

Parameter-robust well-posedness and discretisation for coupled Darcy–Forchheimer and advection–diffusion–reaction equations*

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Abstract

We adapt the recent theory for unique solvability of perturbed saddle-point problems in Banach spaces to the case of parameter-independent stability bounds. This also constitutes an extension to the Brezzi–Braess theory for parameter-robust stability of perturbed saddle-point problems from the Hilbert to the Banach setting. We apply the new abstract result to the mixed formulation of the advection–diffusion–reaction equation and tackle also its coupling with the Darcy–Forchheimer equations. The complete system is shown to be stable irrespective of the model parameters of permeability, Forchheimer coefficient, and reaction modulation. We discretise the problem with mixed finite element methods and show convergence in appropriately parameter-weighted norms. We also design operator-based preconditioners for the full system, utilising weighted norms that provide robustness with respect to some model parameters. We also present a few numerical examples that show the convergence and robustness of the proposed method in some regimes of practical interest.

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1 Introduction

This paper investigates the robust parameter stability of a system of the Darcy–Forchheimer equations coupled with the advection–diffusion–reaction (ADR) equation as presented in [31]. Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, denote a bounded domain with Lipschitz-continuous boundary $\partial\Omega$, on which its outward unit normal is denoted by \mathbf{n} . The boundary is decomposed into two sub-boundaries Γ_u and Γ_p , both with positive $(d - 1)$ –Hausdorff measure. The Darcy–Forchheimer equations are written in their discharge velocity and pressure head formulation \mathbf{u}, p featuring a nonlinear drag term, and the ADR transport using a conservative form in terms of total (diffusive plus advective) flux $\boldsymbol{\zeta}$ and solute concentration φ :

$$\kappa^{-1}\mathbf{u} + \mathbf{F}|\mathbf{u}|\mathbf{u} + \nabla p = \mathbf{f}(\varphi) \quad \text{in } \Omega, \quad (1.1a)$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \quad (1.1b)$$

$$\alpha^{-1}\boldsymbol{\zeta} = \nabla\varphi - \alpha^{-1}\mathbf{u}\varphi \quad \text{in } \Omega, \quad (1.1c)$$

$$\varrho_0\varphi - \operatorname{div} \boldsymbol{\zeta} = m \quad \text{in } \Omega, \quad (1.1d)$$

$$\mathbf{u} \cdot \mathbf{n} = \varphi = 0 \quad \text{on } \Gamma_u, \quad (1.1e)$$

$$p = \boldsymbol{\zeta} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_p. \quad (1.1f)$$

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Here, κ is the hydraulic conductivity (permeability of the porous matrix divided by the fluid viscosity), \mathbf{f} is a forcing vector that depends on the concentration field which satisfies the following condition (cf. [31, Assumption 2.1])

$$\|\mathbf{f}(\varphi)\|_{0,\Omega} \leq \|\mathbf{f}(0)\|_{0,\Omega} + \alpha_1 \|\varphi\|_{0,6,\Omega}, \quad (1.2)$$

where α_1 is a positive constant. In addition, $F > 0$ is the Forchheimer coefficient, $\alpha > 0$ is the diffusivity coefficient, $\varrho_0 > 0$ is the reaction coefficient and m is the concentration source/sink. A variety of discretisations with finite differences and finite element methods can be found in, e.g. [18, 27, 9], or in [1] for the case of singular sources (see also [13, 25] for analysis of similar models in the fully mixed case). Such problems find applications in reservoir modeling and geothermal energy optimisation, for example.

Main contribution. Owing to the form of (1.1), and with the aim of preserving both robustness of the parameters—both at the continuous and discrete levels—the variational formulation of the problem is cast involving sums and intersections of reflexive Banach spaces. The main tool in the theoretical analysis of well-posedness (as well as error estimates) is a new abstract result for perturbed saddle-point problems in reflexive Banach spaces that stands an extension of [15] (to the parameter-robust context) as well as of [10] (to its non-Hilbert counterpart). A more closely related reference that deals with the design of preconditioning for flow-transport couplings, is the recent work [12] addressing robust stability of a linearised model for coupled heat and Biot equations. The main differences here are that, due to the presence of the nonlinear terms and Banach spaces functional structure, we need to take into consideration sums of three non-Hilbert spaces and realise the action of duality maps that are non-standard when compared to the recent literature on operator preconditioning (see, e.g., [6, 12, 26]). It should be noted that, in the context of ADR problems, the presence of the advective term renders the underlying saddle-point structure non-symmetric. The treatment of this term within the design of preconditioners remains challenging for both preconditioning strategies and iterative solvers.

The preconditioners we explore here are inspired by Schur complement and operator preconditioning frameworks for symmetric elliptic operators; however, their extension to the nonsymmetric and nonnormal setting lacks a firm theoretical basis [32, 21]. Additionally, while the individual Darcy–Forchheimer and reaction-diffusion subproblems are uniformly bounded and inf-sup stable, robustness with respect to α and F is partially lost in the coupled system due to the fixed-point coupling of the nonlinear velocity and transport flux. This analytical loss of robustness does not manifest in our observed numerical convergence rates, suggesting it may be an artifact of sub-optimal scaling in the weighted norms or limitations inherent to the fixed-point framework.

Common notation and preliminaries. For $s \geq 0$ and $r \in [1, +\infty]$, we denote by $L^r(\Omega)$ and $W^{s,r}(\Omega)$ the usual Lebesgue and Sobolev spaces endowed with the norms $\|\bullet\|_{0,r,\Omega}$ and $\|\bullet\|_{s,r,\Omega}$, respectively. Note that $W^{0,r}(\Omega) = L^r(\Omega)$. If $r = 2$, we write $H^s(\Omega)$ in place of $W^{s,2}(\Omega)$, and denote the corresponding norm by $\|\bullet\|_{s,\Omega}$. Recall also the $L^r(\Omega)$ - $L^s(\Omega)$ embedding for finite measure domains, i.e., $\|\bullet\|_{0,r,\Omega} \leq |\Omega|^{\frac{1}{r}-\frac{1}{s}} \|\bullet\|_{0,s,\Omega}$, for $1 < r \leq s < \infty$. Also, C_{emb} denotes the constants arising from a Sobolev embedding, which depends only on $|\Omega|$. We denote by \mathbf{H} the vectorial counterpart of a generic scalar functional space H . The $L^2(\Omega)$ inner product for scalar, vector, or tensor valued functions is denoted by $(\bullet, \bullet)_{0,\Omega}$. Moreover, we use for $r, s \in [1, +\infty]$, the Banach space $\mathbf{H}^r(\text{div}_s, \Omega) = \{\mathbf{v} \in \mathbf{L}^r(\Omega) : \text{div } \mathbf{v} \in L^s(\Omega)\}$, endowed with the norm $\|\mathbf{v}\|_{r,\text{div}_s,\Omega}^2 = \|\mathbf{v}\|_{0,r,\Omega}^2 + \|\text{div } \mathbf{v}\|_{0,s,\Omega}^2$, and in particular, for $r = 2, s = 2$, we call it $\mathbf{H}(\text{div}, \Omega)$. Often, we use a subscript Γ_i on the functional space to denote that the (appropriate) traces vanish on the part of the boundary $\Gamma_i \subset \partial\Omega$.

For two Banach spaces $(X, \|\bullet\|_X)$ and $(Y, \|\bullet\|_Y)$, we denote by $\mathcal{L}(X, Y)$ the space of bounded linear maps from X to Y . Let $(Z, \|\bullet\|_Z)$ be another Banach space. The intersection and sum spaces are

$$\begin{aligned} X \cap Y \cap Z &= \{v : v \in X \text{ and } v \in Y \text{ and } v \in Z\}, \\ X + Y + Z &= \{u + v + w : u \in X \text{ and } v \in Y \text{ and } w \in Z\}, \end{aligned}$$

and their respective norms are (see [6], noting that the same norm characterisation done there for Hilbert spaces holds also for the Banach case)

$$\|r\|_{X \cap Y \cap Z} = \|r\|_X + \|r\|_Y + \|r\|_Z, \quad \|z\|_{X+Y+Z} = \inf_{\substack{r=u+v+w, \\ u \in X, v \in Y, w \in Z}} [\|u\|_X + \|v\|_Y + \|w\|_Z].$$

Furthermore, if $X \cap Y \cap Z$ is dense in X and Y and Z , then the following relation (understood as an identification under an isometry) holds

$$(X + Y + Z)' = X' \cap Y' \cap Z'. \quad (1.3)$$

For a fixed $\alpha > 0$, by αX we denote the Banach space whose elements coincide with those in X but are measured with the norm $\|\bullet\|_{\alpha X}^2 = \alpha^2 \|\bullet\|_X^2$. For any scalar function φ , we recall that the sign function (denoted by $\text{sgn}(\varphi)$) takes the value 1, for non-negative φ , and -1 , otherwise. Additionally, $\varphi \text{sgn}(\varphi) = |\varphi|$.

Outline. The remainder of the paper is organised as follows. Section 2 presents the weak formulation for (1.1), specifying the required parameter-weighted spaces. Also we state and prove an abstract theory for robust stability of perturbed saddle-point problems in Banach spaces. In Section 3, we formulate our problem in the context of abstract theory and show, using fixed-point arguments, that there exists a unique solution of the coupled system and that we have a parameter-independent continuous dependence on data. Section 4 deals with the finite element formulation, the robust discrete stability—via a discrete fixed-point argument—and the derivation of robust a priori error estimates. We construct variable block-diagonal operator preconditioners in Section 5 and show their robustness with respect to all model parameters and the discretisation parameter, except the diffusivity coefficient. Finally, Section 6 includes a series of numerical tests showing the convergence of the method in 2D, also demonstrating the performance of the preconditioners.

2 Mixed weak formulation and abstract setting

We now consider the weighted spaces proposed in [17, Section 3] for Darcy velocity and pressure:

$$\mathbf{V} := \kappa^{-\frac{1}{2}} \mathbf{L}^2(\Omega) \cap \mathbf{H}_{\Gamma_u}(\text{div}, \Omega) \cap \mathbf{F}^{\frac{1}{3}} \mathbf{L}^3(\Omega), \quad \text{and} \quad \mathbf{Q} := \mathbf{L}^2(\Omega) + \kappa^{\frac{1}{2}} \mathbf{H}_{\Gamma_p}^1(\Omega) + \mathbf{F}^{-\frac{1}{3}} \mathbf{W}_{\Gamma_p}^{1, \frac{3}{2}}(\Omega),$$

endowed with the following norms

$$\|\mathbf{v}\|_{\mathbf{V}} := \kappa^{-\frac{1}{2}} \|\mathbf{v}\|_{0, \Omega} + \|\text{div } \mathbf{v}\|_{0, \Omega} + \mathbf{F}^{\frac{1}{3}} \|\mathbf{v}\|_{0, 3, \Omega}, \quad \text{and} \quad (2.1a)$$

$$\|q\|_{\mathbf{Q}} := \inf_{s \in \mathbf{H}_{\Gamma_p}^1} (\|q - s\|_{0, \Omega} + \kappa^{\frac{1}{2}} \|\nabla q\|_{0, \Omega} + \mathbf{F}^{-\frac{1}{3}} \|\nabla q\|_{0, \frac{3}{2}, \Omega}), \quad (2.1b)$$

respectively. Denote as follows the trial and test function spaces for flux and concentration (following [10, Section 3])

$$\mathbf{W} := \alpha^{-\frac{1}{2}} \mathbf{L}^2(\Omega) \cap \mathbf{H}_{\Gamma_p}(\text{div}_{\frac{6}{5}}, \Omega), \quad \text{and} \quad \Psi := \alpha^{\frac{1}{2}} \mathbf{H}_{\Gamma_u}^1(\Omega) + \mathbf{L}^6(\Omega),$$

equipped with the norms

$$\|\mathbf{w}\|_{\mathbf{W}} := \alpha^{-\frac{1}{2}} \|\mathbf{w}\|_{0, \Omega} + \|\text{div } \mathbf{w}\|_{0, \frac{6}{5}, \Omega}, \quad \text{and} \quad (2.2a)$$

$$\|\psi\|_{\Psi} := \inf_{s \in \alpha^{\frac{1}{2}} \mathbf{H}_{\Gamma_u}^1(\Omega)} (\|\psi - s\|_{0, 6, \Omega} + \alpha^{\frac{1}{2}} \|\nabla s\|_{0, \Omega}), \quad (2.2b)$$

respectively. The weak formulation of (1.1) consists in finding $(\mathbf{u}, p, \zeta, \varphi) \in \mathbf{V} \times \mathbf{Q} \times \mathbf{W} \times \Psi$ such that

$$a_1(\mathbf{u}, \mathbf{v}) + d_1(\mathbf{u}; \mathbf{u}, \mathbf{v}) + b_1(\mathbf{v}, p) = F^\varphi(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}, \quad (2.3a)$$

$$b_1(\mathbf{u}, q) = 0 \quad \forall q \in \mathbf{Q}, \quad (2.3b)$$

$$a_2(\zeta, \mathbf{w}) + b_2(\mathbf{w}, \varphi) + d_2^{\hat{\mathbf{u}}}(\mathbf{w}, \varphi) = 0 \quad \forall \mathbf{w} \in \mathbf{W}, \quad (2.3c)$$

$$b_2(\zeta, \psi) - c_2(\varphi, \psi) = H(\psi) \quad \forall \psi \in \Psi, \quad (2.3d)$$

where the bilinear forms $a_1 : \mathbf{V} \times \mathbf{V} \rightarrow \mathbb{R}$, $b_1 : \mathbf{V} \times \mathbf{Q} \rightarrow \mathbb{R}$, $a_2 : \mathbf{W} \times \mathbf{W} \rightarrow \mathbb{R}$, $b_2 : \mathbf{W} \times \Psi \rightarrow \mathbb{R}$, $c_2 : \Psi \times \Psi \rightarrow \mathbb{R}$, $d_2^{\hat{\mathbf{u}}} : \mathbf{W} \times \Psi \rightarrow \mathbb{R}$ (for a fixed $\hat{\mathbf{u}}$), linear functionals $F^{\hat{\varphi}} : \mathbf{V} \rightarrow \mathbb{R}$ (for a fixed $\hat{\varphi}$), $H : \Psi \rightarrow \mathbb{R}$, and nonlinear form $d_1 : \mathbf{V} \times \mathbf{V} \times \mathbf{V} \rightarrow \mathbb{R}$, are defined as:

$$\begin{aligned} a_1(\mathbf{u}, \mathbf{v}) &:= \int_{\Omega} \kappa^{-1} \mathbf{u} \cdot \mathbf{v}, & b_1(\mathbf{v}, q) &:= - \int_{\Omega} q \text{div } \mathbf{v}, & d_1(\mathbf{w}; \mathbf{u}, \mathbf{v}) &:= \int_{\Omega} \mathbf{F}|\mathbf{w}| \mathbf{u} \cdot \mathbf{v}, \\ a_2(\zeta, \mathbf{w}) &:= \int_{\Omega} \alpha^{-1} \zeta \cdot \mathbf{w}, & b_2(\mathbf{w}, \psi) &:= \int_{\Omega} \psi \text{div } \mathbf{w}, & c_2(\varphi, \psi) &:= \int_{\Omega} \varrho_0 \varphi \psi, \\ d_2^{\hat{\mathbf{u}}}(\mathbf{w}, \psi) &:= \int_{\Omega} \alpha^{-1} \hat{\mathbf{u}} \cdot \mathbf{w} \psi, & F^{\hat{\varphi}}(\mathbf{v}) &:= \int_{\Omega} \mathbf{f}(\hat{\varphi}) \cdot \mathbf{v}, & H(\psi) &:= - \int_{\Omega} m \psi. \end{aligned}$$

Let \mathbf{X} and Y be reflexive Banach spaces equipped with the norms $\|\bullet\|_{\mathbf{X}}$ and $\|\bullet\|_Y$, respectively. Further, let Y_c be a dense linear subspace of Y . Consider the problem of finding $(\sigma, u) \in \mathbf{X} \times Y$ such that

$$a(\sigma, \tau) + b(\tau, u) = F(\tau) \quad \forall \tau \in \mathbf{X}, \quad (2.4a)$$

$$b(\sigma, v) - t^2 c(u, v) = G(v) \quad \forall v \in Y_c, \quad (2.4b)$$

where $0 \leq t \leq 1$ is a fixed scaling parameter and $a : \mathbf{X} \times \mathbf{X} \rightarrow \mathbb{R}$, $b : \mathbf{X} \times Y \rightarrow \mathbb{R}$, and $c : Y_c \times Y_c \rightarrow \mathbb{R}$ are given bilinear forms. Here, also $(F, G) \in \mathbf{X}' \times Y'$ are given linear forms. Let $\mathbf{B} : \mathbf{X} \rightarrow Y'$ be a bounded linear map induced by $b(\bullet, \bullet)$ as $\mathbf{X} \ni \tau \mapsto \mathbf{B}\tau$ with $\langle \mathbf{B}\tau, v \rangle := b(\tau, v)$ for all $v \in Y$.

We extend the Brezzi–Braess theorem ([10, Theorem 2.1]) to the case of reflexive Banach spaces. For this we adapt [20, Theorem 49.15] to the case of perturbed saddle-point problems. The invertibility of \mathcal{A} will follow as a consequence of the following conditions

$$\text{BNB 1 :} \quad \inf_{\substack{(\sigma, u) \in \mathbf{X} \times Y \\ (\sigma, u) \neq (0, 0)}} \sup_{\substack{(\tau, v) \in \mathbf{X} \times Y \\ (\tau, v) \neq (0, 0)}} \frac{\mathcal{A}(\sigma, u; \tau, v)}{\|(\sigma, u)\|_{\mathbf{X} \times Y} \|(\tau, v)\|_{\mathbf{X} \times Y}} \geq C_{\natural} \quad \forall (\sigma, u), (\tau, v) \in \mathbf{X} \times Y, \quad (2.5a)$$

$$\text{BNB 2 :} \quad \forall (\sigma, u) \in \mathbf{X} \times Y, \quad [\forall (\tau, v) \in \mathbf{X} \times Y, \mathcal{A}(\sigma, u; \tau, v) = 0] \implies [(\sigma, u) = (0, 0)], \quad (2.5b)$$

where C_{\natural} is a positive constant independent of t and $\|(\tau, v)\|_{\mathbf{X} \times Y} := \|\tau\|_{\mathbf{X}} + \|v\|_Y + t|v|_c$, and $|\bullet|_c^2 = c(\bullet, \bullet)$. The main result is discussed below.

Theorem 2.1 (Robust well-posedness in Banach spaces) *Assume that*

- $a(\bullet, \bullet)$, $b(\bullet, \bullet)$, and $c(\bullet, \bullet)$ are bounded with continuity constants $\|a\|$, $\|b\|$ and $\|c\|$, respectively, independent of the scaling parameter t .
- $a(\bullet, \bullet)$ and $c(\bullet, \bullet)$ are symmetric and positive semi-definite.
- *Inf-sup condition in $\mathbf{Z} = \ker(\mathbf{B})$: there exists $C_a > 0$, independent of the parameter t , such that*

$$\sup_{\tau \in \mathbf{Z} \setminus \{0\}} \frac{a(\sigma, \tau)}{\|\tau\|_{\mathbf{X}}} \geq C_a \|\sigma\|_{\mathbf{X}} \quad \forall \sigma \in \mathbf{Z}. \quad (2.6)$$

- *Inf-sup condition: there exists $C_b > 0$, independent of t , such that*

$$\sup_{\tau \in \mathbf{X} \setminus \{0\}} \frac{b(\tau, v)}{\|\tau\|_{\mathbf{X}}} \geq C_b \|v\|_Y \quad \forall v \in Y. \quad (2.7)$$

- *Braess condition: there exists $C_c > 0$, independent of t , such that*

$$\sup_{(\tau, v) \in \mathbf{X} \times Y \setminus \{(0, 0)\}} \frac{a(\sigma, \tau) + b(\sigma, v)}{\|(\tau, v)\|_{\mathbf{X} \times Y}} \geq C_c \|\sigma\|_{\mathbf{X}} \quad \forall \sigma \in \mathbf{X}. \quad (2.8)$$

Then, for each pair $(F, G) \in \mathbf{X}' \times Y'$, the problem (2.4) is uniquely solvable. Moreover, there holds

$$\|(\sigma, u)\|_{\mathbf{X} \times Y} \leq C [\|F\|_{\mathbf{X}'} + \|G\|_{Y'}],$$

where

$$C := \max\left[\frac{1}{MC_c}, 6\beta_1^{-1} + 9C_b^2\beta_1^{-2}\right],$$

does not depend on the scaling parameter t , and $M = \left(1 + \frac{2\|a\|}{C_b}\right)^{-1}$ and $\beta_1 = \min\{C_b, \|a\|\}$.

Proof. We follow closely the analysis in [20, Theorem 49.15] and [11, Lemma 3]. First, we define

$$\mathcal{A}(\sigma, u; \tau, v) := a(\sigma, \tau) + b(\tau, u) + b(\sigma, v) - t^2 c(u, v), \quad \forall (\sigma, u), (\tau, v) \in \mathbf{X} \times Y, \quad (2.9)$$

and then, we focus on proving BNB 1, for which we consider two cases.

Case 1. Let us first assume that

$$\|\sigma\|_{\mathbf{X}} \leq \frac{C_b}{2\|a\|} \{\|u\|_Y + t|u|_c\}. \quad (2.10)$$

Re-writing $\|(\bullet, \bullet)\|_{\mathbf{X} \times Y}$ in the following form, with $\beta_1 = \min\{C_b, \|a\|\}$, and using (2.10), we obtain

$$\begin{aligned} \|(\sigma, u)\|_{\mathbf{X} \times Y} &= \|\sigma\|_{\mathbf{X}} + \|u\|_Y + t|u|_c \leq \beta_1^{-1} (\|a\| \|\sigma\|_{\mathbf{X}} + C_b \|u\|_Y + C_b t|u|_c) \\ &\leq \frac{3}{2} C_b \beta_1^{-1} \{\|u\|_Y + t|u|_c\}. \end{aligned} \quad (2.11)$$

Note that the positive semi-definiteness of $a(\bullet, \bullet)$ yields

$$t|u|_c \leq [a(\sigma, \sigma) + t^2|u|_c^2]^{\frac{1}{2}} = (\mathcal{A}(\sigma, u; \sigma, -u))^{\frac{1}{2}}. \quad (2.12)$$

We continue with the inf-sup condition of $b(\bullet, \bullet)$ and use (2.9). Then, we use the boundedness of $a(\bullet, \bullet)$ and (2.10) to derive the following upper bound of $\|u\|_Y$

$$\begin{aligned} C_b \|u\|_Y &\leq \sup_{\mathbf{0} \neq \tau \in \mathbf{X}} \frac{b(\tau, u)}{\|\tau\|_{\mathbf{X}}} = \sup_{\mathbf{0} \neq \tau \in \mathbf{X}} \frac{\mathcal{A}(\sigma, u; \tau, 0) - a(\sigma, \tau)}{\|\tau\|_{\mathbf{X}}} \\ &\leq \sup_{(\mathbf{0}, 0) \neq (\tau, v) \in \mathbf{X} \times Y} \frac{\mathcal{A}(\sigma, u; \tau, v)}{\|(\tau, v)\|_{\mathbf{X} \times Y}} + \|a\| \|\sigma\|_{\mathbf{X}} \\ &\leq \sup_{(\mathbf{0}, 0) \neq (\tau, v) \in \mathbf{X} \times Y} \frac{\mathcal{A}(\sigma, u; \tau, v)}{\|(\tau, v)\|_{\mathbf{X} \times Y}} + \frac{C_b}{2} \{\|u\|_Y + t|u|_c\}. \end{aligned} \quad (2.13)$$

Note that, (2.13) provides a bound on $\|u\|_Y$ once one subtracts $\frac{C_b \|u\|_Y}{2}$ from both sides first, and then multiplies both sides by $2C_b^{-1}$. Invoking (2.12)-(2.13) in (2.11), we obtain

$$\begin{aligned} \|(\sigma, u)\|_{\mathbf{X} \times Y} &\leq 3\beta_1^{-1} \sup_{(\mathbf{0}, 0) \neq (\tau, v) \in \mathbf{X} \times Y} \frac{\mathcal{A}(\sigma, u; \tau, v)}{\|(\tau, v)\|_{\mathbf{X} \times Y}} + 3C_b \beta_1^{-1} t|u|_c \\ &\leq 3\beta_1^{-1} \sup_{(\mathbf{0}, 0) \neq (\tau, v) \in \mathbf{X} \times Y} \frac{\mathcal{A}(\sigma, u; \tau, v)}{\|(\tau, v)\|_{\mathbf{X} \times Y}} + 3C_b \beta_1^{-1} \mathcal{A}(\sigma, u; \sigma, -u)^{\frac{1}{2}}. \end{aligned} \quad (2.14)$$

We recall that for any three positive real numbers x, y, z such that $x \leq y + z$, an application of the Young's inequality shows $x \leq 2y + \frac{z^2}{x}$ (see, [11, Equation 3.4]). Choose

$$x = \|(\sigma, u)\|_{\mathbf{X} \times Y}, \quad y = 3\beta_1^{-1} \sup_{(\mathbf{0}, 0) \neq (\tau, v) \in \mathbf{X} \times Y} \frac{\mathcal{A}(\sigma, u; \tau, v)}{\|(\tau, v)\|_{\mathbf{X} \times Y}} \quad \text{and} \quad z = 3C_b \beta_1^{-1} \mathcal{A}(\sigma, u; \sigma, -u)^{\frac{1}{2}},$$

in this context, to obtain the following stability bound

$$\begin{aligned} \|(\sigma, u)\|_{\mathbf{X} \times Y} &\leq 6\beta_1^{-1} \sup_{(\mathbf{0}, 0) \neq (\tau, v) \in \mathbf{X} \times Y} \frac{\mathcal{A}(\sigma, u; \tau, v)}{\|(\tau, v)\|_{\mathbf{X} \times Y}} + 9C_b^2 \beta_1^{-2} \frac{\mathcal{A}(\sigma, u; \sigma, -u)}{\|(\sigma, u)\|_{\mathbf{X} \times Y}} \\ &\leq (6\beta_1^{-1} + 9C_b^2 \beta_1^{-2}) \sup_{(\mathbf{0}, 0) \neq (\tau, v) \in \mathbf{X} \times Y} \frac{\mathcal{A}(\sigma, u; \tau, v)}{\|(\tau, v)\|_{\mathbf{X} \times Y}}. \end{aligned}$$

Case 2. On the other hand, we, now, assume that

$$\|\sigma\|_{\mathbf{X}} \geq \frac{C_b}{2\|a\|} \{\|u\|_Y + t|u|_c\}. \quad (2.15)$$

Then, adding $\|\sigma\|_{\mathbf{X}}$ to both sides of (2.15) shows

$$\|\sigma\|_{\mathbf{X}} \geq M \|(\sigma, u)\|_{\mathbf{X} \times Y}, \quad \text{where } M = \left(1 + \frac{2\|a\|}{C_b}\right)^{-1}. \quad (2.16)$$

Multiplying by C_c to both sides of (2.16) and applying the Braess condition, yields

$$\begin{aligned} MC_c \|\!(\boldsymbol{\sigma}, u)\!\|_{\mathbf{X} \times Y} &\leq C_c \|\boldsymbol{\sigma}\|_{\mathbf{X}} \leq \sup_{(\mathbf{0},0) \neq (\boldsymbol{\tau}, v) \in \mathbf{X} \times Y} \frac{a(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(\boldsymbol{\sigma}, v)}{\|\!(\boldsymbol{\tau}, v)\!\|_{\mathbf{X} \times Y}} \\ &\leq \sup_{(\mathbf{0},0) \neq (\boldsymbol{\tau}, v) \in \mathbf{X} \times Y} \frac{\mathcal{A}(\boldsymbol{\sigma}, u; \boldsymbol{\tau}, 0)}{\|\!(\boldsymbol{\tau}, v)\!\|_{\mathbf{X} \times Y}} \leq \sup_{(\mathbf{0},0) \neq (\boldsymbol{\tau}, v) \in \mathbf{X} \times Y} \frac{\mathcal{A}(\boldsymbol{\sigma}, u; \boldsymbol{\tau}, v)}{\|\!(\boldsymbol{\tau}, v)\!\|_{\mathbf{X} \times Y}}. \end{aligned}$$

Combining the two cases, we arrive at the following stability bound

$$\|\!(\boldsymbol{\sigma}, u)\!\|_{\mathbf{X} \times Y} \leq C \sup_{(\mathbf{0},0) \neq (\boldsymbol{\tau}, v) \in \mathbf{X} \times Y} \frac{\mathcal{A}(\boldsymbol{\sigma}, u; \boldsymbol{\tau}, v)}{\|\!(\boldsymbol{\tau}, v)\!\|_{\mathbf{X} \times Y}},$$

where

$$C := \max\left[\frac{1}{MC_c}, 6\beta_1^{-1} + 9C_b^2\beta_1^{-2}\right]. \quad (2.17)$$

We now turn to showing BNB 2. We consider a fixed $(\boldsymbol{\sigma}, u) \in \mathbf{X} \times Y$ for which $\mathcal{A}(\boldsymbol{\sigma}, u; \boldsymbol{\tau}, v) = 0$ for all $(\boldsymbol{\tau}, v) \in \mathbf{X} \times Y$, i.e.,

$$a(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, u) = 0 \quad \forall \boldsymbol{\tau} \in \mathbf{X}, \quad (2.18a)$$

$$b(\boldsymbol{\sigma}, v) - t^2 c(u, v) = 0 \quad \forall v \in Y_c. \quad (2.18b)$$

Our goal is to show that $(\boldsymbol{\sigma}, u) = (\mathbf{0}, 0)$. Substituting $v = -u$ in (2.18b) and then using the positive semi-definiteness of $c(\bullet, \bullet)$, we conclude that $b(\boldsymbol{\sigma}, u) = 0$ (note that, the inf-sup condition of $b(\bullet, \bullet)$ guarantees that, $b(\boldsymbol{\sigma}, u) \geq 0$ for all $(\boldsymbol{\sigma}, u) \in \mathbf{X} \times Y$), which in particular implies that $\boldsymbol{\sigma} \in \ker(\mathbf{B})$. Taking $\boldsymbol{\tau} \in \ker(\mathbf{B})$ in (2.18a), and using (2.6), we infer that $a(\boldsymbol{\sigma}, \boldsymbol{\tau}) = 0$, $\forall \boldsymbol{\tau} \in \mathbf{Z}$, which further implies that $\boldsymbol{\sigma} = \mathbf{0}$ (using (2.6)). Finally, (2.18a) yields $b(\boldsymbol{\tau}, u) = 0$, $\forall \boldsymbol{\tau} \in \mathbf{X}$, and using (2.7), we conclude that $u = 0$. This completes the rest of the proof. \square

Remark 2.1 Unfortunately, we cannot readily derive t -robust stability such as Theorem 2.1 by directly applying [15, Theorem 3.4] with the t -dependent norm $\|\!(\bullet, \bullet)\!\|_{\mathbf{X} \times Y} := \|\bullet\|_{\mathbf{X}} + \|\bullet\|_{Y} + t|\bullet|_c$. This is because the bounds [15, Eq. (3.56)-(3.58)] depend explicitly on the scaling parameter t as follows

$$\begin{aligned} \|\!(\boldsymbol{\sigma}, u)\!\|_{\mathbf{X} \times Y} &\leq \left[\frac{1}{C_b} + \left(1 + \frac{\|a\|}{C_b}\right)\left(\frac{1}{C_a} + \frac{(2 \max(C_1^\sharp, C_2^\sharp))^{\frac{1}{2}}}{t}\right)\right] + \\ &\quad \left[\frac{1}{C_b} + t\left(1 + \frac{\|a\|}{C_b}\right)\left(\frac{1}{C_a} + \frac{(2 \max(C_1^\sharp, C_2^\sharp))^{\frac{1}{2}}}{t}\right)\right] \|F\|_{\mathbf{X}'} + \left[\left(\frac{\|a\|}{C_b} + \left(1 + \frac{\|a\|}{C_b}\right)\frac{1}{C_b} + \right. \right. \\ &\quad \left. \left. \left(\frac{1}{C_a} + \frac{(2 \max(C_1^\sharp, C_2^\sharp))^{\frac{1}{2}}}{t}\right)\right) + \left[\frac{1}{C_b} + t\left(1 + \frac{\|a\|}{C_b}\right)\left(\frac{1}{C_a} + \frac{(2 \max(C_1^\sharp, C_2^\sharp))^{\frac{1}{2}}}{t}\right)\right]\right] \|G\|_{Y'}, \end{aligned}$$

where

$$\begin{aligned} C_1^\sharp &:= \left(1 + \frac{\|a\|}{C_a}\right) \frac{1}{C_a} \frac{1}{C_b} + t^2 \frac{\|c\|}{C_b^2} \left(1 + \frac{\|a\|}{C_a}\right)^2, \quad \text{and} \\ C_2^\sharp &:= \frac{1}{C_b} \left(1 + \frac{\|a\|}{C_a}\right) \left[1 + \frac{\|a\|}{C_b} + t^2 \frac{\|a\|^2 \|c\|}{C_b^3} \left(1 + \frac{\|a\|}{C_a}\right)\right], \end{aligned}$$

and the stability deteriorates with $t \rightarrow 0$. In addition, note that the proof of Theorem 2.1 differs from [11, Lemma 3.1] as the Braess condition (2.8) is different from [11, (3.1)].

3 Existence and uniqueness of weak solution

This section employs a fixed-point framework to split the Darcy–Forchheimer equations from the ADR system. After showing that the two uncoupled systems are well-posed, we construct a fixed-point operator and then prove that it maps a ball into itself, and it satisfies the contraction mapping theorem. The treatment is similar to the analysis in, e.g., [13, 16, 25].

3.1 Unique solvability of the decoupled sub-problems

Now, for a fixed concentration $\hat{\varphi} \in \Psi$, consider the Darcy–Forchheimer sub-system in weak form as follows

$$a_1(\mathbf{u}, \mathbf{v}) + d_1(\mathbf{u}; \mathbf{u}, \mathbf{v}) + b_1(\mathbf{v}, p) = F^{\hat{\varphi}}(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}, \quad (3.1a)$$

$$b_1(\mathbf{u}, q) = 0 \quad \forall q \in \mathbf{Q}. \quad (3.1b)$$

We recall from [17, Theorem 3.3] the following result for the well-posedness of the system (3.1a)–(3.1b).

Theorem 3.1 (Well-posedness of Darcy–Forchheimer equations) *Assume that $\mathbf{f}(\hat{\varphi}) \in \mathbf{L}^2(\Omega)$ for a fixed $\hat{\varphi} \in \Psi$. Then there exists a unique solution $(\mathbf{u}, p) \in \mathbf{V} \times \mathbf{Q}$ to the problem (3.1a)–(3.1b). Furthermore, there is a positive constant C_c^{DF} , independent of the model parameters κ, \mathbf{F} , such that*

$$\|(\mathbf{u}, p)\|_{\mathbf{V} \times \mathbf{Q}} \leq C_c^{\text{DF}} \max\{\|\mathbf{f}(\hat{\varphi})\|_{0, \Omega}, \|\mathbf{f}(\hat{\varphi})\|_{0, \Omega}^2\}.$$

In particular, there is a positive constant $C_{c, \mathbf{u}}^{\text{DF}}$ (see, [17, Equation 3.10]), independent of \mathbf{u}, κ , and \mathbf{F} , such that

$$\|\mathbf{u}\|_{\mathbf{V}} \leq C_{c, \mathbf{u}}^{\text{DF}} \|\mathbf{f}(\hat{\varphi})\|_{0, \Omega}. \quad (3.2)$$

In what follows, we state and prove the unique-solvability result for the ADR equations. Prior to this, we establish the following lemma, which later assists in proving the parameter-robust inf-sup stability.

Lemma 3.2 *There exists an operator \mathcal{S}_c satisfying the following properties:*

$$\begin{aligned} \mathcal{S}_c \in \mathcal{L}(\Psi', \mathbf{W}), \quad \|\mathcal{S}_c\|_{\mathcal{L}(\Psi', \mathbf{W})} \text{ is independent of } \alpha, \varrho_0, \quad \text{and} \\ \langle \text{div}_{\frac{6}{5}}[\mathcal{S}_c(\ell)], \psi \rangle = \langle \ell, \psi \rangle \quad \text{for all } \ell \in \Psi', \psi \in \Psi. \end{aligned} \quad (3.3)$$

Proof. We define \mathcal{S}_c as a bounded linear operator from two different spaces, i.e. $\mathcal{S}_c \in \mathcal{L}(\alpha^{-\frac{1}{2}}(\mathbf{H}_{\Gamma_u}^1(\Omega))', \alpha^{-\frac{1}{2}}\mathbf{L}^2(\Omega))$, and $\mathcal{S}_c \in \mathcal{L}((\mathbf{L}^6(\Omega))', \mathbf{H}_{\Gamma_p}(\text{div}_{\frac{6}{5}}, \Omega))$, and then call upon the duality relation between sum and intersection between Banach spaces, i.e. $\Psi' = [\alpha^{\frac{1}{2}}\mathbf{H}_{\Gamma_u}^1(\Omega) + \mathbf{L}^6(\Omega)]' = \alpha^{-\frac{1}{2}}(\mathbf{H}_{\Gamma_u}^1(\Omega))' \cap (\mathbf{L}^6(\Omega))'$ (see, for instance, (1.3), with $X = \alpha^{\frac{1}{2}}\mathbf{H}_{\Gamma_u}^1(\Omega)$, $Y = \mathbf{L}^6(\Omega)$, and Z being the null space), to conclude that $\mathcal{S}_c \in \mathcal{L}(\Psi', \mathbf{W})$.

The existence of $\mathcal{S}_c \in \mathcal{L}((\mathbf{H}_{\Gamma_u}^1(\Omega))', \mathbf{L}^2(\Omega))$ follows directly from [17, Lemma 3.2]. Moreover, a scaling by $\alpha^{-\frac{1}{2}}$ does not compromise the α -robustness of the boundedness constant of \mathcal{S}_c .

For the existence of $\mathcal{S}_c \in \mathcal{L}((\mathbf{L}^6(\Omega))', \mathbf{H}_{\Gamma_p}(\text{div}_{\frac{6}{5}}, \Omega))$ with $\ell \in (\mathbf{L}^6(\Omega))'$, let $\vartheta \in \mathbf{H}_{\Gamma_p}^1(\Omega)$ be the unique solution of the following elliptic problem with mixed boundary conditions and homogeneous data (owing to, e.g., [29, Chapter 7])

$$\Delta \vartheta = \ell \quad \text{in } \Omega, \quad \nabla \vartheta \cdot \mathbf{n} = 0 \quad \text{in } \Gamma_u, \quad \vartheta = 0 \quad \text{in } \Gamma_p.$$

Next, setting $\tilde{\vartheta} = -\nabla \vartheta \in \mathbf{L}^2(\Omega)$, it is evident that

$$\text{div}(\tilde{\vartheta}) = \Delta \vartheta = \ell \in \mathbf{L}^{\frac{6}{5}}(\Omega), \quad \tilde{\vartheta} \cdot \mathbf{n} = 0, \quad \text{and} \quad \|\tilde{\vartheta}\|_{\text{div}_{\frac{6}{5}}, \Omega} \leq \|\ell\|_{0, \frac{6}{5}, \Omega}.$$

Then it suffices to define $\mathcal{S}_c(\ell) = \tilde{\vartheta}$, which further implies that $\mathcal{S}_c \in \mathcal{L}((\mathbf{L}^6(\Omega))', \mathbf{H}_{\Gamma_p}(\text{div}_{\frac{6}{5}}, \Omega))$. \square

Next, for a fixed advecting velocity $\hat{\mathbf{u}} \in \mathbf{V}$, let us consider the following ADR system in its weak form

$$a_2(\boldsymbol{\zeta}, \mathbf{w}) + b_2(\mathbf{w}, \varphi) + d_2^{\hat{\mathbf{u}}}(\mathbf{w}, \varphi) = 0 \quad \forall \mathbf{w} \in \mathbf{W}, \quad (3.4a)$$

$$b_2(\boldsymbol{\zeta}, \psi) - c_2(\varphi, \psi) = H(\psi) \quad \forall \psi \in \Psi, \quad (3.4b)$$

and we state the following continuous stability result.

Theorem 3.3 (Well-posedness of decoupled advection-diffusion-reaction equations) *Assume that $m \in \mathbf{L}^{\frac{6}{5}}(\Omega)$. Then, for a fixed $\hat{\mathbf{u}} \in \mathbf{V}$, there exist positive constants C_c, C_α^c (depending on α), such that, whenever $\|\hat{\mathbf{u}}\|_{0, 3, \Omega} \leq \frac{C_c}{2C_\alpha^c}$, there exists a unique solution $(\boldsymbol{\zeta}, \varphi) \in \mathbf{W} \times \Psi$ to the problem (3.4a)–(3.4b). Furthermore, there is a positive constant C_c^{ADR} , independent of ϱ_0 , such that*

$$\|(\boldsymbol{\zeta}, \varphi)\|_{\mathbf{W} \times \Psi} \leq C_c^{\text{ADR}} \|m\|_{0, \frac{6}{5}, \Omega}. \quad (3.5)$$

Proof. We first use Theorem 2.1 to show the well-posedness of the following reaction-diffusion system, which is identical to (3.4) without the advection part, i.e., one seeks $(\zeta, \varphi) \in \mathbf{W} \times \Psi$ such that

$$a_2(\zeta, \mathbf{w}) + b_2(\mathbf{w}, \varphi) = 0 \quad \forall \mathbf{w} \in \mathbf{W}, \quad (3.6a)$$

$$b_2(\zeta, \psi) - c_2(\varphi, \psi) = H(\psi) \quad \forall \psi \in \Psi. \quad (3.6b)$$

Boundedness of $a_2(\bullet, \bullet)$, $b_2(\bullet, \bullet)$ and $c_2(\bullet, \bullet)$. Observe that a use of the Cauchy–Schwarz inequality gives the boundedness of the bilinear forms $a_2(\bullet, \bullet)$ and $c_2(\bullet, \bullet)$ as

$$a_2(\zeta, \boldsymbol{\tau}) = \int_{\Omega} \alpha^{-1} \zeta \cdot \boldsymbol{\tau} \leq \alpha^{-\frac{1}{2}} \|\zeta\|_{0,\Omega} \alpha^{-\frac{1}{2}} \|\boldsymbol{\tau}\|_{0,\Omega} \leq \|\zeta\|_{\mathbf{W}} \|\boldsymbol{\tau}\|_{\mathbf{W}} \quad \forall \zeta, \boldsymbol{\tau} \in \mathbf{W},$$

$$c_2(\varphi, \psi) = \int_{\Omega} \varrho_0 \varphi \psi \leq \|\varphi\|_{0,\Omega} \|\psi\|_{0,\Omega} \leq \|\varphi\|_{\Psi_c} \|\psi\|_{\Psi_c} \quad \forall \varphi, \psi \in \Psi_c,$$

In order to show the boundedness of $b_2(\bullet, \bullet)$, we follow a decomposition argument over Ψ . Let $\psi = \psi_0 + \psi_1$, with $\psi_0 \in L^6(\Omega)$ and $\psi_1 \in H_{\Gamma_u}^1(\Omega)$. Then, $b_2(\boldsymbol{\tau}, \psi) = b_2(\boldsymbol{\tau}, \psi_0 + \psi_1) = \langle \operatorname{div} \boldsymbol{\tau}, \psi_0 \rangle - (\alpha^{-\frac{1}{2}} \boldsymbol{\tau}, \alpha^{\frac{1}{2}} \nabla \psi_1)$. Next, a use of the Hölder's with the Cauchy–Schwarz inequality, the fact that $a^2 + b^2 \leq (a + b)^2$, for any two nonnegative real numbers a, b , and then taking the infimum over all decompositions of $\psi \in \Psi$ give the desired boundedness as follows

$$\begin{aligned} b_2(\boldsymbol{\tau}, \psi) &\leq \|\operatorname{div} \boldsymbol{\tau}\|_{0, \frac{6}{5}, \Omega} \|\psi_0\|_{0,6,\Omega} + \|\alpha^{-\frac{1}{2}} \boldsymbol{\tau}\|_{0,\Omega} \|\alpha^{\frac{1}{2}} \nabla \psi_1\|_{0,\Omega} \\ &\leq (\|\operatorname{div} \boldsymbol{\tau}\|_{0, \frac{6}{5}, \Omega}^2 + \|\alpha^{-\frac{1}{2}} \boldsymbol{\tau}\|_{0,\Omega}^2)^{\frac{1}{2}} (\|\psi_0\|_{0,6,\Omega}^2 + \|\alpha^{\frac{1}{2}} \nabla \psi_1\|_{0,\Omega}^2)^{\frac{1}{2}} \\ &\leq (\|\operatorname{div} \boldsymbol{\tau}\|_{0, \frac{6}{5}, \Omega} + \|\alpha^{-\frac{1}{2}} \boldsymbol{\tau}\|_{0,\Omega}) (\|\psi_0\|_{0,6,\Omega} + \|\alpha^{\frac{1}{2}} \nabla \psi_1\|_{0,\Omega}) = \|\boldsymbol{\tau}\|_{\mathbf{W}} \|\psi\|_{\Psi} \quad \forall \boldsymbol{\tau} \in \mathbf{W}, \psi \in \Psi. \end{aligned}$$

Symmetry and positive-semidefiniteness of $a_2(\bullet, \bullet)$ and $c_2(\bullet, \bullet)$. Clearly, $a_2(\bullet, \bullet)$ and $c_2(\bullet, \bullet)$ are symmetric, as well as positive-semidefinite bilinear forms, as

$$a_2(\zeta, \zeta) = \alpha^{-1} \int_{\Omega} \zeta \cdot \zeta \geq 0 \quad \text{and} \quad c_2(\varphi, \varphi) = \varrho_0 \int_{\Omega} \varphi^2 \geq 0 \quad \forall \zeta \in \mathbf{W}, \varphi \in \Psi_c.$$

Inf-sup of $a_2(\bullet, \bullet)$ on \mathbf{Z} . Choosing $\boldsymbol{\tau} = \zeta$ and using the kernel characterisation with definition of the \mathbf{W} -norm, we arrive at

$$\sup_{\boldsymbol{\tau} \in \mathbf{Z} \setminus \{0\}} \frac{a_2(\zeta, \boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\mathbf{W}}} \geq \alpha^{-1} \frac{\int_{\Omega} \zeta \cdot \zeta}{\|\zeta\|_{\mathbf{W}}} = C_{a_2} \|\zeta\|_{\mathbf{W}} \quad \forall \zeta \in \mathbf{Z}.$$

Then, this readily satisfies the second hypothesis of Theorem 2.1 with the constant $C_{a_2} = 1$.

Inf-sup stability of $b_2(\bullet, \bullet)$. From the existence of \mathcal{S}_c defined in Lemma 3.2, it immediately follows that

$$\sup_{\boldsymbol{\tau} \in \mathbf{W}} \frac{\int_{\Omega} \psi \operatorname{div} \boldsymbol{\tau}}{\|\boldsymbol{\tau}\|_{\mathbf{W}}} \geq \sup_{\ell \in \Psi'} \frac{\langle \operatorname{div}_{\frac{6}{5}} [\mathcal{S}_c(\ell)], \psi \rangle}{\|\mathcal{S}_c(\ell)\|_{\mathbf{W}}} = \sup_{\ell \in \Psi'} \frac{\langle \ell, \psi \rangle}{\|\mathcal{S}_c(\ell)\|_{\mathbf{W}}} \geq \|\mathcal{S}_c\|_{\mathcal{L}(\Psi', \mathbf{W})}^{-1} \sup_{\ell \in \Psi'} \frac{\langle \ell, \psi \rangle}{\|\ell\|_{\Psi'}} = C_{b_2} \|\psi\|_{\Psi},$$

where $C_{b_2} = \|\mathcal{S}_c\|_{\mathcal{L}(\Psi', \mathbf{W})}^{-1}$ is independent of α, ϱ_0 , and hence, the inf-sup condition is satisfied.

Braess condition. Now, fixing $\boldsymbol{\tau} = \zeta, \psi = 0$, and observing that $\|(\zeta, 0)\|_{\mathbf{W} \times \Psi} = \|\zeta\|_{\mathbf{W}}$, we arrive at

$$\sup_{(\mathbf{0}, 0) \neq (\boldsymbol{\tau}, \psi) \in \mathbf{W} \times \Psi} \frac{a_2(\zeta, \boldsymbol{\tau}) + b_2(\zeta, \psi)}{\|(\boldsymbol{\tau}, \psi)\|_{\mathbf{W} \times \Psi}} \geq \alpha^{-1} \frac{\int_{\Omega} \zeta \cdot \zeta}{\|(\zeta, 0)\|_{\mathbf{W} \times \Psi}} = \alpha^{-1} \frac{\|\zeta\|_{0,\Omega}^2}{\|\zeta\|_{\mathbf{W}}}. \quad (3.7)$$

Next, taking $\boldsymbol{\tau} = \mathbf{0}$, and $\psi = \operatorname{sgn}(\operatorname{div} \zeta) |\operatorname{div} \zeta|^{\frac{1}{5}} \in L^{\frac{6}{5}}(\Omega)$, we evaluate the following quantities

$$\begin{aligned} b_2(\zeta, \psi) &= \int_{\Omega} |\operatorname{div} \zeta|^{\frac{6}{5}} = \|\operatorname{div} \zeta\|_{0, \frac{6}{5}, \Omega}^{\frac{6}{5}}, \quad \|\psi\|_{0,\Omega}^6 = \int_{\Omega} |\operatorname{div} \zeta|^{\frac{6}{5}} = \|\operatorname{div} \zeta\|_{0, \frac{6}{5}, \Omega}^{\frac{6}{5}}, \\ \|(\boldsymbol{\tau}, \psi)\|_{\mathbf{W} \times \Psi} &= \|(\mathbf{0}, \psi)\|_{\mathbf{W} \times \Psi} \leq \|\psi\|_{\Psi} + \varrho_0 \|\psi\|_{0,6,\Omega} \leq 2 \|\psi\|_{0,6,\Omega}, \end{aligned}$$

where the last inequality follows from (2.2b), and by using $\varrho_0 \leq 1$. Furthermore using the above expressions for $b_2(\zeta, \psi)$, $\|\psi\|_{0,\Omega}$ and $\|(\tau, \psi)\|_{\mathbf{W} \times \Psi}$, we immediately obtain

$$\sup_{(0,0) \neq (\tau, \psi) \in \mathbf{W} \times \Psi} \frac{a_2(\zeta, \tau) + b_2(\zeta, \psi)}{\|(\tau, \psi)\|_{\mathbf{W} \times \Psi}} \geq \frac{1}{2} \frac{\|\operatorname{div} \zeta\|_{0, \frac{6}{5}, \Omega}^{\frac{6}{5}}}{\|\operatorname{div} \zeta\|_{0, \frac{6}{5}, \Omega}^{\frac{1}{5}}} = \frac{1}{2} \|\operatorname{div} \zeta\|_{0, \frac{6}{5}, \Omega}. \quad (3.8)$$

Now, combining (3.7) and (3.8), we arrive at

$$\sup_{(0,0) \neq (\tau, \psi) \in \mathbf{W} \times \Psi} \frac{a_2(\zeta, \tau) + b_2(\zeta, \psi)}{\|(\tau, \psi)\|_{\mathbf{W} \times \Psi}} \geq \max(\alpha^{-1} \frac{\|\zeta\|_{0,\Omega}^2}{\|\zeta\|_{\mathbf{W}}}, \frac{1}{2} \|\operatorname{div} \zeta\|_{0, \frac{6}{5}, \Omega}). \quad (3.9)$$

Set $s := \alpha^{-\frac{1}{2}} \frac{\|\zeta\|_{0,\Omega}}{\|\zeta\|_{\mathbf{W}}} \in [0, 1)$ and use (3.9) to derive the following relation

$$\max(\alpha^{-1} \frac{\|\zeta\|_{0,\Omega}^2}{\|\zeta\|_{\mathbf{W}}}, \|\operatorname{div} \zeta\|_{0, \frac{6}{5}, \Omega}) = (\max_{s \in [0,1)} [s^2, \frac{1}{2} \sqrt{1-s^2}]) \|\zeta\|_{\mathbf{W}}.$$

After denoting

$$\gamma_c := (\max_{s \in [0,1)} [s^2, \frac{1}{2} \sqrt{1-s^2}]) < 1,$$

we can conclude from (3.9) that

$$\sup_{(0,0) \neq (\tau, \psi) \in \mathbf{W} \times \Psi} \frac{a_2(\zeta, \tau) + b_2(\zeta, \psi)}{\|(\tau, \psi)\|_{\mathbf{W} \times \Psi}} \geq \gamma_c \|\zeta\|_{\mathbf{W}}.$$

An appeal to Theorem 2.1 now yields the well-posedness of (3.6), and additionally, we obtain the following inf-sup type estimate for \mathcal{A}

$$\sup_{(0,0) \neq (\tau, \psi) \in \mathbf{W} \times \Psi} \frac{\mathcal{A}(\zeta, \varphi; \tau, \psi)}{\|(\tau, \psi)\|_{\mathbf{W} \times \Psi}} \geq C_c \|\zeta, \varphi\|_{\mathbf{W} \times \Psi}, \quad (3.10)$$

where C_c depends on $\|a_2\|$, C_{b_2} , γ_c (see (2.17) for the explicit expression of C_c).

Next, in order to show the well-posedness of the uncoupled ADR equations, for a fixed $\hat{u} \in \mathbf{V}$, we consider another bilinear form $\mathcal{A}^{\text{ADR}} : \mathbf{W} \times \Psi \rightarrow \mathbb{R}$ defined by

$$\mathcal{A}^{\text{ADR}}(\zeta, \varphi; \tau, \psi) := \mathcal{A}(\zeta, \varphi; \tau, \psi) + d_2^{\hat{u}}(\tau, \varphi), \quad \forall (\zeta, \varphi), (\tau, \psi) \in \mathbf{W} \times \Psi. \quad (3.11)$$

Thus, showing the well-posedness of the system (2.3c)-(2.3d) boils down to analysing the problem of finding $(\zeta, \varphi) \in \mathbf{W} \times \Psi$ satisfying

$$\mathcal{A}^{\text{ADR}}(\zeta, \varphi; \tau, \psi) = H(\psi), \quad \forall \psi \in \Psi.$$

Note that, by the generalised Hölder's inequality with $1/3+1/2+1/6=1$, and the uniform positivity and boundedness of κ , the following estimate readily follows:

$$d_2^{\hat{u}}(\tau, \varphi) \leq C_\alpha^c \|\hat{u}\|_{0,3,\Omega} \|(\tau, \psi)\|_{\mathbf{W} \times \Psi} \|(\zeta, \varphi)\|_{\mathbf{W} \times \Psi}, \quad (3.12)$$

where $C_\alpha^c := \alpha^{-\frac{1}{2}}(1 + C_{\text{emb}}\alpha^{-\frac{1}{2}})$. Invoking the bound of $d_2^{\hat{u}}(\bullet, \bullet)$ (cf. (3.12)), it follows from (3.10) and (3.11) that

$$\sup_{(0,0) \neq (\tau, \psi) \in \mathbf{W} \times \Psi} \frac{\mathcal{A}^{\text{ADR}}(\zeta, \varphi; \tau, \psi)}{\|(\tau, \psi)\|_{\mathbf{W} \times \Psi}} \geq [C_c - C_\alpha^c \|\hat{u}\|_{0,3,\Omega}] \|(\zeta, \varphi)\|_{\mathbf{W} \times \Psi}, \quad \forall (\zeta, \varphi) \in \mathbf{W} \times \Psi.$$

Hence, under the assumption that for each $\hat{u} \in \mathbf{V}$ such that $\|\hat{u}\|_{0,3,\Omega} \leq \frac{C_c}{2C_\alpha^c}$, we arrive at

$$\sup_{(0,0) \neq (\tau, \psi) \in \mathbf{W} \times \Psi} \frac{\mathcal{A}^{\text{ADR}}(\zeta, \varphi; \tau, \psi)}{\|(\tau, \psi)\|_{\mathbf{W} \times \Psi}} \geq \frac{C_c}{2} \|(\zeta, \varphi)\|_{\mathbf{W} \times \Psi}. \quad (3.13)$$

Therefore, the ADR system (cf. (2.3c)-(2.3d)) is well-posed. Additionally, using (3.11) in (3.13), and using the Hölder's inequality to see that $H(\psi) \leq \|m\|_{0, \frac{6}{5}, \Omega} \|\psi\|_{0,6,\Omega}$, we obtain the continuous dependence on the data as follows:

$$\|(\zeta, \varphi)\|_{\mathbf{W} \times \Psi} \leq C_c^{\text{ADR}} \|m\|_{0, \frac{6}{5}, \Omega}, \quad (3.14)$$

with $C_c^{\text{ADR}} := 2C_c^{-1}(1 + C_{\text{emb}}\alpha^{-\frac{1}{2}})$ being independent of ϱ_0 . This concludes the rest of the proof. \square

3.2 Fixed-point analysis

Now, define the following map

$$\mathcal{S}^{\text{DF}} : \Psi \rightarrow \mathbf{V} \times \mathbf{Q}, \quad \hat{\varphi} \mapsto \mathcal{S}^{\text{DF}}(\hat{\varphi}) = (\mathcal{S}_1^{\text{DF}}(\hat{\varphi}), \mathcal{S}_2^{\text{DF}}(\hat{\varphi})) := (\mathbf{u}, p),$$

where $(\mathbf{u}, p) \in \mathbf{V} \times \mathbf{Q}$ is the unique solution of the Darcy–Forchheimer equations for fixed $\hat{\varphi} \in \Psi$, as stated in Theorem 3.1, and $\mathcal{S}_1^{\text{DF}} : \Psi \rightarrow \mathbf{V}$, $\mathcal{S}_2^{\text{DF}} : \Psi \rightarrow \mathbf{Q}$. Further, for a fixed $\hat{\mathbf{u}} \in \mathbf{V}$, set the solution operator associated with the mixed ADR equations as

$$\mathcal{S}^{\text{ADR}} : \mathbf{V} \rightarrow \mathbf{W} \times \Psi, \quad \hat{\mathbf{u}} \mapsto \mathcal{S}^{\text{ADR}}(\hat{\mathbf{u}}) = (\mathcal{S}_1^{\text{ADR}}(\hat{\mathbf{u}}), \mathcal{S}_2^{\text{ADR}}(\hat{\mathbf{u}})) := (\zeta, \varphi),$$

where (ζ, φ) is the unique solution of the ADR equations, as stated in Theorem 3.3, and $\mathcal{S}_1^{\text{ADR}} : \mathbf{V} \rightarrow \mathbf{W}$, $\mathcal{S}_2^{\text{ADR}} : \mathbf{V} \rightarrow \Psi$. These maps are well-defined; the same holds for the following one

$$\mathcal{S} : \mathbf{V} \rightarrow \mathbf{V}, \quad \hat{\mathbf{u}} \mapsto \mathcal{S}(\hat{\mathbf{u}}) := \mathcal{S}_1^{\text{DF}}(\mathcal{S}_2^{\text{ADR}}(\hat{\mathbf{u}})). \quad (3.15)$$

Finding a fixed point \mathbf{u} of \mathcal{S} is therefore equivalent to find a solution to (2.3). We use the Banach contraction mapping theorem, and consider, for a generic $r > 0$, the following closed ball

$$\mathbf{B}_0^r := \{\hat{\mathbf{u}} \in \mathbf{V} : \|\hat{\mathbf{u}}\|_{\mathbf{V}} \leq r\},$$

and proceed next to show that \mathcal{S} maps it to itself and that \mathcal{S} is a contraction. For $\mathbf{u} \in \mathbf{B}_0^r$, using Theorem 3.1, (3.2), (1.2), and Theorem 3.3 shows that

$$\|\mathcal{S}(\mathbf{u})\|_{\mathbf{V}} = \|\mathcal{S}_1^{\text{DF}}(\mathcal{S}_2^{\text{ADR}}(\mathbf{u}))\|_{\mathbf{V}} \leq C_{c,\mathbf{u}}^{\text{DF}} [\|\mathbf{f}(\mathcal{S}_2^{\text{ADR}}(\mathbf{u}))\|_{0,\Omega}] \leq C_{c,\mathbf{u}}^{\text{DF}} [\|\mathbf{f}(0)\|_{0,\Omega} + C_c^{\text{ADR}} \alpha_1 \|m\|_{0,\frac{6}{5},\Omega}]. \quad (3.16)$$

Lemma 3.4 (Ball mapping property) *Under the small data assumption*

$$C_{c,\mathbf{u}}^{\text{DF}} [\|\mathbf{f}(0)\|_{0,\Omega} + C_c^{\text{ADR}} \alpha_1 \|m\|_{0,\frac{6}{5},\Omega}] \leq r < \frac{1}{2}, \quad (3.17)$$

we have that $\mathcal{S}(\mathbf{B}_0^r) \subseteq \mathbf{B}_0^r$.

Proof. Given $\hat{\mathbf{u}} \in \mathbf{B}_0^r$, a use of (3.15) with (3.16) and (3.17) yields $\|\mathcal{S}(\hat{\mathbf{u}})\|_{\mathbf{V}} = \|\mathcal{S}_1^{\text{DF}}(\mathcal{S}_2^{\text{ADR}}(\hat{\mathbf{u}}))\|_{\mathbf{V}} \leq r < \frac{1}{2}$, and hence, $\mathcal{S}(\mathbf{B}_0^r) \subseteq \mathbf{B}_0^r$, which concludes the proof. \square

Lemma 3.5 (Lipschitz-continuity) *There exists a positive constant $L_{\mathcal{S}}$ such that*

$$\|\mathcal{S}(\mathbf{u}_1) - \mathcal{S}(\mathbf{u}_2)\|_{\mathbf{V}} \leq L_{\mathcal{S}} \|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathbf{V}} \quad \forall \mathbf{u}_1, \mathbf{u}_2 \in \mathbf{V}. \quad (3.18)$$

Proof. It is enough to show the Lipschitz-continuity of the maps $\mathcal{S}_1^{\text{DF}}$ and $\mathcal{S}_2^{\text{ADR}}$, as in that case, using the Lipschitzness of the individual maps, one simply obtains the following

$$\begin{aligned} \|\mathcal{S}(\mathbf{u}_1) - \mathcal{S}(\mathbf{u}_2)\|_{\mathbf{V}} &= \|\mathcal{S}_1^{\text{DF}}(\mathcal{S}_2^{\text{ADR}}(\mathbf{u}_1)) - \mathcal{S}_1^{\text{DF}}(\mathcal{S}_2^{\text{ADR}}(\mathbf{u}_2))\|_{\mathbf{V}} \\ &\leq L_{\mathcal{S}_1} \|\mathcal{S}_2^{\text{ADR}}(\mathbf{u}_1) - \mathcal{S}_2^{\text{ADR}}(\mathbf{u}_2)\|_{\mathbf{V}} \leq L_{\mathcal{S}_1} L_{\mathcal{S}_2} \|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathbf{V}}, \end{aligned}$$

with respective Lipschitz constants, $L_{\mathcal{S}_1}$ and $L_{\mathcal{S}_2}$ (to be specified below) associated with the maps $\mathcal{S}_1^{\text{DF}}$ and $\mathcal{S}_2^{\text{ADR}}$. We now proceed to show the Lipschitzness of $\mathcal{S}_1^{\text{DF}}$ and $\mathcal{S}_2^{\text{ADR}}$.

Lipschitz-continuity of $\mathcal{S}_1^{\text{DF}}$. Let $\varphi_1, \varphi_2 \in \Psi$ be two elements in Ψ . Fixing φ_1 and φ_2 results in two solutions $(\mathbf{u}_1, p_1), (\mathbf{u}_2, p_2) \in \mathbf{V} \times \mathbf{Q}$ corresponding to (2.3a)-(2.3b). Now, denote $\mathcal{S}_1^{\text{DF}}(\varphi_1) := \mathbf{u}_1$ and $\mathcal{S}_1^{\text{DF}}(\varphi_2) := \mathbf{u}_2$. Substituting $v = \mathbf{u}_1 - \mathbf{u}_2, q = p_1 - p_2$ into (2.3a), (2.3b), using (1.2), and an application of the uniformly strong monotonicity of $a_1(\bullet, \bullet) + d_1(\bullet, \bullet, \bullet)$ in the new equation obtained by subtracting (2.3b) from (2.3a) implies

$$\|\mathcal{S}_1^{\text{DF}}(\varphi_1) - \mathcal{S}_1^{\text{DF}}(\varphi_2)\|_{\mathbf{V}} = \|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathbf{V}} \leq L_{\mathcal{S}_1} \|\varphi_1 - \varphi_2\|_{\Psi}, \quad (3.19)$$

where $L_{\mathcal{S}_1} := \alpha_{\text{DF}}^{-\frac{1}{2}} \alpha_1 C_{\text{emb}}^2$, α_{DF} being the strong monotonicity constant (see, [17, Equation 3.8]), and α_1 is the local Lipschitz continuity constant for the Darcy–Forchheimer system (see, [17, Equation 3.7]), which can be further bounded by $\|\mathbf{f}(0)\|_{0,\Omega}$, $\alpha_1, C_c^{\text{ADR}}$, and $\|m\|_{0,\frac{6}{5},\Omega}$ (cf. (3.2) and (1.2)).

Lipschitz-continuity of $\mathcal{S}_2^{\text{ADR}}$. Let $\mathbf{u}_1, \mathbf{u}_2 \in \mathbf{V}$ be two elements in \mathbf{V} . Fixing $\mathbf{u}_1, \mathbf{u}_2$ in (2.3c),(2.3d) produces the solutions $(\zeta_1, \varphi_1), (\zeta_2, \varphi_2) \in \mathbf{W} \times \Psi$ corresponding to (2.3c)-(2.3d). Let us denote $\varphi := \varphi_1 - \varphi_2$. Subtracting (2.3d) from (2.3c), adding $\int_{\Omega} \mathbf{u}_1 \cdot \mathbf{w} \varphi_2, \int_{\Omega} \mathbf{u}_2 \cdot \mathbf{w} \varphi_1$, and subtracting $2 \int_{\Omega} \mathbf{u}_2 \cdot \mathbf{w} \varphi_2$, and using Theorem 3.3 we obtain

$$\frac{1}{C_c^{\text{ADR}}} \|(\zeta, \varphi)\|_{\mathbf{W} \times \Psi} \leq \sup_{\substack{(\mathbf{w}, \psi) \in \mathbf{W} \times \Psi \\ (\mathbf{w}, \psi) \neq (\mathbf{0}, \mathbf{0})}} \frac{-\int_{\Omega} ((\mathbf{u}_1 - \mathbf{u}_2) \cdot \mathbf{w}) \varphi_2 + \int_{\Omega} (\mathbf{u}_2 \cdot \mathbf{w}) \varphi}{\|(\mathbf{w}, \psi)\|_{\mathbf{W} \times \Psi}}.$$

Finally, using Hölder's inequality and (3.5), we conclude the following

$$\begin{aligned} \frac{1}{C_c^{\text{ADR}}} \|(\zeta, \varphi)\|_{\mathbf{W} \times \Psi} &\leq \sup_{\substack{(\mathbf{w}, \psi) \in \mathbf{W} \times \Psi \\ (\mathbf{w}, \psi) \neq (\mathbf{0}, \mathbf{0})}} \frac{F^{-\frac{1}{3}} \|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathbf{V}} \|(\mathbf{w}, \psi)\|_{\mathbf{W} \times \Psi} \|m\|_{0, \frac{6}{5}, \Omega}}{\|(\mathbf{w}, \psi)\|_{\mathbf{W} \times \Psi}} \\ &+ \sup_{\substack{(\mathbf{w}, \psi) \in \mathbf{W} \times \Psi \\ (\mathbf{w}, \psi) \neq (\mathbf{0}, \mathbf{0})}} \frac{F^{-\frac{1}{3}} \|\mathbf{u}_2\|_{\mathbf{V}} \|(\mathbf{w}, \psi)\|_{\mathbf{W} \times \Psi} \|(\zeta, \varphi)\|_{\mathbf{W} \times \Psi}}{\|(\mathbf{w}, \psi)\|_{\mathbf{W} \times \Psi}}, \end{aligned}$$

which further implies that

$$\|(\zeta, \varphi)\|_{\mathbf{W} \times \Psi} \leq \frac{2C_{\alpha}^c}{C_c(C_{\alpha}^c - F^{-\frac{1}{3}})} \|m\|_{0, \frac{6}{5}, \Omega} \|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathbf{V}}, \quad (3.20)$$

where the above inequality makes sense when $C_{\alpha}^c F^{\frac{1}{3}} > 1$. This concludes the Lipschitz-continuity of \mathcal{S} by invoking the bounds from (3.19) and (3.20) as

$$\begin{aligned} \|\mathcal{S}(\mathbf{u}_1) - \mathcal{S}(\mathbf{u}_2)\|_{\mathbf{V}} &= \|\mathcal{S}_1^{\text{DF}}(\mathcal{S}_2^{\text{ADR}}(\mathbf{u}_1)) - \mathcal{S}_1^{\text{DF}}(\mathcal{S}_2^{\text{ADR}}(\mathbf{u}_2))\|_{\mathbf{V}} \\ &\leq \alpha_{\text{DF}}^{-1} \alpha_1 \|\mathcal{S}_2^{\text{ADR}}(\mathbf{u}_1) - \mathcal{S}_2^{\text{ADR}}(\mathbf{u}_2)\|_{\Psi} \leq \frac{2C_{\alpha}^c}{C_c(C_{\alpha}^c - 1)} \|m\|_{0, \frac{6}{5}, \Omega} \alpha_{\text{DF}}^{-\frac{1}{2}} \alpha_1 \|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathbf{V}}, \end{aligned} \quad (3.21)$$

with the Lipschitz constant $L_{\mathcal{S}} := \frac{2C_{\alpha}^c}{C_c(C_{\alpha}^c - F^{-\frac{1}{3}})} \|m\|_{0, \frac{6}{5}, \Omega} \alpha_{\text{DF}}^{-\frac{1}{2}} \alpha_1$. This completes the rest of the proof. \square

As a consequence of the previous analysis, we can state the following main result.

Theorem 3.6 (Well-posedness of the fully-coupled continuous problem) *Suppose that*

$$C_{c, \mathbf{u}}^{\text{DF}} [\|\mathbf{f}(0)\|_{0, \Omega} + C_c^{\text{ADR}} \alpha_1 \|m\|_{0, \frac{6}{5}, \Omega}] \leq r < \frac{1}{2}, \quad \text{and that} \quad L_{\mathcal{S}} < 1.$$

Then the coupled problem (2.3) has a unique solution $(\mathbf{u}, p, \zeta, \varphi) \in \mathbf{V} \times \mathbf{Q} \times \mathbf{W} \times \Psi$. Moreover, there holds

$$\begin{aligned} \|\mathbf{u}\|_{\mathbf{V}} + \|p\|_{\mathbf{Q}} + \|\zeta\|_{\mathbf{W}} + \|\varphi\|_{\Psi} &\leq C_c^{\text{ADR}} \|m\|_{0, \frac{6}{5}, \Omega} + C_{c, \mathbf{u}}^{\text{DF}} \max \left\{ C_{c, \mathbf{u}}^{\text{DF}} [\|\mathbf{f}(0)\|_{0, \Omega} + C_c^{\text{ADR}} \alpha_1 \|m\|_{0, \frac{6}{5}, \Omega}], \right. \\ &\quad \left. (C_{c, \mathbf{u}}^{\text{DF}} [\|\mathbf{f}(0)\|_{0, \Omega} + C_c^{\text{ADR}} \alpha_1 \|m\|_{0, \frac{6}{5}, \Omega}])^2 \right\}, \end{aligned}$$

where the constants do not depend on the model parameters κ, F, ϱ_0 .

Proof. We first recall that Lemma 3.4 guaranties that \mathcal{S} maps \mathbf{W} to itself. Then, bearing in mind the Lipschitz-continuity of $\mathcal{S} : \mathbf{W} \rightarrow \mathbf{W}$ given by Lemma 3.5 along with the fact that $L_{\mathcal{S}} < 1$, a direct application of the Banach fixed-point theorem yields the existence of a unique fixed point $\mathbf{u} \in \mathbf{V}$. Hence, a unique solution of the system (2.3). In addition, the *a priori* estimate is a direct consequence of the continuous dependence on data provided by Theorems 3.1 and 3.3. \square

Remark 3.1 *Let us briefly remark that, due to the coupling between the Darcy–Forchheimer velocity and the transport equation, our fixed-point framework does not allow us to readily obtain α -independent estimates for $\|\hat{\mathbf{u}}\|_{\mathbf{V}}$. This manifests itself in the hypothesis of Theorem 3.3 as well as in the Lipschitz constant $L_{\mathcal{S}}$ (cf. (3.21)).*

4 Finite element discretisation

Let \mathcal{T}_h be a regular simplicial mesh defined in a bounded Lipschitz domain Ω , consisting of tetrahedral elements (or triangular elements in two dimensions) K with diameter h_K . We define the mesh size as $h := \max(h_K : K \in \mathcal{T}_h)$. Given an integer $k \geq 0$ and a generic element $K \in \mathcal{T}_h$, we denote by $\mathbb{P}_k(K)$ the space of polynomials defined locally in K with a degree at most k , and denote by $\mathbb{P}_k(\mathcal{T}_h)$ its global counterpart. In addition, let $\mathbb{RT}_k(\mathcal{T}_h)$ denote the $\mathbf{H}_{\Gamma_\bullet}(\text{div}, \Omega)$ -discrete conforming subspace defined by the Raviart–Thomas elements of order k , i.e.,

$$\mathbb{RT}_k(\mathcal{T}_h) := \{\mathbf{u} \in \mathbf{H}_{\Gamma_\bullet}(\text{div}_s, \Omega) : \mathbf{u}|_K = [\mathbb{P}_k(K)]^d \oplus \tilde{\mathbb{P}}_k(K)\mathbf{x}, \mathbf{x} \in \Omega \subset \mathbb{R}^d, K \in \mathcal{T}_h\},$$

where $\tilde{\mathbb{P}}_k(K)$ is the space of all polynomials over K of degree exactly equal to k . We follow [17, Section 4] for constructing discrete spaces for the system (2.3). Define conforming discrete weighted subspaces, $\mathbf{V}_h \subseteq \mathbf{V}$ and $\mathbf{W}_h \subseteq \mathbf{W}$ as follows

$$\mathbf{V}_h := \kappa^{-\frac{1}{2}} \mathbf{L}_h^2(\Omega) \cap \mathbf{H}_{h, \Gamma_u}(\text{div}, \Omega) \cap \mathbf{F}^{\frac{1}{3}} \mathbf{L}_h^3(\Omega), \quad (4.1a)$$

$$\mathbf{W}_h := \alpha^{-\frac{1}{2}} \mathbf{L}_h^2(\Omega) \cap \mathbf{H}_{h, \Gamma_p}(\text{div}_{\frac{6}{5}}, \Omega), \quad (4.1b)$$

where $\mathbf{L}_h^r(\Omega)$, $\mathbf{H}_{h, \Gamma_\bullet}(\text{div}, \Omega)$, $\mathbf{H}_{h, \Gamma_\bullet}(\text{div}_s, \Omega)$ -norms correspond to functions in $\mathbb{RT}_k(\mathcal{T}_h)$ equipped with $\mathbf{L}^r(\Omega)$, for $r = 2, 3$, $\mathbf{H}_{\Gamma_\bullet}(\text{div}, \Omega)$, and $\mathbf{H}_{\Gamma_\bullet}(\text{div}_{\frac{6}{5}}, \Omega)$. Next, define the discrete gradient operator $\nabla_h : \mathbb{P}_k(\mathcal{T}_h) \rightarrow \mathbb{RT}_k(\mathcal{T}_h)$ (cf. [4, Equation 3.1]) by

$$\sum_{K \in \mathcal{T}_h} \int_K (\nabla_h q_h) \cdot \mathbf{v}_h = - \sum_{K \in \mathcal{T}_h} \int_K (\text{div } \mathbf{v}_h) q_h \quad \forall q_h \in \mathbb{P}_k(\mathcal{T}_h), \text{ and } \forall \mathbf{v}_h \in \mathbb{RT}_k(\mathcal{T}_h). \quad (4.2)$$

Additionally, define the following broken Sobolev spaces

$$\mathbf{W}^{1,r}(\mathcal{T}_h) := \{p_h \in L^2(\Omega) : p_h|_K \in W^{1,r}(K) \quad \forall K \in \mathcal{T}_h\}, \quad \text{for } r = 2, \frac{3}{2},$$

and whenever needed, we add a subscript Γ_\bullet denoting a part on the boundary, where the value of the function is prescribed to zero. These spaces are equipped with DG (broken Sobolev) seminorms $|\bullet|_{1, \mathcal{T}_h}$ and $|\bullet|_{1,r, \mathcal{T}_h}$, defined as

$$\begin{aligned} \mathbf{H}^1(\mathcal{T}_h) \ni \zeta &\mapsto |\zeta|_{1, \mathcal{T}_h}^2 := \left(\sum_{K \in \mathcal{T}_h} \|\nabla \zeta\|_{0,K}^2 + \sum_{F \in \mathcal{F}_h} h_F^{-1} \|[[\zeta]]\|_{0,F}^2 \right)^{\frac{1}{2}}, \\ \mathbf{W}^{1,r}(\mathcal{T}_h) \ni \xi &\mapsto |\xi|_{1,r, \mathcal{T}_h}^r := \left(\sum_{K \in \mathcal{T}_h} \|\nabla \xi\|_{0,r,K}^r + \sum_{F \in \mathcal{F}_h} h_F^{1-r} \|[[\xi]]\|_{0,r,F}^r \right)^{\frac{1}{r}}, \end{aligned}$$

respectively, where \mathcal{F}_h denotes the set of facets in the mesh \mathcal{T}_h . Further, define the following weighted spaces for the pressure, as the sum of the infinite-dimensional (but broken) spaces $L^2(\Omega)$, $\mathbf{H}_{\Gamma_p}^1(\mathcal{T}_h)$ and $\mathbf{W}_{\Gamma_p}^{1, \frac{3}{2}}(\mathcal{T}_h)$, and the concentration, as the sum of $L^6(\Omega)$ and $\mathbf{H}_{\Gamma_u}^1(\mathcal{T}_h)$:

$$\widehat{\mathbf{Q}}(\mathcal{T}_h) := L^2(\Omega) + \kappa^{\frac{1}{2}} \mathbf{H}_{\Gamma_p}^1(\mathcal{T}_h) + \mathbf{F}^{-\frac{1}{3}} \mathbf{W}_{\Gamma_p}^{1, \frac{3}{2}}(\mathcal{T}_h), \quad (4.3a)$$

$$\widehat{\mathbf{\Psi}}(\mathcal{T}_h) := L^6(\Omega) + \alpha^{\frac{1}{2}} \mathbf{H}_{\Gamma_u}^1(\mathcal{T}_h). \quad (4.3b)$$

Define broken weighted norms on $\widehat{\mathbf{Q}}(\mathcal{T}_h)$ and $\widehat{\mathbf{\Psi}}(\mathcal{T}_h)$ (see, [17, Section 4]) as

$$\|q\|_{\widehat{\mathbf{Q}}(\mathcal{T}_h)} := \inf_{\substack{q=q_1+q_2+q_3 \\ (q_1, q_2, q_3) \in L^2(\Omega) \times \mathbf{H}_{\Gamma_p}^1(\mathcal{T}_h) \times \mathbf{W}_{\Gamma_p}^{1, \frac{3}{2}}(\mathcal{T}_h)}} \left(\|q_1\|_{0, \Omega} + \kappa^{\frac{1}{2}} |q_2|_{1, \mathcal{T}_h} + \mathbf{F}^{-\frac{1}{3}} |q_3|_{1, \frac{3}{2}, \mathcal{T}_h} \right), \quad (4.4)$$

$$\|\psi\|_{\widehat{\mathbf{\Psi}}(\mathcal{T}_h)} := \inf_{\substack{\psi=\psi_1+\psi_2 \\ (\psi_1, \psi_2) \in L^6(\Omega) \times \mathbf{H}_{\Gamma_u}^1(\mathcal{T}_h)}} \left(\|\psi_1\|_{0,6, \Omega} + \alpha^{\frac{1}{2}} |\psi_2|_{1, \mathcal{T}_h} \right). \quad (4.5)$$

Finally, we set discrete spaces \mathbf{Q}_h and Ψ_h (non-conforming subspaces of \mathbf{Q} and Ψ , respectively) as

$$\mathbf{Q}_h := L_h^2(\Omega) + \kappa^{\frac{1}{2}} H_{h,\Gamma_p}^1(\Omega) + F^{-\frac{1}{3}} W_{h,\Gamma_p}^{1,\frac{3}{2}}(\Omega), \quad (4.6a)$$

$$\Psi_h := L_h^6(\Omega) + \alpha^{\frac{1}{2}} H_{h,\Gamma_u}^1(\Omega), \quad (4.6b)$$

where $L_h^r(\Omega)$, $H_h^1(\Omega)$, and $W_h^{1,\frac{3}{2}}(\Omega)$ are polynomial spaces having discrete functions from $\mathbb{P}_k(\mathcal{T}_h)$, and equipped with the $L^r(\Omega)$ norm (for $r = 2, 6$) and the broken $H^1(\mathcal{T}_h)$, $W^{1,\frac{3}{2}}(\mathcal{T}_h)$ seminorms, respectively. We equip \mathbf{Q}_h and Ψ_h with the $\|\bullet\|_{\widehat{\mathbf{Q}}(\mathcal{T}_h)}$ and $\|\bullet\|_{\widehat{\Psi}(\mathcal{T}_h)}$ norms, respectively. Next, set discrete bilinear forms $b_{1,h} : \mathbf{V}_h \times \mathbf{Q}_h \rightarrow \mathbb{R}$, $b_{2,h} : \mathbf{W}_h \times \Psi_h \rightarrow \mathbb{R}$ and the discrete linear functional $H_h : \Psi_h \rightarrow \mathbb{R}$ as follows

$$b_{1,h}(\mathbf{v}_h, q_h) := \int_{\Omega} \operatorname{div} \mathbf{v}_h q_h, \quad b_{2,h}(\boldsymbol{\zeta}_h, \psi_h) := \int_{\Omega} \operatorname{div} \boldsymbol{\zeta}_h \psi_h, \quad H_h(\psi_h) := - \int_{\Omega} m \psi_h,$$

for all $(\mathbf{v}_h, q_h, \boldsymbol{\zeta}_h, \psi_h) \in \mathbf{V}_h \times \mathbf{Q}_h \times \mathbf{W}_h \times \Psi_h$. Then, with the FE spaces (4.1a), (4.1b), (4.6a) and (4.6b) the mixed FE formulation of (2.3) consists in finding $(\mathbf{u}_h, p_h, \boldsymbol{\zeta}_h, \varphi_h) \in \mathbf{V}_h \times \mathbf{Q}_h \times \mathbf{W}_h \times \Psi_h$ such that

$$a_1(\mathbf{u}_h, \mathbf{v}_h) + d_1(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) + b_{1,h}(\mathbf{v}_h, p_h) = F^{\varphi_h}(\mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \quad (4.7a)$$

$$b_{1,h}(\mathbf{u}_h, q_h) = 0 \quad \forall q_h \in \mathbf{Q}_h, \quad (4.7b)$$

$$a_2(\boldsymbol{\zeta}_h, \mathbf{w}_h) + b_{2,h}(\mathbf{w}_h, \varphi_h) + d_2^{\mathbf{u}_h}(\mathbf{w}_h, \varphi_h) = 0 \quad \forall \mathbf{w}_h \in \mathbf{W}_h, \quad (4.7c)$$

$$b_{2,h}(\boldsymbol{\zeta}_h, \psi_h) - c_{2,h}(\varphi_h, \psi_h) = H_h(\psi_h) \quad \forall \psi_h \in \Psi_h. \quad (4.7d)$$

A direct discrete analysis based on $\mathbf{V}_h := (\mathbb{RT}_k(\mathcal{T}_h), \|\bullet\|_{\operatorname{div},\Omega})$, $\mathbf{W}_h := (\mathbb{RT}_k(\mathcal{T}_h), \|\bullet\|_{\operatorname{div}_{\frac{5}{6}},\Omega})$, $\mathbf{Q}_h := (\mathbb{P}_k(\mathcal{T}_h), \|\bullet\|_{0,\Omega})$, and $\Psi_h := (\mathbb{P}_k(\mathcal{T}_h), \|\bullet\|_{0,6,\Omega})$, endowed with the standard norms, does not lead to parameter-robust inf-sup and Braess conditions. The parameter robustness is instead obtained by adopting the weighted discrete spaces (4.1a)-(4.1b), (4.6a)-(4.6b).

4.1 Robust discrete solvability

Consider the following system for a fixed $\hat{\varphi}_h \in \Psi_h$

$$a_1(\mathbf{u}_h, \mathbf{v}_h) + d_1(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) + b_{1,h}(\mathbf{v}_h, p_h) = F^{\hat{\varphi}_h}(\mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \quad (4.8a)$$

$$b_{1,h}(\mathbf{u}_h, q_h) = 0 \quad \forall q_h \in \mathbf{Q}_h, \quad (4.8b)$$

and recall from [17, Theorem 4.3] the following result.

Theorem 4.1 (Well-posedness of discrete Darcy–Forchheimer equations) *Assume that $\mathbf{f}(\hat{\varphi}_h) \in \mathbf{L}^2(\Omega)$. Then there exists a unique solution $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times \mathbf{Q}_h$ to (4.7a)-(4.7b). Furthermore, there is a positive constant C_d^{DF} , independent of the parameters κ, F, h , such that*

$$\|(\mathbf{u}_h, p_h)\|_{\mathbf{V} \times \widehat{\mathbf{Q}}(\mathcal{T}_h)} \leq C_d^{\text{DF}} \max(\|\mathbf{f}(\hat{\varphi}_h)\|_{0,\Omega}, \|\mathbf{f}(\hat{\varphi}_h)\|_{0,\Omega}^2).$$

Additionally, there exists a positive constant $C_{d,\mathbf{u}_h}^{\text{DF}}$, independent of κ, F, \mathbf{u}_h , and h , such that

$$\|\mathbf{u}_h\|_{\mathbf{V}} \leq C_{d,\mathbf{u}_h}^{\text{DF}} \|\mathbf{f}(\hat{\varphi}_h)\|_{0,\Omega}. \quad (4.9)$$

We next prove the following lemma, which is crucial in deducing the discrete inf-sup stability of $b_{2,h}(\bullet, \bullet)$.

Lemma 4.2 *There exists an operator \mathcal{S}_d satisfying the following properties:*

$$\begin{aligned} \mathcal{S}_d &\in \mathcal{L}(\Psi'_h, \mathbf{W}_h), \quad \|\mathcal{S}_d\|_{\mathcal{L}(\Psi'_h, \mathbf{W}_h)} \text{ is independent of } \alpha, \varrho_0, h \quad \text{and} \\ \langle \operatorname{div}_{\frac{5}{6}}[\mathcal{S}_d(\ell)], \psi_h \rangle &= \langle \ell, \psi_h \rangle \quad \text{for all } \ell \in \Psi'_h, \psi_h \in \Psi_h. \end{aligned} \quad (4.10)$$

Proof. First, construct \mathcal{S}_d in such a way that it satisfies the following properties

$$\mathcal{S}_d \in \mathcal{L}\left(\left(\mathbf{H}_{h,\Gamma_u}^1(\Omega)\right)', \mathbf{L}_h^2(\Omega)\right), \quad \text{and} \quad \mathcal{S}_d \in \mathcal{L}\left(\left(\mathbf{L}_h^6(\Omega)\right)', \mathbf{H}_{h,\Gamma_p}(\text{div}_{\frac{6}{5}}, \Omega)\right),$$

respectively, and then using the fact that (cf. (1.3)) $\Psi'_h = \left(\mathbf{H}_{h,\Gamma_u}^1(\Omega) + \mathbf{L}_h^6(\Omega)\right)' = \left(\mathbf{H}_{h,\Gamma_u}^1(\Omega)\right)' \cap \left(\mathbf{L}_h^6(\Omega)\right)'$, we conclude that $\mathcal{S}_d \in \mathcal{L}\left(\Psi'_h, \mathbf{W}_h\right)$. The existence of $\mathcal{S}_d \in \mathcal{L}\left(\left(\mathbf{H}_{h,\Gamma_u}^1(\Omega)\right)', \mathbf{L}_h^2(\Omega)\right)$ follows directly from [17, Lemma 4.1].

Next, we aim to construct $\mathcal{S}_d \in \mathcal{L}\left(\left(\mathbf{L}_h^6(\Omega)\right)', \mathbf{H}_{h,\Gamma_p}(\text{div}_{\frac{6}{5}}, \Omega)\right)$. Let $q_h \in \mathbf{L}_h^6(\Omega)$, and \mathcal{O} be a convex bounded domain containing $\bar{\Omega}$. Now, define

$$\ell := \begin{cases} |q_h|^4 q_h, & \text{in } \Omega, \\ 0, & \text{in } \mathcal{O} \setminus \bar{\Omega}. \end{cases} \quad (4.11)$$

Note that, $\ell \in \mathbf{L}^{\frac{6}{5}}(\mathcal{O})$, satisfying the following

$$\|\ell\|_{0,\frac{6}{5},\Omega} = \||q_h|^4 q_h\|_{0,\frac{6}{5},\Omega} = \|q_h\|_{0,6,\Omega}^5. \quad (4.12)$$

It now follows from [23, Corollary 1] that, there exists a unique $z \in \mathbf{W}_{\Gamma_u}^{1,\frac{6}{5}}(\mathcal{O}) \cap \mathbf{W}^{2,\frac{6}{5}}(\mathcal{O})$ satisfying the following boundary value problem

$$\Delta z = \ell \quad \text{in } \Omega, \quad z = 0 \quad \text{on } \Gamma_u, \quad \nabla z \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_p, \quad (4.13)$$

and additionally, there exists a positive constant C_{reg} (depending only on d and Ω) such that z satisfies the following stability estimate (the last equality is due to (4.12))

$$\|z\|_{2,\frac{6}{5},\mathcal{O}} \leq C_{\text{reg}} \|\ell\|_{0,\frac{6}{5},\Omega} = C_{\text{reg}} \|q_h\|_{0,6,\Omega}^5. \quad (4.14)$$

Next, set $\varsigma := (\nabla z)|_{\Omega} \in \mathbf{W}^{1,\frac{6}{5}}(\Omega)$, and note from (4.13) and (4.14) that

$$\text{div}(\varsigma) = \ell = |q_h|^4 q_h \quad \text{in } \Omega, \quad (4.15)$$

$$\|\varsigma\|_{1,\frac{6}{5},\Omega} \leq \|z\|_{2,\frac{6}{5},\Omega} \leq C_{\text{reg}} \|q_h\|_{0,6,\Omega}^5. \quad (4.16)$$

In turn, let us now define $\varsigma_h := \Pi_h^k(\varsigma)$ (Π_h^k being the Raviart–Thomas projection operator satisfying [14, Equation 5.38, Equation 5.39]). Further, using the boundedness of the projection operator, (4.16), and finally the continuous injection of $\mathbf{W}^{1,\frac{6}{5}}(\Omega)$ into $\mathbf{L}^2(\Omega)$, we obtain the following estimate (the absorbed constant being independent of h)

$$\|\varsigma_h\|_{0,\Omega} = \|\Pi_h^k(\varsigma)\|_{0,\Omega} \leq C_{\text{reg}} \|q_h\|_{0,6,\Omega}^5. \quad (4.17)$$

In addition, from [14, Equation (5.35)] and (4.15), we obtain the following commutativity of the divergence operator and the projection operator

$$\text{div}(\varsigma_h) = \text{div}(\Pi_h^k(\varsigma)) = \Pi_h^k(\text{div}(\varsigma)) = \Pi_h^k(|q_h|^4 q_h), \quad (4.18)$$

and further, using (4.15), and the boundedness of Π_h^k , it follows that

$$\|\text{div}(\varsigma_h)\|_{0,\frac{6}{5},\Omega} \leq \|q_h\|_{0,6,\Omega}^5. \quad (4.19)$$

Then, it suffices to define $\mathcal{S}_d(\ell) := \varsigma_h$, which further implies that $\mathcal{S}_d \in \mathcal{L}\left(\left(\mathbf{L}_h^6(\Omega)\right)', \mathbf{H}_{h,\Gamma_p}(\text{div}_{\frac{6}{5}}, \Omega)\right)$. This completes the rest of the proof. \square

We now consider the following discrete ADR system for a fixed $\hat{\mathbf{u}}_h \in \mathbf{V}_h$

$$a_2(\zeta_h, \mathbf{w}_h) + b_{2,h}(\mathbf{w}_h, \varphi_h) + d_2^{\hat{\mathbf{u}}_h}(\mathbf{w}_h, \varphi_h) = 0 \quad \forall \mathbf{w}_h \in \mathbf{W}_h, \quad (4.20a)$$

$$b_{2,h}(\zeta_h, \psi_h) - c_{2,h}(\varphi_h, \psi_h) = H_h(\psi_h) \quad \forall \psi_h \in \Psi_h, \quad (4.20b)$$

and employ the discrete version of Theorem 2.1 to discuss its parameter-robust unique solvability.

Theorem 4.3 (Well-posedness of discrete advection-diffusion-reaction equations) Assume that $m \in L^{\frac{6}{5}}(\Omega)$. Then, for a fixed $\hat{\mathbf{u}}_h \in \mathbf{V}_h$, there exist positive constants C_d, C_α^d (depending on α), such that, whenever $\|\hat{\mathbf{u}}_h\|_{0,3,\Omega} \leq \frac{C_d}{2C_\alpha^d}$, there exists a unique solution $(\zeta_h, \varphi_h) \in \mathbf{W}_h \times \Psi_h$ to the problem (4.20). Furthermore, there is a positive constant C_d^{ADR} independent of h , and ϱ_0 , such that

$$\|(\zeta_h, \varphi_h)\|_{\mathbf{W} \times \widehat{\Psi}(\mathcal{T}_h)} \leq C_d^{\text{ADR}} \|m\|_{0,\frac{6}{5},\Omega}.$$

Proof. The boundedness of $a_2(\bullet, \bullet), b_{2,h}(\bullet, \bullet), c_{2,h}(\bullet, \bullet)$, and symmetry and positive definiteness of $a_2(\bullet, \bullet)$ and $c_{2,h}(\bullet, \bullet)$, and the inf-sup stability of $a_2(\bullet, \bullet)$ in $\mathbf{Z}_h := \{\tau_h \in \mathbf{W}_h : b_{2,h}(\tau_h, \psi_h) = 0, \forall \psi_h \in \Psi_h\}$ follow analogously as in Theorem 3.3. Therefore, to complete the rest of the proof, it is enough to prove the discrete inf-sup stability of $b_{2,h}(\bullet, \bullet)$ and the discrete Braess condition.

Discrete inf-sup condition. From the existence of \mathcal{S}_d defined in Lemma 4.2, it follows that

$$\begin{aligned} \sup_{\mathbf{\tau}_h \in \mathbf{W}_h} \frac{\langle \text{div } \mathbf{\tau}_h, \psi_h \rangle}{\|\mathbf{\tau}_h\|_{\mathbf{W}}} &\geq \sup_{\mathbf{\ell} \in \Psi'_h} \frac{\langle \text{div } \frac{6}{5} \mathcal{S}_d(\mathbf{\ell}), \psi_h \rangle}{\|\mathcal{S}_d(\mathbf{\ell})\|_{\mathbf{W}}} = \sup_{\mathbf{\ell} \in \Psi'_h} \frac{\langle \mathbf{\ell}, \psi_h \rangle}{\|\mathcal{S}_d(\mathbf{\ell})\|_{\mathbf{W}}} \\ &\geq \|\mathcal{S}_d\|_{\mathcal{L}(\Psi'_h, \mathbf{W}_h)}^{-1} \sup_{\mathbf{\ell} \in \Psi'_h} \frac{\langle \mathbf{\ell}, \psi_h \rangle}{\|\mathbf{\ell}\|_{\Psi'_h}} = \tilde{C}_b \|\psi_h\|_{\widehat{\Psi}(\mathcal{T}_h)}, \end{aligned}$$

where $\tilde{C}_b = \|\mathcal{S}_d\|_{\mathcal{L}(\Psi'_h, \mathbf{W}_h)}^{-1}$, is independent of α, ϱ_0, h , and hence the discrete stability of inf-sup is achieved.

Discrete Braess condition. Setting $\mathbf{w}_h = \zeta_h, \psi_h = 0$, and exploiting $\|\zeta_h\|_{\mathbf{W}} = \|(\zeta_h, 0)\|_{\mathbf{W} \times \widehat{\Psi}(\mathcal{T}_h)}$, one obtains

$$\sup_{(\mathbf{0},0) \neq (\mathbf{w}_h, \psi_h) \in \mathbf{W}_h \times \Psi_h} \frac{a_2(\zeta_h, \mathbf{w}_h) + b_{2,h}(\zeta_h, \psi_h)}{\|(\mathbf{w}_h, \psi_h)\|_{\mathbf{W} \times \widehat{\Psi}(\mathcal{T}_h)}} \geq \frac{\alpha^{-1} \int_{\Omega} \zeta_h \cdot \zeta_h}{\|(\zeta_h, 0)\|_{\mathbf{W} \times \widehat{\Psi}(\mathcal{T}_h)}} = \frac{\alpha^{-1} \|\zeta_h\|_{0,\Omega}^2}{\|\zeta_h\|_{\mathbf{W}}}. \quad (4.21)$$

Next, taking the particular values $\mathbf{w}_h = \mathbf{0}$, and $\psi_h = \text{sgn}(\text{div } \zeta_h) |\text{div } \zeta_h|^{\frac{1}{5}}$, one obtains the following

$$\text{(LHS)}_{(4.21)} \geq \frac{a_2(\zeta_h, \mathbf{0}) + b_{2,h}(\zeta_h, \text{sgn}(\text{div } \zeta_h) |\text{div } \zeta_h|^{\frac{1}{5}})}{\|(\mathbf{0}, \text{sgn}(\text{div } \zeta_h) |\text{div } \zeta_h|^{\frac{1}{5}})\|_{\mathbf{W} \times \widehat{\Psi}(\mathcal{T}_h)}} \geq \frac{1}{2} \frac{\|\text{div } \zeta_h\|_{0,\frac{6}{5},\Omega}^{\frac{6}{5}}}{\|\text{div } \zeta_h\|_{0,\frac{6}{5},\Omega}^{\frac{1}{5}}} = \frac{1}{2} \|\text{div } \zeta_h\|_{0,\frac{6}{5},\Omega}. \quad (4.22)$$

Now, let us combine (4.21) and (4.22) to obtain the following

$$\sup_{(\mathbf{0},0) \neq (\mathbf{w}_h, \psi_h) \in \mathbf{W}_h \times \Psi_h} \frac{a_2(\zeta_h, \mathbf{w}_h) + b_{2,h}(\zeta_h, \psi_h)}{\|(\mathbf{w}_h, \psi_h)\|_{\mathbf{W} \times \widehat{\Psi}(\mathcal{T}_h)}} \geq \max\left(\frac{\alpha^{-1} \|\zeta_h\|_{0,\Omega}^2}{\|\zeta_h\|_{\mathbf{W}}}, \frac{1}{2} \|\text{div } \zeta_h\|_{0,\frac{6}{5},\Omega}\right). \quad (4.23)$$

Denoting $\gamma := \frac{\alpha^{-\frac{1}{2}} \|\zeta_h\|_{0,\Omega}}{\|\zeta_h\|_{\mathbf{W}}} \in [0, 1)$, and using (4.23), we establish the following inequality

$$\max\left(\frac{\alpha^{-1} \|\zeta_h\|_{0,\Omega}^2}{\|\zeta_h\|_{\mathbf{W}}}, \|\text{div } \zeta_h\|_{0,\frac{6}{5},\Omega}\right) \geq \|\zeta_h\|_{\mathbf{W}} \left(\max_{t \in [0,1)} [\gamma^2, \frac{1}{2} \sqrt{1-\gamma^2}]\right).$$

On the other hand, if we denote $\gamma_d := \left(\max_{t \in [0,1)} [\gamma^2, \frac{1}{2} \sqrt{1-\gamma^2}]\right)$, we can readily conclude that

$$\sup_{(\mathbf{0},0) \neq (\mathbf{w}_h, \psi_h) \in \mathbf{W}_h \times \Psi_h} \frac{a_2(\zeta_h, \mathbf{w}_h) + b_{2,h}(\zeta_h, \psi_h)}{\|(\mathbf{w}_h, \psi_h)\|_{\mathbf{W} \times \widehat{\Psi}(\mathcal{T}_h)}} \geq \gamma_d \|\zeta_h\|_{\mathbf{W}}.$$

An appeal to Theorem 2.1 now yields the well-posedness of the discrete reaction-diffusion system (4.7c) and (4.7d) without the advection part. Additionally, the following estimate holds

$$\sup_{(\mathbf{0},0) \neq (\tau_h, \psi_h) \in \mathbf{W}_h \times \Psi_h} \frac{\mathcal{A}(\zeta_h, \varphi_h; \tau_h, \psi_h)}{\|(\tau_h, \psi_h)\|_{\mathbf{W} \times \widehat{\Psi}(\mathcal{T}_h)}} \geq C_d \|(\zeta_h, \varphi_h)\|_{\mathbf{W} \times \widehat{\Psi}(\mathcal{T}_h)}, \quad (4.24)$$

where C_d is the positive constant from (2.17), independent of the model and discretisation parameters.

Finally, for the well-posedness of the uncoupled discrete ADR equations, for a fixed $\hat{\mathbf{u}}_h \in \mathbf{V}_h$, we consider another bilinear form $\mathcal{A}_d^{\text{ADR}} : \mathbf{W}_h \times \Psi_h \rightarrow \mathbb{R}$ defined by

$$\mathcal{A}_d^{\text{ADR}}(\zeta_h, \varphi_h; \boldsymbol{\tau}_h, \psi_h) := \mathcal{A}(\zeta_h, \varphi_h; \boldsymbol{\tau}_h, \psi_h) + d_2^{\hat{\mathbf{u}}_h}(\boldsymbol{\tau}_h, \varphi_h), \quad \forall (\zeta_h, \varphi_h), (\boldsymbol{\tau}_h, \psi_h) \in \mathbf{W}_h \times \Psi_h. \quad (4.25)$$

Thus, to show the well-posedness of (4.7c)–(4.7d), we now analyse the problem: Find $(\zeta_h, \varphi_h) \in \mathbf{W}_h \times \Psi_h$ such that

$$\mathcal{A}_d^{\text{ADR}}(\zeta_h, \varphi_h; \boldsymbol{\tau}_h, \psi_h) = H_h(\psi_h), \quad \forall \psi_h \in \Psi_h.$$

From the boundedness of $d_2^{\hat{\mathbf{u}}_h}(\bullet, \bullet)$,

$$d_2^{\hat{\mathbf{u}}_h}(\boldsymbol{\tau}_h, \varphi_h) \leq C_\alpha^d \|\hat{\mathbf{u}}_h\|_{0,3,\Omega} \|(\boldsymbol{\tau}_h, \psi_h)\|_{\mathbf{W} \times \hat{\Psi}(\mathcal{T}_h)} \|(\zeta_h, \varphi_h)\|_{\mathbf{W} \times \hat{\Psi}(\mathcal{T}_h)}, \quad (4.26)$$

where $C_\alpha^d = \alpha^{-\frac{1}{2}}(1 + C_{\text{emb}}\alpha^{-\frac{1}{2}})$, combined with (4.24) and (4.25), we can assert that

$$\sup_{(\mathbf{0},0) \neq (\boldsymbol{\tau}_h, \psi_h) \in \mathbf{W}_h \times \Psi_h} \frac{\mathcal{A}_d^{\text{ADR}}(\zeta_h, \varphi_h; \boldsymbol{\tau}_h, \psi_h)}{\|(\boldsymbol{\tau}_h, \psi_h)\|_{\mathbf{W} \times \hat{\Psi}(\mathcal{T}_h)}} \geq [C_d - C_\alpha^d \|\hat{\mathbf{u}}_h\|_{0,3,\Omega}] \|(\zeta_h, \varphi_h)\|_{\mathbf{W} \times \hat{\Psi}(\mathcal{T}_h)},$$

for all $(\zeta_h, \varphi_h) \in \mathbf{W}_h \times \Psi_h$. Hence, under the assumption that $\|\hat{\mathbf{u}}_h\|_{0,3,\Omega} \leq \frac{C_d}{2C_\alpha^d}$, we arrive at

$$\sup_{(\mathbf{0},0) \neq (\boldsymbol{\tau}_h, \psi_h) \in \mathbf{W}_h \times \Psi_h} \frac{\mathcal{A}_d^{\text{ADR}}(\zeta_h, \varphi_h; \boldsymbol{\tau}_h, \psi_h)}{\|(\boldsymbol{\tau}_h, \psi_h)\|_{\mathbf{W} \times \hat{\Psi}(\mathcal{T}_h)}} \geq \frac{C_d}{2} \|(\zeta_h, \varphi_h)\|_{\mathbf{W} \times \hat{\Psi}(\mathcal{T}_h)}. \quad (4.27)$$

Therefore, the discrete ADR system (4.7c)–(4.7d) is well-posed. Additionally, invoking (4.25) in (4.27), and applying Hölder's inequality, gives the dependence on data $\|(\zeta_h, \varphi_h)\|_{\mathbf{W} \times \hat{\Psi}(\mathcal{T}_h)} \leq C_d^{\text{ADR}} \|m\|_{0, \frac{6}{5}, \Omega}$, where $C_d^{\text{ADR}} := 2C_d^{-1}(1 + C_{\text{emb}}\alpha^{-\frac{1}{2}})$, which concludes the proof. \square

4.2 Fixed-point analysis in discrete framework

Analogous to the continuous case (cf. (3.15)), we now define the fixed-point operator in the discrete framework as follows

$$\mathcal{S}_h : \mathbf{V}_h \rightarrow \mathbf{V}_h, \quad \hat{\mathbf{u}}_h \mapsto \mathcal{S}_h(\hat{\mathbf{u}}_h) := \mathcal{S}_{1,h}^{\text{DF}}(\mathcal{S}_{2,h}^{\text{ADR}}(\hat{\mathbf{u}}_h)), \quad \hat{\mathbf{u}}_h \in \mathbf{V}_h, \quad (4.28)$$

where the maps $\mathcal{S}_{1,h}^{\text{DF}}$ and $\mathcal{S}_{2,h}^{\text{ADR}}$ are defined over the respective discrete spaces as follows

$$\mathcal{S}_h^{\text{DF}} : \Psi_h \rightarrow \mathbf{V}_h \times Q_h, \quad \hat{\varphi}_h \mapsto \mathcal{S}_h^{\text{DF}}(\hat{\varphi}_h) = (\mathcal{S}_{1,h}^{\text{DF}}(\hat{\varphi}_h), \mathcal{S}_{2,h}^{\text{DF}}(\hat{\varphi}_h)) := (\mathbf{u}_h, p_h),$$

where $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ is the unique solution of the Darcy–Forchheimer equations as stated in Theorem 4.1. Further, we define the solution operator associated with the mixed ADR equations as

$$\mathcal{S}_h^{\text{ADR}} : \mathbf{V}_h \rightarrow \mathbf{W}_h \times \Psi_h, \quad \hat{\mathbf{u}}_h \mapsto \mathcal{S}_h^{\text{ADR}}(\hat{\mathbf{u}}_h) = (\mathcal{S}_{1,h}^{\text{ADR}}(\hat{\mathbf{u}}_h), \mathcal{S}_{2,h}^{\text{ADR}}(\hat{\mathbf{u}}_h)) := (\zeta_h, \varphi_h),$$

where (ζ_h, φ_h) is the unique solution of the ADR equations, as stated in Theorem 4.3.

A parallel set of arguments can be applied here likewise the continuous case to show that the discrete fixed-point map \mathcal{S}_h follows the subsequent properties

- For a positive real number $0 < \tilde{r} < \frac{1}{2}$, the map \mathcal{S}_h preserves the ball-mapping property, that is, $\mathcal{S}_h(B_0^{\tilde{r}}) \subseteq B_0^{\tilde{r}}$, under the small data assumption

$$C_{d,\mathbf{u}_h}^{\text{DF}} (\|\mathbf{f}(0)\|_{0,\Omega} + C_d^{\text{ADR}} \alpha_1 \|m\|_{0, \frac{6}{5}, \Omega}) \leq \tilde{r} < \frac{1}{2},$$

where $C_{d,\mathbf{u}_h}^{\text{DF}}$ comes from (4.9).

- \mathcal{S}_h is Lipschitz continuous with the Lipschitz constant $L_S^d := \frac{2C_\alpha^d}{C_d(C_\alpha^d - F^{-\frac{1}{3}})} \|m\|_{0, \frac{6}{5}, \Omega} \alpha_{\text{DF}}^{-\frac{1}{2}} \alpha_1$.

We next state the discrete stability theorem for the fully-coupled discrete DF-ADR system similar to the continuous case as in Theorem 3.6.

Theorem 4.4 (Well-posedness of the fully-coupled discrete problem) *Suppose that*

$$C_{d,\mathbf{u}_h}^{\text{DF}} (\|\mathbf{f}(0)\|_{0,\Omega} + C_d^{\text{ADR}} \alpha_1 \|m\|_{0,\frac{6}{5},\Omega}) < \frac{1}{2},$$

and that $L_S^d < 1$. Then, the coupled problem (4.7) has a unique solution $(\mathbf{u}_h, p_h, \boldsymbol{\zeta}_h, \varphi_h) \in \mathbf{V}_h \times Q_h \times \mathbf{W}_h \times \Psi_h$. Moreover, there are

$$\begin{aligned} \|\mathbf{u}_h\|_{\mathbf{V}_h} + \|p_h\|_{\widehat{Q}(\mathcal{T}_h)} + \|\boldsymbol{\zeta}_h\|_{\mathbf{W}_h} + \|\varphi_h\|_{\widehat{\Psi}(\mathcal{T}_h)} &\leq C_d^{\text{ADR}} \|m\|_{0,\frac{6}{5},\Omega} + C_{d,\mathbf{u}_h}^{\text{DF}} \max [C_{d,\mathbf{u}_h}^{\text{DF}} [\|\mathbf{f}(0)\|_{0,\Omega} \\ &+ C_d^{\text{ADR}} \alpha_1 \|m\|_{0,\frac{6}{5},\Omega}], (C_{d,\mathbf{u}_h}^{\text{DF}} [\|\mathbf{f}(0)\|_{0,\Omega} + C_d^{\text{ADR}} \alpha_1 \|m\|_{0,\frac{6}{5},\Omega})^2], \end{aligned}$$

where the constants do not depend on the model parameters κ, F, ϱ_0 , nor the discretisation parameter h .

In the next section, we estimate the error between the exact and approximate solutions in the proposed weighted-norms, showing that the convergence is robust with respect to the material parameters.

4.3 A priori error analysis

We start this section by proving the following parameter-robust *a priori* error estimate for the coupled system (1.1).

Theorem 4.5 (Quasi-optimality) *Let $(\mathbf{u}, p, \boldsymbol{\zeta}, \varphi)$ uniquely solve the system (2.3) and $(\mathbf{u}_h, p_h, \boldsymbol{\zeta}_h, \varphi_h)$ uniquely solve the system (4.7). Additionally, suppose that the assumptions of Theorem 3.6 and Theorem 4.4 hold. Also, suppose that the following relation holds*

$$F^{-\frac{1}{3}} (1 + C_{\text{emb}} \alpha^{-\frac{1}{2}}) C_c^{\text{ADR}} C_d^{\text{ADR}} \|m\|_{0,\frac{6}{5},\Omega} \leq \frac{1}{2},$$

with α_{DF} being the uniform strong monotonicity constant (cf. [17, Equation (4.9)]). Then there exists a positive constant $C_{C\acute{e}a}$ such that the following quasi-optimal error estimate holds

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{V}} + \|p - p_h\|_{\widehat{Q}(\mathcal{T}_h)} + \|\boldsymbol{\zeta} - \boldsymbol{\zeta}_h\|_{\mathbf{W}} + \|\varphi - \varphi_h\|_{\widehat{\Psi}(\mathcal{T}_h)} \\ \leq C_{C\acute{e}a} (\text{dist}(\mathbf{u}, \mathbf{V}_h) + (\text{dist}(\mathbf{u}, \mathbf{V}_h))^2 + \text{dist}(p, Q_h) + \text{dist}(\boldsymbol{\zeta}, \mathbf{W}_h) + \text{dist}(\varphi, \Psi_h)), \end{aligned} \quad (4.29)$$

where $C_{C\acute{e}a}$ is independent of κ, F, ϱ_0 , and h .

Proof. Let $\varphi \in \Psi$ solve the system (3.4) for $\hat{\mathbf{u}} = \mathbf{u} \in \mathbf{V}$, and $\varphi_h \in \Psi_h$ solve (4.20) for $\hat{\mathbf{u}}_h = \mathbf{u}_h \in \mathbf{V}_h$. Subtracting (4.8) from (3.1), with $\hat{\varphi} = \varphi$ and $\hat{\varphi}_h = \varphi_h$, we obtain the following system of equations

$$a_1(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) + d_1(\mathbf{u}; \mathbf{u}, \mathbf{v}_h) - d_1(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) + b_{1,h}(\mathbf{v}_h, p - p_h) = F^\varphi(\mathbf{v}_h) - F^{\varphi_h}(\mathbf{v}_h), \quad (4.30a)$$

$$b_{1,h}(\mathbf{u} - \mathbf{u}_h, q_h) = 0, \quad (4.30b)$$

for all $\mathbf{v}_h \in \mathbf{V}_h$ and $q_h \in Q_h$. Let $\mathbf{u}_h^* \in \mathbf{V}_h$ such that $\mathbf{u}_h - \mathbf{u}_h^* \in \{\mathbf{v}_h \in \mathbf{V}_h : b_{1,h}(\mathbf{v}_h, q_h) = 0, \forall q_h \in Q_h\}$. From the uniform strong monotonicity of $a_1(\bullet, \bullet) + d_1(\bullet; \bullet, \bullet)$ on the kernel of $b_{1,h}$ (see [17, Equation 3.8]), we deduce

$$\alpha_{\text{DF}} \|\mathbf{u}_h - \mathbf{u}_h^*\|_{\mathbf{V}}^2 \leq a_1(\mathbf{u}_h, \mathbf{u}_h - \mathbf{u}_h^*) - a_1(\mathbf{u}_h^*, \mathbf{u}_h - \mathbf{u}_h^*) + d_1(\mathbf{u}_h; \mathbf{u}_h, \mathbf{u}_h - \mathbf{u}_h^*) - d_1(\mathbf{u}_h^*; \mathbf{u}_h^*, \mathbf{u}_h - \mathbf{u}_h^*). \quad (4.31)$$

Further, adding and subtracting $a_1(\mathbf{u}, \mathbf{u}_h - \mathbf{u}_h^*) + d_1(\mathbf{u}; \mathbf{u}, \mathbf{u}_h - \mathbf{u}_h^*)$ to the above inequality, using (4.30a), (1.2), and the local Lipschitz continuity of $a_1(\bullet, \bullet) + d_1(\bullet; \bullet, \bullet)$ (see [17, Equation 3.7]), and lastly using the triangle inequality the following bound is obtained

$$\begin{aligned} \alpha_{\text{DF}} \|\mathbf{u}_h - \mathbf{u}_h^*\|_{\mathbf{V}} &\leq \|p - p_h\|_{\widehat{Q}(\mathcal{T}_h)} + \|F^\varphi - F^{\varphi_h}\|_{\mathbf{V}'} + (1 + \|\mathbf{u}\|_{\mathbf{V}} + \|\mathbf{u}_h^*\|_{\mathbf{V}}) \|\mathbf{u} - \mathbf{u}_h^*\|_{\mathbf{V}} \\ &\leq \|p - p_h\|_{\widehat{Q}(\mathcal{T}_h)} + (1 + C_{\text{emb}} \alpha^{-\frac{1}{2}}) \alpha_1 \|\varphi - \varphi_h\|_{\widehat{\Psi}(\mathcal{T}_h)} + (1 + \|\mathbf{u}\|_{\mathbf{V}} + \|\mathbf{u}_h^*\|_{\mathbf{V}}) \|\mathbf{u} - \mathbf{u}_h^*\|_{\mathbf{V}} \\ &\leq \|p - p_h\|_{\widehat{Q}(\mathcal{T}_h)} + (1 + C_{\text{emb}} \alpha^{-\frac{1}{2}}) \alpha_1 \|\varphi - \varphi_h\|_{\widehat{\Psi}(\mathcal{T}_h)} + (1 + 2\|\mathbf{u}\|_{\mathbf{V}}) \|\mathbf{u} - \mathbf{u}_h^*\|_{\mathbf{V}} + \|\mathbf{u} - \mathbf{u}_h^*\|_{\mathbf{V}}^2. \end{aligned}$$

Finally, using triangle inequality and the fact that $\|\mathbf{u} - \mathbf{u}_h^*\|_{\mathbf{V}} \leq (1 + \frac{1}{\beta_d}) \text{dist}(\mathbf{u}, \mathbf{V}_h)$ (cf. [24, Theorem 2.6]), where β_d is the continuous inf-sup constant corresponding to the Darcy–Forchheimer model, we arrive at the following error estimate for the velocity

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{V}} &\leq \frac{1}{\alpha_{\text{DF}}} \text{dist}(p, \mathbf{Q}_h) + (1 + C_{\text{emb}} \alpha^{-\frac{1}{2}}) \frac{\alpha_1}{\alpha_{\text{DF}}} \|\varphi - \varphi_h\|_{\widehat{\Psi}(\mathcal{T}_h)} \\ &\quad + [1 + \frac{1}{\alpha_{\text{DF}}} (1 + \frac{1}{\beta_d}) (1 + \|\mathbf{u}\|_{\mathbf{V}})] \text{dist}(\mathbf{u}, \mathbf{V}_h) + \frac{1}{\alpha_{\text{DF}}} (1 + \frac{1}{\beta_d}) \text{dist}(\mathbf{u}, \mathbf{V}_h) \\ &\leq C_G (\text{dist}(\mathbf{u}, \mathbf{V}_h) + \text{dist}(\mathbf{u}, \mathbf{V}_h)^2 + \text{dist}(p, \mathbf{Q}_h)) + (1 + C_{\text{emb}} \alpha^{-\frac{1}{2}}) \frac{\alpha_1}{\alpha_{\text{DF}}} \|\varphi - \varphi_h\|_{\widehat{\Psi}_h}, \end{aligned} \quad (4.32)$$

where $C_G := \max(1 + \frac{1}{\alpha_{\text{DF}}} (1 + \frac{1}{\beta_d}) (1 + \|\mathbf{u}\|_{\mathbf{V}}), 1 + \frac{1}{\alpha_{\text{DF}}} (1 + \frac{1}{\beta_d}), \frac{1}{\alpha_{\text{DF}}})$. Next, for the pressure estimate, let us consider an arbitrary element $p_h^* \in \mathbf{Q}_h$, use (4.30a), and the discrete inf-sup stability of the pressure, and lastly, Lipschitz-continuity of $a_1(\bullet, \bullet) + d_1(\bullet, \bullet, \bullet)$ to conclude the following

$$\begin{aligned} \beta_d \|p_h - p_h^*\|_{\widehat{\mathbf{Q}}(\mathcal{T}_h)} &\leq \sup_{\mathbf{0} \neq \mathbf{v}_h \in \mathbf{V}_h} \frac{b_{1,h}(\mathbf{v}_h, p_h - p_h^*)}{\|\mathbf{v}_h\|_{\mathbf{V}}} \\ &\leq \sup_{\mathbf{0} \neq \mathbf{v}_h \in \mathbf{V}_h} \frac{b_{1,h}(\mathbf{v}_h, p - p_h^*) + b_{1,h}(\mathbf{v}_h, p_h - p)}{\|\mathbf{v}_h\|_{\mathbf{V}}} \leq \sup_{\mathbf{0} \neq \mathbf{v}_h \in \mathbf{V}_h} \frac{b_{1,h}(\mathbf{v}_h, p - p_h^*)}{\|\mathbf{v}_h\|_{\mathbf{V}}} \\ &\quad + \sup_{\mathbf{0} \neq \mathbf{v}_h \in \mathbf{V}_h} \frac{F^\varphi(\mathbf{v}_h) - F^{\varphi_h}(\mathbf{v}_h) - a_1(\mathbf{u}_h - \mathbf{u}_h^*, \mathbf{v}_h) - d_1(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) + d_1(\mathbf{u}_h^*; \mathbf{u}_h^*, \mathbf{v}_h)}{\|\mathbf{v}_h\|_{\mathbf{V}}} \\ &\leq \|p - p_h^*\|_{\widehat{\mathbf{Q}}(\mathcal{T}_h)} + (1 + C_{\text{emb}} \alpha^{-\frac{1}{2}}) \alpha_1 \|\varphi - \varphi_h\|_{\widehat{\Psi}(\mathcal{T}_h)} + (1 + 2\|\mathbf{u}\|_{\mathbf{V}}) \|\mathbf{u} - \mathbf{u}_h^*\|_{\mathbf{V}} + \|\mathbf{u} - \mathbf{u}_h^*\|_{\mathbf{V}}^2. \end{aligned}$$

Finally, a use of the triangle inequality yields the error estimate for pressure

$$\begin{aligned} \|p - p_h\|_{\widehat{\mathbf{Q}}(\mathcal{T}_h)} &\leq (1 + \frac{1}{\beta_d} + C_G) \text{dist}(p, \mathbf{Q}_h) + (1 + C_{\text{emb}} \alpha^{-\frac{1}{2}}) (\frac{\alpha_1}{\alpha_{\text{DF}}} + \frac{\alpha_1}{\beta_d}) \|\varphi - \varphi_h\|_{\widehat{\Psi}(\mathcal{T}_h)} \\ &\quad + C_G (\text{dist}(\mathbf{u}, \mathbf{V}_h) + \text{dist}(\mathbf{u}, \mathbf{V}_h)^2) \\ &\leq \tilde{C}_G (\text{dist}(\mathbf{u}, \mathbf{V}_h) + \text{dist}(\mathbf{u}, \mathbf{V}_h)^2 + \text{dist}(p, \mathbf{Q}_h) + \|\varphi - \varphi_h\|_{\widehat{\Psi}(\mathcal{T}_h)}), \end{aligned} \quad (4.33)$$

where $\tilde{C}_G := \max((1 + \frac{1}{\beta_d} + C_G), (1 + C_{\text{emb}} \alpha^{-\frac{1}{2}}) (\frac{\alpha_1}{\alpha_{\text{DF}}} + \frac{\alpha_1}{\beta_d}))$.

Next, subtracting (4.20) from (3.4), it is not difficult to observe that $(\zeta^* - \zeta_h, \varphi^* - \varphi_h) \in \mathbf{W} \times \widehat{\Psi}(\mathcal{T}_h)$ solves the following system of equations for any arbitrary element $(\zeta_h^*, \varphi_h^*) \in \mathbf{W}_h \times \Psi_h(\mathbf{u} - \mathbf{u}_h)$ (by substituting, in particular, $\hat{\mathbf{u}} = \mathbf{u}$, $\hat{\mathbf{u}}_h = \mathbf{u}_h$ in (3.1), (4.8), respectively). Hence,

$$\begin{aligned} a_2(\zeta_h^* - \zeta_h, \mathbf{w}_h) + b_{2,h}(\mathbf{w}_h, \varphi_h^* - \varphi_h) + d_2^{\mathbf{u}_h}(\mathbf{w}_h, \varphi_h^* - \varphi_h) &= \mathcal{H}_{1,h}(\mathbf{w}_h) \quad \forall \mathbf{w}_h \in \mathbf{W}_h, \\ b_{2,h}(\zeta_h^* - \zeta_h, \psi_h) - c_{2,h}(\varphi_h^* - \varphi_h, \psi_h) &= \mathcal{H}_{2,h}(\psi_h) \quad \forall \psi_h \in \Psi_h, \end{aligned}$$

where

$$\begin{aligned} \mathcal{H}_{1,h}(\mathbf{w}_h) &:= a_2(\zeta_h^* - \zeta, \mathbf{w}_h) + b_{2,h}(\mathbf{w}_h, \varphi_h^* - \varphi) + d_2^{\mathbf{u}_h}(\mathbf{w}_h, \varphi_h^* - \varphi) - d_2^{\mathbf{u} - \mathbf{u}_h}(\mathbf{w}_h, \varphi), \\ \text{and } \mathcal{H}_{2,h}(\psi_h) &:= b_{2,h}(\zeta_h^* - \zeta, \psi_h) + c_{2,h}(\varphi - \varphi_h^*, \psi_h). \end{aligned}$$

Finally, invoking (4.27), together with the discrete stability estimate for $(\zeta_h - \zeta_h^*, \varphi_h - \varphi_h^*)$, and applying the triangle inequality, we obtain

$$\begin{aligned} \|(\zeta - \zeta_h, \varphi - \varphi_h)\|_{\mathbf{W} \times \widehat{\Psi}(\mathcal{T}_h)} &\leq C_{\text{Céa}}^{\text{ADR}} (\text{dist}(\zeta, \mathbf{W}_h) + \text{dist}(\varphi, \Psi_h)) \\ &\quad + \mathbb{F}^{-\frac{1}{3}} (1 + C_{\text{emb}} \alpha^{-\frac{1}{2}}) C_c^{\text{ADR}} C_d^{\text{ADR}} \|m\|_{0, \frac{\mathbb{g}}{5}, \Omega} \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{V}}, \end{aligned} \quad (4.34)$$

where

$$\begin{aligned} C_{\text{Céa}}^{\text{ADR}} &:= C_d^{\text{ADR}} (3 + C_{\text{emb}} (1 + C_{\text{emb}} \alpha^{-\frac{1}{2}})^2 + (\mathbb{F}^{-\frac{1}{3}} (1 + C_{\text{emb}} \alpha^{-\frac{1}{2}}) \\ &\quad \times C_{d, \mathbf{u}_h}^{\text{DF}} (\|\mathbf{f}(\mathbf{0})\|_{0, \Omega} + \alpha_1 \alpha^{-\frac{1}{2}} (1 + C_{\text{emb}})) C_d^{\text{ADR}} \|m\|_{0, \frac{\mathbb{g}}{5}, \Omega})). \end{aligned} \quad (4.35)$$

Next, invoking (4.34) in (4.32), we arrive at the following error estimate for velocity

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{V}} &\leq C_G (\text{dist}(\mathbf{u}, \mathbf{V}_h) + \text{dist}(\mathbf{u}, \mathbf{V}_h)^2 + \text{dist}(p, Q_h)) \\ &\quad + C_{C_{\acute{e}a}}^{\text{ADR}} (1 + C_{\text{emb}} \alpha^{-\frac{1}{2}}) \frac{\alpha_1}{\alpha_{\text{DF}}} (\text{dist}(\boldsymbol{\zeta}, \mathbf{W}_h) + \text{dist}(\varphi, \Psi_h)) \\ &\quad + (1 + C_{\text{emb}} \alpha^{-\frac{1}{2}}) \frac{\alpha_1}{\alpha_{\text{DF}}} \mathbb{F}^{-\frac{1}{3}} (1 + C_{\text{emb}} \alpha^{-\frac{1}{2}}) C_c^{\text{ADR}} C_d^{\text{ADR}} \|m\|_{0, \frac{6}{5}, \Omega} \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{V}}, \end{aligned} \quad (4.36)$$

and hence,

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{V}} \leq C_{\mathbf{u}}^{\text{C}\acute{e}a} (\text{dist}(\mathbf{u}, \mathbf{V}_h) + \text{dist}(\mathbf{u}, \mathbf{V}_h)^2 + \text{dist}(p, Q_h) + \text{dist}(\boldsymbol{\zeta}, \mathbf{W}_h) + \text{dist}(\varphi, \Psi_h)), \quad (4.37)$$

where $C_{\mathbf{u}}^{\text{C}\acute{e}a} := \max(C_G, \frac{\alpha_1}{\alpha_{\text{DF}}} (1 + C_{\text{emb}} \alpha^{-\frac{1}{2}}) C_{C_{\acute{e}a}}^{\text{ADR}}) (1 - \mathbb{F}^{-\frac{1}{3}} (1 + C_{\text{emb}} \alpha^{-\frac{1}{2}}) C_c^{\text{ADR}} C_d^{\text{ADR}} \|m\|_{0, \frac{6}{5}, \Omega})^{-1}$. Remember that, from the hypothesis of Theorem 4.5, $(1 - \mathbb{F}^{-\frac{1}{3}} (1 + C_{\text{emb}} \alpha^{-\frac{1}{2}}) C_c^{\text{ADR}} C_d^{\text{ADR}} \|m\|_{0, \frac{6}{5}, \Omega}) > 0$.

Invoking the bound of (4.37) and (4.34) in (4.33) we obtain the error estimate for pressure as follows

$$\|p - p_h\|_{\hat{Q}(\mathcal{T}_h)} \leq C_p^{\text{C}\acute{e}a} (\text{dist}(\mathbf{u}, \mathbf{V}_h) + \text{dist}(\mathbf{u}, \mathbf{V}_h)^2 + \text{dist}(p, Q_h) + \text{dist}(\boldsymbol{\zeta}, \mathbf{W}_h) + \text{dist}(\varphi, \Psi_h)), \quad (4.38)$$

where $C_p^{\text{C}\acute{e}a} := \max(\tilde{C}_G, C_{C_{\acute{e}a}}^{\text{ADR}}, C_d^{\text{ADR}} C_{\mathbf{u}}^{\text{C}\acute{e}a} \|m\|_{0, \frac{6}{5}, \Omega})$. Finally, using (4.37) and (4.38) in (4.34), we obtain

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{V}} + \|p - p_h\|_{\hat{Q}(\mathcal{T}_h)} + \|\boldsymbol{\zeta} - \boldsymbol{\zeta}_h\|_{\mathbf{W}} + \|\varphi - \varphi_h\|_{\hat{\Psi}(\mathcal{T}_h)} \\ \leq C_{C_{\acute{e}a}} (\text{dist}(\mathbf{u}, \mathbf{V}_h) + \text{dist}(\mathbf{u}, \mathbf{V}_h)^2 + \text{dist}(p, Q_h) + \text{dist}(\boldsymbol{\zeta}, \mathbf{W}_h) + \text{dist}(\varphi, \Psi_h)), \end{aligned}$$

where $C_{C_{\acute{e}a}} := \max(\mathbb{F}^{-\frac{1}{3}} (1 + C_{\text{emb}} \alpha^{-\frac{1}{2}}) C_c^{\text{ADR}} C_d^{\text{ADR}} \|m\|_{0, \frac{6}{5}, \Omega} C_{\mathbf{u}}^{\text{C}\acute{e}a}, C_p^{\text{C}\acute{e}a}, C_{C_{\acute{e}a}}^{\text{ADR}})$, is independent of the model parameters κ , ϱ_0 , and the discretisation parameter h . This concludes the rest of the proof. \square

Remark 4.1 *Let us remark that, similarly as in [17, Corollary 4.5], it is possible to derive pressure-robust and concentration-robust error estimates for velocity and flux (because of the discrete velocity and flux kernels being subspaces of the continuous ones) as follows*

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{V}} \leq C_V (\text{dist}(\mathbf{u}, \mathbf{V}_h) + \text{dist}(\mathbf{u}, \mathbf{V}_h)^2), \quad \text{and} \quad \|\boldsymbol{\zeta} - \boldsymbol{\zeta}_h\|_{\mathbf{W}} \leq C_F \text{dist}(\boldsymbol{\zeta}, \mathbf{W}_h), \quad (4.39a)$$

where the constants C_V and C_F are independent of κ , ϱ_0 , and h .

In order to provide a theoretical convergence rate, we recall some useful approximation properties. Let $t \in (1, \infty)$ and denote by $\Pi_h : \mathbf{W}^{l,t}(\Omega) \cap \mathbf{H}(\text{div}_{1,t}, \Omega) \rightarrow \mathbb{RT}_k(\mathcal{T}_h)$ the Fortin operator and by $P_h : W^{l,t}(K) \rightarrow \mathbb{P}_k(K)$, the local orthogonal L^2 projection. Then, as a standard consequence of the Bramble–Hilbert lemma, optimal approximation, and inverse inequalities on polynomial spaces, we can assert that

$$\star \|\mathbf{v} - \Pi_h \mathbf{v}\|_{m,t,\Omega} \leq C_1 h^{l-m} |\mathbf{v}|_{l,t,\Omega} \quad \forall \mathbf{v} \in \mathbf{W}^{l,t}(\Omega), \quad 1 \leq l \leq k+1, \quad (4.40a)$$

$$\begin{aligned} \star \|\text{div}(\mathbf{v} - \Pi_h \mathbf{v})\|_{m,t,\Omega} &\leq C_2 h^{l-m} |\text{div} \mathbf{v}|_{l,t,\Omega} \quad \forall \mathbf{v} \in \mathbf{W}^{1,t}(\Omega), \quad \text{with} \\ &\quad \text{div}(\mathbf{v}) \in W^{1,t}(\Omega) \quad \text{and} \quad 0 \leq l \leq k+1, \end{aligned} \quad (4.40b)$$

$$\star \|q - P_h q\|_{m,t,\Omega} \leq C_2 h^{l-m} |q|_{l,t,\Omega} \quad \forall q \in W^{l,t}(\Omega), \quad 0 \leq l \leq k+1, \quad (4.40c)$$

$$\star \|\nabla(q - P_h q)\|_{0,t,K} \leq C_3 h_K^l |q|_{l+1,t,K} \quad \forall q \in W^{l+1,t}(K), \quad K \in \mathcal{T}_h, \quad l \leq k, \quad (4.40d)$$

for $0 \leq m \leq l$, where $C_1, C_2 > 0$ are independent of h and $C_3 > 0$ is independent of h_K (see [25, Sections 4.1 and 4.5] for the first three, and [19] for the fourth one).

To conclude this section, the following theorem provides the theoretical rate of convergence of the mixed finite element scheme (2.3), under suitable regularity on the exact solution.

Theorem 4.6 (Convergence rates) *Let $(\mathbf{u}, p, \boldsymbol{\zeta}, \varphi) \in \mathbf{V} \times Q \times \mathbf{W} \times \Psi$ be the solution of the continuous problem (2.3) with the following regularity conditions*

$$\mathbf{u} \in \kappa^{-\frac{1}{2}} \mathbf{H}^l(\Omega) \cap \mathbb{F}^{\frac{1}{3}} \mathbf{W}^{l,3}(\Omega), \quad \text{div} \mathbf{u} \in \mathbf{H}^l(\Omega), \quad \text{and} \quad p = p_1 + p_2 + p_3,$$

$$\begin{aligned} & \text{with } p_1 \in \mathbf{H}^l(\Omega), \quad p_2 \in \kappa^{\frac{1}{2}}\mathbf{H}^{l+1}(\Omega), \quad \text{and } p_3 \in \mathbf{F}^{-\frac{1}{3}}\mathbf{W}^{l+1, \frac{3}{2}}(\Omega), \\ & \zeta \in \alpha^{-\frac{1}{2}}\mathbf{H}^l(\Omega) \cap \mathbf{H}(\operatorname{div}_{\frac{6}{5}}, \Omega), \quad \operatorname{div} \zeta \in \mathbf{W}^{l, \frac{6}{5}}(\Omega), \quad \text{and } \varphi = \varphi_1 + \varphi_2, \\ & \text{with } \varphi_1 \in \alpha^{\frac{1}{2}}\mathbf{H}^{l+1}(\Omega), \quad \varphi_2 \in \mathbf{W}^{l+1, 6}(\Omega), \end{aligned}$$

where $1 \leq l \leq k+1$. Then, there exist positive constants C_{\sharp}^i , $i = 1, \dots, 4$, independent of $h, \kappa, \mathbf{F}, \alpha$, and ϱ_0 , such that

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{V}} + \|p - p_h\|_{\widehat{\mathcal{Q}}(\mathcal{T}_h)} + \|\zeta - \zeta_h\|_{\mathbf{W}} + \|\varphi - \varphi_h\|_{\widehat{\Psi}(\mathcal{T}_h)} &\leq C_{\sharp}^1 h^l [\kappa^{-\frac{1}{2}} \|\mathbf{u}\|_{l, \Omega} + \|\operatorname{div} \mathbf{u}\|_{l, \Omega} \\ &+ \mathbf{F}^{\frac{1}{3}} \|\mathbf{u}\|_{l, 3, \Omega}] + (C_{\sharp}^1)^2 h^{2l} [\kappa^{-\frac{1}{2}} \|\mathbf{u}\|_{l, \Omega} + \|\operatorname{div} \mathbf{u}\|_{l, \Omega} + \mathbf{F}^{\frac{1}{3}} \|\mathbf{u}\|_{l, 3, \Omega}]^2 \\ &+ C_{\sharp}^2 h^l [|p_1|_{l, \Omega} + \kappa^{\frac{1}{2}} |p_2|_{l+1, \Omega} + \mathbf{F}^{-\frac{1}{3}} |p_3|_{l+1, \frac{3}{2}, \Omega}] \\ &+ C_{\sharp}^3 h^l \{\alpha^{-\frac{1}{2}} \|\zeta\|_{l, \Omega} + \|\operatorname{div} \zeta\|_{l, \frac{6}{5}, \Omega}\} + C_{\sharp}^4 h^l \{\alpha^{\frac{1}{2}} |\varphi_1|_{l+1, \Omega} + |\varphi_2|_{l+1, 6, \Omega}\}. \end{aligned}$$

Proof. Using the approximation properties (4.40a), (4.40b), (4.40c), and (4.40d) in (4.37) and (4.38), we obtain the following convergence rates for velocity and pressure

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{V}} &\leq C_V \operatorname{dist}(\mathbf{u}, \mathbf{V}_h) \leq C_V \max(2C_1, C_2) h^l (\kappa^{-\frac{1}{2}} \|\mathbf{u}\|_{l, \Omega} + \|\operatorname{div} \mathbf{u}\|_{l, \Omega} + \mathbf{F}^{\frac{1}{3}} \|\mathbf{u}\|_{l, 3, \Omega}) \\ &+ (\max(2C_1, C_2) h^l (\kappa^{-\frac{1}{2}} \|\mathbf{u}\|_{l, \Omega} + \|\operatorname{div} \mathbf{u}\|_{l, \Omega} + \mathbf{F}^{\frac{1}{3}} \|\mathbf{u}\|_{l, 3, \Omega}))^2, \quad \text{and} \\ \|p - p_h\|_{\widehat{\mathcal{Q}}(\mathcal{T}_h)} &\leq C_p^{C\acute{e}a} (|p_1 - P_h p_1|_{0, \Omega} + \kappa^{\frac{1}{2}} |p_2 - P_h p_2|_{1, \mathcal{T}_h} + \mathbf{F}^{-\frac{1}{3}} |p_3 - P_h p_3|_{1, \frac{3}{2}, \mathcal{T}_h}) \\ &\leq C_p^{C\acute{e}a} \max(C_2, C_3, C_4) h^l (|p_1|_{l, \Omega} + \kappa^{\frac{1}{2}} |p_2|_{l+1, \Omega} + \mathbf{F}^{-\frac{1}{3}} |p_3|_{l+1, \frac{3}{2}, \Omega}), \end{aligned}$$

where C_4 is same as in [17, Theorem 4.6]. Next, for the error estimate for flux, at first take $m = 0$ and $t = 0$ in (4.40a), and then, $m = 0$ and $t = \frac{6}{5}$ in (4.40b), which lead to

$$\|\zeta - \Pi_h \zeta\|_{0, \Omega} \leq C_1 C_V h^l |\zeta|_{l, \Omega}, \quad \text{and} \quad \|\operatorname{div}(\zeta - \Pi_h \zeta)\|_{0, \frac{6}{5}, \Omega} \leq C_2 C_V h^l |\operatorname{div} \mathbf{u}|_{l, \frac{6}{5}, \Omega}.$$

In turn, owing to Remark 4.1, we derive the error bound on the flux as follows

$$\|\zeta - \zeta_h\|_{\mathbf{W}} \leq C_F \operatorname{dist}(\zeta, \mathbf{W}_h) \leq C_F \max(C_1, C_2) h^l (\alpha^{-\frac{1}{2}} \|\zeta\|_{l, \Omega} + \|\operatorname{div} \zeta\|_{0, \frac{6}{5}, \Omega}).$$

Lastly, for the concentration we employ the decomposition $\varphi = \varphi_1 + \varphi_2$, and the properties of the sum norm (4.5), and apply (4.40c) with $m = 0$ and $t = 2$, and (4.40d) with $t = 0$ and $t = 6$, summed over $K \in \mathcal{T}_h$. We combine that with the following trace inequality to control the jump term in the broken norms

$$h_F^{1-t} \|\llbracket \varphi_j - P_h \varphi_j \rrbracket\|_{0, t, F}^t = h_F^{1-t} \|\llbracket P_h \varphi_j \rrbracket\|_{0, t, F}^t \leq C_5 h^{lt} |\varphi_j|_{l+1, r, \omega_F}^t,$$

where ω_F stands for the usual face patch. Therefore, we readily obtain the following bound

$$\begin{aligned} \|\varphi - P_h \varphi\|_{\widehat{\Psi}(\mathcal{T}_h)} &\leq C_{C\acute{e}a} (\|\varphi_1 - P_h \varphi_1\|_{0, 6, \Omega} + \alpha^{\frac{1}{2}} |\varphi_2 - P_h \varphi_2|_{1, \mathcal{T}_h}) \\ &\leq C_{C\acute{e}a} \max(C_2, C_3, C_5) h^l (|\varphi_1|_{l, 6, \Omega} + \alpha^{\frac{1}{2}} |\varphi_2|_{l+1, \Omega}), \end{aligned}$$

where $C_5 > 0$ is independent of all the parameters. This completes the proof. \square

In the next section we define a variable operator preconditioner for the Newton linearisation of (2.3).

5 Operator preconditioning

This section discusses preconditioners for the monolithic coupled system arising from the weighted norms introduced in the previous sections which is robust with respect to all the model parameters expect for coefficient of diffusion.

First, for $0 < s < \infty$, we denote by $\mathcal{J}_s : L^s(\Omega) \rightarrow (L^s(\Omega))'$ the canonical duality map defined as

$$\langle \mathcal{J}_s(\theta_1), \theta_2 \rangle := \int_{\Omega} |\theta_1|^{s-2} \theta_1 \theta_2, \quad \forall \theta_1, \theta_2 \in L^s(\Omega). \quad (5.1)$$

Similarly, mimicking the operator notation $-\nabla \operatorname{div}$ that induces the div-div bilinear form, we define $-\nabla \mathcal{J}_s(\operatorname{div})$ as

$$\langle -\nabla \mathcal{J}_s(\operatorname{div})(\zeta_1), \zeta_2 \rangle := \langle \mathcal{J}_s(\operatorname{div} \zeta_1), \operatorname{div} \zeta_2 \rangle, \quad \forall \zeta_1, \zeta_2 \text{ such that } \operatorname{div} \zeta_i \in L^s(\Omega). \quad (5.2)$$

Denote by $\mathcal{H}_1(\mathbf{u})$ the Gâteaux derivative of the nonlinear operator $d_1(\bullet; \bullet, \bullet)$, i.e., $|\mathbf{u}| \mathbf{u}$ at the state \mathbf{u} in the direction of $\delta \mathbf{u}$, and $\mathcal{H}_2(\ast)$ by the linearisation of $d_2^\ast(\bullet, \bullet)$ around \ast . The linearised system corresponding to (1.1) takes the following form

$$\begin{aligned} [\kappa^{-1} + \mathbf{F}\mathcal{H}_1(\hat{\mathbf{u}})]\mathbf{u} + \nabla p &= \mathbf{f}(\hat{\varphi}) \quad \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0 \quad \text{in } \Omega, \\ \alpha^{-1}\zeta - \nabla \varphi + \alpha^{-1}\mathcal{H}_2(\hat{\mathbf{u}})\varphi &= \mathbf{0} \quad \text{in } \Omega, \\ \operatorname{div} \zeta - \varrho_0 \varphi &= -m \quad \text{in } \Omega, \end{aligned} \quad (5.3)$$

with the same boundary conditions as in (1.1). In addition, consider the Newton–Raphson linearisation, which starting from an initial guess $\mathbf{u}^0, p^0, \zeta^0, \varphi^0$, and iterating on $m = 1, 2, \dots$ until convergence, seeks velocity, pressure, flux, and concentration increments $(\delta \mathbf{u}, \delta p, \delta \zeta, \delta \varphi)$ in the space

$$\begin{aligned} \underline{\mathbf{V}} \times \underline{\mathbf{Q}} \times \underline{\mathbf{W}} \times \underline{\Psi} &:= [\kappa^{-\frac{1}{2}} \mathbf{L}^2(\Omega) \cap \mathbf{H}_{\Gamma_u}(\operatorname{div}, \Omega) \cap \{\mathbf{F}\mathcal{H}_1(\hat{\mathbf{u}})\}^{\frac{1}{3}} \mathbf{L}^3(\Omega)] \\ &\times [\mathbf{L}^2(\Omega) + \kappa^{\frac{1}{2}} \mathbf{H}_{\Gamma_p}^1(\Omega) + \{\mathbf{F}\mathcal{H}_1(\hat{\mathbf{u}})\}^{-\frac{1}{3}} \mathbf{W}_{\Gamma_p}^{1, \frac{3}{2}}(\Omega)] \\ &\times [(\mathcal{H}_2^{-1}(\hat{\mathbf{u}})\alpha)^{-\frac{1}{2}} \mathbf{L}^2 \cap \mathbf{H}_{\Gamma_p}(\operatorname{div}_{\frac{6}{5}}, \Omega)] \times [(\mathcal{H}_2^{-1}(\hat{\mathbf{u}})\alpha)^{\frac{1}{2}} \mathbf{H}^1(\Omega) + \mathbf{L}^6(\Omega)], \end{aligned} \quad (5.4)$$

such that the following holds in $\underline{\mathbf{V}}' \times \underline{\mathbf{Q}}' \times \underline{\mathbf{W}}' \times \underline{\Psi}'$

$$\begin{pmatrix} [\kappa^{-1} + \mathbf{F}\mathcal{H}(\mathbf{u}^m)]\mathcal{I} & \nabla & 0 & 0 \\ -\operatorname{div} & 0 & 0 & 0 \\ \alpha^{-1}\varphi^m & 0 & \alpha^{-1}\mathcal{I} & -\nabla + \alpha^{-1}\mathbf{u}^m \\ 0 & 0 & \operatorname{div} & -\varrho_0\mathcal{I} \end{pmatrix} \begin{pmatrix} \delta \mathbf{u} \\ \delta p \\ \delta \zeta \\ \delta \varphi \end{pmatrix} = \begin{pmatrix} F - \{\kappa^{-1} + |\mathbf{u}^m|\}(\mathbf{u}^m) - \nabla p^m \\ G + \operatorname{div} \mathbf{u}^m \\ -\alpha^{-1}\zeta^m + \nabla \varphi^m - \alpha^{-1}\mathbf{u}^m \varphi^m \\ H - \operatorname{div} \zeta^m - \varrho_0 \varphi^m \end{pmatrix}. \quad (5.5)$$

Here, \mathcal{I} denotes the identity operator at the continuous level, but for the discrete case it stands for the corresponding mass matrix (either for velocity or pressure or flux or concentration). Further, we note that, in weak form, the action of the operator $\mathbf{F}\mathcal{H}(\mathbf{u}^m)\mathcal{I} : \underline{\mathbf{V}} \rightarrow \underline{\mathbf{V}}'$ reads

$$\langle \mathbf{F}\mathcal{H}(\mathbf{u}^m)(\delta \mathbf{u}), \mathbf{v} \rangle := \int_{\Omega} \mathbf{F}|\mathbf{u}^m| \delta \mathbf{u} \cdot \mathbf{v} + \int_{\Omega} \mathbf{F}|\mathbf{u}^m|^{-1}(\mathbf{u}^m \cdot \delta \mathbf{u})(\mathbf{u}^m \cdot \mathbf{v}),$$

and

$$\langle \mathcal{H}_2(\mathbf{u}^m)\boldsymbol{\tau}, \varphi \rangle := \int_{\Omega} (\mathbf{u}^m \cdot \boldsymbol{\tau}) \varphi.$$

Then, update the solution as $(\mathbf{u}^{m+1}, p^{m+1}, \zeta^{m+1}, \varphi^{m+1}) = (\mathbf{u}^m + \delta \mathbf{u}, p^m + \delta p, \zeta^m + \delta \zeta, \varphi^m + \delta \varphi)$. Next, we follow the approach recently advanced in [12, Section 3] to propose a robust block-diagonal preconditioner for (5.5). We construct block-diagonal preconditioners for the Darcy–Forchheimer velocity and pressure blocks following [17]. On the other hand, and differently from [12] and [17], we readily note that the sub-blocks associated with the ARD equations consist of a perturbed, *non-symmetric* saddle-point of the form

$$\begin{pmatrix} \alpha^{-1}\mathbb{A} & \mathbb{B}^\ast + \alpha^{-1}\mathbb{D} \\ \mathbb{B} & -\varrho_0\mathbb{C} \end{pmatrix}, \quad (5.6)$$

where we are abusing of notation to emphasise the parameter dependence of the different blocks. Let us first concentrate on the discrete case since it simplifies the discussion as $\mathbf{u}_h^m \in \mathbf{L}^\infty(\Omega)$ and so the Banach structure is not required. In this case, $\mathbb{A} : \mathbf{H}(\operatorname{div}, \Omega) \rightarrow \mathbf{H}(\operatorname{div}, \Omega)'$ denotes the $\mathbf{L}^2(\Omega)$ mass map in $\mathbf{H}(\operatorname{div}, \Omega)$, $\mathbb{B} : \mathbf{H}(\operatorname{div}, \Omega) \rightarrow \mathbf{L}^2(\Omega)'$ is the divergence operator, \mathbb{D} is the *mixed* advection coupling operator ($\mathbb{D}\varphi \approx \mathbf{u}^m \varphi$), and $\mathbb{C} : \mathbf{L}^2(\Omega) \rightarrow \mathbf{L}^2(\Omega)'$ is the scalar mass map in $\mathbf{L}^2(\Omega)$. Therefore a typical Riesz map preconditioner, also based on a Schur complement approximation, is motivated by [8, Sect. 3.1] (see also [6, Section 1] for the symmetric, unperturbed case). The guiding principle is to scale the flux space with α^{-1} , the concentration space with ϱ_0 , and ensure that the Schur complement is spectrally equivalent

to a properly weighted operator. In this case, a block-diagonal preconditioner reads as follows (using a complete Schur complement structure for the (2,2)-block and an incomplete one for the (1,1)-block)

$$\text{diag}(\mathcal{R}, \mathcal{S}) = \begin{pmatrix} (\alpha^{-1}\mathbb{A} + \mathbb{B}^*\mathbb{B})^{-1} & 0 \\ 0 & ([1 + \varrho_0]\mathbb{C})^{-1} + (\varrho_0\mathbb{C} + \alpha\mathbb{B}\mathbb{A}^{-1}\mathbb{B}^* + \mathbb{B}\mathbb{A}^{-1}\mathbb{D})^{-1} \end{pmatrix}, \quad (5.7)$$

or, equivalent Schur complement structure for the (1,1)-block (and note that it is also possible to derive block-triangular versions, see for e.g., [30]). For the the advection term the preconditioner part $\mathbb{B}\mathbb{A}^{-1}\mathbb{D}$ is not symmetric and it corresponds to perturbation with respect to the diffusion term $\mathbb{B}\mathbb{A}^{-1}\mathbb{B}^*$ if \mathbf{u}^m is bounded. However, for $\alpha \ll 1$, the advection term \mathbb{D} becomes dominant and the Schur complement changes from diffusion–reaction to transport–reaction and it can destroy the spectral equivalence. For a fully computable preconditioner associated with (5.7), we approximate the top diagonal block by $(\alpha^{-1}\mathcal{I} - \alpha^{-1}\nabla(\text{div}))^{-1}$, where $\mathcal{I} - \nabla(\text{div})$ is the Raviart–Thomas full matrix. Finally, inspired by (5.7), and considering another formulation inspired by the primal formulation, we propose the following preconditioners ($\mathcal{B}_{\text{DFARD}}^{1,1}$, $\mathcal{B}_{\text{DFARD}}^{1,2}$, $\mathcal{B}_{\text{DFARD}}^{2,1}$, and $\mathcal{B}_{\text{DFARD}}^{2,2}$) for the coupled monolithic problem (5.3), where

$$\mathcal{B}_{\text{DFARD}}^{i,j} := \begin{pmatrix} \mathcal{P}_i & 0 & 0 & 0 \\ 0 & \mathcal{Q}_i & 0 & 0 \\ 0 & 0 & \mathcal{R}_j & 0 \\ 0 & 0 & 0 & \mathcal{S}_j \end{pmatrix}, \quad (5.8)$$

with

$$\begin{aligned} \mathcal{P}_1 &:= ([\kappa^{-1} + \text{FH}(\hat{\mathbf{u}})]\mathcal{I} - [\kappa^{-1} + \text{FH}(\hat{\mathbf{u}})]\nabla \text{div})^{-1}, & \text{and} & \quad \mathcal{Q}_1 := [\kappa^{-1} + \text{FH}(\hat{\mathbf{u}})]^{-1}\mathcal{I}^{-1}, \\ \mathcal{P}_2 &:= ([\kappa^{-1} + \text{FH}(\hat{\mathbf{u}})]\mathcal{I} - \nabla \text{div})^{-1}, & \text{and} & \quad \mathcal{Q}_2 := \mathcal{I}^{-1} + (-\kappa\Delta)^{-1} - ([\text{FH}(\hat{\mathbf{u}})]^{-1}\Delta_{\frac{3}{2}})^{-1}, \\ \mathcal{R}_1 &:= (\alpha^{-1}\mathcal{I} + \nabla \cdot \text{div} + \alpha^{-1}\hat{\mathbf{u}}\nabla \cdot)^{-1}, & \text{and} & \quad \mathcal{S}_1 := (1 + \varrho_0)\mathcal{I}^{-1} + (\varrho_0\mathcal{I} - \alpha\Delta)^{-1}, \\ \mathcal{R}_2 &:= (\alpha^{-1}\mathcal{I} + \nabla \cdot \text{div})^{-1}, & \text{and} & \quad \mathcal{S}_2 := (1 + \varrho_0)\mathcal{I}^{-1} + (\varrho_0\mathcal{I} - \alpha\Delta + \mathbf{u}^m \cdot \nabla)^{-1}. \end{aligned}$$

6 Numerical results

In this section, we illustrate the accuracy and performance of the proposed scheme and operator preconditioning through several numerical experiments. We verify that the convergence rates are robust. The numerical implementations are based on the open-source finite element libraries Gridap [5] and FEniCS [2].

Example 1 (convergence against smooth solutions in 2D). We first verify convergence with unity parameters $\rho_0 = \alpha = \kappa = \alpha_0 = \alpha_1 = 1$. The domain is the unit square $\Omega = (0, 1)^2$, with $\Gamma_{\mathbf{u}}$ the bottom and left edges and Γ_p the remainder of the boundary. The manufactured velocity, pressure, and concentration are

$$\mathbf{u} = \begin{pmatrix} \cos(\pi x) \sin(\pi y) \\ -\sin(\pi x) \cos(\pi y) \end{pmatrix}, \quad p = \sin(\pi x) \sin(\pi y), \quad \varphi = \cos(2\pi x) \cos(2\pi y).$$

The Darcy–Forchheimer velocity is divergence-free, and the forcing term is chosen as $\mathbf{f}(\varphi) = (\alpha_0 + \alpha_1\varphi)\mathbf{g}$, where $\mathbf{g} = (0, 1)^\top$, which clearly fulfills the condition (1.2). With these solutions we construct an exact total flux, and additional source terms (so that the momentum balance is satisfied) and non-homogeneous essential and natural boundary conditions required for the convergence assessment. The error history associated with the proposed mixed finite element method on a sequence of successively refined partitions of the domain are presented in Table 6.1, where we also tabulate rates of error decay computed as $\text{rate} = \log(e_{(\bullet)}/\tilde{e}_{(\bullet)})[\log(h/\tilde{h})]^{-1}$, where e, \tilde{e} denote errors generated on two consecutive meshes of sizes h and \tilde{h} , respectively. The results indicate an optimal $O(h^{k+1})$ convergence in all fields, where we are using the natural norms, without parameter-weighting, and also the parameter-weighted norms. Sample numerical solutions (obtained with the second-order scheme and on the finest mesh resolution) are portrayed in Figure 6.1.

Example 2 (convergence relative to fine-mesh reference solutions). Let $\Omega = (0, 2) \times (0, 1)$ and $(\mathbf{f}, g, m) = (\mathbf{0}, 0, 0)$. The boundary conditions are $\mathbf{u} \cdot \mathbf{n} = -0.25 y(1 - y)$ and $\varphi = 2$ on the left edge, $\mathbf{u} \cdot \mathbf{n} = \boldsymbol{\zeta} \cdot \mathbf{n} = 0$ on the top and bottom

DoF	h	$\ \mathbf{u} - \mathbf{u}_h\ _{3,\text{div},\Omega}$	rate	$\ p - p_h\ _{0,\Omega}$	rate	$\ \zeta - \zeta_h\ _{\text{div}_6/5,\Omega}$	rate	$\ \varphi - \varphi_h\ _{0,6,\Omega}$	rate	it
Errors and convergence rates for $k = 0$										
40	0.7071	5.07e-01	★	2.44e-01	★	3.50e+01	★	7.09e-01	★	5
160	0.3536	2.92e-01	0.797	1.29e-01	0.921	1.88e+01	0.900	3.43e-01	1.050	6
640	0.1768	1.51e-01	0.946	6.52e-02	0.982	9.68e+00	0.955	2.02e-01	0.764	6
2560	0.0884	7.64e-02	0.987	3.27e-02	0.996	4.88e+00	0.988	1.03e-01	0.967	5
10240	0.0442	3.83e-02	0.997	1.64e-02	0.999	2.45e+00	0.997	5.19e-02	0.992	5
40960	0.0221	1.91e-02	0.999	8.18e-03	1.000	1.22e+00	0.999	2.60e-02	0.998	6
Errors and convergence rates for $k = 1$										
128	0.7071	1.80e-01	★	7.62e-02	★	1.92e+01	★	3.14e-01	★	7
512	0.3536	4.78e-02	1.914	1.97e-02	1.954	5.20e+00	1.884	1.23e-01	1.356	7
2048	0.1768	1.24e-02	1.943	4.96e-03	1.986	1.37e+00	1.923	3.33e-02	1.883	8
8192	0.0884	3.15e-03	1.979	1.24e-03	1.997	3.47e-01	1.985	8.54e-03	1.962	7
32768	0.0442	7.92e-04	1.993	3.11e-04	1.999	8.68e-02	1.997	2.15e-03	1.991	8
131072	0.0221	1.98e-04	1.998	7.78e-05	2.000	2.17e-02	1.999	5.38e-04	1.998	8
Errors and convergence rates for $k = 0$										
40	0.7071	9.83e-01	★	6.05e-02	★	3.50e+01	★	3.66e-01	★	6
160	0.3536	5.56e-01	0.824	1.94e-02	1.640	1.88e+01	0.900	1.77e-01	1.050	7
640	0.1768	2.88e-01	0.947	4.96e-03	1.969	9.68e+00	0.955	5.14e-02	1.782	7
2560	0.0884	1.46e-01	0.986	1.25e-03	1.990	4.88e+00	0.988	1.33e-02	1.948	7
10240	0.0442	7.29e-02	0.996	3.13e-04	1.996	2.45e+00	0.997	3.36e-03	1.987	7
40960	0.0221	3.65e-02	0.999	7.84e-05	1.998	1.22e+00	0.999	8.42e-04	1.997	7
Errors and convergence rates for $k = 1$										
128	0.7071	3.31e-01	★	1.55e-01	★	1.92e+01	★	3.68e-01	★	7
512	0.3536	8.86e-02	1.902	4.68e-02	1.730	5.20e+00	1.884	1.81e-01	1.019	8
2048	0.1768	2.30e-02	1.946	1.23e-02	1.931	1.37e+00	1.923	5.61e-02	1.041	9
8192	0.0884	5.82e-03	1.981	3.11e-03	1.981	3.47e-01	1.985	8.82e-02	1.938	8
32768	0.0442	1.46e-03	1.994	7.80e-04	1.994	8.68e-02	1.997	5.81e-03	1.984	8
131072	0.0221	3.66e-04	1.998	1.95e-04	1.998	2.17e-02	1.999	1.46e-03	1.994	9

Table 6.1: Example 1. Error history against smooth solutions for different polynomial degrees ($k = 0, 1$), and iteration count for the nonlinear Newton–Raphson solver. Here we use unit parameters, and the non-weighted norms (top rows), and the parameter-weighted norms (bottom rows) for all fields.

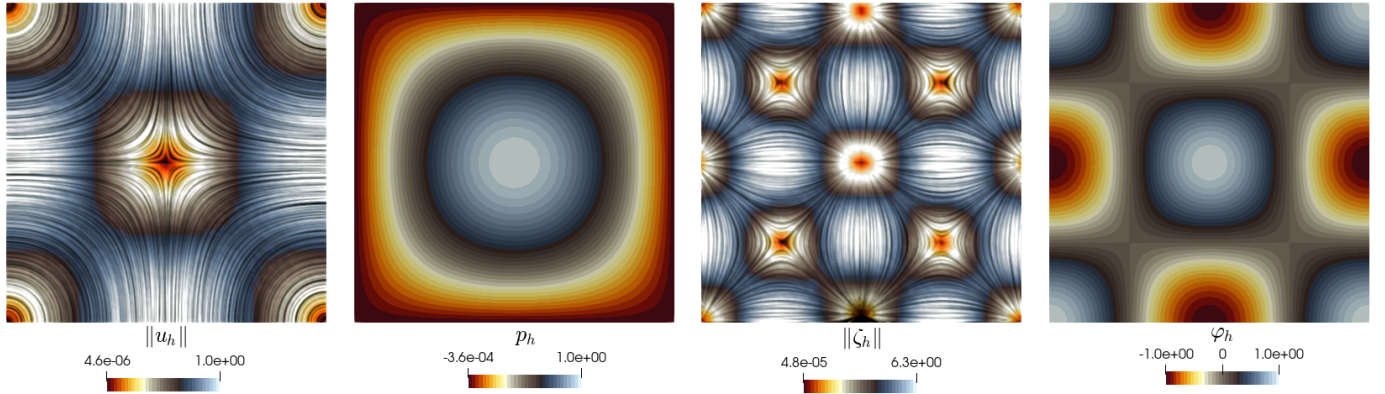


Figure 6.1: Example 1. Approximate velocity magnitude and line integral contours, pressure profile, flux magnitude and line integral contours, and concentration, all computed with a second-order scheme.

edges, $p = 0$ and $\varphi = 1$ on the right edge. Following a similar test in [17], we consider the following heterogeneous permeability

$$\kappa(\mathbf{x}) = \kappa_0 \left(1 + \exp\left(-\frac{1}{2} [5 - 4x + 7y + \cos x]^2\right) \right).$$

Convergence rates are computed against reference solutions obtained via additional mesh refinements. Errors are reported in both the standard norms of Table 6.1 and the weighted norms \mathbf{V} , $\widehat{\mathbf{Q}}$, \mathbf{W} , $\widehat{\Psi}$. While the experimental order of convergence seems affected by the extreme parameter values when using the usual norms, the optimality is restored when using the weighted norms. Optimal rates are observed for all variables and both polynomial degrees. The Newton–Raphson iteration counts (Table 6) remain low even for extreme parameter values. We show the approximate solutions on a coarse mesh in Figure 6.2.

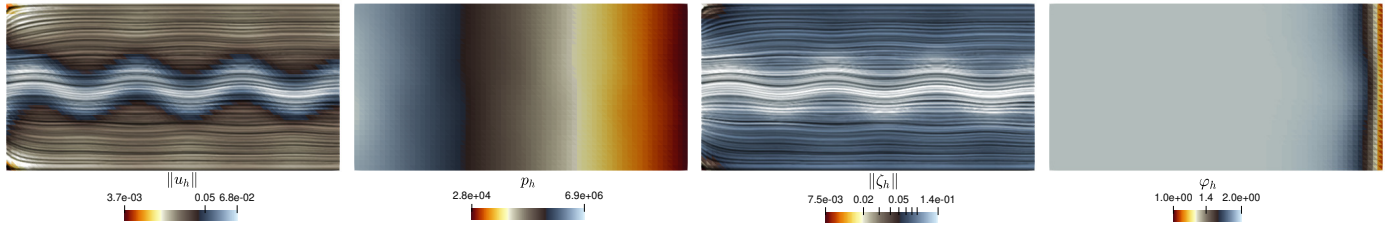


Figure 6.2: Example 2. Approximate velocity magnitude and line integral contours, pressure profile, flux magnitude and line integral contours, and concentration, all computed with a lowest-order scheme on a coarse.

DoF	h	$\ \mathbf{u} - \mathbf{u}_h\ _{3,\text{div}}$	rate	$\ p - p_h\ _{0,\Omega}$	rate	$\ \zeta - \zeta_h\ _{\text{div}_{\frac{\mathbf{g}}{\mathbf{g}}},\Omega}$	rate	$\ \varphi - \varphi_h\ _{0,6,\Omega}$	rate	it
72	0.7071	5.18e-03	*	1.02e+05	*	1.22e+00	*	3.71e+01	*	5
168	0.3536	4.25e-03	0.285	3.86e+04	1.403	1.04e+00	0.222	2.04e+01	0.862	5
480	0.1768	4.56e-03	-0.101	1.78e+04	1.122	1.00e+00	0.060	1.11e+01	0.880	5
1584	0.0884	2.77e-03	0.718	6.29e+03	1.497	5.17e-01	0.952	4.11e+00	1.432	5
5712	0.0442	1.53e-03	0.857	1.93e+03	1.707	2.83e-01	0.868	2.70e+00	0.605	5
21648	0.0221	1.19e-03	0.383	1.53e+03	0.465	2.25e-01	0.182	2.14e+00	0.278	5
DoF	h	$\ \mathbf{u} - \mathbf{u}_h\ _{\mathbf{v}}$	rate	$\ p - p_h\ _{\hat{Q}_h}$	rate	$\ \zeta - \zeta_h\ _{\mathbf{w}}$	rate	$\ \varphi - \varphi_h\ _{\hat{\Psi}_h}$	rate	it
72	0.7071	1.44e+01	*	7.54e+00	*	9.90e+00	*	6.77e+00	*	5
168	0.3536	6.95e+00	1.075	3.10e+00	1.281	6.71e+00	0.562	4.53e+00	0.579	5
480	0.1768	3.65e+00	0.905	1.57e+00	0.980	4.21e+00	0.673	2.70e+00	0.746	5
1584	0.0884	1.64e+00	1.153	5.40e-01	1.542	1.94e+00	1.119	1.15e+00	1.229	5
5712	0.0442	5.26e-01	1.642	2.93e-01	0.883	7.63e-01	1.343	6.81e-01	0.961	5
21648	0.0221	2.18e-01	1.297	1.17e-01	1.630	3.17e-01	1.405	3.50e-01	0.985	5

Table 6.2: Example 2. Error history relative to a fine-mesh reference solution: errors on successively refined grids, convergence rates, for $k = 0$. Newton–Raphson iteration counts corresponding to the nonlinear solver are also reported. Here we are using extreme values of the parameters $\kappa_0 = 10^{-8}$, $F = 10^3$, $\alpha = 5 \cdot 10^{-3}$, $\rho_0 = 10^{-5}$, and both the usual norms (top row), the weighted norms (bottom row) for velocity, pressure, flux, and concentration, respectively.

Example 3 (operator preconditioning). In this example, we test the robustness of the preconditioners $\mathcal{B}_{\text{DFARD}}^{i,j}$, $i, j = 1, 2$ for the fully-coupled linearised Darcy–Forchheimer and ADR system. We stick to the domain $\Omega = (0, 1)^2$, the lowest order finite element family, and mixed boundary conditions. We perform a sequence of experiments in which the permeability of the material κ , the Forchheimer coefficient F , the diffusivity coefficient α , the reaction coefficient ϱ_0 , and the mesh size h vary by several orders of magnitude.

Note that in this context $-\Delta$ and $\mathbf{u}^m \cdot \nabla$ are the discrete diffusion and convection operators, and they both require stabilisation. To ensure robust invertibility, the classical consistency (average-times-jump) terms found in standard Symmetric Interior Penalty Galerkin (SIPG) methods [3] are omitted. This isolates a purely positive-definite interface penalty operator parametrised by the stabilisation penalty factor $\beta = 10$ and the local characteristic mesh sizes h_e

$$\langle -\alpha \Delta \varphi_h, \psi_h \rangle = \alpha \beta \sum_{K \in \mathcal{T}_h} \int_K \nabla \varphi_h \cdot \nabla \psi_h + \sum_{F \in \mathcal{F}_h} \int_F \frac{\alpha \beta}{h_F} \llbracket \varphi_h \rrbracket \llbracket \psi_h \rrbracket.$$

Hyperbolic stability for the advection term is governed by a pointwise upwind numerical flux along the interior interfaces and boundaries, utilizing the velocity field \mathbf{u}^m . By defining the positive and negative part operators as $[x]_+ = \max(x, 0)$ and $[x]_- = \min(x, 0)$, respectively, the corresponding block is

$$\begin{aligned} \langle (\mathbf{u}^m \cdot \nabla) \varphi_h, \psi_h \rangle &= - \sum_{K \in \mathcal{T}_h} \int_K (\mathbf{u}_h^m \cdot \nabla \psi_h) \varphi_h + \sum_{F \in \mathcal{F}_h} \int_F ([(\mathbf{u}^m)^+ \cdot \mathbf{n}_F^+]_+ \varphi^+ + [(\mathbf{u}^m)^+ \cdot \mathbf{n}_F^+]_- \varphi^-) \llbracket \psi \rrbracket \\ &\quad + \sum_{F \in \Gamma_{\text{out}} \cup \Gamma_{\text{in}}} \int_F [\mathbf{u}^m \cdot \mathbf{n}_F]_+ \varphi \psi, \end{aligned}$$

where $\llbracket \phi \rrbracket = \phi^+ - \phi^-$ denotes the standard jump operator across the facet F associated with the outward unit normal \mathbf{n}_F^+ . This retains the essential upwind stability and diffusion scales while keeping a diagonally dominant structure [22, 28]. Even with this, we note that the tested preconditioners are not robust in the advection-dominated regime (large mesh Péclet number $\text{Pe}_h = \frac{|\hat{\mathbf{u}}| h}{\alpha} \gg 1$). Therefore, we restrict our attention to the diffusion-dominated regime, where $\text{Pe}_h \ll 1$,

ρ_0	κ	α	F	h		
				2^{-2}	2^{-3}	2^{-4}
1	10^{-8}	10^{-9}	10^6	1.8 (6)	1.8 (6)	1.8 (7)
	10^{-4}	10^{-5}	1	1.8 (10)	1.8 (12)	1.8 (14)
	1	10^{-1}	10^{-6}	2.9 (27)	3.0 (28)	3.0 (28)
10^{-5}	10^{-8}	10^{-9}	10^6	1.2 (8)	1.3 (10)	1.6 (14)
	10^{-4}	10^{-5}	1	3.6 (31)	4.0 (35)	4.2 (35)
	1	10^{-1}	10^{-6}	2.1 (22)	2.1 (22)	2.1 (22)
10^{-9}	10^{-8}	10^{-9}	10^6	3.6 (24)	4.1 (26)	4.3 (28)
	10^{-4}	10^{-5}	1	3.7 (32)	4.1 (35)	4.2 (35)
	1	10^{-1}	10^{-6}	2.1 (22)	2.1 (22)	2.1 (22)

ρ_0	κ	α	F	h		
				2^{-2}	2^{-3}	2^{-4}
1	10^{-8}	10^{-9}	10^6	1.8 (6)	1.8 (6)	1.8 (7)
	10^{-4}	10^{-5}	1	1.8 (10)	1.8 (12)	1.8 (14)
	1	10^{-1}	10^{-6}	2.9 (28)	3.0 (29)	3.0 (29)
10^{-5}	10^{-8}	10^{-9}	10^6	1.2 (8)	1.3 (10)	1.6 (14)
	10^{-4}	10^{-5}	1	3.6 (32)	4.0 (36)	4.2 (36)
	1	10^{-1}	10^{-6}	2.1 (23)	2.0 (23)	2.0 (23)
10^{-9}	10^{-8}	10^{-9}	10^6	3.6 (25)	4.1 (27)	4.3 (28)
	10^{-4}	10^{-5}	1	3.7 (32)	4.1 (35)	4.2 (37)
	1	10^{-1}	10^{-6}	2.1 (23)	2.0 (23)	2.0 (23)

ρ_0	κ	α	F	h		
				2^{-2}	2^{-3}	2^{-4}
1	10^{-8}	10^{-9}	10^6	3.7 (21)	4.1 (25)	4.4 (25)
	10^{-4}	10^{-5}	1	3.6 (30)	3.8 (32)	3.5 (35)
	1	10^{-1}	10^{-6}	2.9 (29)	3.0 (29)	3.0 (29)
10^{-5}	10^{-8}	10^{-9}	10^6	3.7 (25)	4.1 (29)	4.4 (32)
	10^{-4}	10^{-5}	1	3.7 (34)	4.1 (37)	4.2 (37)
	1	10^{-1}	10^{-6}	1.9 (21)	1.9 (21)	1.9 (20)
10^{-9}	10^{-8}	10^{-9}	10^6	3.7 (25)	4.1 (27)	4.4 (29)
	10^{-4}	10^{-5}	1	3.7 (33)	4.1 (37)	4.2 (37)
	1	10^{-1}	10^{-6}	1.9 (21)	1.9 (21)	1.9 (20)

ρ_0	κ	α	F	h		
				2^{-2}	2^{-3}	2^{-4}
1	10^{-8}	10^{-9}	10^6	3.7 (21)	4.1 (25)	4.4 (25)
	10^{-4}	10^{-5}	1	3.6 (30)	3.8 (32)	3.5 (35)
	1	10^{-1}	10^{-6}	2.9 (29)	3.0 (30)	3.0 (31)
10^{-5}	10^{-8}	10^{-9}	10^6	3.7 (25)	4.1 (29)	4.4 (32)
	10^{-4}	10^{-5}	1	3.6 (34)	4.1 (37)	4.2 (37)
	1	10^{-1}	10^{-6}	1.8 (21)	1.8 (21)	1.8 (22)
10^{-9}	10^{-8}	10^{-9}	10^6	3.7 (25)	4.1 (27)	4.4 (29)
	10^{-4}	10^{-5}	1	3.7 (34)	4.1 (37)	4.2 (37)
	1	10^{-1}	10^{-6}	1.8 (21)	1.8 (21)	1.8 (22)

Table 6.3: Example 3. Condition numbers (and MINRES iteration counts) corresponding to the preconditioners ((5.8)) $\mathcal{B}_{\text{DFARD}}^{1,1}$ (top left), $\mathcal{B}_{\text{DFARD}}^{1,2}$ (top right), $\mathcal{B}_{\text{DFARD}}^{2,1}$ (bottom left), and $\mathcal{B}_{\text{DFARD}}^{2,2}$ (bottom right), under different parameter regimes of the coupled Darcy–Forchheimer and ADR model.

in which the preconditioners exhibit the desired robustness, see Table 6.3. The performance of the preconditioners is assessed through condition numbers of the preconditioned system and the number of MINRES iterations needed to converge, all against mesh refinement.

Example 4 (lid-driven cavity flow). Next, we present a numerical simulation following the classical lid-driven cavity problem for the Darcy–Forchheimer-transport coupling [31]. The domain is $\Omega = (0, 1)^2$ (discretised with a triangular mesh representing 82176 DoFs for the lowest-order method), the model parameters are taken as follows $\kappa = 1$, $F = 20$, $\alpha = 1$, $\mathbf{f}(\varphi) = (10\varphi, 10\varphi)^\dagger$, $m = 0$, $\varrho_0 = 0$, and as boundary conditions we impose $\mathbf{u} \cdot \mathbf{n} = 0$ everywhere on $\partial\Omega$ and so for the sake of pressure uniqueness, we impose $\int_{\Omega} p = 0$ (with a real Lagrange multiplier). We also set $\varphi = 20$ on the top edge and $\varphi = 0$ on the remainder of the boundary. In the first two rows of Figure 6.3 we present the solution profiles, which match the results of [31]. We also carried out an extension to 3D of the same test, taking $\Omega = (0, 1)^3$ (discretised with a mesh representing 2323200 DoFs), $\mathbf{f}(\varphi) = (10\varphi, 10\varphi, 10\varphi)^\dagger$, and maintained all other parameters unchanged. The obtained numerical solutions are shown in the bottom row of the same figure. For both cases we have used the lowest-order method, and the Newton–Raphson iterative scheme (using Hager–Zhang line search) took seven iterations to reach the prescribed tolerance of 10^{-6} . At least in this flow-transport regime and with the chosen parametric values, the discrepancies between 3D and 2D simulations are unnoticeable.

Example 5 (transient miscible displacement). To conclude, we apply a time-dependent version of (1.1) (replacing the reaction term $\varrho_0\varphi$ by $\varrho_0\partial_t\varphi$ in the concentration equation and the diffusion $-\varepsilon\Delta\varphi$ by $-\varrho_0\varepsilon\Delta\varphi$, where ϱ_0 is the porosity) to the classical problem of viscous fingering in a Hele–Shaw cell. Here φ is interpreted as the solute concentration (in mol/m³) of a resident fluid being displaced by a second, less viscous fluid (water).

The viscosity of the fluid mixture follows an Arrhenius law $\mu(\varphi) = \mu_1 \exp(R \frac{\varphi}{\varphi_2})$, where μ_1 is the viscosity of the displacing fluid (water), $\varphi_2 = 6500$ mol/m³ is a reference concentration, and $R = \ln r$ is the log-mobility ratio. The viscosity ratio is $r = \mu(\varphi = 1)/\mu_1 = 2$, hence $R = \ln 2$. The hydraulic conductivity is then $\kappa = K/\mu(\varphi)$, with K the intrinsic permeability; the body force $\mathbf{f}(\varphi)$ is set to zero. The computational domain is the channel $\Omega = (0, 0.03) \times (0, 0.08)$ m². The initial concentration field $\varphi(x, y, 0)$ is a step function from 0 (water) on top of a parabolic profile located near the top boundary $y = 0.0675 + 25(x - 0.015)^2$, to φ_2 (resident fluid) below it. As water

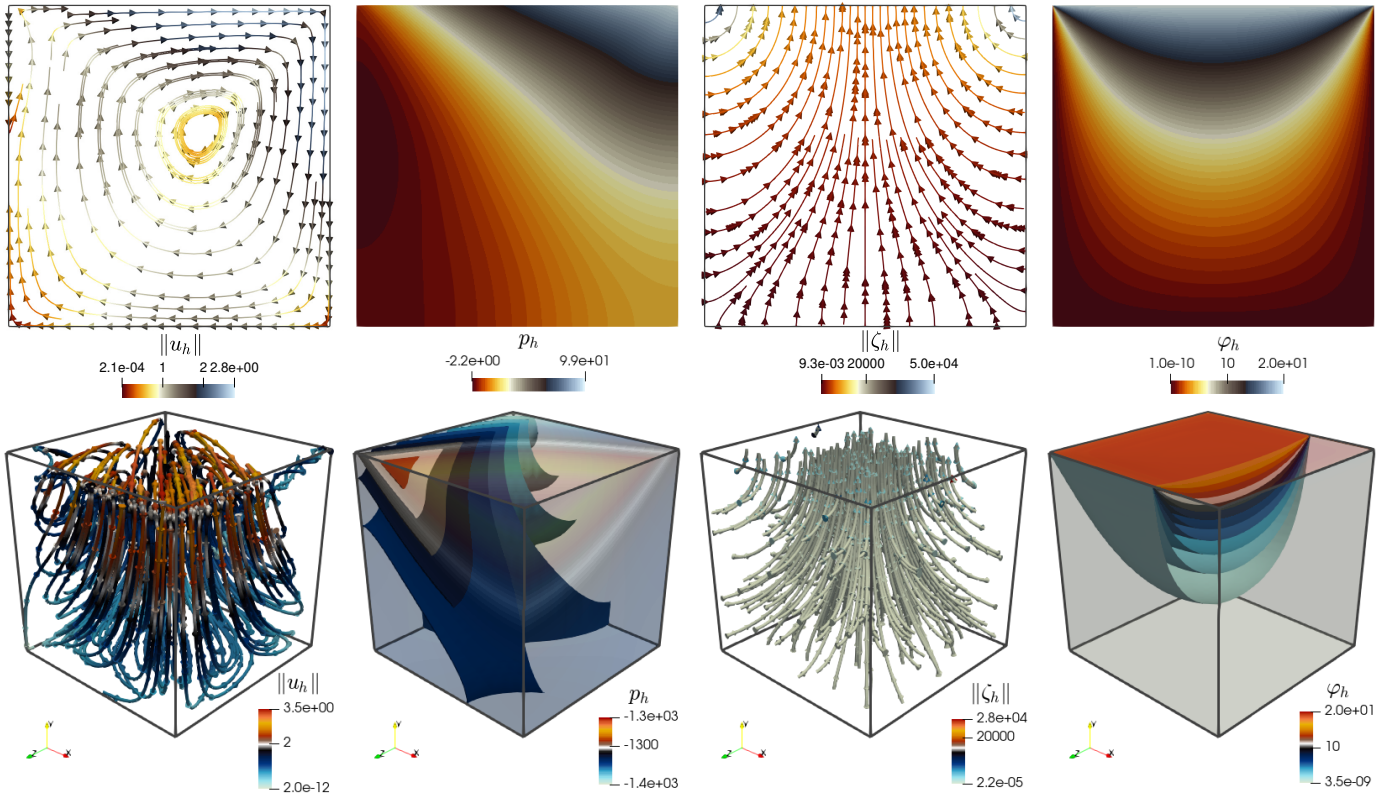


Figure 6.3: Example 4. Numerical solutions (discharge velocity streamlines, pressure profile, total flux, and concentration) for the lid-driven cavity problem. Parameters are taken as in [31] for the 2D case (top and middle rows), and we also show its extension to 3D (bottom row).

is injected from the top boundary (setting a prescribed normal discharge inflow velocity $\mathbf{u} \cdot \mathbf{n} = -9 \cdot 10^{-4}$ m/s and fixed concentration $\varphi = 0$), the resident fluid is displaced and the viscous fingering instability develops spontaneously. No artificial random perturbation is added; the unstructured mesh provides sufficient asymmetry to trigger the instability near the initial interface. On the vertical walls we impose $\mathbf{u} \cdot \mathbf{n} = 0$ and $\zeta \cdot \mathbf{n} = 0$. On the outlet (bottom segment) we set $p = 0$ and zero diffusive flux (not total flux), this means that we need to include the remainder terms $-\varepsilon \langle \varphi, \mathbf{w} \cdot \mathbf{n} \rangle_{\text{out}}$ and $\langle (\zeta + \varphi \mathbf{u}) \cdot \mathbf{n}, \mathbf{w} \cdot \mathbf{n} \rangle_{\text{out}}$ in equation (2.3c).

The model parameters (diffusivity $\alpha = 4 \cdot 10^{-8}$ m²/s, porosity $\varrho_0 = 0.5$, permeability $K = 10^{-6}$ m², reference concentration $\varphi_2 = 6500$ mol/m³, the constant mobility ratio $R = 2$, and water viscosity $\mu_1 = 1$ mPa·s) are taken from [7, 25]; the source term m in (1.1) is zero. The domain is discretised with an unstructured mesh of 37745 triangles. The time derivative is discretised using backward Euler’s method with constant time step $\Delta t = 0.01$ s, and the simulation runs until the final time $t = 18$ s. The numerical results (snapshots of the approximate solutions at four different times, showing in this case only velocity and concentration) are displayed in Figure 6.4. They reproduce the characteristic viscous fingering patterns observed in Hele–Shaw displacements. For this test the maximum number of nonlinear iterations to convergence was four (and required at the initial time steps).

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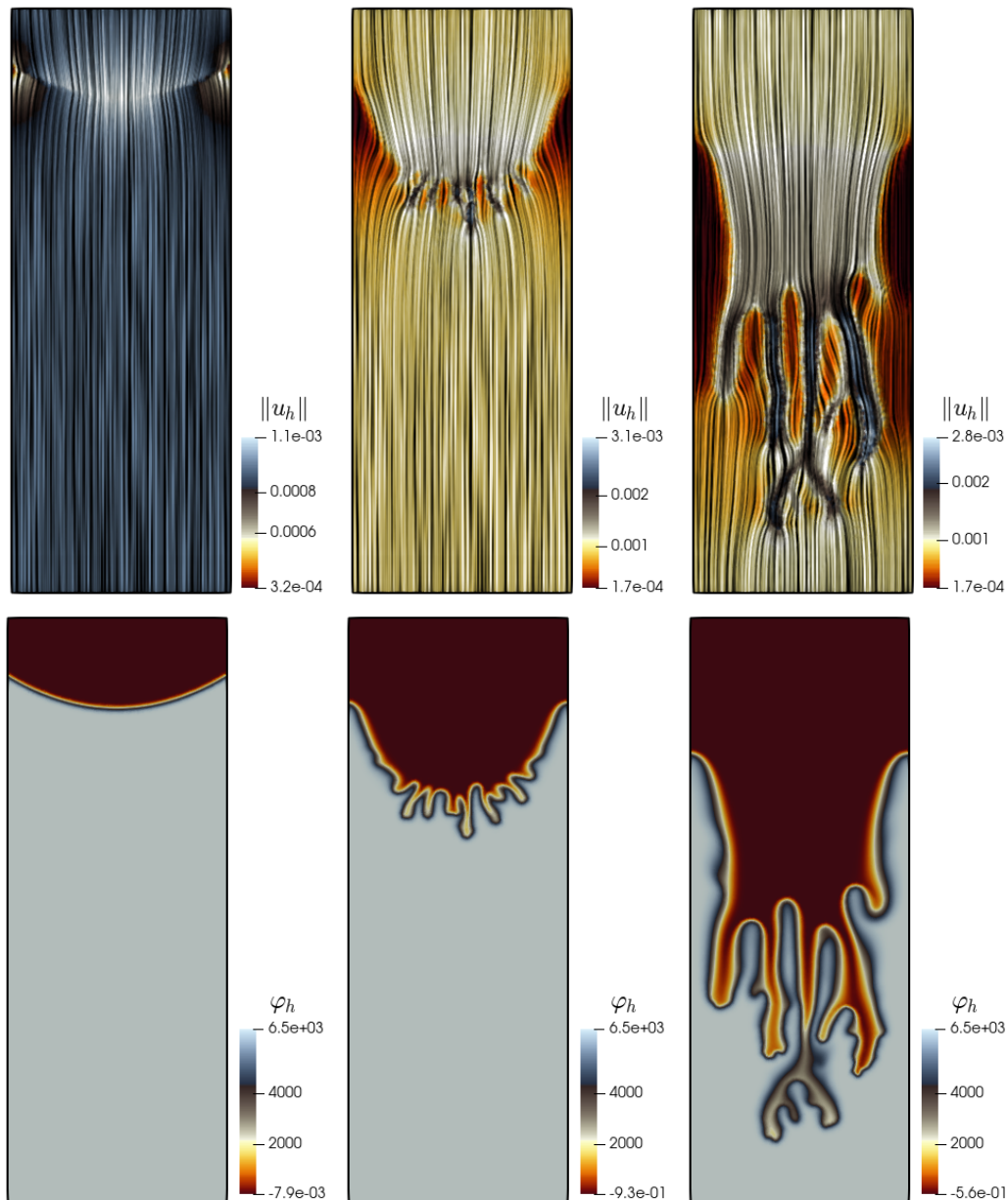


Figure 6.4: Example 5. Snapshots of the approximate discharge flux (top) and concentration profile (bottom) at times $t = 0.01$ (left), $t = 6$ (centre), and $t = 18$ (right).

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A Preconditioners for sub-systems

Robust preconditioners for the Darcy–Forchheimer system. Let us consider the Darcy–Forchheimer system alone from (1.1)

$$\begin{aligned}
 \kappa^{-1}\mathbf{u} + \mathbf{F}|\mathbf{u}|\mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega, \\
 \operatorname{div} \mathbf{u} &= 0 && \text{in } \Omega, \\
 \mathbf{u} \cdot \mathbf{n} &= 0 && \text{on } \Gamma_{\mathbf{u}}, \\
 p &= 0 && \text{on } \Gamma_p.
 \end{aligned} \tag{A.1}$$

The following two (κ, \mathbf{F}, h) -robust preconditioners for (A.1) have been mentioned in [17, Section 5], and the robustness of these preconditioners can be seen in Table A.1.

$$\mathcal{B}_{\text{DF}}^1 := \begin{pmatrix} (\kappa^{-1}\mathcal{I} + \mathbf{F}\mathcal{H}(\mathbf{u})\mathcal{I} - \nabla \operatorname{div})^{-1} & 0 \\ 0 & \mathcal{I}^{-1} + (-\kappa\Delta)^{-1} - ([\mathbf{F}\mathcal{H}(\mathbf{u})]^{-1}\Delta_{\frac{2}{3}})^{-1} \end{pmatrix}, \tag{A.2}$$

κ	F	h		
		2^{-2}	2^{-3}	2^{-4}
10^{-10}	10^{-10}	3.7 (23)	4.1 (31)	4.4 (33)
	1	5.7 (23)	4.1 (31)	4.4 (33)
	10^5	3.7 (23)	4.1 (31)	4.4 (33)
10^{-5}	10^{-10}	3.7 (25)	4.1 (31)	4.2 (33)
	1	3.7 (25)	4.1 (31)	4.2 (33)
	10^5	3.7 (27)	4.1 (31)	4.3 (33)
1	10^{-10}	1.2 (11)	1.2 (11)	1.2 (11)
	1	1.2 (11)	1.2 (11)	1.2 (11)
	10^5	4.4 (27)	4.6 (33)	4.5 (38)

κ	F	h		
		2^{-2}	2^{-3}	2^{-4}
10^{-10}	10^{-10}	1.2 (1)	1.2 (1)	1.2 (1)
	1	1.2 (1)	1.2 (1)	1.2 (1)
	10^5	1.2 (1)	1.2 (1)	1.2 (1)
10^{-5}	10^{-10}	1.2 (6)	1.2 (6)	1.2 (6)
	1	1.2 (6)	1.2 (6)	1.2 (6)
	10^5	1.2 (8)	1.2 (8)	1.2 (8)
1	10^{-10}	1.2 (6)	1.2 (6)	1.2 (6)
	1	1.2 (8)	1.2 (8)	1.2 (8)
	10^5	1.4 (13)	1.4 (13)	1.3 (12)

Table A.1: Condition numbers of the preconditioned systems ((A.2) on left, and (A.3) on right) with MINRES iterations in parentheses for different values of κ , F, and h .

α	ϱ_0	h		
		2^{-2}	2^{-3}	2^{-4}
10^{-9}	10^{-9}	3.7 (17)	4.1 (19)	4.4 (21)
	10^{-6}	1.2 (10)	1.7 (13)	2.5 (18)
	10^{-3}	1.0 (4)	1.0 (4)	1.0 (4)
	1	1.5 (3)	1.5 (3)	1.5 (3)
10^{-3}	10^{-9}	3.1 (24)	3.3 (27)	3.4 (29)
	10^{-6}	3.1 (24)	3.3 (27)	3.4 (29)
	10^{-3}	3.0 (23)	3.0 (25)	3.1 (29)
	1	1.9 (12)	2.3 (15)	2.6 (19)
1	10^{-9}	1.2 (11)	1.2 (11)	1.2 (11)
	10^{-6}	1.2 (11)	1.2 (11)	1.2 (11)
	10^{-3}	1.2 (11)	1.2 (11)	1.2 (11)
	1	2.6 (14)	2.6 (14)	2.6 (14)

Table A.2: Condition numbers of the preconditioned system (corresponding to A.5) with MINRES iterations in parentheses for different parameters and mesh sizes.

and

$$\mathcal{B}_{\text{DF}}^2 := \begin{pmatrix} ([\kappa^{-1}\mathcal{I} + \text{FH}(\mathbf{u})] (\mathcal{I} - \nabla \text{div}))^{-1} & 0 \\ 0 & ([\kappa^{-1}\mathcal{I} + \text{FH}(\mathbf{u})]^{-1} \mathcal{I})^{-1} \end{pmatrix}, \quad (\text{A.3})$$

where $\mathcal{H}(\mathbf{u})$ is the Gâteaux derivative of the nonlinear operator $|\mathbf{u}|\mathbf{u}$ in the state \mathbf{u} in the direction of $\delta\mathbf{u}$, and $\Delta_{\frac{3}{2}}$ is the $\frac{3}{2}$ -Laplace operator associated with the space $W_{\Gamma_p}^{1, \frac{3}{2}}(\Omega)$, defined as

$$\langle \Delta_{\frac{3}{2}} \phi, \psi \rangle := \int_{\Omega} |\nabla \phi|^{-\frac{1}{2}} \nabla \phi \cdot \nabla \psi, \quad \forall \psi \in W^{1,3}(\Omega).$$

A robust preconditioner for the reaction-diffusion system. Consider the mixed reaction-diffusion system

$$\begin{aligned} \alpha^{-1} \zeta - \nabla \varphi &= \mathbf{0} && \text{in } \Omega, \\ \text{div } \zeta - \varrho_0 \varphi &= -m && \text{in } \Omega, \\ \varphi &= 0 && \text{on } \Gamma_{\mathbf{u}}, \\ \zeta \cdot \mathbf{n} &= 0 && \text{on } \Gamma_p. \end{aligned} \quad (\text{A.4})$$

The natural choice of a (α, φ, h) -robust preconditioner for (A.4) (following [10, Equation 3.8]) is the following

$$\mathcal{B}_{\text{RD}} := \begin{pmatrix} (\frac{1}{\alpha} \mathcal{I} - \nabla(\text{div}))^{-1} & 0 \\ 0 & ([1 + \varrho_0] \mathcal{I})^{-1} + (\varrho_0 \mathcal{I} - \alpha \Delta)^{-1} \end{pmatrix}. \quad (\text{A.5})$$

The condition numbers and the MINRES iteration counts are reported in Table A.2.

Robust preconditioners for the advection-diffusion-reaction system. Consider now the ADR system

$$\begin{aligned} \alpha^{-1} \zeta - \nabla \varphi + \alpha^{-1} \hat{\mathbf{u}} \varphi &= \mathbf{0} && \text{in } \Omega, \\ \text{div } \zeta - \varrho_0 \varphi &= -m && \text{in } \Omega, \\ \varphi &= 0 && \text{on } \Gamma_{\mathbf{u}}, \\ \zeta \cdot \mathbf{n} &= 0 && \text{on } \Gamma_p. \end{aligned} \quad (\text{A.6})$$

$ \hat{\mathbf{u}} $	ϱ_0	α	h		
			2^{-2}	2^{-3}	2^{-4}
10^{-10}	10^{-8}	10^{-9}	3.3 (26)	3.9 (31)	4.2 (32)
		10^{-6}	3.7 (19)	4.2 (26)	4.4 (28)
		10^{-3}	3.1 (20)	3.3 (26)	3.4 (28)
		1	1.2 (11)	1.2 (11)	1.2 (11)
	10^{-4}	10^{-9}	1.0 (5)	1.0 (6)	1.0 (7)
		10^{-6}	2.2 (20)	3.0 (29)	3.7 (33)
		10^{-3}	3.1 (20)	3.2 (25)	3.3 (28)
		1	1.2 (11)	1.2 (11)	1.2 (11)
	1	10^{-9}	1.5 (3)	1.5 (3)	1.5 (3)
		10^{-6}	1.5 (3)	1.5 (3)	1.3 (5)
		10^{-3}	1.9 (11)	2.6 (14)	2.6 (18)
		1	2.6 (13)	2.6 (13)	2.6 (13)
10^{-5}	10^{-8}	10^{-9}	1567.6 (63)	2232.1 (248)	3189.9 (★)
		10^{-6}	64.1 (42)	55.2(74)	6.0 (61)
		10^{-3}	3.1 (20)	3.3 (26)	3.3 (29)
		1	1.2 (11)	1.2 (11)	1.2 (11)
	10^{-4}	10^{-9}	3.9 (32)	64.6 (128)	105.6 (521)
		10^{-6}	3.3 (31)	4.8 (48)	1.5 (5)
		10^{-3}	3.1(20)	3.2 (26)	3.3 (29)
		1	1.2 (11)	1.2 (11)	1.2 (11)
	1	10^{-9}	1.5 (3)	1.5 (4)	1.5 (5)
		10^{-6}	1.5 (3)	1.5 (5)	1.5 (5)
		10^{-3}	1.9 (11)	2.3 (14)	2.6 (18)
		1	2.6 (13)	2.6 (13)	2.6 (13)
10^{-1}	10^{-8}	10^{-9}	1.4e+07 (61)	1.1e+07 (243)	8.6e+06 (991)
		10^{-6}	22040.3 (64)	17537.9 (250)	12259.2 (998)
		10^{-3}	21.1 (65)	190.7 (230)	356.6 (377)
		1	1.2 (11)	1.2 (11)	1.2 (11)
	10^{-4}	10^{-9}	24970.7 (62)	30598.7 (245)	44066.2 (988)
		10^{-6}	3711.7 (64)	4484.1 (248)	3435.6 (996)
		10^{-3}	21.0 (65)	186.5 (230)	345.8 (377)
		1	1.2 (11)	1.2 (11)	1.2 (11)
	1	10^{-9}	1.9 (43)	7.4 (143)	9.8 (494)
		10^{-6}	1.9 (43)	7.5 (143)	9.4 (491)
		10^{-3}	1.8 (39)	6.0 (84)	18.2 (126)
		1	2.6 (14)	2.5 (14)	2.5 (13)

Table A.3: Condition numbers and MINRES iteration counts in parenthesis for different values of α and mesh sizes h corresponding to the preconditioner (A.7). The symbol ★ indicates that the MINRES iteration count exceeds 1000.

The preconditioners (5.8) for the full system already include the blocks needed for the sub-system (A.6). Here, we vary the magnitude of the Darcy–Forchheimer velocity $\hat{\mathbf{u}}$ to illustrate the deterioration in performance of these preconditioners in the advection-dominated regime. Specifically, we restrict our attention to $\text{diag}(\mathcal{R}_2, \mathcal{S}_2)$ from (5.8), noting that the remaining preconditioners exhibit similar behaviour:

$$\mathcal{B}_{\text{DFARD}}^{1,2} := \begin{pmatrix} (\alpha^{-1}\mathcal{I} + \nabla \text{div})^{-1} & 0 \\ 0 & (1 + \varrho_0)\mathcal{I}^{-1} + (\varrho_0\mathcal{I} - \alpha\Delta + \hat{\mathbf{u}} \cdot \nabla)^{-1} \end{pmatrix}. \quad (\text{A.7})$$

A conclusion from Table A.3 is that whenever $\frac{|\hat{\mathbf{u}}|h}{\alpha} \ll 1$, both the condition numbers of the preconditioned system and the MINRES iteration counts remain well controlled. Outside this regime, however, the performance of the preconditioner deteriorates and becomes inconsistent. Possible remedies include a better stabilisation for the advection term $\mathbf{u}^m \varphi$ (pure upwind in general will not yield robust preconditioners for extreme regimes). It is well established [21] that classical discretisations of convection-dominated systems give rise to highly nonnormal discrete operators. Such nonnormality is not adequately reflected by the eigenvalues of the preconditioned systems alone; instead, their behavior deteriorates markedly by increasing Péclet number and mesh refinement. As a consequence, Krylov subspace methods will struggle for this type of problems.

Another difficulty already mentioned in Section 5 is the nonsymmetric nature of the underlying discrete operators. Operator preconditioning and the usual Schur complement structures are not yet systematically developed. Here we have adopted a structure motivated by symmetric elliptic operators, which do not seem to provide definitive or robust conclusions. See [32] for a detailed review of the challenges associated with preconditioning nonsymmetric systems.