/2}\|_{pw}^2$, and $\mu = \mu^* k/T$, and use that $mk \leq T$ (which implies $e^{\frac{mk}{T} \mu^*} \leq e^{\mu^*}$) and the bounds from (4.21) to obtain

$$\|\bar{\partial}_t \zeta^{m+1/2}\|^2 + \|\zeta^{m+1/2}\|_{pw}^2 + \|\theta^{m+1/2}\|_{pw}^2 \lesssim h^{2\sigma} + k^4. \quad (5.40)$$

Inequality (5.40) plus triangle inequality (applied thrice) show $\|\bar{\partial}_t(u^{m+1/2} - u_M^{m+1/2})\| + \|u^{m+1/2} - u_M^{m+1/2}\|_{pw} + \|v^{m+1/2} - v_M^{m+1/2}\|_{pw} \leq \|\bar{\partial}_t \rho^{m+1/2}\| + \|\rho^{m+1/2}\|_{pw} + \|\chi^{m+1/2}\|_{pw} + \|\bar{\partial}_t \zeta^{m+1/2}\| + \|\zeta^{m+1/2}\|_{pw} + \|\theta^{m+1/2}\|_{pw}$. Applying Lemma 3.2 for the first three terms and (5.40) to the remaining terms, we conclude the proof.

6 Numerical experiments

In this section we present the outcomes of our numerical experiments performed using the fully-discrete scheme described in Section 2, with the goal to confirm the theoretical findings established in Theorems 2.3 and 2.5. The numerical realization of the fully-discrete formulation (2.10)-(2.12) is carried out in an open-source finite element library FreeFem++ [26]. All simulations were performed on a workstation equipped with a 32-core processor, 2×32 GB

RAM, a 2 TB enterprise HDD, and an NVIDIA RTX 4000 Ada 20 GB GDDR6 GPU, housed in a convertible tower cabinet. A Newton's iterative procedure is utilized to solve the nonlinear system arising at each time level, under a user-specified tolerance of $\text{tol} := 10^{-7}$. The details of Newton's iterations are given in the Table 6.1. Here (u_M^n, v_M^n) denotes the i^{th} Newton iterate for the discrete solution at time level t_n , obtained by linearizing (2.10) and (2.13) about the previous iterate (u_M^{n-1}, v_M^{n-1}) . At each time step t_n with $n \geq 1$, the initial guess for the Newton loop is taken to be the converged solution (u_M^{n-1}, v_M^{n-1}) from the previous time step, with the solution at $t = t_0$ itself obtained directly from the linear system (2.12) and hence requiring no Newton iteration. The loop terminates, i.e., the range of i closes, as soon as the increment between successive iterates falls below tol . For the tests in this article, Newton's method takes

Newton iterations ($\text{tol} = 10^{-7}$)

Initial step corresponding to (2.13):

$$\begin{aligned} & 2k^{-2}(u_M^1 - u_M^0 - ku_1, \varphi_M)_{L^2(\Omega)} + \frac{1}{2}a_{\text{pw}}(u_M^1 + u_M^0, \varphi_M) + \frac{1}{4}b_{\text{pw}}(u_M^1, \varphi_M, v_M^1 + v_M^0) \\ & + \frac{1}{4}b_{\text{pw}}(u_M^1 + u_M^0, \varphi_M, v_M^1) = -\frac{1}{4}b_{\text{pw}}(u_M^0, \varphi_M, v_M^0) + \frac{1}{4}b_{\text{pw}}(u_M^1, \varphi_M, v_M^1) + (f^{1/2}, \varphi_M)_{L^2(\Omega)}, \\ & a_{\text{pw}}(v_M^1, \psi_M) - b_{\text{pw}}(u_M^1, u_M^1, \psi_M) = -\frac{1}{2}b_{\text{pw}}(u_M^1, u_M^1, \psi_M) + (g^1, \psi_M)_{L^2(\Omega)}. \end{aligned}$$

General step ($n \geq 1$) corresponding to (2.10):

$$\begin{aligned} & k^{-2}(u_M^{n+1} - 2u_M^n + u_M^{n-1}, \varphi_M)_{L^2(\Omega)} + \frac{1}{4}a_{\text{pw}}(u_M^{n+1} + 2u_M^n + u_M^{n-1}, \varphi_M) + \frac{1}{16}b_{\text{pw}}(u_M^{n+1}, \varphi_M, v_M^{n+1} + 2v_M^n + v_M^{n-1}) \\ & + \frac{1}{16}b_{\text{pw}}(u_M^{n+1} + 2u_M^n + u_M^{n-1}, \varphi_M, v_M^{n+1}) = -\frac{1}{16}b_{\text{pw}}(2u_M^n + u_M^{n-1}, \varphi_M, 2v_M^n + v_M^{n-1}) + \frac{1}{16}b_{\text{pw}}(u_M^{n+1}, \varphi_M, v_M^{n+1}) + (f^{n,1/4}, \varphi_M)_{L^2(\Omega)}, \\ & \frac{1}{2}a_{\text{pw}}(v_M^{n+1} + v_M^n, \psi_M) - \frac{1}{4}b_{\text{pw}}(u_M^{n+1}, u_M^{n+1} + u_M^n, \psi_M) = \frac{1}{8}b_{\text{pw}}(u_M^n, u_M^n, \psi_M) - \frac{1}{8}b_{\text{pw}}(u_M^{n+1}, u_M^{n+1}, \psi_M) + (g^{n+1/2}, \psi_M)_{L^2(\Omega)}. \end{aligned}$$

Table 6.1: Newton's iterative procedure for (2.10) and (2.13).

a maximum of four iterations per time step to converge.

The following notation is adopted for the convergence history tables in the rest of this subsection:

$$\begin{aligned} \text{Error}_0(u) &= \max_{0 \leq n \leq N} \|u^n - u_M^n\|, \quad \text{Error}_0(v) = \max_{0 \leq n \leq N} \|v^n - v_M^n\|, \\ \text{Error}_2(u) &= \max_{0 \leq n \leq N-1} \|u^{n+1/2} - u_M^{n+1/2}\|_{\text{pw}}, \quad \text{Error}_2(v) = \max_{0 \leq n \leq N-1} \|v^{n+1/2} - v_M^{n+1/2}\|_{\text{pw}}. \end{aligned}$$

Example 1. (Convergence for smooth exact solution) We choose the right-hand side load functions (f, g) such that

$$\begin{aligned} f(x, y, t) &= u_{tt} + \Delta^2 u - [u, v], \\ g(x, y, t) &= \Delta^2 v + \frac{1}{2}[u, u], \end{aligned}$$

is satisfied on the domain $\Omega \times [0, T] = (0, 1)^2 \times [0, 1]$ with the exact solution (u, v) given by

$$u(x, y, t) = (1/10 \sin(2\pi t) \sin(\pi x) \sin(\pi y))^2 \quad \text{and} \quad v(x, y, t) = \exp(t)(x(x-1)y(y-1))^2.$$

To compute the numerical convergence rates both in spatial and time discretization, we apply $k = h/4$ at each discretization step. Table 6.2 shows the errors and experimental convergence rates for the variables u_M^n and v_M^n . The computational order of convergences in L^2 and energy norms are quasi-optimal and verify the theoretical results obtained in Theorem 2.3 and 2.5.

Example 2. (2D-3D plate deflection comparison) Let $\Omega \subset \mathbb{R}^2$ be the mid-surface (assumed to lie in alignment with the xy -axis) of a thin, isotropic, flat plate $\hat{\Omega} = [0, 1] \times [0, 1] \times [-d/2, d/2]$ with a uniform thickness d . Building upon [19], given a space-time dependent loading $(\hat{f}_1(t), \hat{f}_2(t), \hat{f}_3(t)) = \mathbf{f}(t) : \hat{\Omega} \rightarrow \mathbb{R}^3$, the dynamic 3D von Kármán model seeks the displacement field $\mathbf{u} = (\hat{u}_i)_{i=1}^3$ and the Piola–Kirchhoff stress tensor $\boldsymbol{\sigma} = (\hat{\sigma}_{ij})_{i,j=1}^3$ such that

$$\mathbf{u}_{tt} - \partial_j (\hat{\sigma}_{ij} + \hat{\sigma}_{kj} \partial_k \hat{u}_i) = \mathbf{f}(\hat{\mathbf{x}}, t) \quad \text{for } (\hat{\mathbf{x}}, t) \in \hat{\Omega} \times (0, T], \quad (6.1a)$$

h	Error ₀ (u)	Rate	Error ₂ (u)	Rate	Error ₀ (v)	Rate	Error ₂ (v)	Rate
0.707	1.47e-02	★	2.89e-01	★	6.94e-03	★	1.90e-01	★
0.354	3.37e-03	2.124	1.14e-01	1.346	3.82e-03	0.859	1.39e-01	0.452
0.177	8.24e-04	2.032	6.01e-02	0.921	1.14e-03	1.749	7.72e-02	0.849
0.088	2.14e-04	1.948	3.08e-02	0.965	3.00e-04	1.925	3.99e-02	0.952
0.044	5.18e-05	2.042	1.55e-02	0.992	7.60e-05	1.979	2.02e-02	0.984
0.022	1.28e-05	2.020	7.76e-03	0.996	1.91e-05	1.994	1.01e-02	0.995
0.011	3.18e-06	2.006	3.88e-03	0.999	4.77e-06	1.999	5.07e-03	0.998
0.006	7.94e-07	2.003	1.94e-03	1.000	1.19e-06	2.000	2.54e-03	0.999

Table 6.2: Convergence history in L^2 and energy norms with smooth exact solution (u, v) .

$$\left(\frac{1+\nu}{E}\right) \hat{\sigma}_{ij} - \frac{\nu}{E} \hat{\sigma}_{ll} \delta_{ij} = \frac{1}{2} (\partial_i \hat{u}_j + \partial_j \hat{u}_i + \partial_i \hat{u}_l \partial_j \hat{u}_l) \quad \text{for } (\hat{\mathbf{x}}, t) \in \hat{\Omega} \times [0, T], \quad (6.1b)$$

$$\text{with the boundary conditions, } \hat{u}_3 = 0, \text{ and } \hat{u}_1, \hat{u}_2 \text{ are independent of } z \text{ on } \partial\Omega \times [d/2, d/2] \times (0, T], \quad (6.1c)$$

$$(\hat{\sigma}_{i3} + \hat{\sigma}_{l3} \partial_l \hat{u}_i) = 0 \text{ on } \Omega \times \{\pm d/2\} \times (0, T], \quad \frac{1}{d} \int_{-d/2}^{d/2} (\hat{\sigma}_{rs} + \hat{\sigma}_{ls} \partial_l \hat{u}_s) n_r = 0 \text{ on } \partial\Omega \times (0, T] \quad (6.1d)$$

with $i, j, l \in \{1, 2, 3\}$ and $r, s \in \{1, 2\}$. Here, E is Young's modulus, ν is the Poisson's ratio, and $\mathbf{n} = (n_1, n_2)$ is the unit outer normal to $\partial\Omega$.

A dimensional reduction analysis in [19] for static case derives a 2D model (6.2) from the 3D model (6.1) as: given the load function $f = \frac{1}{d} \int_{-d/2}^{d/2} \hat{f}_3 \, dz$, seek transverse displacement averaged over thickness, $u = \frac{1}{d} \int_{-d/2}^{d/2} \hat{u}_3 \, dz$, and the Airy stress function v such that [19, eqn. 6.11-6.15]

$$u_{tt} + \frac{d^2 E}{12(1-\nu^2)} \Delta^2 u - [u, v] = f(\mathbf{x}, t) \quad \text{for } (\mathbf{x}, t) \in \Omega \times (0, T] \quad (6.2a)$$

$$\frac{1}{E} \Delta^2 v + \frac{1}{2} [u, u] = 0 \quad (\text{for } \mathbf{x}, t) \in \Omega \times [0, T] \quad (6.2b)$$

$$u = \frac{\partial u}{\partial \mathbf{n}} = 0, \quad v = \frac{\partial v}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega \times (0, T]. \quad (6.2c)$$

Our objective is to illustrate that the 2D von Kármán model (6.2), effectively approximates the 3D von Kármán model (6.1) in the sense that if U^n is the approximation of the displacement u of the 2D model (6.2) at time $t = t_n$ computed with the discrete formulation (2.10) (with the coefficients $\frac{d^2 E}{12(1-\nu^2)}$ (resp. $1/E$) in bilinear forms of (2.10a) (resp. (2.10b)) and $\hat{\mathbf{U}}^n$ is the discrete solution of (6.1) at $t = t_n$, then U^n approximates $\frac{1}{d} \int_{-d/2}^{d/2} \hat{\mathbf{U}}_3^n \, dz$ with $\hat{\mathbf{U}}^n = (\hat{U}_1^n, \hat{U}_2^n, \hat{U}_3^n)$. To achieve this, we solve the 3D system (6.1) using continuous H^1 -conforming Lagrange finite element space: $(\mathcal{P}_1(\mathcal{T}))^3$ for displacement $\mathbf{u}$, and $(\mathcal{P}_0(\mathcal{T}))^9$ -dG for stress. The temporal discretization is done using the modified Newmark scheme. Homogeneous Dirichlet boundary conditions are set on all the sides except the surfaces $z = -d/2, d/2$ where the plate is assumed traction free. The 3D load function reads

$$\mathbf{f} = \left(0, 0, \frac{1}{100} t \sin(\pi x) \sin(\pi y)\right). \quad (6.3)$$

For 2D, we utilize $f = \frac{1}{d} \int_{-d/2}^{d/2} \hat{f}_3^n \, dz$ to obtain $f = dz = \frac{1}{100} t \sin(\pi x) \sin(\pi y)$. The initial conditions are set to zero in both 2D and 3D cases. The elastic parameters take typical values for copper plates from [36]. Let $T = 1$, $\Delta t = 1/20$ and consider the cells $\hat{\Omega}_c = [5/64, 6/64] \times [5/64, 6/64] \times [-d/2, d/2]$ and $\Omega_c = [5/64, 6/64] \times [5/64, 6/64]$. At time $t = t_n$, we will use the following output quantities

$$U_{3D}^n := \frac{1}{|\hat{\Omega}_c|} \int_{\hat{\Omega}_c} \hat{U}_3 \, d\hat{\mathbf{x}} \text{ for } \hat{\mathbf{x}} = (x, y, z) \text{ and } U_{2D}^n := \frac{1}{|\Omega_c|} \int_{\Omega_c} U^n \, d\mathbf{x} \text{ for } \mathbf{x} = (x, y).$$

The simulations in Figure 6.1 reveal that as the plate thickness d decreases the results of the 2D model approximate those of the 3D model. As expected, the computational efficiency is significantly improved: The 2D model takes approximately 25% of time than that of 3D one.

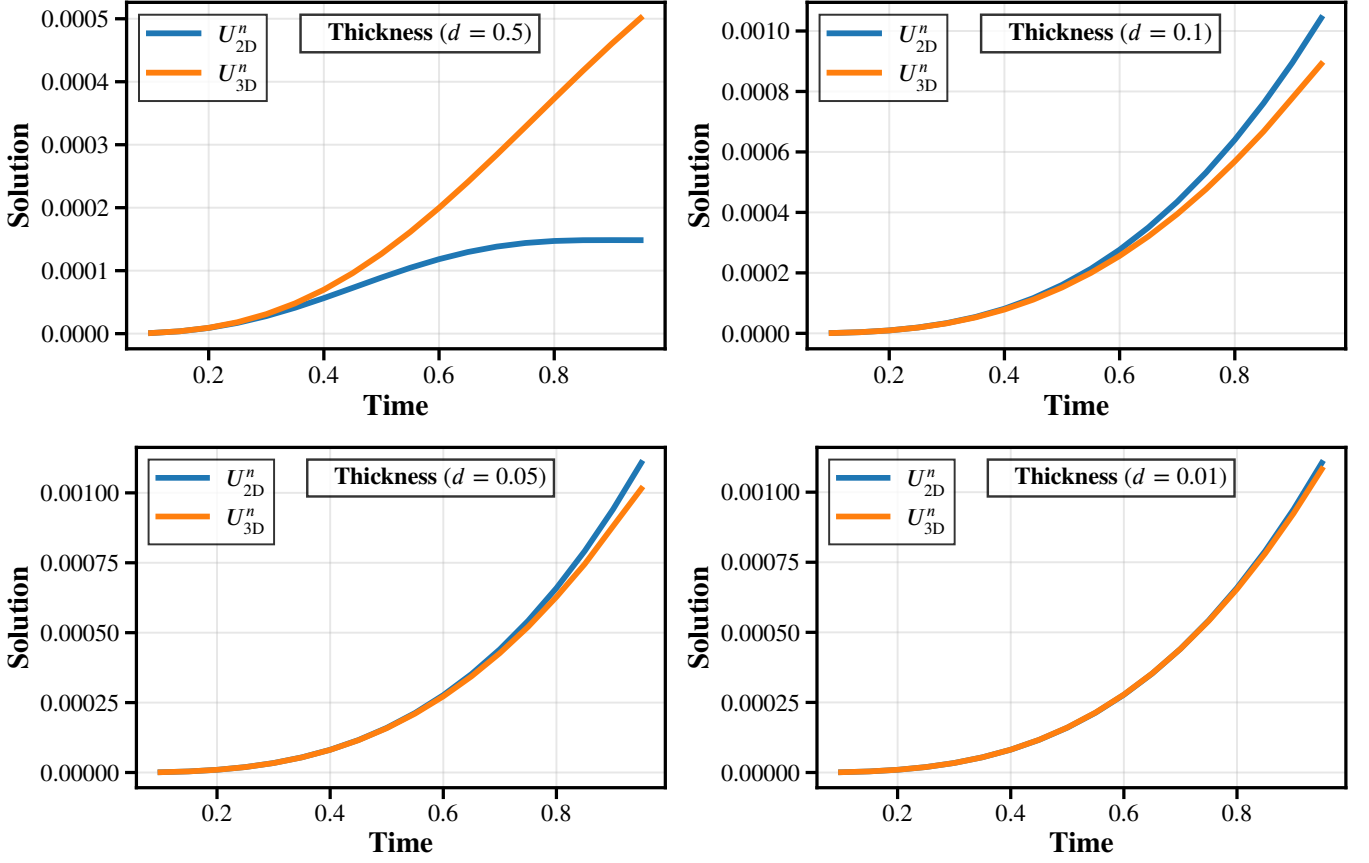


Figure 6.1: Example 2. 2D displacement U_{2D}^n and 3D displacement U_{3D}^n vs time t_n with plate thickness $d = 0.5, 0.1, 0.05, 0.01$.

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