

A stabilization-free mixed virtual element approximation for unsteady non-Newtonian pseudoplastic Stokes flows

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In this work, we propose and analyze a mixed virtual element method within a Banach space framework to numerically investigate the unsteady motion of non-Newtonian pseudoplastic Stokes flows. Motivated by the growing interest in non-Newtonian fluid problems where the stress response plays a pivotal role, our formulation introduces additional unknowns, including the rate of strain and the symmetric stress tensor. This leads to a mixed variational formulation that incorporates the velocity, strain rate, and stress within a Banach space setting. We establish the well-posedness of the weak solution and derive stability estimates using classical results from the theory of nonlinear monotone operators. The discretization in both space and time is performed using an $\mathbb{H}(\mathbf{div})$ -conforming virtual element method and the implicit Euler scheme, respectively. Specifically, the pseudostress is approximated using a virtual element subspace of $\mathbb{H}(\mathbf{div}; \Omega)$, while piecewise polynomial subspaces of degree j are employed for approximating the velocity and the rate of strain tensor. The nonlinear term is treated implicitly in the time discretization, and the resulting scheme is shown to be well-posed and unconditionally stable. A rigorous convergence analysis is provided for all unknowns in their natural norms, establishing optimal convergence rates with respect to both the spatial mesh size and the time step. Finally, a series of numerical experiments is presented to validate the accuracy and effectiveness of the proposed method.

Keywords: Time-dependent pseudoplastic flows, mixed virtual element methods, wellposedness, stability and convergence

Mathematics Subject Classifications (2020): 65M30, 65M12

1. Introduction

Flow behavior is significantly influenced by the viscosity of the fluid, which varies across different fluids depending on the relationship between shear stress and shear rate. Based on their response to shear stress, fluids are commonly classified as either Newtonian or non-

Newtonian. Newtonian fluids exhibit a linear relationship between shear stress and shear rate, while non-Newtonian fluids display a nonlinear relationship between these quantities. The latter class encompasses a wide range of practical fluids encountered in applications such as the petroleum industry, chemical and pharmaceutical processes, (bio)polymer manufacturing, and food production. To model such fluids, constitutive laws such as the power-law^{21,47} and Carreau–Yasuda models^{54,6} are frequently employed. These applications have driven a growing interest in the development of robust numerical methods for simulating non-Newtonian fluid flows across diverse configurations and computational domains.

To the best of the authors' knowledge, Baranger and Najib⁶, as well as Sandri⁵⁶, were among the first to carry out a rigorous analysis of finite element methods (FEMs) for incompressible flows with viscosity described by the Carreau or power-law models. Subsequently, Du and Gunzburger³⁵ investigated FEMs for non-Newtonian flows governed by the Ladyzhenskaya viscosity law. Later, Barrett *et al.*^{7,8} derived error estimates for velocity and pressure in suitable quasi-norms for models based on either the Carreau or power-law viscosity. These results were later improved in Refs. 15, 49, where optimal error estimates were established for shear-thinning fluids, particularly those characterized by a power-law exponent $r < 2$. In the context of mixed FEMs for non-Newtonian flows, Gatica *et al.*^{38,39} developed a mixed FEM for nonlinear Stokes-type models in quasi-Newtonian fluids, introducing both the flux and the tensor gradient of the velocity field as additional unknowns. Their work includes comprehensive *a priori* and *a posteriori* error analyses for the corresponding lowest-order Galerkin scheme. Similarly, Howell⁵⁰ proposed a dual-mixed finite element formulation for both linear and nonlinear Stokes systems, incorporating the velocity, its gradient, and the pseudostress tensor, and provided a detailed error analysis. Other notable contributions to the numerical approximation of non-Newtonian flows include Refs. 41, 43, 17, 25, 34, 48, 52, which further expand the range of models and numerical techniques applicable to these complex fluid dynamics problems.

In recent years, there has been a growing interest within the scientific community in the development of numerical methods for solving partial differential equations on polygonal and polytopal meshes in two- and three-dimensional domains, respectively. A major advancement in this direction is the Virtual Element Method (VEM), originally introduced in Ref. 9 for the primal conforming Poisson problem as a generalization of the classical H^1 -conforming finite element method. This framework was later extended to mixed formulations by generalizing mixed finite elements¹⁶ to $\mathbf{H}(\text{div})$ -conforming vector fields, resulting in the mixed virtual element method, as proposed in Ref. 20 and further developed in Refs. 10, 11. Owing to its strong theoretical foundation and flexibility in handling general meshes, the mixed VEM has been successfully employed to solve a broad range of linear and nonlinear problems, particularly in fluid mechanics. These applications include both velocity-pressure formulations^{13,14,4,3} and pseudostress-velocity-based approaches^{22,23,44,37}. In particular, the work in Ref. 22 introduced a mixed VEM for the Stokes problem based on a pseudostress-velocity formulation, effectively extending the mixed finite element framework of Ref. 40 to the virtual element setting. Similarly, in Ref. 23, this approach was applied to the Brinkman equation using a pseudostress formulation. Subsequent studies have further generalized this methodology to nonlinear models in fluid mechanics, including quasi-Newtonian Stokes flows²⁴, Navier–Stokes equations⁴⁴, and Boussinesq systems^{37,45,46}.

In the present contribution, we study the unsteady flow of an incompressible pseudoplas-

tic (shear-thinning) non-Newtonian fluid, governed by a time-dependent coupled system of nonlinear partial differential equations that arise in a variety of real-life applications such as biomedical engineering (e.g., blood flow in microcirculation), polymer extrusion, and enhanced oil recovery. These problems are often characterized by evolving or highly irregular geometries and nonlinear rheological behavior, which pose significant challenges for classical discretization techniques. Although a broad literature exists on non-Newtonian flows using finite elements^{7,8,15,34,38,39,49,50,52}, discontinuous Galerkin^{53,62,55}, and finite volume methods⁴⁸, there is still a need for robust and flexible schemes that perform well on general polygonal meshes. The aim of this work is to extend the study of the VEM recently conducted in Ref. 3 by developing and analyzing a mixed VEM scheme for the unsteady model, with particular emphasis on its stability, convergence, and numerical performance. The VEM is particularly well-suited for unsteady pseudoplastic Stokes flows: it handles arbitrary-shaped elements, is robust to mesh distortion, and produces conforming approximations of velocity and pressure. This is especially relevant for pseudoplastic flows, where high gradients and local shear-thinning behavior can lead to numerical instability.

According to the above bibliographic discussion, the primary objective of this work is to develop and analyze a mixed virtual element method within a Banach space framework for the unsteady motion governed by the non-Newtonian Stokes equations. To this end, we introduce the symmetric stress and rate-of-strain tensors as auxiliary unknowns. The pressure variable is subsequently eliminated through the incompressibility constraint and later recovered via a post-processing technique. The spatial discretization of both the auxiliary and primal variables is performed using virtual and non-virtual spaces as described in Ref. 37, while the temporal discretization is carried out using the backward Euler method. A notable feature of our discrete formulation is its complete independence from any stabilization term; the only dependence on inf-sup conditions arises in the discrete setting. We establish the well-posedness of both the continuous and discrete problems by employing classical results from the theory of nonlinear monotone operators and derive stability estimates for all unknowns without imposing assumptions beyond those stated in the problem formulation. In contrast to the approach in Ref. 3, which focuses on the steady-state problem, our analysis for the time-dependent case is significantly simplified by relying solely on the proof of an inf-sup condition. Moreover, we derive optimal error estimates in natural norms for all unknowns, achieving convergence rates of the same order, with particular emphasis on the challenging case $\delta = 0$, which corresponds to the power-law model. The current analysis can be further extended to the more general case $\delta > 0$, for instance by combining it with techniques from Refs. 7, 8. To the best of the authors' knowledge, this work represents the first arbitrary-order polytopal mixed approximation for non-Newtonian flow models in the context of fluid mechanics.

It is worth noting that the three-field formulation, involving velocity, strain rate, and symmetric stress tensor, is designed to:

- take into account the nonlinearity without the need to invert the strain rate-shear stress law (which cannot be done explicitly for the general Carreau model).
- impose the symmetry of the stress tensor without the need of introducing a strongly symmetric virtual element space as in Ref. 5.
- obtaining a mixed VE scheme which does not require a stabilization in the discretization of the nonlinear term.

The remainder of the paper is organized as follows. Section 2 introduces the non-Newtonian Stokes model and presents its variational formulation. The solvability of the continuous problem is analyzed in Section 3 using classical results from nonlinear monotone operator theory. In Section 4, we develop the mixed VEM, following the framework of Ref. 44. This section comprises four main components: the mesh assumptions, the definition of local and global mixed virtual element spaces, the construction of projection operators, and the derivation of discrete bilinear forms. Section 5 addresses the well-posedness of the discrete problem by employing a discrete counterpart of the nonlinear monotone operator theory developed in Section 3, and establishes stability bounds for the discrete solution. To this end, we derive key estimates for the bilinear forms and prove a discrete inf-sup condition. In Section 6, we perform an *a priori* error analysis. Finally, in Section 7, we present a series of two-dimensional numerical experiments, both with and without manufactured solutions, to demonstrate the accuracy, robustness, and flexibility of the proposed mixed VEM.

1.1. Preliminaries

Sobolev and Banach Spaces Let $\Omega \subset \mathbb{R}^n$, with $n \in \{2, 3\}$, be a Lipschitz domain with polyhedral boundary Γ , and let \mathbf{v} denote the outward unit normal vector on Γ . We consider that Γ is disjointly split in two sub-boundaries Γ_D and Γ_N both having positive $(n-1)$ -Hausdorff measure $|\Gamma_D| \cdot |\Gamma_N| > 0$. In addition, due to an auxiliary result associated with a discrete inf-sup condition we require that Γ_N is convex. We use standard notation for Lebesgue spaces $L^p(\Omega)$ and Sobolev spaces $W^{r,p}(\Omega)$, where $r \geq 0$ and $p \in [1, +\infty)$. The associated norms are denoted by $\|\cdot\|_{0,p;\Omega}$ and $\|\cdot\|_{r,p;\Omega}$, respectively. Note that $W^{0,p}(\Omega) = L^p(\Omega)$, and when $p = 2$, we write $H^r(\Omega)$ instead of $W^{r,2}(\Omega)$. The norm and seminorm in this case are denoted as $\|\cdot\|_{r,\Omega}$ and $|\cdot|_{r,\Omega}$, respectively. On another note, for any given scalar functional space M , we denote its vector- and tensor-valued counterparts as \mathbf{M} and \mathbb{M} . As customary, \mathbb{I} represents the identity tensor in $\mathbb{R}^{n \times n}$, and the notation $|\cdot|$ refers to the Euclidean norm in \mathbb{R}^n . For $\rho, \varrho \in (1, +\infty)$ such that $\varrho \leq \rho$, we define the Banach space by $\mathbf{H}^\rho(\operatorname{div}_{\varrho}; \Omega) := \left\{ \mathbf{v} \in \mathbf{L}^\rho(\Omega) : \operatorname{div}(\mathbf{v}) \in L^\varrho(\Omega) \right\}$, which is equipped with the norm $\|\mathbf{v}\|_{\rho, \operatorname{div}_{\varrho}; \Omega} := \|\mathbf{v}\|_{0,\rho;\Omega} + \|\operatorname{div}(\mathbf{v})\|_{0,\varrho;\Omega}$. The space $\mathbb{H}^\rho(\operatorname{div}_{\varrho}; \Omega)$ is defined to consist of matrix-valued functions whose rows belong to $\mathbf{H}^\rho(\operatorname{div}_{\varrho}; \Omega)$, and it is endowed with the norm $\|\cdot\|_{\rho, \operatorname{div}_{\varrho}; \Omega}$. Here, the divergence operator div is understood to act row-wise on tensor. Moreover, for any tensor field $\boldsymbol{\tau}$, the corresponding deviatoric part is defined as $\operatorname{dev}(\boldsymbol{\tau}) := \boldsymbol{\tau} - \frac{1}{n} \operatorname{tr}(\boldsymbol{\tau}) \mathbb{I}$.

Moreover, let V be a separable Banach space endowed with the norm $\|\cdot\|_V$. Given $1 \leq p \leq \infty$ and $t_F > 0$, we define $L^p(0, t_F; V)$ as the space of classes of functions $f : (0, t_F) \rightarrow V$ that are Bochner measurable and such that $\|f\|_{L^p(0, t_F; V)} < \infty$, with

$$\|f\|_{L^p(0, t_F; V)}^p := \int_0^{t_F} \|f(t)\|_V^p dt \quad \forall 1 \leq p < \infty \quad \text{and} \quad \|f\|_{L^\infty(0, t_F; V)} := \operatorname{ess\,sup}_{t \in [0, t_F]} \|f(t)\|_V.$$

Generalized Green's identity We present key concepts and auxiliary results related to integration by parts, commonly referred to as the generalized Green's identity (see Proposition 3.58 in Ref. 32). Especially, proceeding as in Corollary B. 57 of Ref. 36, it is easy to show that, given $\rho, r \in (1, +\infty)$ conjugate to each other, there holds

$$\langle \boldsymbol{\tau} \mathbf{v}, \mathbf{v} \rangle_\Gamma = \int_\Omega \left\{ \boldsymbol{\tau} : \nabla \mathbf{v} + \mathbf{v} \cdot \operatorname{div}(\boldsymbol{\tau}) \right\} \quad \forall (\boldsymbol{\tau}, \mathbf{v}) \in \mathbb{H}^\rho(\operatorname{div}_\rho; \Omega) \times \mathbf{W}^{1,r}(\Omega), \quad (1.1)$$

where $\langle \cdot, \cdot \rangle_\Gamma$ stands for the duality pairing between $\mathbf{W}^{-1/\rho, \rho}(\Gamma)$ and $\mathbf{W}^{1/\rho, r}(\Gamma)$. We now present the analytical tools necessary for this generalization. Given $\ell, j \in (1, \infty)$ conjugate to each other, we denote by $\gamma_{0, \ell}$ the trace operator from $\mathbf{W}^{1, \ell}(\Omega)$ into $\mathbf{W}^{1/j, \ell}(\Gamma)$, which is linear, bounded and surjective with kernel $\mathbf{W}_0^{1, \ell}(\Omega)$ (see for instance Theorem B.52 of Ref. 36)). In addition, from Corollary B.53 of Ref. 36, we recall the existence of a positive constant C_ℓ such that for each $\xi \in \mathbf{W}^{1/j, \ell}(\Gamma)$, there exists \mathbf{v}_ξ satisfying

$$\gamma_{0, \ell}(\mathbf{v}_\xi) = \xi \quad \text{and} \quad \|\mathbf{v}_\xi\|_{1, \ell; \Omega} \leq C_\rho \|\xi\|_{1/j, \ell; \Gamma}. \quad (1.2)$$

Theorem 1.1. *Let $n = 2$, $r < 2$, and let $\rho, \varrho \in (1, \infty)$ be such that ρ and r are conjugate to each other, and $\varrho \leq \rho$. Then there exists a linear, bounded, and surjective operator $\gamma_\mathbf{v} : \mathbb{H}^\rho(\operatorname{div}_\varrho; \Omega) \rightarrow \mathbf{W}^{-1/\rho, \rho}(\Gamma)$ such that for each $\tau \in \mathbb{W}^{1, \rho}(\Omega)$, $\gamma_\mathbf{v}(\tau)$ is identified, by means of the duality pairing between $\mathbf{L}^\rho(\Gamma)$ and $\mathbf{L}^r(\Gamma)$, with $\gamma_{0, \rho}(\tau)\mathbf{v}$, where $\gamma_{0, \rho}(\tau)$ denotes the component-wise application of $\gamma_{0, \rho}$ to τ . Moreover, for every $\tau \in \mathbb{H}^\rho(\operatorname{div}_\varrho; \Omega)$, where $\varrho \leq \rho$, there hold*

$$\langle \gamma_\mathbf{v}(\tau), \gamma_{0, r}(\mathbf{v}) \rangle_\Gamma = \int_\Omega \left\{ \tau : \nabla \mathbf{v} + \mathbf{v} \cdot \operatorname{div}(\tau) \right\} \quad \forall \mathbf{v} \in \mathbf{W}^{1, r}(\Omega). \quad (1.3)$$

Proof. It is a slight modification of the proof of Theorem 3.54 in Ref. 32. \square

2. Continuous problem

In this section we introduce the model problem and derive its corresponding weak formulation.

2.1. The model problem

Let Ω be as introduced at the beginning of Section 1.1, and assume $n = 2$. Let Γ_D and Γ_N denote the inlet and outlet boundaries, respectively, such that $\bar{\Gamma}_D \cup \bar{\Gamma}_N = \Gamma$ and $\Gamma_D \cap \Gamma_N = \emptyset$. Consider given a function $\mu : \Omega \rightarrow \mathbb{R}$. Throughout the following discussion, we assume the existence of positive constants $\underline{\mu}, \bar{\mu} \in \mathbb{R}$ such that, almost everywhere in Ω ,

$$0 < \underline{\mu} \leq \mu \leq \bar{\mu}.$$

Let t_F denote the final time, and define $J := (0, t_F]$, $\Omega_t := \Omega \times J$, $\Gamma_{D, t} := \Gamma_D \times J$, and $\Gamma_{N, t} := \Gamma_N \times J$. Let $\mathbf{f} : \Omega_t \rightarrow \mathbb{R}^n$ denote the volumetric source term. The unsteady nonlinear Stokes problem reads as follows³: Find the velocity field $\mathbf{u} : \Omega_t \rightarrow \mathbb{R}^n$ and the pressure field $p : \Omega_t \rightarrow \mathbb{R}$ such that

$$\partial_t \mathbf{u} - \operatorname{div}(\boldsymbol{\vartheta}(\boldsymbol{\varepsilon}(\mathbf{u}))) + \nabla p = \mathbf{f} \quad \text{in } \Omega_t, \quad (2.1a)$$

$$\operatorname{div}(\mathbf{u}) = 0 \quad \text{in } \Omega_t, \quad (2.1b)$$

$$\boldsymbol{\sigma} \mathbf{v} = \mathbf{0} \quad \text{on } \Gamma_{N, t} \quad \text{and} \quad \mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_{D, t}, \quad (2.1c)$$

$$\mathbf{u}(\cdot, 0) = \mathbf{u}_0 \quad \text{in } \Omega, \quad (2.1d)$$

where the nonlinear shear stress-strain rate $\boldsymbol{\vartheta}(\boldsymbol{\varepsilon}(\mathbf{u}))$ is determined by the Carreau-Yasuda law $\boldsymbol{\vartheta}$, which is defined as

$$\boldsymbol{\vartheta}(\mathbf{r}) := \mu (\delta^\alpha + |\mathbf{r}|^\alpha)^{\frac{r-2}{\alpha}} \mathbf{r} \quad \forall \mathbf{r} \in \mathbb{R}^{n \times n}, \quad (2.2)$$

parameter $\alpha \in (0, \infty)$ represents the local transition flow behavior index, $r \in (1, \infty)$ is the flow behavior index, and $\delta \in [0, \infty)$ is the degeneracy parameter. The tensor $\boldsymbol{\varepsilon}(\mathbf{u})$ denotes the symmetric part of the velocity gradient, defined as $\boldsymbol{\varepsilon}(\mathbf{u}) = (\nabla \mathbf{u} + (\nabla \mathbf{u})^\top)/2$. The nonlinear shear stress–strain rate relation appearing in the first term on the left-hand side of (2.1a) corresponds to the Carreau–Yasuda model, which generalizes the Carreau model^{54,6} when $\alpha = 2$. Moreover, the classical power-law model^{21,47} can be recovered from (2.1a) by setting $\delta = 0$. Shear-thinning behavior, commonly observed in real fluids governed by such constitutive laws, corresponds to the case $r < 2$. It is worth noting that for $r = 2$, problem (2.1) reduces to the standard Stokes system for Newtonian fluids. For brevity, we restrict our attention to the pseudoplastic regime with $r < 2$, which is both prevalent in practical applications and more challenging from an analytical standpoint. Nonetheless, analogous results can be obtained for the dilatant case $r > 2$ by appropriately adapting the arguments presented below.

Next, to derive a mixed formulation for (2.1), we follow the approach outlined in Ref. 42. Specifically, we introduce the strain-rate and stress tensors, defined respectively as follows:

$$\mathbf{t} := \nabla \mathbf{u}, \quad \text{and} \quad \boldsymbol{\sigma} := \boldsymbol{\vartheta}(\mathbf{t}_{\text{sym}}) - p\mathbb{I} \quad \text{in} \quad \Omega_t, \quad (2.3)$$

where $\mathbf{t}_{\text{sym}} := (\mathbf{t} + \mathbf{t}^\top)/2$ is the symmetric part of \mathbf{t} .

Thus, applying the trace operator to the tensor $\boldsymbol{\sigma}$, and using the condition (2.1b), we obtain

$$p = -\frac{1}{2} \text{tr}(\boldsymbol{\sigma}) \quad \text{in} \quad \Omega_t. \quad (2.4)$$

Thus, substituting (2.4) back into (2.3) and performing some straightforward calculations, we find out that the problem (2.1) can be equivalently rewritten as follows: Find $(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u})$, belong to the appropriate spaces specified in the following section, such that

$$\mathbf{t} = \nabla \mathbf{u} \quad \text{in} \quad \Omega_t, \quad (2.5a)$$

$$\text{dev}(\boldsymbol{\sigma}) = \boldsymbol{\vartheta}(\mathbf{t}_{\text{sym}}) \quad \text{in} \quad \Omega_t, \quad (2.5b)$$

$$\partial_t \mathbf{u} - \text{div}(\boldsymbol{\sigma}) = \mathbf{f} \quad \text{in} \quad \Omega_t, \quad (2.5c)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma_{D,t} \quad \boldsymbol{\sigma} \boldsymbol{\nu} = \mathbf{0} \quad \text{on} \quad \Gamma_{N,t}, \quad (2.5d)$$

$$\mathbf{u}(\cdot, 0) = \mathbf{u}_0 \quad \text{in} \quad \Omega. \quad (2.5e)$$

We end this section by noting that the stress-strain law given by (2.5b) satisfies the following assumption (see Appendix A of Ref. 17).

Assumption 2.1. The shear stress-strain rate law $\boldsymbol{\vartheta}$ presented in (2.5b) is a Caratheodory function that satisfies $\boldsymbol{\vartheta}(\mathbf{0}) = \mathbf{0}$, and for a fixed $r \in (1, \infty)$ the following properties hold:

- (i) (*Hölder continuity*). there exists a constant $L_{\boldsymbol{\vartheta}} > 0$, depending only on $\bar{\mu}$, α , r , satisfying

$$|\boldsymbol{\vartheta}(\mathbf{t}) - \boldsymbol{\vartheta}(\mathbf{s})| \leq L_{\boldsymbol{\vartheta}} \left(\delta^r + |\mathbf{t}|^r + |\mathbf{s}|^r \right)^{\frac{r-2}{r}} |\mathbf{t} - \mathbf{s}|, \quad (2.6)$$

- (ii) (*Strong monotonicity*). there exists a constant $\alpha_{\boldsymbol{\vartheta}} > 0$, depending only on $\underline{\mu}$, α , r , satisfying

$$(\boldsymbol{\vartheta}(\mathbf{t}) - \boldsymbol{\vartheta}(\mathbf{s})) : (\mathbf{t} - \mathbf{s}) \geq \alpha_{\boldsymbol{\vartheta}} \left(\delta^r + |\mathbf{t}|^r + |\mathbf{s}|^r \right)^{\frac{r-2}{r}} |\mathbf{t} - \mathbf{s}|^2. \quad (2.7)$$

2.2. The variational formulation

We begin by noting from (2.5b) that the term $\boldsymbol{\vartheta}(\mathbf{t}_{\text{sym}}) := \mu (\delta^\alpha + |\mathbf{t}_{\text{sym}}|^\alpha)^{\frac{r-2}{\alpha}} \mathbf{t}_{\text{sym}}$ constitutes a key expression that must be carefully analyzed in order to determine the appropriate function spaces for both the unknown and the test functions. Indeed, by multiplying the aforementioned equation with an arbitrary test tensor \mathbf{s} associated with \mathbf{t}_{sym} , we formally obtain

$$\int_{\Omega} \boldsymbol{\vartheta}(\mathbf{t}_{\text{sym}}) : \mathbf{s} - \int_{\Omega} \text{dev}(\boldsymbol{\sigma}) : \mathbf{s} = 0 \quad \text{for a.e } t \in \mathbf{J}. \quad (2.8)$$

Next, let ρ denote the conjugate exponent of r , i.e., $\rho := \frac{r}{r-1}$ for $r > 1$. Recalling from (2.2) that $\boldsymbol{\vartheta}(\mathbf{0}) = \mathbf{0}$, and applying the continuity property of $\boldsymbol{\vartheta}$ (cf. (2.6)) together with Hölder's inequality, we deduce that

$$\begin{aligned} \left| \int_{\Omega} \boldsymbol{\vartheta}(\mathbf{t}_{\text{sym}}) : \mathbf{s} \right| &= \left| \int_{\Omega} (\boldsymbol{\vartheta}(\mathbf{t}_{\text{sym}}) - \boldsymbol{\vartheta}(\mathbf{0})) : \mathbf{s} \right| \leq L_{\boldsymbol{\vartheta}} \int_{\Omega} (\delta^r + |\mathbf{t}_{\text{sym}}|^r)^{\frac{r-2}{r}} |\mathbf{t}_{\text{sym}}| |\mathbf{s}| \\ &\leq L_{\boldsymbol{\vartheta}} \left\{ \int_{\Omega} (\delta^r + |\mathbf{t}_{\text{sym}}|^r)^{\frac{\rho(r-2)}{r}} |\mathbf{t}_{\text{sym}}|^{\rho} \right\}^{1/\rho} \|\mathbf{s}\|_{0,r;\Omega}. \end{aligned} \quad (2.9)$$

Now, observing that $\delta \geq 0$ and $r < 2$, it follows that $\delta^r + |\mathbf{t}_{\text{sym}}|^r \geq |\mathbf{t}_{\text{sym}}|^r$, thus yielding

$$(\delta^r + |\mathbf{t}_{\text{sym}}|^r)^{\frac{r-2}{r}} \leq |\mathbf{t}_{\text{sym}}|^{r-2}. \quad (2.10)$$

In this way, substituting (2.10) into (2.9) and recalling that $\rho(r-1) = r$, we arrive at

$$\left| \int_{\Omega} \boldsymbol{\vartheta}(\mathbf{t}_{\text{sym}}) : \mathbf{s} \right| \leq L_{\boldsymbol{\vartheta}} \|\mathbf{t}_{\text{sym}}\|_{0,r;\Omega}^{r-1} \|\mathbf{s}\|_{0,r;\Omega}, \quad (2.11)$$

which shows that for $t \in \mathbf{J}$ the first term in (2.8) is well-defined for $\mathbf{t}(t)$, $\mathbf{s} \in \mathbb{L}^r(\Omega)$. More precisely, due to the free trace property of \mathbf{t} , we actually need to look for it in the space

$$\mathbb{L}_{\text{tr}}^r(\Omega) := \left\{ \mathbf{s} \in \mathbb{L}^r(\Omega) : \text{tr}(\mathbf{s}) = 0 \right\}.$$

In addition, thanks to the decomposition $\mathbb{L}^r(\Omega) = \mathbb{L}_{\text{tr}}^r(\Omega) \oplus L^r(\Omega)\mathbb{I}$, it is easily seen that imposing (2.8) with $\mathbf{s} \in \mathbb{L}^r(\Omega)$ is equivalent to doing it with $\mathbf{s} \in \mathbb{L}_{\text{tr}}^r(\Omega)$. Finally, it is readily seen that the second term of (2.8), which becomes $\int_{\Omega} \boldsymbol{\sigma} : \mathbf{s}$ for all $\mathbf{s} \in \mathbb{L}_{\text{tr}}^r(\Omega)$, is well-defined if $\boldsymbol{\sigma}(t)$ for $t \in \mathbf{J}$ is sought in $\mathbb{L}^\rho(\Omega)$. In summary, for all $t \in \mathbf{J}$, eq. (2.8) can be equivalently reformulated as follows:

$$\int_{\Omega} \boldsymbol{\vartheta}(\mathbf{t}_{\text{sym}}) : \mathbf{s} - \int_{\Omega} \boldsymbol{\sigma} : \mathbf{s} = 0 \quad \forall \mathbf{s} \in \mathbb{L}_{\text{tr}}^r(\Omega). \quad (2.12)$$

Furthermore, multiplying (2.5a) by $\boldsymbol{\tau} \in \mathbb{L}^s(\Omega)$, we get

$$\int_{\Omega} \mathbf{t} : \boldsymbol{\tau} = \int_{\Omega} \nabla \mathbf{u} : \boldsymbol{\tau} \quad \text{for a.e } t \in \mathbf{J}, \quad (2.13)$$

from which it only remains to look at the term on the right-hand. To this end, and aiming to apply the more general integration by parts formula given by (1.3) (cf. Theorem 1.1), we first notice that, having $r \in (1, 2)$, there holds $\rho \geq 2$, and thus $\varrho := 2 \leq \rho$. Therefore, employing the aforementioned formula with $\boldsymbol{\tau} \in \mathbb{H}^\rho(\text{div}; \Omega)$ and $\mathbf{u}(t) \in \mathbf{W}_{0,\Gamma_D}^{1,r}(\Omega)$ for $t \in \mathbf{J}$, we deduce from (2.13) that

$$\int_{\Omega} \mathbf{t} : \boldsymbol{\tau} + \int_{\Omega} \mathbf{u} \cdot \text{div}(\boldsymbol{\tau}) = 0 \quad \forall \boldsymbol{\tau} \in \mathbb{H}_{0,\Gamma_N}^\rho(\text{div}; \Omega), \quad (2.14)$$

where

$$\begin{aligned} \mathbb{H}_{0,\Gamma_N}^\rho(\mathbf{div}; \Omega) &:= \left\{ \boldsymbol{\tau} \in \mathbb{H}^\rho(\mathbf{div}; \Omega) : \boldsymbol{\tau} \boldsymbol{\nu} = \mathbf{0} \text{ on } \Gamma_N \right\} \text{ and} \\ \mathbb{H}^\rho(\mathbf{div}; \Omega) &:= \left\{ \boldsymbol{\tau} \in \mathbb{L}^\rho(\Omega) : \mathbf{div}(\boldsymbol{\tau}) \in \mathbb{L}^2(\Omega) \right\}. \end{aligned}$$

We stress here that the second term on the left-hand side of (2.14) confirms that it suffices to look for $\mathbf{u}(t) \in \mathbb{L}^2(\Omega)$ for $t \in \mathbb{J}$. Furthermore, looking for the unknown $\boldsymbol{\sigma}(t)$ in $\mathbb{H}_{0,\Gamma_N}^\rho(\mathbf{div}; \Omega)$ as well, we realize that (2.5c) can be weakly imposed as

$$\int_{\Omega} \partial_t \mathbf{u} \cdot \mathbf{v} - \int_{\Omega} \mathbf{div}(\boldsymbol{\sigma}) \cdot \mathbf{v} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \text{ for a.e } t \in \mathbb{J}, \quad (2.15)$$

for any $\mathbf{v} \in \mathbb{L}^2(\Omega)$, which requires to assume from now on that $\mathbf{f}(t) \in \mathbb{L}^2(\Omega)$ for $t \in \mathbb{J}$.

According to the above, by choosing the spaces $\mathbf{X} := \mathbb{L}_{\text{tr}}^r(\Omega)$, $\mathbf{Y} := \mathbb{H}_{0,\Gamma_N}^\rho(\mathbf{div}; \Omega)$, $\mathbf{Q} := \mathbb{L}^2(\Omega)$ endowed with their respective norms $\|\cdot\|_{\mathbf{X}} := \|\cdot\|_{0,r;\Omega}$, $\|\cdot\|_{\mathbf{Y}} := \|\cdot\|_{\rho;\mathbf{div};\Omega}$, $\|\cdot\|_{\mathbf{Q}} := \|\cdot\|_{0,\Omega}$, we arrive at the variational problem: For a.e $t \in \mathbb{J}$, find $\mathbf{t}(t) \in \mathbf{X}$, $\boldsymbol{\sigma}(t) \in \mathbf{Y}$ and $\mathbf{u}(t) \in \mathbf{Q}$, such that $\mathbf{u}(0) = \mathbf{u}_0$ and

$$[a(\mathbf{t}), \mathbf{s}] + [b_1(\mathbf{s}), \boldsymbol{\sigma}] = 0 \quad \forall \mathbf{s} \in \mathbf{X}, \quad (2.16a)$$

$$[b_1(\mathbf{t}), \boldsymbol{\tau}] + [b_2(\mathbf{u}), \boldsymbol{\tau}] = 0 \quad \forall \boldsymbol{\tau} \in \mathbf{Y}, \quad (2.16b)$$

$$\frac{\partial}{\partial t} [c(\mathbf{u}), \mathbf{v}] + [b_2(\mathbf{v}), \boldsymbol{\sigma}] = [F, \mathbf{v}] \quad \forall \mathbf{v} \in \mathbf{Q}, \quad (2.16c)$$

where the operators $a : \mathbf{X} \rightarrow \mathbf{X}'$, $b_1 : \mathbf{X} \rightarrow \mathbf{Y}'$, $b_2 : \mathbf{Q} \rightarrow \mathbf{Y}'$, $c : \mathbf{Q} \rightarrow \mathbf{Q}'$, are defined, respectively, as

$$[a(\mathbf{t}), \boldsymbol{\tau}] := \int_{\Omega} \boldsymbol{\vartheta}(\mathbf{t}_{\text{sym}}) : \mathbf{s} \quad \forall \mathbf{t}, \mathbf{s} \in \mathbf{X}, \quad (2.17a)$$

$$[b_1(\mathbf{s}), \boldsymbol{\tau}] := - \int_{\Omega} \boldsymbol{\tau} : \mathbf{s} \quad \forall (\mathbf{s}, \boldsymbol{\tau}) \in \mathbf{X} \times \mathbf{Y}, \quad (2.17b)$$

$$[b_2(\mathbf{v}), \boldsymbol{\tau}] := - \int_{\Omega} \mathbf{div}(\boldsymbol{\tau}) \cdot \mathbf{v} \quad \forall (\mathbf{v}, \boldsymbol{\tau}) \in \mathbf{Q} \times \mathbf{Y}, \quad (2.17c)$$

$$[c(\mathbf{u}), \mathbf{v}] := \int_{\Omega} \mathbf{u} \cdot \mathbf{v} \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{Q}. \quad (2.17d)$$

In turn, $F \in \mathbf{Q}'$ is the bounded linear functional defined by

$$[F, \mathbf{v}] := \int_{\Omega} \mathbf{f} \cdot \mathbf{v}. \quad (2.18)$$

In all the terms above, $[\cdot, \cdot]$ denotes the duality pairing induced by the corresponding operators. Let us define the global unknown and space:

$$\tilde{\mathbf{u}} := (\mathbf{u}, \mathbf{t}) \in \mathbf{V} := \mathbf{Q} \times \mathbf{X}, \quad (2.19)$$

where \mathbf{V} is endowed with the norm

$$\|\tilde{\mathbf{v}}\|_{\mathbf{V}}^2 = \|(\mathbf{v}, \mathbf{s})\|_{\mathbf{V}}^2 = \|\mathbf{v}\|_{\mathbf{Q}}^2 + \|\mathbf{s}\|_{\mathbf{X}}^2 \quad \forall \tilde{\mathbf{v}} := (\mathbf{v}, \mathbf{s}) \in \mathbf{V}.$$

Now, thanks to the above notation, it is easy to see that (2.16) can be rewritten equivalently as: Find $(\tilde{\mathbf{u}}, \boldsymbol{\sigma}) : [0, t_F] \rightarrow \mathbf{V} \times \mathbf{Y}$ such that

$$\left[\frac{\partial}{\partial t} C(\tilde{\mathbf{u}}(t)), \tilde{\mathbf{v}} \right] + [\mathcal{A}(\tilde{\mathbf{u}}(t)), \tilde{\mathbf{v}}] + [\mathcal{B}(\tilde{\mathbf{v}}), \boldsymbol{\sigma}(t)] = [\mathcal{F}(t), \tilde{\mathbf{v}}] \quad \forall \tilde{\mathbf{v}} \in \mathbf{V}, \quad (2.20a)$$

$$[\mathcal{B}(\tilde{\mathbf{u}}(t)), \boldsymbol{\tau}] = 0 \quad \forall \boldsymbol{\tau} \in \mathbf{Y}, \quad (2.20b)$$

where, the operators $\mathcal{A} : \mathbf{V} \rightarrow \mathbf{V}'$, $\mathcal{B} : \mathbf{V} \rightarrow \mathbf{Y}'$ and $\mathcal{C} : \mathbf{V} \rightarrow \mathbf{V}'$ are defined by

$$[\mathcal{A}(\tilde{\mathbf{w}}), \tilde{\mathbf{v}}] := [a(\mathbf{r}, \mathbf{s})] \quad \forall \tilde{\mathbf{w}} = (\mathbf{w}, \mathbf{r}), \tilde{\mathbf{v}} = (\mathbf{v}, \mathbf{s}) \in \mathbf{V}, \quad (2.21a)$$

$$[\mathcal{B}(\tilde{\mathbf{v}}), \boldsymbol{\tau}] := [b_1(\mathbf{s}, \boldsymbol{\tau})] + [b_2(\mathbf{v}, \boldsymbol{\tau})] \quad \forall (\tilde{\mathbf{v}}, \boldsymbol{\tau}) = ((\mathbf{v}, \mathbf{s}), \boldsymbol{\tau}) \in \mathbf{V} \times \mathbf{Y}, \quad (2.21b)$$

$$[\mathcal{C}(\tilde{\mathbf{w}}), \tilde{\mathbf{v}}] := [c(\mathbf{w}, \mathbf{v})] \quad \forall \tilde{\mathbf{w}}, \tilde{\mathbf{v}} \in \mathbf{V}, \quad (2.21c)$$

whereas the functional $\mathcal{F} \in \mathbf{V}'$ is set as

$$[\mathcal{F}, \tilde{\mathbf{v}}] := [F, \mathbf{v}]. \quad (2.22)$$

3. Well-posedness of the continuous problem

In this section, we proceed in a manner similar to Ref. 30 to establish the existence of a solution to problem (2.20). More precisely, we recall several auxiliary results that will be employed in the subsequent analysis.

3.1. Some abstract results

We begin by recalling a fundamental result from Theorem IV.6.1(b) in Ref. 58, which will be instrumental in establishing the existence of a solution to problem (2.20).

Theorem 3.1. *Given V_n a seminorm space, which obtained from a symmetric and non-negative bilinear form $n(\cdot, \cdot)$, we let $\mathcal{N} : V_n \rightarrow V'_n$ be the bounded linear operator induced by n , which is defined by*

$$\mathcal{N}x(y) = n(x, y) \quad \forall x, y \in V_n.$$

In addition, let D be a dense subspace of V_n , $\mathcal{M} : D \rightarrow V'_n$ be linear operator and $N(\mathcal{N})$ and $N(\mathcal{M})$ be the respective null spaces of operators \mathcal{N} and \mathcal{M} . Assume that:

i) \mathcal{M} is monotone, that is,

$$[\mathcal{M}(x) - \mathcal{M}(y), x - y] \geq 0 \quad \forall x, y \in D.$$

ii) $N(\mathcal{N}) \cap D \subset N(\mathcal{M})$ and $\mathcal{N} + \mathcal{M} : D \rightarrow V'_n$ is onto.

Then for every $f \in W^{1,1}(J; V'_n)$ and $u_0 \in D$ there exists a solution of $\partial_t(\mathcal{N}u)(t) + \mathcal{M}u(t) = f(t)$, $t > 0$, with $(\mathcal{N}u)(0) = \mathcal{N}u_0$.

Furthermore, to verify hypothesis ii) in Theorem 3.1, we invoke the following abstract result, whose proof is an adaptation of Theorem 3.1 27 to our context.

Theorem 3.2. *Let X_1 , X_2 and Y be separable and reflexive Banach spaces, X_1 and X_2 being uniformly convex, and set $X = X_1 \times X_2$. Let $\mathcal{A} : X \rightarrow X'$ be a nonlinear operator, $\mathcal{B} \in \mathcal{L}(X, Y')$, and let K be the kernel of \mathcal{B} , that is,*

$$K := \left\{ v \in X : [\mathcal{B}(v), q] = 0 \quad \forall q \in Y' \right\}.$$

Assume that

(i) \mathcal{A} is hemi-continuous, that is, for each $u, v \in X$

$$J : \mathbb{R} \rightarrow \mathbb{R}, \quad t \rightarrow J(t) = [\mathcal{A}(u + tv), v] \quad \text{is continuous.}$$

(ii) there exist constants $L > 0$ and $p_1, p_2 \in (1, 2]$, such that

$$\|\mathcal{A}(u) - \mathcal{A}(v)\|_{X'} \leq L \left\{ \|u_1 - v_1\|_{X_1}^{p_1-1} + \|u_2 - v_2\|_{X_2}^{p_2-1} \right\},$$

for all $u = (u_1, u_2), v = (v_1, v_2) \in X$.

(iii) the family of operators $\left\{ \mathcal{A}(\cdot + t) : K \rightarrow K' : t \in X \right\}$ is uniformly strictly monotone, that is there exist $\gamma > 0$ and $p_1, p_2 \in (1, 2]$, such that

$$[\mathcal{A}(u+t) - \mathcal{A}(v+t), u-v] \geq \gamma \left\{ \|u_1 - v_1\|_{X_1}^{p_1} + \frac{\|u_2 - v_2\|_{X_2}^2}{1 + (\|u_2\|_{X_2} + \|v_2\|_{X_2})^{2-p_2}} \right\},$$

for all $t \in X$, and for all $u = (u_1, u_2), v = (v_1, v_2) \in K$.

(iv) there exist $\beta > 0$ such that

$$\sup_{v \in X} \frac{[\mathcal{B}(v), q]}{\|v\|_X} \geq \beta \|q\|_Y \quad \forall q \in Y.$$

Then, for each $(\mathcal{F}, \mathcal{G}) \in X' \times Y'$ there exists a unique $(u, p) \in X \times Y$ such that

$$\begin{aligned} [\mathcal{A}(u), v] + [\mathcal{B}(v), p] &= \mathcal{F}(v) \quad \forall v \in X, \\ [\mathcal{B}(u), q] &= \mathcal{G}(q) \quad \forall q \in Y. \end{aligned}$$

Proof. The proof is similar to the proofs of Theorem 3.4 in Ref. 29 and Theorem 3.1 in Ref. 27. \square

Then, we rewrite the problem (2.20) in notations of Theorem 3.1. Specifically, we define the operators

$$\mathcal{N} := \begin{pmatrix} \mathcal{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} : V_n \rightarrow V'_n \quad \text{and} \quad \mathcal{M} := \begin{pmatrix} \mathcal{A} & \mathcal{B}' \\ -\mathcal{B} & \mathbf{0} \end{pmatrix} : D \rightarrow V'_n, \quad (3.1)$$

with

$$V_n := \mathbf{V} \times \mathbf{Y}, \quad u := \begin{pmatrix} \vec{u} \\ \sigma \end{pmatrix}, \quad V'_n := (\mathbf{L}^2(\Omega) \times \{\mathbf{0}\}) \times \{\mathbf{0}\}, \quad (3.2)$$

and

$$D := \left\{ (\vec{u}, \sigma) \in V_n : \mathcal{M}(\vec{u}, \sigma) \in V'_n \right\}. \quad (3.3)$$

Finally, we establish the stability properties of the operators \mathcal{N} and \mathcal{M} . We begin by noting that the operators \mathcal{B} and \mathcal{C} , as well as the functional \mathcal{F} , are linear. Moreover, by applying Hölder's and Cauchy–Schwarz inequalities, we obtain

$$|[\mathcal{B}(\vec{v}), \tau]| \leq \|\vec{v}\|_{\mathbf{V}} \|\tau\|_{\mathbf{Y}} \quad \forall (\vec{v}, \tau) \in \mathbf{V} \times \mathbf{Y}, \quad (3.4a)$$

$$|[\mathcal{C}(\vec{w}), \vec{v}]| \leq \|\mathbf{w}\|_{\mathbf{Q}} \|\mathbf{v}\|_{\mathbf{Q}} \leq \|\vec{w}\|_{\mathbf{V}} \|\vec{v}\|_{\mathbf{V}}, \quad [\mathcal{C}(\vec{v}), \vec{v}] = \|\mathbf{v}\|_{\mathbf{Q}}^2 \quad \forall \vec{w}, \vec{v} \in \mathbf{V}, \quad (3.4b)$$

$$|[\mathcal{F}, \vec{v}]| \leq \|\mathbf{f}\|_{\mathbf{Q}} \|\vec{v}\|_{\mathbf{V}} \quad \forall \vec{v} \in \mathbf{V}. \quad (3.4c)$$

This implies that the operators \mathcal{B} and \mathcal{F} are bounded and continuous, while the operator \mathcal{N} is bounded, continuous, and monotone. Furthermore, by applying the estimate (2.11), it

follows that the nonlinear operator \mathcal{A} is also bounded, with an upper bound $C_{\mathcal{A}} := L_{\boldsymbol{\theta}}$, that is

$$|[\mathcal{A}(\vec{\mathbf{w}}), \vec{\mathbf{v}}]| \leq C_{\mathcal{A}} \|\mathbf{t}\|_{\mathbf{X}}^{r-1} \|\mathbf{s}\|_{\mathbf{X}} \leq C_{\mathcal{A}} \|\mathbf{t}\|_{\mathbf{X}}^{r-1} \|\vec{\mathbf{v}}\|_{\mathbf{V}}. \quad (3.5)$$

This result, together with (3.4a), implies that the operator \mathcal{M} is bounded and continuous.

Next, we verify hypothesis ii) of Theorem 3.1 to establish the well-posedness of problem (2.20). To this end, we consider the resolvent system associated with (2.20): find $(\vec{\mathbf{u}}, \boldsymbol{\sigma}) \in \mathbf{V} \times \mathbf{Y}$ such that

$$[(C + \mathcal{A})(\vec{\mathbf{u}}), \vec{\mathbf{v}}] + [\mathcal{B}(\vec{\mathbf{v}}), \boldsymbol{\sigma}] = [\mathcal{F}, \vec{\mathbf{v}}] \quad \forall \vec{\mathbf{v}} \in \mathbf{V}, \quad (3.6a)$$

$$[\mathcal{B}(\vec{\mathbf{u}}), \boldsymbol{\tau}] = 0 \quad \forall \boldsymbol{\tau} \in \mathbf{Y}. \quad (3.6b)$$

3.2. The well-posedness of (3.6)

In this section, we apply Theorem 3.2 to establish the existence and uniqueness of a solution to problem (3.6). We begin by observing that, since the space $\mathbb{L}^r(\Omega)$ is uniformly convex and separable for $r \in (1, 2]$, the spaces \mathbf{V} and \mathbf{Y} inherit these properties and are therefore also uniformly convex and separable.

Next, we proceed to verify hypothesis ii) of Theorem 3.2, which asserts a continuity bound for the nonlinear operator $C + \mathcal{A}$.

Lemma 3.1. *Let $r \in (1, 2]$. Then, there exists L_{nn} , depending on $L_{\boldsymbol{\theta}}$, ρ , r , such that*

$$\|(C + \mathcal{A})(\vec{\mathbf{u}}) - (C + \mathcal{A})(\vec{\mathbf{v}})\|_{\mathbf{V}'} \leq L_{\text{nn}} \left\{ \|\mathbf{u} - \mathbf{v}\|_{\mathbf{Q}} + \|\mathbf{t} - \mathbf{s}\|_{\mathbf{X}}^{r-1} \right\}. \quad (3.7)$$

Proof. Let $\vec{\mathbf{u}} = (\mathbf{u}, \mathbf{t})$ and $\vec{\mathbf{v}} = (\mathbf{v}, \mathbf{s}) \in \mathbf{V} := \mathbf{Q} \times \mathbf{X}$. Then, owing to the linearity of the operator C and using the definition of \mathcal{A} (cf. (2.21a)) together with the boundedness estimate (3.4b), we obtain

$$\begin{aligned} |[(C + \mathcal{A})(\vec{\mathbf{u}}), \vec{\mathbf{w}}]| &\leq |[C(\vec{\mathbf{u}} - \vec{\mathbf{v}}), \vec{\mathbf{w}}]| + |[\mathcal{A}(\vec{\mathbf{u}}) - \mathcal{A}(\vec{\mathbf{v}}), \vec{\mathbf{w}}]| \\ &\leq \|\mathbf{u} - \mathbf{v}\|_{\mathbf{Q}} \|\mathbf{w}\|_{\mathbf{Q}} + |[a(\mathbf{t}) - a(\mathbf{s}), \mathbf{r}]|. \end{aligned} \quad (3.8)$$

To estimate the second term, we apply the definition of the bilinear form a given in (2.17a), together with Assumption 2.1, which yields

$$\begin{aligned} |[a(\mathbf{t}) - a(\mathbf{s}), \mathbf{r}]| &\leq \left| \int_{\Omega} \boldsymbol{\vartheta}(\mathbf{t}_{\text{sym}}) - \boldsymbol{\vartheta}(\mathbf{s}_{\text{sym}}) : \mathbf{r} \right| \\ &\leq \left(\int_{\Omega} |\boldsymbol{\vartheta}(\mathbf{t}_{\text{sym}}) - \boldsymbol{\vartheta}(\mathbf{s}_{\text{sym}})|^{\rho} \right)^{1/\rho} \|\mathbf{r}\|_{\mathbf{X}} \\ &\leq L_{\boldsymbol{\theta}} \left(\int_{\Omega} (\delta^r + |\mathbf{t}_{\text{sym}} - \mathbf{s}_{\text{sym}}|^r)^{\frac{\rho(r-2)}{r}} |\mathbf{t}_{\text{sym}} - \mathbf{s}_{\text{sym}}|^{\rho} \right)^{1/\rho} \|\mathbf{r}\|_{\mathbf{X}}, \end{aligned}$$

from which, and proceeding in a manner analogous to the derivation of (2.11), we obtain

$$|[a(\mathbf{t}) - a(\mathbf{s}), \mathbf{r}]| \leq L_{\boldsymbol{\theta}} \|\mathbf{t} - \mathbf{s}\|_{\mathbf{X}}^{r-1} \|\mathbf{r}\|_{\mathbf{X}}. \quad (3.9)$$

Thus, replacing back (3.9) into (3.8), we obtain (3.7) with $L_{\text{nn}} := \max\{1, L_{\boldsymbol{\theta}}\}$. \square

Next, before verifying hypothesis iii) of Theorem 3.2 for the operator $C + \mathcal{A}$, we observe that, following an argument similar to that in Ref. 31, the kernel of the operator \mathcal{B} , denoted

by \mathbf{K} , can be characterized as

$$\mathbf{K} := \left\{ \vec{\mathbf{v}} = (\mathbf{v}, \mathbf{s}) \in \mathbf{V} : \nabla \mathbf{v} = \mathbf{s} \quad \text{and} \quad \mathbf{v} \in \mathbf{W}_{0,\Gamma_D}^{1,r}(\Omega) \right\}. \quad (3.10)$$

Lemma 3.2. *Let $r \in (1, 2]$. The family of operators $\left\{ (C + \mathcal{A})(\cdot + \vec{\mathbf{z}}) : \mathbf{K} \rightarrow \mathbf{K}' : \vec{\mathbf{z}} \in \mathbf{V} \right\}$ is uniformly strictly monotone, that is, there exists a constant $\alpha_{\text{nn}} > 0$, depending on $\alpha_{\mathcal{A}}$, r , δ and $|\Omega|$, such that*

$$[(C + \mathcal{A})(\vec{\mathbf{u}} + \vec{\mathbf{z}}) - (C + \mathcal{A})(\vec{\mathbf{v}} + \vec{\mathbf{z}}), \vec{\mathbf{u}} - \vec{\mathbf{v}}] \geq \alpha_{\text{nn}} \left\{ \|\mathbf{u} - \mathbf{v}\|_{\mathbf{Q}}^2 + \frac{\|\mathbf{t} - \mathbf{s}\|_{\mathbf{X}}^2}{1 + (\|\mathbf{t}\|_{\mathbf{X}} + \|\mathbf{s}\|_{\mathbf{X}})^{2-r}} \right\}, \quad (3.11)$$

for all $\vec{\mathbf{z}} = (\mathbf{z}, \mathbf{r}) \in \mathbf{V}$ and $\vec{\mathbf{u}} = (\mathbf{u}, \mathbf{t}), \vec{\mathbf{v}} = (\mathbf{v}, \mathbf{s}) \in \mathbf{K}$.

Proof. Given $\vec{\mathbf{z}} = (\mathbf{z}, \mathbf{r}) \in \mathbf{V}$ and $\vec{\mathbf{u}} = (\mathbf{u}, \mathbf{t}), \vec{\mathbf{v}} = (\mathbf{v}, \mathbf{s}) \in \mathbf{K}$, from the definitions of \mathcal{A} , C (cf. (2.21a), (2.21c)), we have

$$[(C + \mathcal{A})(\vec{\mathbf{u}} + \vec{\mathbf{z}}) - (C + \mathcal{A})(\vec{\mathbf{v}} + \vec{\mathbf{z}}), \vec{\mathbf{u}} - \vec{\mathbf{v}}] = \|\mathbf{u} - \mathbf{v}\|_{\mathbf{Q}}^2 + [a(\mathbf{t} + \mathbf{r}) - a(\mathbf{s} + \mathbf{r}), \mathbf{t} - \mathbf{s}].$$

To derive a lower bound for the second term on the right-hand side of the above expression, we apply Assumption 2.1 (cf. (2.7)) together with Hölder's inequality with conjugate exponents $(\frac{2}{2-r}, \frac{2}{r})$, yielding

$$\begin{aligned} \alpha_{\mathcal{A}} \|\mathbf{t}_{\text{sym}} - \mathbf{s}_{\text{sym}}\|_{\mathbf{X}}^2 &\leq \alpha_{\mathcal{A}} \left(\int_{\Omega} (|\mathbf{t}_{\text{sym}} - \mathbf{s}_{\text{sym}}|^2)^{r/2} \right)^{2/r} \\ &\leq \left(\int_{\Omega} (\delta^r + |\mathbf{t}_{\text{sym}}|^r + |\mathbf{s}_{\text{sym}}|^r)^{\frac{2-r}{2}} \left((\boldsymbol{\vartheta}(\mathbf{t}_{\text{sym}}) - \boldsymbol{\vartheta}(\mathbf{s}_{\text{sym}})) : (\mathbf{t}_{\text{sym}} - \mathbf{s}_{\text{sym}}) \right)^{r/2} \right)^{2/r} \\ &\leq \left(\int_{\Omega} (\delta^r + |\mathbf{t}_{\text{sym}}|^r + |\mathbf{s}_{\text{sym}}|^r) \right)^{\frac{2-r}{r}} \int_{\Omega} (\boldsymbol{\vartheta}(\mathbf{t}_{\text{sym}}) - \boldsymbol{\vartheta}(\mathbf{s}_{\text{sym}})) : (\mathbf{t}_{\text{sym}} - \mathbf{s}_{\text{sym}}) \\ &\leq \left(|\Omega| \delta^r + \|\mathbf{t}_{\text{sym}}\|_{\mathbf{X}}^r + \|\mathbf{s}_{\text{sym}}\|_{\mathbf{X}}^r \right)^{\frac{2-r}{r}} [a(\mathbf{t}) - a(\mathbf{s}), \mathbf{t} - \mathbf{s}], \end{aligned}$$

which, thanks to the inequality $(a + b)^\alpha \leq a^\alpha + b^\alpha$ for $a, b > 0$ and $\alpha \in [0, 1]$, implies that

$$\begin{aligned} &\alpha_{\mathcal{A}} \|\mathbf{t}_{\text{sym}} - \mathbf{s}_{\text{sym}}\|_{\mathbf{X}}^2 \\ &\leq \left(|\Omega|^{\frac{2-r}{r}} \delta^{2-r} + (\|\mathbf{t}_{\text{sym}}\|_{\mathbf{X}}^r + \|\mathbf{s}_{\text{sym}}\|_{\mathbf{X}}^r)^{\frac{2-r}{r}} \right) [a(\mathbf{t}) - a(\mathbf{s}), \mathbf{t} - \mathbf{s}] \\ &\leq \max \left\{ |\Omega|^{\frac{2-r}{r}} \delta^{2-r}, 1 \right\} \left(1 + (\|\mathbf{t}_{\text{sym}}\|_{\mathbf{X}} + \|\mathbf{s}_{\text{sym}}\|_{\mathbf{X}})^{2-r} \right) [a(\mathbf{t}) - a(\mathbf{s}), \mathbf{t} - \mathbf{s}]. \end{aligned} \quad (3.12)$$

On the other hand, we know from (3.10) that $\nabla \mathbf{v} = \mathbf{s}$, $\nabla \mathbf{u} = \mathbf{t}$ and $\mathbf{v} \in \mathbf{W}_{0,\Gamma_D}^{1,r}(\Omega)$. Hence, by applying Körn's inequality and the fact that $r < 2$ we obtain

$$\begin{aligned} \frac{\|\mathbf{t}_{\text{sym}} - \mathbf{s}_{\text{sym}}\|_{\mathbf{X}}^2}{1 + (\|\mathbf{t}_{\text{sym}}\|_{\mathbf{X}} + \|\mathbf{s}_{\text{sym}}\|_{\mathbf{X}})^{2-r}} &\geq \frac{\|\mathbf{t}_{\text{sym}} - \mathbf{s}_{\text{sym}}\|_{\mathbf{X}}^2}{1 + (\|\mathbf{t}\|_{\mathbf{X}} + \|\mathbf{s}\|_{\mathbf{X}})^{2-r}} \\ &\geq \frac{\|\epsilon(\mathbf{u} - \mathbf{v})\|_{\mathbf{X}}^2}{1 + (\|\mathbf{t}\|_{\mathbf{X}} + \|\mathbf{s}\|_{\mathbf{X}})^{2-r}} \\ &\geq \frac{1}{2} \frac{|\mathbf{u} - \mathbf{v}|_{1,r;\Omega}^2}{1 + (\|\mathbf{t}\|_{\mathbf{X}} + \|\mathbf{s}\|_{\mathbf{X}})^{2-r}} \\ &\geq \frac{1}{2} \frac{\|\mathbf{t} - \mathbf{s}\|_{\mathbf{X}}^2}{1 + (\|\mathbf{t}\|_{\mathbf{X}} + \|\mathbf{s}\|_{\mathbf{X}})^{2-r}}. \end{aligned} \quad (3.13)$$

Thus, replacing (3.13) back into (3.12), implies (3.11) with $\alpha_{\text{nn}} := \min \left\{ \frac{\alpha_{\mathcal{A}}}{2} \max \left\{ |\Omega|^{\frac{2-r}{r}} \delta^{2-r}, 1 \right\}, 1 \right\}$. \square

In turn, in order to prove that the operator \mathcal{B} satisfies *iv*) of Theorem 3.2, we first need to provide a useful estimate for tensors in $\mathbb{H}_{0,\Gamma_N}^\rho(\mathbf{div}; \Omega)$.

Lemma 3.3. *Let $r \in [1, 2)$ and $|\Gamma_N| > 0$. Then, there exists a constant $\widehat{C}_\rho > 0$, depending on n , $|\Omega|$ and norm of the continuous injection $\mathbf{i}_{r,2}$, such that*

$$\|\boldsymbol{\tau}\|_{0,\rho;\Omega} \leq \widehat{C}_\rho \left(\|\boldsymbol{\tau}^d\|_{0,\rho;\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,\Omega} \right) \quad \forall \boldsymbol{\tau} \in \mathbf{Y}. \quad (3.14)$$

Proof. We have shown this result in another work that is not available yet²⁸, so we add here a sketch of the proof. Specifically, for $\boldsymbol{\tau} \in \mathbf{Y} = \mathbb{H}_{0,\Gamma_N}^\rho(\mathbf{div}; \Omega)$, we consider the decomposition

$$\begin{aligned} \mathbb{H}_{0,\Gamma_N}^\rho(\mathbf{div}; \Omega) &= \mathring{\mathbb{H}}_{0,\Gamma_N}^\rho(\mathbf{div}; \Omega) + \mathbb{R}\mathbb{I}, \quad \text{where} \\ \mathring{\mathbb{H}}_{0,\Gamma_N}^\rho(\mathbf{div}; \Omega) &:= \left\{ \boldsymbol{\tau} \in \mathbb{H}_{0,\Gamma_N}^\rho(\mathbf{div}; \Omega) : \int_{\Omega} \text{tr}(\boldsymbol{\tau}) = 0 \right\}. \end{aligned} \quad (3.15)$$

Then, $\boldsymbol{\tau}$ can be written uniquely as $\boldsymbol{\tau} = \widetilde{\boldsymbol{\tau}} + c\mathbb{I}$ with $\widetilde{\boldsymbol{\tau}} \in \mathring{\mathbb{H}}_{0,\Gamma_N}^\rho(\mathbf{div}; \Omega)$ and $c := -\frac{1}{|\Gamma_N|} \int_{\Gamma_N} (\widetilde{\boldsymbol{\tau}}\boldsymbol{\nu}) \cdot \boldsymbol{\nu} \in \mathbb{R}$. In particular, $\widetilde{\boldsymbol{\tau}}$ has zero mean trace. This decomposition allows us to separate the estimate of the trace into a zero-mean part and a constant part.

Next, using the decomposition $L^r(\Omega) = L_0^r(\Omega) \oplus \mathbb{R}$, we write the L^ρ -norm of $\text{tr}(\boldsymbol{\tau})$ in dual form. In this way, we obtain

$$\|\text{tr}(\boldsymbol{\tau})\|_{0,\rho;\Omega} = \sup_{\mathbf{v} \in L^r(\Omega)} \frac{\int_{\Omega} \mathbf{v} \text{tr}(\boldsymbol{\tau})}{\|\mathbf{v}\|_{0,r;\Omega}} = \|\text{tr}(\widetilde{\boldsymbol{\tau}})\|_{0,\rho;\Omega} + |c| |\Omega|^{1-1/r}. \quad (3.16)$$

To estimate the zero-mean part, let $\widetilde{\mathbf{v}} \in L_0^r(\Omega)$. By the surjectivity of the divergence operator on zero-mean functions (cf. Theorem 7.7 of Ref. 1), there exists $\widetilde{\mathbf{z}}_{\mathbf{v}} \in \mathbf{W}_{0,\Gamma_D}^{1,r}(\Omega)$ such that

$$\text{div}(\widetilde{\mathbf{z}}_{\mathbf{v}}) = \widetilde{\mathbf{v}} \quad \text{and} \quad \|\widetilde{\mathbf{z}}_{\mathbf{v}}\|_{1,r;\Omega} \leq C_r \|\widetilde{\mathbf{v}}\|_{0,r;\Omega}. \quad (3.17)$$

Then, using the identity $\text{div}(\widetilde{\mathbf{z}}_{\mathbf{v}}) = \text{tr}(\nabla \widetilde{\mathbf{z}}_{\mathbf{v}})$, the decomposition of a tensor into its deviatoric and volumetric parts, and the integration by parts formula (1.3), one obtains

$$\int_{\Omega} \widetilde{\mathbf{v}} \text{tr}(\widetilde{\boldsymbol{\tau}}) = -n \left(\int_{\Omega} \mathbf{div}(\widetilde{\boldsymbol{\tau}}) \cdot \widetilde{\mathbf{z}}_{\mathbf{v}} + \int_{\Omega} \mathbf{dev}(\widetilde{\boldsymbol{\tau}}) : \nabla \widetilde{\mathbf{z}}_{\mathbf{v}} \right).$$

Applying Hölder's inequality, the second bound in (3.17), and the continuous embedding $\mathbf{W}^{1,r}(\Omega) \hookrightarrow \mathbf{L}^2(\Omega)$, we arrive at

$$\left| \int_{\Omega} \widetilde{\mathbf{v}} \text{tr}(\widetilde{\boldsymbol{\tau}}) \right| \leq n C_r \|\widetilde{\mathbf{v}}\|_{0,r;\Omega} \left(\|\mathbf{dev}(\widetilde{\boldsymbol{\tau}})\|_{0,\rho;\Omega} + \|\mathbf{div}(\widetilde{\boldsymbol{\tau}})\|_{0,\Omega} \right).$$

Taking the supremum over all $\widetilde{\mathbf{v}} \in L_0^r(\Omega)$, we conclude that

$$\|\text{tr}(\widetilde{\boldsymbol{\tau}})\|_{0,\rho;\Omega} \leq n C_r \left(\|\mathbf{dev}(\widetilde{\boldsymbol{\tau}})\|_{0,\rho;\Omega} + \|\mathbf{div}(\widetilde{\boldsymbol{\tau}})\|_{0,\Omega} \right). \quad (3.18)$$

It remains to control the coefficient c in (3.16). By its definition and Hölder's inequality, c can be bounded by the normal trace of $\boldsymbol{\tau}$ on Γ_N . On the other hand, using the definition of the dual norm of the normal trace, the surjectivity of the trace operator, Green's formula,

and the continuous embedding $\mathbf{W}^{1,r}(\Omega) \hookrightarrow \mathbf{L}^2(\Omega)$, one concludes that there exists a constant \tilde{C} , depending on C_r , n , and the norm of the continuous injection $\mathbf{i}_{r,2}$ such that

$$|c| \leq \tilde{C} \left(\|\mathbf{dev}(\bar{\boldsymbol{\tau}})\|_{0,\rho;\Omega} + \|\mathbf{div}(\bar{\boldsymbol{\tau}})\|_{0,\Omega} \right). \quad (3.19)$$

Thus, upon inserting (3.18) and (3.19) into (3.16), we obtain

$$\|\mathrm{tr}(\boldsymbol{\tau})\|_{0,\rho;\Omega} \leq \tilde{C}_\rho \left(\|\mathbf{dev}(\boldsymbol{\tau})\|_{0,\rho;\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,\Omega} \right). \quad (3.20)$$

Finally, since $\boldsymbol{\tau} = \mathbf{dev}(\boldsymbol{\tau}) + \frac{1}{n} \mathrm{tr}(\boldsymbol{\tau})\mathbb{I}$, the triangle inequality immediately gives (3.14) and the proof is complete. \square

Then, we are in position to prove the following result.

Lemma 3.4. *There exists a constant $\beta > 0$, depending only on $|\Omega|$ and the norm of the continuous embedding $\mathbf{i}_{r,2} : \mathbf{W}^{1,r}(\Omega) \rightarrow \mathbf{L}^2(\Omega)$, such that*

$$\sup_{\mathbf{0} \neq \tilde{\mathbf{v}} \in \mathbf{V}} \frac{[\mathcal{B}(\tilde{\mathbf{v}}), \boldsymbol{\tau}]}{\|\tilde{\mathbf{v}}\|_{\mathbf{V}}} \geq \beta \|\boldsymbol{\tau}\|_{\mathbf{Y}} \quad \forall \boldsymbol{\tau} \in \mathbf{Y}. \quad (3.21)$$

Proof. The proof is a modification of Lemma 3.3 in Ref. 31. More precisely, given $\mathbf{dev}(\boldsymbol{\tau}) \neq \mathbf{0}$ we set $\tilde{\mathbf{s}} := |\mathbf{dev}(\boldsymbol{\tau})|^{\rho-2} \mathbf{dev}(\boldsymbol{\tau})$, and notice that $\|\tilde{\mathbf{s}}\|_{\mathbf{X}} = \|\mathbf{dev}(\boldsymbol{\tau})\|_{0,\rho;\Omega}^\rho$ and $\mathrm{tr}(\tilde{\mathbf{s}}) = |\mathbf{dev}(\boldsymbol{\tau})|^{\rho-2} \mathrm{tr}(\mathbf{dev}(\boldsymbol{\tau})) = 0$, which says that $\tilde{\mathbf{s}} \in \mathbf{L}_{\mathrm{tr}}^r(\Omega)$, and additionally there holds

$$\int_{\Omega} \boldsymbol{\tau} : \tilde{\mathbf{s}} = \|\mathbf{dev}(\boldsymbol{\tau})\|_{0,\rho;\Omega}^\rho = \|\mathbf{dev}(\boldsymbol{\tau})\|_{0,\rho;\Omega} \|\tilde{\mathbf{s}}\|_{\mathbf{X}}. \quad (3.22)$$

Then, employing (3.22) we find that

$$\sup_{\mathbf{0} \neq \tilde{\mathbf{v}} \in \mathbf{V}} \frac{[\mathcal{B}(\tilde{\mathbf{v}}), \boldsymbol{\tau}]}{\|\tilde{\mathbf{v}}\|_{\mathbf{V}}} \geq \frac{[\mathcal{B}(\mathbf{0}, \tilde{\mathbf{s}}), \boldsymbol{\tau}]}{\|\tilde{\mathbf{s}}\|_{\mathbf{X}}} = \frac{\int_{\Omega} \tilde{\mathbf{s}} : \boldsymbol{\tau}}{\|\tilde{\mathbf{s}}\|_{\mathbf{X}}} = \|\mathbf{dev}(\boldsymbol{\tau})\|_{0,\rho;\Omega}. \quad (3.23)$$

In turn, denoting by $\boldsymbol{\tau}_j$ the j -th row of $\boldsymbol{\tau}$ for $j = 1, \dots, n$, we now set $\tilde{\mathbf{v}} = (\tilde{\mathbf{v}}, \mathbf{0}) \in \mathbf{V}$, with $\tilde{\mathbf{v}} := (\mathrm{div}(\boldsymbol{\tau}_j))_{j=1, \dots, n} \in \mathbf{L}^2(\Omega)$. Then, it follows that

$$\sup_{\mathbf{0} \neq \tilde{\mathbf{v}} \in \mathbf{V}} \frac{[\mathcal{B}(\tilde{\mathbf{v}}), \boldsymbol{\tau}]}{\|\tilde{\mathbf{v}}\|_{\mathbf{V}}} \geq \frac{[\mathcal{B}(\tilde{\mathbf{v}}, \mathbf{0}), \boldsymbol{\tau}]}{\|\tilde{\mathbf{v}}\|_{\mathbf{Q}}} = \|\mathbf{div}(\boldsymbol{\tau})\|_{0,\Omega},$$

which, together with (3.23) implies

$$\sup_{\mathbf{0} \neq \tilde{\mathbf{v}} \in \mathbf{V}} \frac{[\mathcal{B}(\tilde{\mathbf{v}}), \boldsymbol{\tau}]}{\|\tilde{\mathbf{v}}\|_{\mathbf{V}}} \geq \|\mathbf{dev}(\boldsymbol{\tau})\|_{0,\rho;\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,\Omega}. \quad (3.24)$$

Finally, by combining (3.14) and (3.24) we arrive at (3.21) with $\beta = \min \left\{ \frac{1}{2}, \frac{1}{2\tilde{C}_\rho} \right\}$. \square

We are now ready to establish the main result of this section.

Lemma 3.5. *Let $r \in (1, 2]$. Then, for each $\mathbf{f} \in \mathbf{L}^2(\Omega)$, there exists a unique solution $(\tilde{\mathbf{u}}, \boldsymbol{\sigma}) = ((\mathbf{u}, \mathbf{t}), \boldsymbol{\sigma}) \in \mathbf{V} \times \mathbf{Y}$ to (3.6).*

Proof. As a consequence of Lemmas 3.1, 3.2, and 3.4, we conclude that the operators $\mathcal{A} + \mathcal{C}$ and \mathcal{B} satisfy the assumptions of Theorem 3.2. Hence, a direct application of this abstract result yields the desired conclusion. \square

We end this section by presenting a suitable result on the initial condition, which is essential for the application of Theorem 3.1.

Lemma 3.6. *Let \mathbf{M} be the subspace of $\mathbf{W}_{0,\Gamma_D}^{1,r}(\Omega)$*

$$\mathbf{M} := \left\{ \mathbf{v} \in \mathbf{W}_{0,\Gamma_D}^{1,r}(\Omega) : \mathbf{div}(\boldsymbol{\vartheta}(\boldsymbol{\epsilon}(\mathbf{v}))) \in \mathbf{L}^2(\Omega) \quad \text{and} \quad \mathbf{div}(\mathbf{v}) = 0 \text{ in } \Omega \right\}, \quad (3.25)$$

and assume that $\mathbf{u}_0 \in \mathbf{Q} \cap \mathbf{M}$. Then, there exists $(\mathbf{t}_0, \boldsymbol{\sigma}_0) \in \mathbf{X} \times \mathbf{Y}$ such that $\vec{\mathbf{u}}_0 := (\mathbf{u}_0, \mathbf{t}_0)$ and $\boldsymbol{\sigma}_0$ satisfy

$$\begin{pmatrix} \mathcal{A} & \mathcal{B}' \\ -\mathcal{B} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \vec{\mathbf{u}}_0 \\ \boldsymbol{\sigma}_0 \end{pmatrix} \in (\mathbf{L}^2(\Omega) \times \{\mathbf{0}\}) \times \{\mathbf{0}\}. \quad (3.26)$$

Proof. Given $\mathbf{u}_0 \in \mathbf{Q} \cap \mathbf{M}$, we define

$$\mathbf{t}_0 := \nabla \mathbf{u}_0 \quad \text{and} \quad \boldsymbol{\sigma}_0 := \boldsymbol{\vartheta}(\boldsymbol{\epsilon}(\mathbf{u}_0)), \quad (3.27)$$

which satisfy

$$\text{tr}(\mathbf{t}_0) = 0, \quad \mathbf{div}(\boldsymbol{\sigma}_0) = \mathbf{div}(\boldsymbol{\vartheta}(\boldsymbol{\epsilon}(\mathbf{u}_0))). \quad (3.28)$$

Notice that $\mathbf{t}_0 \in \mathbf{X}$, and $\boldsymbol{\sigma}_0 \in \mathbf{Y}$. Next, by applying integration by parts to the identity $\mathbf{t}_0 = \nabla \mathbf{u}_0$ and following a similar approach to that used in (2.13), we get

$$[\mathcal{B}(\vec{\mathbf{u}}_0), \boldsymbol{\tau}] = 0 \quad \forall \boldsymbol{\tau} \in \mathbf{Y}.$$

Therefore, given $\mathbf{u}_0 \in \mathbf{Q} \cap \mathbf{M}$, where \mathbf{M} is defined in (3.25), by multiplying the second rows in (3.27) and (3.28) by the respective test functions $\mathbf{s} \in \mathbf{X}$ and $\mathbf{v} \in \mathbf{Q}$, we find that

$$\begin{pmatrix} \mathcal{A} & \mathcal{B}' \\ -\mathcal{B} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \vec{\mathbf{u}}_0 \\ \boldsymbol{\sigma}_0 \end{pmatrix} = \begin{pmatrix} \vec{F}_0 \\ 0 \end{pmatrix}, \quad (3.29)$$

where $\vec{F}_0 = (f_0, 0)$ and

$$(f_0, \mathbf{v}) := - \int_{\Omega} \mathbf{div}(\boldsymbol{\vartheta}(\boldsymbol{\epsilon}(\mathbf{u}_0))) \cdot \mathbf{v}. \quad (3.30)$$

Also, we have

$$|(f_0, \mathbf{v})| \leq \left\| \mathbf{div}(\boldsymbol{\vartheta}(\boldsymbol{\epsilon}(\mathbf{u}_0))) \right\|_{0,\Omega} \|\mathbf{v}\|_{0,\Omega}. \quad (3.31)$$

Thus, $\vec{F}_0 \in \mathbf{L}^2(\Omega) \times \{\mathbf{0}\}$, which implies (3.26), completing the proof. \square

3.3. Main result

In this section, we establish the well-posedness of problem (2.20).

Theorem 3.3. *For each $\mathbf{f} \in W^{1,1}(\mathbf{J}; \mathbf{L}^2(\Omega))$ and every compatible initial data $(\vec{\mathbf{u}}_0, \boldsymbol{\sigma}_0) = ((\mathbf{u}_0, \mathbf{t}_0), \boldsymbol{\sigma}_0)$ derived in Lemma 3.6, there exists a unique $(\vec{\mathbf{u}}, \boldsymbol{\sigma}) = ((\mathbf{u}, \mathbf{t}), \boldsymbol{\sigma}) : [0, t_F] \rightarrow \mathbf{V} \times \mathbf{Y}$ solution to (2.20), such that $\mathbf{u} \in W^{1,\infty}(\mathbf{J}; \mathbf{Q})$ and $((\mathbf{u}(0), \mathbf{t}(0)), \boldsymbol{\sigma}(0)) = ((\mathbf{u}_0, \mathbf{t}_0), \boldsymbol{\sigma}_0)$.*

Proof. We note that the structure of problem (3.6) is the same as in Theorem 3.1, based on the definitions given in (3.1)-(3.3). Additionally, the operator \mathcal{N} is linear, symmetric, and monotone due to the definition of \mathcal{C} (cf. (3.4b)), while \mathcal{M} is monotone thanks to the strictly

monotonicity of \mathcal{A} (cf. Lemma 3.2). Alternatively, Lemma 3.5 allows us to deduce that for the given $(\hat{F}, \mathbf{0}) \in V'_n$ with $\hat{F} = (\hat{\mathbf{f}}, \mathbf{0})$, there exists a unique $(\tilde{\mathbf{u}}, \boldsymbol{\sigma}) = ((\mathbf{u}, \mathbf{t}), \boldsymbol{\sigma}) \in \mathbf{V} \times \mathbf{Y}$ such that

$$(\hat{F}, \mathbf{0}) = (\mathcal{N} + \mathcal{M})(\tilde{\mathbf{u}}, \boldsymbol{\sigma}),$$

which implies $\mathcal{N} + \mathcal{M} : D \rightarrow V'_n$ is onto, where D is defined in (3.3). Lastly, by taking $\mathbf{u}_0 \in \mathbf{Q} \cap \mathbf{M}$, where \mathbf{M} is defined in (3.25), a direct application of Lemma 3.6 allows us to determine $(\mathbf{t}_0, \boldsymbol{\sigma}_0) \in \mathbf{X} \times \mathbf{Y}$ such that $(\tilde{\mathbf{u}}_0, \boldsymbol{\sigma}_0) \in D$. Consequently, by utilizing Theorem 3.1, we can conclude that for each $t \in \mathbf{J}$ there exists a solution $(\tilde{\mathbf{u}}(t), \boldsymbol{\sigma}(t)) = ((\mathbf{u}(t), \mathbf{t}(t)), \boldsymbol{\sigma}(t)) \in \mathbf{V} \times \mathbf{Y}$ to (2.20) such that $\mathbf{u} \in W^{1,\infty}(\mathbf{J}; \mathbf{Q})$, with initial value $\mathbf{u}(0) = \mathbf{u}_0$.

We now proceed to establish the uniqueness of the solution to (2.20). To this end, let $(\tilde{\mathbf{u}}_i, \boldsymbol{\sigma}_i)$ for $i = 1, 2$ be two solutions corresponding to the same input data. By choosing the test functions in (2.20) as $(\tilde{\mathbf{v}}, \boldsymbol{\tau}) = (\tilde{\mathbf{u}}_1 - \tilde{\mathbf{u}}_2, \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2) \in \mathbf{V} \times \mathbf{Y}$, we find that

$$\frac{1}{2} \partial_t \|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathbf{Q}}^2 + [\mathcal{A}(\tilde{\mathbf{u}}_1) - \mathcal{A}(\tilde{\mathbf{u}}_2), \tilde{\mathbf{u}}_1 - \tilde{\mathbf{u}}_2] = 0.$$

From this, and using the fact that $\tilde{\mathbf{u}}_1 - \tilde{\mathbf{u}}_2 \in \mathbf{K}$ together with the strict monotonicity of \mathcal{A} (cf. (3.11)), we conclude that

$$\frac{1}{2} \partial_t \|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathbf{Q}}^2 + \alpha_{\text{nn}} \frac{\|\mathbf{t}_1 - \mathbf{t}_2\|_{\mathbf{X}}^2}{1 + (\|\mathbf{t}_1\|_{\mathbf{X}} + \|\mathbf{t}_2\|_{\mathbf{X}})^{2-r}} \leq 0. \quad (3.32)$$

Thus, by integrating (3.32) over time from 0 to $t \in (0, t_F]$, and utilizing the condition $\mathbf{u}_1(0) = \mathbf{u}_2(0)$, we get $\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_{\mathbf{Q}} = 0$ and $\int_0^t \|\mathbf{t}_1(s) - \mathbf{t}_2(s)\|_{\mathbf{X}}^2 ds = 0$ and, as a result, $\mathbf{u}_1(t) = \mathbf{u}_2(t)$ and $\mathbf{t}_1(t) = \mathbf{t}_2(t)$ for all $t \in (0, t_F]$. Applying the inf-sup condition (cf. (3.21)) to the operator \mathcal{B} , along with the first equation in (2.20), yields

$$\begin{aligned} \|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\|_{\mathbf{Y}} &\leq \frac{1}{\beta} \sup_{\mathbf{0} \neq \tilde{\mathbf{v}} \in \mathbf{V}} \frac{[\mathcal{B}(\tilde{\mathbf{v}}), \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2]}{\|\tilde{\mathbf{v}}\|_{\mathbf{V}}} \\ &\leq \frac{1}{\beta} \sup_{\mathbf{0} \neq \tilde{\mathbf{v}} \in \mathbf{V}} \frac{-\frac{\partial}{\partial t} [C(\tilde{\mathbf{u}}_1 - \tilde{\mathbf{u}}_2), \tilde{\mathbf{v}}] - [\mathcal{A}(\tilde{\mathbf{u}}_1) - \mathcal{A}(\tilde{\mathbf{u}}_2), \tilde{\mathbf{v}}]}{\|\tilde{\mathbf{v}}\|_{\mathbf{V}}} = 0. \end{aligned}$$

This implies that $\boldsymbol{\sigma}_1(t) = \boldsymbol{\sigma}_2(t)$ for all $t \in (0, t_F]$, and consequently, (2.20) has a unique solution. \square

The following result provides the stability bound for solution of (2.20).

Theorem 3.4. *Let $r \in (1, 2]$. Assume that $\mathbf{u}_0 \in \mathbf{Q} \cap \mathbf{M}$ satisfies (3.26). Then, there exists a constant C_{stab} , depending on α_{nn} , r , β , the norm of the continuous injection $\mathbf{i}_{r,2} : \mathbf{W}^{1,r}(\Omega) \rightarrow \mathbf{L}^2$, such that*

$$\begin{aligned} \|\mathbf{u}(t)\|_{\mathbf{Q}}^2 + \int_0^t \left(\|\mathbf{t}(s)\|_{\mathbf{X}}^r + \|\mathbf{u}(s)\|_{\mathbf{Q}}^r + \|\boldsymbol{\sigma}(s)\|_{\mathbf{Y}}^2 \right) ds \\ \leq C_{\text{stab}} \left\{ \|\mathbf{u}_0\|_{\mathbf{Q}}^2 + \int_0^t \left(\|\mathbf{f}(s)\|_{0,\Omega}^{r/(r-1)} + \|\mathbf{f}(s)\|_{0,\Omega}^2 \right) ds \right\} := \mathcal{N}(\mathbf{f}, \mathbf{u}_0), \end{aligned} \quad (3.33)$$

Proof. We start by selecting $(\tilde{\mathbf{v}}, \boldsymbol{\tau}) = (\tilde{\mathbf{u}}(t), \boldsymbol{\sigma}(t))$ in (2.20), to obtain

$$\frac{1}{2} \partial_t \|\mathbf{u}\|_{\mathbf{Q}}^2 + |[\mathcal{A}(\tilde{\mathbf{u}}), \tilde{\mathbf{u}}]| \leq |[\mathcal{F}, \tilde{\mathbf{u}}]|.$$

As a result of the monotonicity of \mathcal{A} (cf. (3.11)) and the boundedness of \mathcal{F} (cf. (3.4c)), it is clear that

$$\frac{1}{2} \partial_t \|\mathbf{u}\|_{\mathbf{Q}}^2 + \alpha_{\text{nn}} \frac{\|\mathbf{t}\|_{\mathbf{X}}^2}{1 + \|\mathbf{t}\|_{\mathbf{X}}^{2-r}} \leq \|\mathbf{f}\|_{0,\Omega} \|\mathbf{u}\|_{\mathbf{Q}}. \quad (3.34)$$

We observe that the quantity $\|\mathbf{t}\|_{\mathbf{X}}$ satisfies one of the following two alternatives:

$$\|\mathbf{t}\|_{\mathbf{X}} \geq 1 \quad \text{or} \quad \|\mathbf{t}\|_{\mathbf{X}} < 1.$$

In the first case, it follows from (3.34) that

$$\partial_t \|\mathbf{u}\|_{\mathbf{Q}}^2 + \alpha_{\text{nn}} \|\mathbf{t}\|_{\mathbf{X}}^r \leq 2 \|\mathbf{f}\|_{0,\Omega} \|\mathbf{u}\|_{\mathbf{Q}}. \quad (3.35)$$

Furthermore, by inspecting the second row of (2.20), we observe that $\bar{\mathbf{u}}(t) \in \mathbf{K}$ (cf. (3.10)), from which it follows that $\mathbf{t}(t) = \nabla \mathbf{u}(t)$ and $\mathbf{u}(t) \in \mathbf{W}_{0,\Gamma_D}^{1,r}(\Omega)$. Invoking the continuous embedding $\mathbf{i}_{r,2} : \mathbf{W}^{1,r}(\Omega) \rightarrow \mathbf{L}^2$, we deduce that

$$\|\mathbf{t}\|_{\mathbf{X}}^r = \|\nabla \mathbf{u}\|_{\mathbf{X}}^r \geq \frac{1}{\|\mathbf{i}_{r,2}\|^r} \|\mathbf{u}\|_{\mathbf{Q}}^r. \quad (3.36)$$

It then suffices to combine (3.36) with (3.35), and apply Young's inequality

$$ab \leq \frac{\alpha}{p} a^p + \frac{\alpha^{-q/p}}{q} b^q \quad \forall p, q \geq 0 \quad \text{and} \quad \frac{1}{p} + \frac{1}{q} = 1, \quad (3.37)$$

to obtain

$$\partial_t \|\mathbf{u}\|_{\mathbf{Q}}^2 + \frac{1}{2 \|\mathbf{i}_{r,2}\|^r} \alpha_{\text{nn}} \|\mathbf{u}\|_{\mathbf{Q}}^r + \frac{1}{2} \alpha_{\text{nn}} \|\mathbf{t}\|_{\mathbf{X}}^r \leq \frac{\alpha_1}{r} \|\mathbf{u}\|_{\mathbf{Q}}^r + 4 \frac{\alpha_1^{-1/(r-1)}}{r/(r-1)} \|\mathbf{f}\|_{0,\Omega}^{r/(r-1)}.$$

From the above bound, by selecting α_1 such that $\frac{\alpha_1}{r} \leq \frac{1}{4 \|\mathbf{i}_{r,2}\|^r} \alpha_{\text{nn}}$ and integrating over time from 0 to $t \in (0, t_F]$, we obtain

$$\|\mathbf{u}(t)\|_{\mathbf{Q}}^2 + \int_0^t \left(\|\mathbf{t}(s)\|_{\mathbf{X}}^r + \|\mathbf{u}(s)\|_{\mathbf{Q}}^r \right) ds \leq C_1 \left\{ \|\mathbf{u}_0\|_{\mathbf{Q}}^2 + \int_0^t \|\mathbf{f}(s)\|_{0,\Omega}^{r/(r-1)} ds \right\}. \quad (3.38)$$

On the other hand, in the second case, invoking (3.34) and applying (3.36) with $r = 2$, together with Young's inequality once more, we conclude that

$$\partial_t \|\mathbf{u}\|_{\mathbf{Q}}^2 + \frac{1}{2 \|\mathbf{i}_{r,2}\|^2} \alpha_{\text{nn}} \|\mathbf{u}\|_{\mathbf{Q}}^2 + \frac{1}{2} \alpha_{\text{nn}} \|\mathbf{t}\|_{\mathbf{X}}^2 \leq \alpha_2 \|\mathbf{u}\|_{\mathbf{Q}}^2 + \frac{1}{\alpha_2} \|\mathbf{f}\|_{0,\Omega}^2,$$

where, by selecting α_2 sufficiently small such that $\alpha_2 \leq \frac{1}{4 \|\mathbf{i}_{r,2}\|^2} \alpha_{\text{nn}}$ and integrating over the time interval $[0, t]$ for any $t \in (0, t_F]$, we readily obtain

$$\|\mathbf{u}(t)\|_{\mathbf{Q}}^2 + \int_0^t \left(\|\mathbf{t}(s)\|_{\mathbf{X}}^2 + \|\mathbf{u}(s)\|_{\mathbf{Q}}^2 \right) ds \leq C_2 \left\{ \|\mathbf{u}_0\|_{\mathbf{Q}}^2 + \int_0^t \|\mathbf{f}(s)\|_{0,\Omega}^2 ds \right\}. \quad (3.39)$$

Thus, combining estimates (3.38) and (3.39), and using the fact that $r < 2$, we obtain

$$\begin{aligned} & \|\mathbf{u}(t)\|_{\mathbf{Q}}^2 + \int_0^t \left(\|\mathbf{t}(s)\|_{\mathbf{X}}^r + \|\mathbf{u}(s)\|_{\mathbf{Q}}^r \right) ds \\ & \leq \tilde{C} \left\{ \|\mathbf{u}_0\|_{\mathbf{Q}}^2 + \int_0^t \left(\|\mathbf{f}(s)\|_{0,\Omega}^{r/(r-1)} + \|\mathbf{f}(s)\|_{0,\Omega}^2 \right) ds \right\}. \end{aligned} \quad (3.40)$$

Finally, to derive the *a priori* estimate for the pseudostress, we invoke the inf-sup condition for \mathcal{B} given in (3.21), in conjunction with the first row of (2.20) and the stability bounds for the operators \mathcal{F} , \mathcal{C} , and \mathcal{A} (cf. (3.4c), (3.4b), and (3.5)), which together yield

$$\begin{aligned} \beta \|\boldsymbol{\sigma}\|_{\mathbf{Y}} &\leq \sup_{\mathbf{0} \neq \vec{\mathbf{v}} \in \mathbf{V}} \frac{[\mathcal{B}(\vec{\mathbf{v}}), \boldsymbol{\sigma}]}{\|\vec{\mathbf{v}}\|_{\mathbf{V}}} = \sup_{\mathbf{0} \neq \vec{\mathbf{v}} \in \mathbf{V}} \frac{[\mathcal{F}, \vec{\mathbf{v}}] - \left[\frac{\partial}{\partial t} \mathcal{C}(\vec{\mathbf{u}}), \vec{\mathbf{v}} \right] - [\mathcal{A}(\vec{\mathbf{u}}), \vec{\mathbf{v}}]}{\|\vec{\mathbf{v}}\|_{\mathbf{V}}} \\ &\leq C_0 \left(\|\mathbf{f}\|_{0,\Omega} + \|\partial_t \mathbf{u}\|_{\mathbf{Q}} + \|\mathbf{t}\|_{\mathbf{X}}^{r-1} \right). \end{aligned}$$

Then, squaring both sides, integrating over the interval $[0, t]$ for any $t \in (0, t_F]$, we obtain

$$\int_0^t \|\boldsymbol{\sigma}(s)\|_{\mathbf{Y}}^2 ds \leq C_0^2 \int_0^t \left(\|\mathbf{f}(s)\|_{0,\Omega}^2 + \|\partial_t \mathbf{u}(s)\|_{\mathbf{Q}}^2 + \|\mathbf{t}(s)\|_{\mathbf{X}}^{2(r-1)} \right) ds. \quad (3.41)$$

Next, in order to derive an upper bound for the second term in (3.41), we differentiate the second equation of (2.20) with respect to time and choose the test functions as $(\vec{\mathbf{v}}, \boldsymbol{\tau}) = ((\partial_t \mathbf{u}, \partial_t \mathbf{t}), \boldsymbol{\sigma})$, which implies that

$$\left[\frac{\partial}{\partial t} \mathcal{C}(\vec{\mathbf{u}}), \partial_t \vec{\mathbf{u}} \right] = [\mathcal{F}, \partial_t \vec{\mathbf{u}}] - [\mathcal{A}(\vec{\mathbf{u}}), \partial_t \vec{\mathbf{u}}].$$

This result, together with the continuity of \mathcal{A} as stated in (3.7) and the Young inequality, yields

$$\|\partial_t \mathbf{u}\|_{\mathbf{Q}}^2 \leq \frac{1}{2} \|\partial_t \mathbf{u}\|_{\mathbf{Q}}^2 + \left(\|\mathbf{f}\|_{0,\Omega}^2 + L_{\text{m}}^2 \|\mathbf{t}\|_{\mathbf{X}}^{2(r-1)} \right).$$

Integrating over the interval $[0, t]$ for any $t \in (0, t_F]$, substituting the result back into (3.41) and employing (3.40) yields

$$\int_0^t \|\boldsymbol{\sigma}(s)\|_{\mathbf{Y}}^2 ds \leq C_4 \left\{ \|\mathbf{u}_0\|_{\mathbf{Q}}^2 + \int_0^t \left(\|\mathbf{f}(s)\|_{0,\Omega}^{r/(r-1)} + \|\mathbf{f}(s)\|_{0,\Omega}^2 \right) ds \right\},$$

which, combined with the estimate in (3.40), implies (3.33). \square

4. The discrete setting

In this section, we develop an $\mathbb{H}(\mathbf{div})$ -conforming virtual element method for the approximation of the mixed problem (2.5). To this end, we begin by introducing a polygonal decomposition of the domain Ω in Section 4.1. Sections 4.2 and 4.3 are then devoted to the definition of suitable discrete spaces, projection operators, and discrete bilinear forms. Based on these ingredients, the discrete formulation of the method is presented in Section 4.5. It is emphasized that the presentation is restricted to the two-dimensional case.

4.1. Polygonal decomposition and notation

We begin by considering a sequence $\{\Omega_h\}_{h>0}$ of partitions of Ω into general polygons T , where each polygon T has a diameter denoted by h_T and a number of edges denoted by d_T , respectively. As usual, we set $h := \max_{T \in \Omega_h} h_T$, and let N_h^{ed} and N_h^{el} be the number of edges and elements, respectively, and \mathbf{n}_e^T be the unit outward normal on edge $e \subset \partial T$. Also, we denote the edges of ∂T by e , its length by $h_e := |e|$ and the set of edges e of Ω_h by Γ_h . For any $l \in \mathbb{N}$ and any mesh object $\varpi \in \Omega_h \cup \Gamma_h$, let $\mathbb{P}_l(\varpi)$, $\mathbf{P}_l(\varpi)$, $\mathbb{P}_l(\varpi)$ be the space of scalar, vectorial and matrix polynomials defined on ϖ of degree less than or equal to l ,

respectively (with the extended notation $P_{-1}(\varpi) = \{0\}$). The dimension of such spaces, for each $T \in \Omega_h$ and $e \in \Gamma_h$, are

$$\dim(P_l(T)) = \pi_l^{\text{el}} := \frac{(l+1)(l+2)}{2}, \quad \dim(\mathbf{P}_l(T)) = 2\pi_l^{\text{el}}, \quad \dim(\mathbb{P}_l(T)) = 4\pi_l^{\text{el}},$$

and

$$\dim(P_l(e)) = \pi_l^{\text{ed}} := l+1, \quad \dim(\mathbf{P}_l(e)) = 2\pi_l^{\text{ed}}, \quad \dim(\mathbb{P}_l(e)) = 4\pi_l^{\text{ed}}.$$

Also, for any $l \in N$ we introduce the broken space

$$P_\ell(\Omega_h) := \left\{ v \in L^2(\Omega) : v|_T \in P_\ell(T), \quad \forall T \in \Omega_h \right\}.$$

In addition, we suppose that $\{\Omega_h\}_h$ satisfies the following mesh-regularity assumptions:

Assumption 4.1. There exists a positive constant γ such that for any $T \in \{\Omega_h\}_h$:

- T is star-shaped with respect to every point of a disk with radius $\geq \gamma h_T$;
- every edge $e \subset \partial T$ of cell T has length $\geq \gamma h_T$.

It is worth noting that, although the above assumptions are not overly restrictive in many practical applications, they could potentially be further relaxed by combining the present analysis with the methodologies developed in Refs. 12, 18, 19.

4.2. Projection operators

In this section, we closely follow the approach presented in Section 3.1 of Ref. 37 to introduce the polynomial projection operator, which constitutes a fundamental component in the construction of the virtual element method. We begin by defining the L^2 -projection operator $\mathcal{P}_\ell^T : L^1(T) \rightarrow P_\ell(T)$, which, for any $v \in L^1(T)$, is characterized as the solution to the following variational problem:

$$\int_T (\mathcal{P}_\ell^T(v) - v) q = 0 \quad \forall q \in P_\ell(T). \quad (4.1)$$

Under Assumption 4.1, the boundedness and approximation properties of the operator \mathcal{P}_ℓ^T are established in Lemma 3.1 of Ref. 44 and can be stated as follows.

Lemma 4.1. *Let $p \geq 1$, and ℓ, s, m be integers such that $\ell \geq 0$ and $0 \leq m \leq s \leq \ell + 1$. It holds, for all $v \in W^{s,p}(T)$, and $T \in \Omega_h$:*

- Boundedness. *there exists a constant M_ℓ , depending only on ℓ and γ , such that there holds*

$$|\mathcal{P}_\ell^T(v)|_{s,p;T} \leq M_\ell |v|_{s,p;T}. \quad (4.2)$$

- Approximation. *there exists a constant C_ℓ , depending only on ℓ and γ , and hence independent of T , such that*

$$|v - \mathcal{P}_\ell^T(v)|_{m,p;T} \leq C_\ell h_T^{s-m} |v|_{s,p;T}. \quad (4.3)$$

We remark that the scaled projection operator \mathcal{P}_ℓ^T can be naturally extended to its vector- and tensor-valued counterparts, denoted by $\mathcal{P}_\ell^T : L^1(T) \rightarrow \mathbf{P}_\ell(T)$ and $\mathcal{P}_\ell^T : L^1(T) \rightarrow \mathbb{P}_\ell(T)$,

respectively. Moreover, the boundedness and approximation estimates given in (4.2) and (4.3) remain valid for both \mathcal{P}_ℓ^T and \mathcal{P}_ℓ^h .

Finally, for any element $T \in \Omega_h$ and functions $v \in L^1(\Omega)$, $\boldsymbol{\eta} \in \mathbf{L}^1(\Omega)$, $\boldsymbol{\tau} \in \mathbf{L}^1(\Omega)$, the global projection operators \mathcal{P}_ℓ^h , \mathcal{P}_ℓ^h and \mathcal{P}_ℓ^h are defined by

$$\mathcal{P}_\ell^h(v)|_T = \mathcal{P}_\ell^T(v|_T), \quad \mathcal{P}_\ell^h(\boldsymbol{\eta})|_T = \mathcal{P}_\ell^T(\boldsymbol{\eta}|_T) \quad \text{and} \quad \mathcal{P}_\ell^h(\boldsymbol{\tau})|_T = \mathcal{P}_\ell^T(\boldsymbol{\tau}|_T).$$

4.3. Discrete spaces

We now introduce the $\mathbb{H}(\mathbf{div})$ -conforming virtual element subspace as described in Ref. 44. To this end, we begin by recalling the following notational:

- $\mathbf{rot}(\boldsymbol{\tau}) := (\partial_{x_1}\tau_{12} - \partial_{x_2}\tau_{11}, \partial_{x_1}\tau_{22} - \partial_{x_2}\tau_{21})^\top$.
- $\mathcal{G}_\ell(T) := \nabla \mathbf{P}_{\ell+1}(T) \subset \mathbb{P}_\ell(T)$.
- $\mathbb{P}_\ell(T) = \mathcal{G}_\ell(T) \oplus \mathcal{G}_\ell(T)^\perp$, where $\mathcal{G}_\ell(T)^\perp$ is the L^2 -orthogonal of $\mathcal{G}_\ell(T)$ in $\mathbb{P}_\ell(T)$.

For integer $k \geq 0$, we define

$$\mathbf{Y}_k^T := \left\{ \boldsymbol{\tau} \in \mathbb{H}^\rho(\mathbf{div}; T) \cap \mathbb{H}(\mathbf{rot}; T) : \begin{array}{l} (\boldsymbol{\tau} \mathbf{n}_e^T)|_e \in \mathbf{P}_k(e) \quad \text{for each edge } e \text{ of } \partial T \\ \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{P}_k(T) \quad \text{and} \quad \mathbf{rot}(\boldsymbol{\tau}) \in \mathbf{P}_{k-1}(T) \end{array} \right\}.$$

Next, for $\boldsymbol{\tau} \in \mathbf{Y}_k^T$, we introduce the following local degrees of freedom:

- the edge moments

$$\mathbf{D1}(\boldsymbol{\tau}) := \int_e \boldsymbol{\tau} \mathbf{n}_e^T \cdot \mathbf{q} \quad \forall \mathbf{q} \in \mathbf{P}_k(e), \quad (4.4a)$$

- the element moments of the gradient

$$\mathbf{D2}(\boldsymbol{\tau}) := \int_T \boldsymbol{\tau} : \boldsymbol{\xi} \quad \forall \boldsymbol{\xi} \in \mathcal{G}_{k-1}(T), \quad (4.4b)$$

- the element moments

$$\mathbf{D3}(\boldsymbol{\tau}) := \int_T \boldsymbol{\tau} : \boldsymbol{\xi} \quad \forall \boldsymbol{\xi} \in \mathcal{G}_k(T)^\perp. \quad (4.4c)$$

As demonstrated in Ref. 22, the degrees of freedom $\mathbf{D1}$ – $\mathbf{D3}$, defined in (4.4a)–(4.4c), ensure unisolvency for every function in \mathbf{Y}_k^T . Moreover, the quantities

$$\mathcal{P}_k^T(\boldsymbol{\tau}) \quad \text{and} \quad \mathbf{div}(\boldsymbol{\tau}) \quad \forall \boldsymbol{\tau} \in \mathbf{Y}_k^T,$$

are computable using these degrees of freedom. The corresponding global virtual element subspace of \mathbf{Y} is then defined as follows:

$$\mathbf{Y}_h := \left\{ \boldsymbol{\tau} \in \mathbb{H}^\rho(\mathbf{div}; \Omega) : \boldsymbol{\tau}|_T \in \mathbf{Y}_k^T \quad \forall T \in \Omega_h \right\} \cap \mathbf{Y}.$$

To approximate the strain rate tensor and the velocity, we employ piecewise polynomial spaces defined as follows

$$\mathbf{X}_h := \mathbb{P}_k(\Omega_h) \cap \mathbf{X}, \quad \text{and} \quad \mathbf{Q}_h = \mathbf{P}_k(\Omega_h).$$

We end this section by presenting the discrete counterparts of the notations introduced in (2.19), given as follows

$$\vec{\mathbf{u}}_h := (\mathbf{u}_h, \mathbf{t}_h) \in \mathbf{V}_h := \mathbf{Q}_h \times \mathbf{X}_h, \quad \vec{\mathbf{v}}_h := (\mathbf{v}_h, \mathbf{s}_h) \in \mathbf{V}_h.$$

4.4. Interpolation estimates

To define an interpolation operator in the local space \mathbf{Y}_k^T , we denote by χ_i^T the operator associated with the i -th local degree of freedom for each element $T \in \Omega_h$, where $i = 1, \dots, n_k^T$. Owing to the definition of this space, it is readily seen that for every sufficiently smooth function $\boldsymbol{\tau} \in \mathbb{W}^{1,1}(T)$, there exists a unique operator $\boldsymbol{\Pi}_k^T(\boldsymbol{\tau}) \in \mathbf{Y}_k^T$ such that

$$\chi_i^T(\boldsymbol{\tau} - \boldsymbol{\Pi}_k^T(\boldsymbol{\tau})) = 0 \quad \forall i = 1, \dots, n_k^T. \quad (4.5)$$

Furthermore, following the arguments presented in Ref. 22 (see also Ref. 44), the interpolation operator satisfies the standard approximation estimate:

$$\|\boldsymbol{\tau} - \boldsymbol{\Pi}_k^T(\boldsymbol{\tau})\|_{0,p;T} \lesssim h_T^j |\boldsymbol{\tau}|_{j,p;T} \quad \forall \boldsymbol{\tau} \in \mathbb{W}^{j,p}(T), \quad (4.6)$$

where $j \in [1, k+1]$ and $p \in (1, \infty)$.

We also recall the following commutativity property

$$\operatorname{div}(\boldsymbol{\Pi}_k^T(\boldsymbol{\tau})) = \mathcal{P}_k^T(\operatorname{div}(\boldsymbol{\tau})) \quad \forall \boldsymbol{\tau} \in \mathbb{W}^{1,1}(T). \quad (4.7)$$

Now, applying Lemma 4.1 (cf. (4.3)) in conjunction with the commutativity property (4.7), we deduce that for each $\boldsymbol{\tau} \in \mathbb{W}^{1,1}(T)$ such that $\operatorname{div}(\boldsymbol{\tau}) \in \mathbf{H}^j(T)$ with $j \in [0, k+1]$, the following estimate holds (see, e.g., Eq. (3.14) in Ref. 44):

$$\|\operatorname{div}(\boldsymbol{\tau} - \boldsymbol{\Pi}_k^T(\boldsymbol{\tau}))\|_{0,T} \lesssim h_T^j |\operatorname{div}(\boldsymbol{\tau})|_{j,T}. \quad (4.8)$$

As a consequence of the local approximation properties given in (4.6), (4.8), and Lemma 4.1, we readily obtain the following global estimates for each integer $j \in [1, k+1]$:

($\mathbf{AP}_h^{\mathbf{t}}$) for any $\mathbf{t} \in \mathbf{X} \cap \mathbb{W}^{j,r}(\Omega)$ there hold

$$\|\mathbf{t} - \mathcal{P}_k^h(\mathbf{t})\|_{\mathbf{X}} \lesssim h^j |\mathbf{t}|_{j,r;\Omega},$$

($\mathbf{AP}_h^{\boldsymbol{\sigma}}$) for any $\boldsymbol{\tau} \in \mathbf{Y} \cap \mathbb{W}^{j,\rho}(\Omega)$ such that $\operatorname{div}(\boldsymbol{\tau}) \in \mathbf{H}^j(\Omega)$, there hold

$$\|\boldsymbol{\tau} - \boldsymbol{\Pi}_k^h(\boldsymbol{\tau})\|_{\mathbf{Y}} \lesssim h^j \left\{ |\boldsymbol{\tau}|_{j,\rho;\Omega} + |\operatorname{div}(\boldsymbol{\tau})|_{j,\Omega} \right\},$$

($\mathbf{AP}_h^{\mathbf{u}}$) for any $\mathbf{v} \in \mathbf{H}^j(\Omega)$ there hold

$$\|\mathbf{v} - \mathcal{P}_k^h(\mathbf{v})\|_{\mathbf{Q}} \lesssim h^j |\mathbf{v}|_{j,\Omega}.$$

4.5. The fully-discrete scheme

To formulate the discrete scheme for problem (2.16), we begin by introducing computable discrete counterparts of the operators associated with the virtual element space, as required. In particular, based on the definitions of the discrete spaces, the linear and nonlinear operators (cf.(2.17)), and the associated functionals (cf.(2.18)), it is evident that only the discrete version of the inf-sup operator needs to be explicitly defined. This term is discretized via the operator $\mathcal{B}_h^T : \mathbf{V}_h \rightarrow \mathbf{Y}_h^T$ defined as follows:

$$[\mathcal{B}_h^T(\vec{\mathbf{v}}_h), \boldsymbol{\tau}_h] := - \int_T \operatorname{div}(\boldsymbol{\tau}_h) \cdot \mathbf{v}_h - \int_T \mathcal{P}_k^T(\boldsymbol{\tau}_h) : \mathbf{s}_h \quad \forall (\vec{\mathbf{v}}_h, \boldsymbol{\tau}_h) \in \mathbf{V}_h \times \mathbf{Y}_h. \quad (4.9)$$

In addition, as is customary, the global operator is defined by

$$[\mathcal{B}_h(\mathbf{s}_h), \boldsymbol{\tau}_h] := \sum_{T \in \Omega_h} [\mathcal{B}_h^T(\mathbf{s}_h), \boldsymbol{\tau}_h].$$

Finally, by discretizing in time using the backward Euler method, which include introducing a sequence of time steps $t_n = n\Delta t$, $n = 1, \dots, N$ with constant step-size $\Delta t = t_F/N$ and denoting $f^n := f(\cdot, t_n)$, $\delta_t f^n := (f^n - f^{n-1})/\Delta t$ for a generic function f , combined to mixed VEM with considering the above discrete form, we construct the following fully-discrete mixed VE scheme: Find $(\tilde{\mathbf{u}}_h^n, \boldsymbol{\sigma}_h^n) \in \mathbf{V}_h \times \mathbf{Y}_h$, for each $n = 1, \dots, N$, such that

$$[\delta_t C(\tilde{\mathbf{u}}_h^n), \tilde{\mathbf{v}}_h] + [\mathcal{A}(\tilde{\mathbf{u}}_h^n), \tilde{\mathbf{v}}_h] + [\mathcal{B}_h(\tilde{\mathbf{v}}_h), \boldsymbol{\sigma}_h^n] = [\mathcal{F}^n, \tilde{\mathbf{v}}_h] \quad \forall \tilde{\mathbf{v}}_h \in \mathbf{V}_h, \quad (4.10a)$$

$$[\mathcal{B}_h(\tilde{\mathbf{u}}_h^n), \boldsymbol{\tau}_h] = 0 \quad \forall \boldsymbol{\tau}_h \in \mathbf{Y}_h. \quad (4.10b)$$

The initial condition is prescribed by selecting $(\tilde{\mathbf{u}}_h^0, \boldsymbol{\sigma}_h^0) := ((\mathbf{u}_h^0, \mathbf{t}_h^0), \boldsymbol{\sigma}_h^0) \in \mathbf{V}_h \times \mathbf{Y}_h$ as a suitable approximation of $(\tilde{\mathbf{u}}_0, \boldsymbol{\sigma}_0)$, satisfying the following condition:

$$\begin{aligned} [\mathcal{A}(\tilde{\mathbf{u}}_h^0), \tilde{\mathbf{v}}_h] + [\mathcal{B}_h(\tilde{\mathbf{v}}_h), \boldsymbol{\sigma}_h^0] &= [F^0, \tilde{\mathbf{v}}_h] & \forall \tilde{\mathbf{v}}_h \in \mathbf{V}_h, \\ [\mathcal{B}_h(\tilde{\mathbf{u}}_h^0), \boldsymbol{\tau}_h] &= 0 & \forall \boldsymbol{\tau}_h \in \mathbf{Y}_h, \end{aligned} \quad (4.11)$$

with $F^0 \in \mathbf{L}^2(\Omega) \times \{\mathbf{0}\}$ defined in (3.30). This choice is made to guarantee that the discrete initial datum is compatible with Lemma 3.6, thereby enabling the application of Theorem 3.1. It is worth noting that the well-posedness of problem (4.11) can be established by arguments analogous to those used in the proof of Lemma 3.5.

5. Discrete solvability analysis

In this section, following the approach of Section 3, we establish the well-posedness of the fully discrete scheme (4.10) by utilizing the discrete analogues of Theorems 3.1 and 3.2. To this end, since \mathcal{B}_h is the sole discrete operator appearing in (4.10), we begin by analyzing its stability properties.

5.1. Discrete inf-sup condition

In this context, we concentrate on establishing the discrete inf-sup condition for the operator \mathcal{B}_h . To this end, we first recall the abstract results presented in Lemmas 5.1 and 5.2 of Ref. 31, which will serve as a fundamental tool for this purpose.

Lemma 5.1. *Let U, V, V_1, V_2 and W be reflexive Banach spaces with V_1 and V_2 being closed subspaces of V such that $V = V_1 \oplus V_2$, and assume that the norm of V can be redefined, equivalently, but with constants independent of V_1 and V_2 , as $\|v\| := \|v_1\| + \|v_2\|$ for any $v = v_1 + v_2 \in V$, with $v_i \in V_i$ for $i \in \{1, 2\}$. In addition, let $\mathcal{B} \in \mathcal{L}(U \times V, W')$ be a linear operator, and define the following subspaces:*

$$\begin{aligned} Z &:= \left\{ (u, v) \in U \times V : [\mathcal{B}(u, v), w] = 0 \quad \forall w \in W \right\}, \quad \text{and} \\ W_0 &:= \left\{ w \in W : [\mathcal{B}(u, v_2), w] = 0 \quad \forall (u, v_2) \in U \times V_2 \right\}. \end{aligned} \quad (5.1)$$

The following statements are equivalent

(i) there exists positive constants β_1, β_2 such that

$$\sup_{\mathbf{0} \neq (u, v) \in U \times V} \frac{[\mathcal{B}(u, v), w]}{\|(u, v)\|} \geq \beta_1 \|w\| \quad \forall w \in W,$$

and

$$\|v_1\| \geq \beta_2 \|(u, v_2)\| \quad \forall (u, v) \in Z.$$

(ii) there exist positive constants β_3, β_4 such that

$$\begin{aligned} \sup_{0 \neq w \in W} \frac{[\mathcal{B}(u, v_2), w]}{\|w\|} &\geq \beta_3 \|(u, v_2)\| \quad \forall (u, v_2) \in U \times V_2, \quad \text{and} \\ \sup_{0 \neq v_1 \in V_1} \frac{[\mathcal{B}(0, v_1), w]}{\|v_1\|} &\geq \beta_4 \|w\| \quad \forall w \in W_0. \end{aligned} \quad (5.2)$$

The following lemma states sufficient conditions for the inf-sup condition of the operator $\mathcal{B} \in \mathcal{L}(U \times V_2, W')$ (cf. the first row of (5.2)).

Lemma 5.2. *In addition to the hypotheses and notations of Lemma 5.1, we introduce the subspace*

$$W_1 := \left\{ w \in W : [\mathcal{B}(u, 0), w] = 0 \quad \forall u \in U \right\},$$

and assume that there exist positive constants β_5, β_6 such that

$$\begin{aligned} \sup_{0 \neq w \in W} \frac{[\mathcal{B}(u, 0), w]}{\|w\|} &\geq \beta_5 \|u\| \quad \forall u \in U, \quad \text{and} \\ \sup_{0 \neq w \in W_1} \frac{[\mathcal{B}(0, v_2), w]}{\|w\|} &\geq \beta_6 \|v_2\| \quad \forall v_2 \in V_2. \end{aligned} \quad (5.3)$$

Then, there holds the inf-sup condition given in the first row of (5.2).

We are now in a position to apply the two preceding results to establish the discrete inf-sup condition for the operator \mathcal{B}_h . To this end, we begin by identifying the spaces U , V , and W in Lemmas 5.1 and 5.2 with

$$U := \mathbf{Q}_h, \quad V := \mathbf{X}_h, \quad \text{and} \quad W := \mathbf{Y}_h, \quad (5.4)$$

and accordingly define

$$\mathbf{X}_{h, \text{sym}} := \left\{ \mathbf{s} \in \mathbf{X}_h : \mathbf{s}^T - \mathbf{s} = \mathbf{0} \right\} \quad \text{and} \quad \mathbf{X}_{h, \text{skw}} := \left\{ \mathbf{s} \in \mathbf{X}_h : \mathbf{s}^T + \mathbf{s} = \mathbf{0} \right\}.$$

We then decompose the space \mathbf{X}_h as $\mathbf{X}_h = \mathbf{X}_{h, \text{sym}} \oplus \mathbf{X}_{h, \text{skw}}$, and observe from the orthogonality of $\mathbf{X}_{h, \text{sym}}$ and $\mathbf{X}_{h, \text{skw}}$ that for any $\mathbf{s} = \mathbf{s}_{\text{sym}} + \mathbf{s}_{\text{skw}} \in \mathbf{X}_h$, the following identity holds

$$\|\mathbf{s}\|_{\mathbf{X}} = \|\mathbf{s}_{\text{sym}}\|_{\mathbf{X}} + \|\mathbf{s}_{\text{skw}}\|_{\mathbf{X}}.$$

From this, we deduce that the space V , along with the spaces U and W defined in (5.4), satisfies the hypotheses of Lemmas 5.1 and 5.2. Accordingly, by considering the relevant product spaces and introducing the notations

$$\begin{aligned} \vec{\mathbf{v}}_1 &:= (\mathbf{v}, \mathbf{s}_{\text{skw}}) \in \mathbf{V}_{1, h} := \mathbf{Q}_h \times \mathbf{X}_{h, \text{skw}} \quad \text{and} \\ \vec{\mathbf{v}}_2 &:= (\mathbf{0}, \mathbf{s}_{\text{sym}}) \in \mathbf{V}_{2, h} := \{\mathbf{0}\} \times \mathbf{X}_{h, \text{sym}}, \end{aligned} \quad (5.5)$$

in accordance with Lemma 5.1, the discrete inf-sup condition for $\mathcal{B}_h \in \mathcal{L}(\mathbf{V}_h, \mathbf{Y}'_h)$ can be established by verifying the discrete inf-sup conditions for the operators $\mathcal{B}_h \in \mathcal{L}(\mathbf{V}_{1, h}, \mathbf{Y}'_h)$ and $\mathcal{B}_h \in \mathcal{L}(\mathbf{V}_{2, h}, W'_0)$, as indicated in (5.2). Here, W_0 denotes the kernel of the adjoint

operator associated with $\mathcal{B}_h \in \mathcal{L}(\mathbf{V}_{1,h}, \mathbf{Y}'_h)$, and, using (5.1) and (5.4)–(5.5), it is redefined as follows:

$$\begin{aligned} W_0 &:= \left\{ \boldsymbol{\tau} \in \mathbf{Y}_h : [\mathcal{B}_h(\vec{\mathbf{v}}_1), \boldsymbol{\tau}] = 0 \quad \forall \vec{\mathbf{v}}_1 \in \mathbf{V}_{1,h} \right\} \\ &= \left\{ \boldsymbol{\tau} \in \mathbf{Y}_h : \int_{\Omega_h} \mathcal{P}_k^T(\boldsymbol{\tau}) : \mathbf{s}_{\text{skw}} = 0 \quad \text{and} \quad \int_{\Omega_h} \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\tau}) = 0 \quad \forall (\mathbf{v}, \mathbf{s}_{\text{skw}}) \in \mathbf{V}_{1,h} \right\}. \end{aligned} \quad (5.6)$$

On the other hand, applying Lemma 5.2 with the notation specified by

$$\vec{\mathbf{v}}_3 := (\mathbf{v}, \mathbf{0}) \in \mathbf{V}_{3,h} := \mathbf{Q}_h \times \{\mathbf{0}\} \quad \text{and} \quad \vec{\mathbf{v}}_4 := (\mathbf{0}, \mathbf{s}_{\text{skw}}) \in \mathbf{V}_{4,h} := \{\mathbf{0}\} \times \mathbf{X}_{h,\text{skw}}, \quad (5.7)$$

we deduce that establishing the discrete inf-sup condition for $\mathcal{B}_h \in \mathcal{L}(\mathbf{V}_{1,h}, \mathbf{Y}'_h)$ reduces to verifying the inf-sup conditions for the operators $\mathcal{B}_h \in \mathcal{L}(\mathbf{V}_{3,h}, \mathbf{Y}'_h)$ and $\mathcal{B}_h \in \mathcal{L}(\mathbf{V}_{1,h}, \mathbf{W}'_1)$, as stated in (5.3), where \mathbf{W}_1 denotes the corresponding kernel associated with the adjoint operator $\mathcal{B}_h \in \mathcal{L}(\mathbf{V}_{3,h}, \mathbf{Y}'_h)$, that is

$$W_1 := \left\{ \boldsymbol{\tau} \in \mathbf{Y}_h : [\mathcal{B}_h(\vec{\mathbf{v}}_3), \boldsymbol{\tau}] = 0 \quad \forall \vec{\mathbf{v}}_3 = (\mathbf{v}, \mathbf{0}) \in \mathbf{V}_{3,h} \right\}.$$

In view of (5.4) and the notations introduced in (5.7), the space W_1 can be redefined as follows:

$$W_1 = \left\{ \boldsymbol{\tau} \in \mathbf{Y}_h : \int_{\Omega_h} \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\tau}) = 0 \quad \forall \mathbf{v} \in \mathbf{Q} \right\}. \quad (5.8)$$

We observe that, due to the inclusion $\mathbf{div}(\mathbf{Y}_h) \subseteq \mathbf{V}_h$, it follows directly from (5.6) and (5.8) that

$$\begin{aligned} W_0 &= \left\{ \boldsymbol{\tau} \in \mathbf{Y}_h : \int_{\Omega_h} \mathcal{P}_k^T(\boldsymbol{\tau}) : \mathbf{s}_{\text{skw}} = 0 \quad \forall \mathbf{s}_{\text{skw}} \in \mathbf{X}_{h,\text{skw}} \quad \text{and} \quad \mathbf{div}(\boldsymbol{\tau}) = \mathbf{0} \quad \text{in} \quad \Omega \right\}, \\ W_1 &= \left\{ \boldsymbol{\tau} \in \mathbf{Y}_h : \mathbf{div}(\boldsymbol{\tau}) = \mathbf{0} \quad \text{in} \quad \Omega \right\}. \end{aligned} \quad (5.9)$$

In summary, to establish the discrete inf-sup condition for $\mathcal{B}_h \in \mathcal{L}(\mathbf{V}_h, \mathbf{Y}'_h)$, it suffices to verify the following three conditions: $\mathcal{B}_h \in \mathcal{L}(\mathbf{V}_{3,h}, \mathbf{Y}'_h)$, $\mathcal{B}_h \in \mathcal{L}(\mathbf{V}_{2,h}, \mathbf{W}'_0)$, and $\mathcal{B}_h \in \mathcal{L}(\mathbf{V}_{4,h}, \mathbf{W}'_1)$. We begin with establishing the discrete inf-sup condition for the operator $\mathcal{B}_h : \mathbf{V}_{3,h} \rightarrow \mathbf{Y}'_h$.

Lemma 5.3. *There exists a positive constant $\beta_{5,d}$, independent of h , such that*

$$\sup_{\boldsymbol{\tau}_h \in \mathbf{Y}_h} \frac{[\mathcal{B}_h(\mathbf{v}_h, \mathbf{0}), \boldsymbol{\tau}_h]}{\|\boldsymbol{\tau}_h\|_{\mathbf{Y}}} \geq \beta_{5,h} \|\mathbf{v}_h\|_{\mathbf{Q}} \quad \forall \vec{\mathbf{v}}_{3,h} = (\mathbf{v}_h, \mathbf{0}) \in \mathbf{V}_{3,h}. \quad (5.10)$$

Proof. In light of (3.21), it suffices to demonstrate the existence of a Fortin operator. Specifically, it is required to construct an element $\boldsymbol{\tau}_h \in \mathbf{Y}_h$ such that

$$[\mathcal{B}_h(\mathbf{v}_h, \mathbf{0}), \boldsymbol{\tau}_h] = [\mathcal{B}(\mathbf{v}_h, \mathbf{0}), \boldsymbol{\tau}] \quad \forall \mathbf{v}_h \in \mathbf{Q}_h,$$

with

$$\|\boldsymbol{\tau}_h\|_{\mathbf{Y}} \leq C \|\boldsymbol{\tau}\|_{\mathbf{Y}},$$

for some constant $C > 0$, independent of h . To do that, we proceed similarly to the proof of Lemma 3.9 in Ref. 2. Let $\mathcal{D} \subset \mathbb{R}^2$ be a bounded convex Lipschitz domain such that $\Omega \subset \mathcal{D}$,

$\Gamma_N \subset \partial\mathcal{D}$ and assume that such a convex extension exists. Given $\boldsymbol{\tau} \in \mathbf{Y} = \mathbb{H}_{0,\Gamma_N}^\rho(\mathbf{div}; \Omega)$, we then set

$$\mathbf{h}_\tau := \begin{cases} \mathbf{div}(\boldsymbol{\tau}) & \text{in } \Omega, \\ -\frac{1}{|\mathcal{D} \setminus \bar{\Omega}|} \int_{\Omega} \mathbf{div}(\boldsymbol{\tau}) & \text{in } \mathcal{D} \setminus \Omega, \end{cases}$$

Since $\mathbf{h}_\tau \in \mathbf{L}^2(\Omega)$, a well-known result on regularity of elliptic problems (see, e.g., Ref. 51, which requires that Γ_N is convex) implies that there exists a unique weak solution $\mathbf{w} \in \mathbf{H}^2(\mathcal{D})$ of the boundary value problem

$$\Delta \mathbf{w} = \mathbf{h}_\tau \quad \text{in } \mathcal{D}, \quad \nabla \mathbf{w} \boldsymbol{\nu} = \mathbf{0} \quad \text{on } \partial\mathcal{D}, \quad \int_{\Omega} \mathbf{w} = 0,$$

that satisfies

$$\|\mathbf{w}\|_{2,\mathcal{D}} \leq C_1 \|\mathbf{h}_\tau\|_{0,\mathcal{D}} = C_1 \|\mathbf{div}(\boldsymbol{\tau})\|_{0,\Omega},$$

where $C_1 > 0$ is independent of \mathbf{w} .

Then we let $\widehat{\boldsymbol{\tau}} = (\nabla \mathbf{w})_\Omega \in \mathbb{H}^1(\Omega)$, and observe that $\mathbf{div}(\widehat{\boldsymbol{\tau}}) = \mathbf{div}(\boldsymbol{\tau})$ in Ω , $\widehat{\boldsymbol{\tau}} \boldsymbol{\nu} = \mathbf{0}$ on Γ_N (since $\Gamma_N \subseteq \partial\mathcal{D}$), and

$$\|\widehat{\boldsymbol{\tau}}\|_{1,\Omega} \leq \|\mathbf{w}\|_{2,\Omega} \leq \|\mathbf{w}\|_{2,\mathcal{D}} \leq C_1 \|\mathbf{div}(\boldsymbol{\tau})\|_{0,\Omega}. \quad (5.11)$$

We can now define the Fortin operator $\boldsymbol{\Pi}^{\mathcal{F}} : \mathbb{H}^1(\Omega) \rightarrow \mathbf{Y}_h$ as

$$\boldsymbol{\Pi}^{\mathcal{F}}(\boldsymbol{\tau}) := \boldsymbol{\Pi}_k^h(\widehat{\boldsymbol{\tau}}).$$

In turn, by applying the triangle inequality and using the continuous embedding $\mathbf{i}_\rho : \mathbb{H}^1(\Omega) \rightarrow \mathbb{L}^\rho(\Omega)$, it follows that

$$\begin{aligned} \|\boldsymbol{\Pi}^{\mathcal{F}}(\boldsymbol{\tau})\|_{0,\rho;\Omega} &\leq \|\boldsymbol{\Pi}_k^h(\widehat{\boldsymbol{\tau}})\|_{0,\rho;\Omega} \leq \|\widehat{\boldsymbol{\tau}}\|_{0,\rho;\Omega} + \|\boldsymbol{\Pi}_k^h(\widehat{\boldsymbol{\tau}}) - \widehat{\boldsymbol{\tau}}\|_{0,\rho;\Omega} \\ &\leq \|\mathbf{i}_\rho\| \|\widehat{\boldsymbol{\tau}}\|_{1,\Omega} + \|\boldsymbol{\Pi}_k^h(\widehat{\boldsymbol{\tau}}) - \widehat{\boldsymbol{\tau}}\|_{0,\rho;\Omega}. \end{aligned} \quad (5.12)$$

Moreover, for each $T \in \Omega_h$, the use of the inverse Lebesgue embedding together with the interpolation estimate (cf. Eq. (4.6) with $p = 2$ and $j = 1$) yields

$$\|\boldsymbol{\Pi}_k^h(\widehat{\boldsymbol{\tau}}) - \widehat{\boldsymbol{\tau}}\|_{0,\rho;T} \leq C h_T^{\frac{1}{2} - \frac{1}{\rho}} \|\widehat{\boldsymbol{\tau}}\|_{1,T},$$

from which, summing over all $T \in \Omega_h$, and substituting the resulting bound into (5.12), along with (5.11), leads to

$$\|\boldsymbol{\Pi}^{\mathcal{F}}(\boldsymbol{\tau})\|_{0,\rho;\Omega} \leq C_1 \left(\|\mathbf{i}_\rho\| + C |\Omega|^{\frac{1}{2} - \frac{1}{\rho}} \right) \|\mathbf{div}(\boldsymbol{\tau})\|_{0,\Omega} := \widehat{C} \|\mathbf{div}(\boldsymbol{\tau})\|_{0,\Omega}. \quad (5.13)$$

On the other hand, applying the identities (4.7) and (5.11) yields

$$\mathbf{div}(\boldsymbol{\Pi}^{\mathcal{F}}(\boldsymbol{\tau})) = \mathbf{div}(\boldsymbol{\Pi}_k^h(\widehat{\boldsymbol{\tau}})) = \mathcal{P}_k^h(\mathbf{div}(\widehat{\boldsymbol{\tau}})) = \mathcal{P}_k^h(\mathbf{div}(\boldsymbol{\tau})), \quad (5.14)$$

which thanks to (4.2) also gives

$$\|\mathbf{div}(\boldsymbol{\Pi}^{\mathcal{F}}(\boldsymbol{\tau}))\|_{0,\Omega} \leq M_k \|\mathbf{div}(\boldsymbol{\tau})\|_{0,\Omega}. \quad (5.15)$$

Now, squaring both sides of (5.13) and (5.15), and subsequently adding the resulting inequalities, we obtain

$$\|\boldsymbol{\Pi}^{\mathcal{F}}(\boldsymbol{\tau})\|_{\mathbf{Y}} \leq \left(\widehat{C} + M_k \right) \|\boldsymbol{\tau}\|_{\mathbf{Y}}. \quad (5.16)$$

Finally, combining (5.14) and (5.16), we deduce

$$\begin{aligned} \sup_{\boldsymbol{\tau}_h \in \mathbf{Y}_h} \frac{[\mathcal{B}_h(\mathbf{v}_h, \mathbf{0}), \boldsymbol{\tau}_h]}{\|\boldsymbol{\tau}_h\|_{\mathbf{Y}}} &\geq \sup_{\boldsymbol{\tau} \in \mathbf{Y}} \frac{[\mathcal{B}_h(\mathbf{v}_h, \mathbf{0}), \boldsymbol{\Pi}^{\mathcal{F}}(\boldsymbol{\tau})]}{\|\boldsymbol{\Pi}^{\mathcal{F}}(\boldsymbol{\tau})\|_{\mathbf{Y}}} \\ &\geq \sup_{\boldsymbol{\tau} \in \mathbf{Y}} \frac{1}{\widehat{C} + M_k} \frac{\int_{\Omega} \boldsymbol{\mathcal{P}}_k^h(\mathbf{div}(\boldsymbol{\tau})) \cdot \mathbf{v}_h}{\|\boldsymbol{\tau}\|_{\mathbf{Y}}} = \sup_{\boldsymbol{\tau} \in \mathbf{Y}} \frac{1}{\widehat{C} + M_k} \frac{[\mathcal{B}(\mathbf{v}_h, \mathbf{0}), \boldsymbol{\tau}]}{\|\boldsymbol{\tau}\|_{\mathbf{Y}}}, \end{aligned}$$

which, by a modification of Lemma 3.6 in Ref. 26, gives (5.10) with $\beta_{5,d} := ((\widehat{C} + M_k)(1 + \|\mathbf{i}_\rho\|_{C_{reg}}))^{-1}$. \square

We are now in a position to establish the next result, namely, the discrete inf-sup condition for $\mathcal{B}_h \in \mathcal{L}(\mathbf{V}_{2,h}, W_0')$, where the spaces $\mathbf{V}_{2,h}$ and W_0 are defined in the first and second rows of (5.5) and (5.9), respectively.

Lemma 5.4. *There exists a positive constant $\beta_{4,d}$, independent of h , such that*

$$\sup_{0 \neq \mathbf{s}_{h,\text{sym}} \in \mathbf{X}_{h,\text{sym}}} \frac{[\mathcal{B}_h(\mathbf{0}, \mathbf{s}_{h,\text{sym}}), \boldsymbol{\tau}_h]}{\|\mathbf{s}_{h,\text{sym}}\|_{\mathbf{X}}} \geq \beta_{4,d} \|\boldsymbol{\tau}_h\|_{\mathbf{Y}} \quad \forall \boldsymbol{\tau}_h \in W_0. \quad (5.17)$$

Proof. Fix $T \in \Omega_h$ and let $\boldsymbol{\tau}_h|_T \in \mathbf{Y}_k^T$ satisfy $\mathbf{div}(\boldsymbol{\tau}_h) = \mathbf{0}$ in T (cf. W_1 in (5.9)). Since $\boldsymbol{\tau}_h \in \mathbf{Y}_h$, by definition of the local space we also have $\mathbf{rot}(\boldsymbol{\tau}_h) \in \mathbf{P}_{k-1}(T)$ and $(\boldsymbol{\tau}_h \boldsymbol{\nu})|_e \in [\mathbf{P}_k(e)]^2$ for every edge $e \subset \partial T$. We use the polynomial decomposition (cf. the third proposition of Eq. (2.10) in Ref. 10), i.e., $[\mathbf{P}_k(T)]^2 = \mathcal{G}_k \oplus \mathcal{G}_k^\perp$, $\mathcal{G}_k := \nabla \mathbf{P}_{k+1}(T)$, and the fact that $\mathbf{rot} : \mathcal{G}_k^\perp \rightarrow \mathbf{P}_{k-1}(T)$ is an isomorphism. Hence, there exists a unique polynomial field $\mathbf{q}_T \in \mathcal{G}_k^\perp$ such that

$$\mathbf{rot}(\mathbf{q}_T) = \mathbf{rot}(\boldsymbol{\tau}_h) \quad \text{in } T. \quad (5.18)$$

Now we *complete* \mathbf{q}_T by a gradient term in order to match the normal trace. Namely, we seek $\chi_T \in \mathbf{P}_{k+1}(T)$ such that, setting

$$\mathbf{p}_T := \mathbf{q}_T + \nabla \chi_T \in [\mathbf{P}_k(T)]^2,$$

we have

$$(\mathbf{p}_T \boldsymbol{\nu}) = (\boldsymbol{\tau}_h \boldsymbol{\nu}) \quad \text{on } \partial T. \quad (5.19)$$

Since adding a gradient does not change the rotation, (5.18) implies

$$\mathbf{rot}(\mathbf{p}_T) = \mathbf{rot}(\boldsymbol{\tau}_h) \quad \text{in } T.$$

Moreover, because $\mathbf{div}(\boldsymbol{\tau}_h) = \mathbf{0}$ in T , the boundary data in (5.19) satisfy the compatibility condition

$$\int_{\partial T} (\boldsymbol{\tau}_h \boldsymbol{\nu}) = \int_T \mathbf{div}(\boldsymbol{\tau}_h) = 0,$$

so that the Neumann problem determining χ_T is solvable (up to an additive constant), and therefore \mathbf{p}_T is well-defined (with, e.g., the normalization $\int_T \chi_T = 0$). Define $\boldsymbol{\delta} := \boldsymbol{\tau}_h - \mathbf{p}_T$. Then, by construction,

$$\mathbf{rot}(\boldsymbol{\delta}) = \mathbf{0} \text{ in } T, \quad (\boldsymbol{\delta} \boldsymbol{\nu}) = \mathbf{0} \text{ on } \partial T, \quad \mathbf{div}(\boldsymbol{\delta}) = \mathbf{div}(\boldsymbol{\tau}_h) - \mathbf{div}(\mathbf{p}_T) = -\Delta \chi_T \text{ in } T.$$

On the other hand, since $\mathbf{rot}(\boldsymbol{\delta}) = \mathbf{0}$ in T and $(\boldsymbol{\delta}\boldsymbol{\nu}) = \mathbf{0}$ on ∂T , a standard potential argument on simply connected T yields $\boldsymbol{\delta} = \nabla\psi$ for some $\psi \in H^1(T)$ with $\partial_\nu\psi = 0$ on ∂T . Taking the divergence gives $\Delta\psi = \mathbf{div}(\boldsymbol{\delta}) = -\Delta\chi_T$ in T , hence $\Delta(\psi + \chi_T) = 0$ in T with homogeneous Neumann boundary condition on ∂T , which implies that $\psi + \chi_T$ is constant. Using the normalizations (e.g. $\int_T \chi_T = 0$ and $\int_T \psi = 0$) we obtain $\psi = -\chi_T$ and therefore $\boldsymbol{\delta} = \nabla\psi = -\nabla\chi_T$. But then, by (5.19), $(\boldsymbol{\delta}\boldsymbol{\nu}) = \mathbf{0}$ implies $(\nabla\chi_T)\boldsymbol{\nu} = \mathbf{0}$ on ∂T , and the Neumann problem for χ_T yields χ_T constant, hence $\nabla\chi_T = \mathbf{0}$ and finally $\boldsymbol{\delta} \equiv \mathbf{0}$ in T .

Consequently, $\boldsymbol{\tau}_h|_T = \mathbf{p}_T \in [\mathbf{P}_k(T)]^2$. Since T is arbitrary, the conclusion holds element-wise: $\boldsymbol{\tau}_h|_T \in [\mathbf{P}_k(T)]^2$ for all $T \in \Omega_h$. This result, combined with the first identity in (5.9), namely $\int_{\Omega_h} \mathbf{dev}(\boldsymbol{\tau}_h) : \mathbf{s}_{h,\text{skw}} = 0$, enables us to define $\widetilde{\mathbf{s}}_{h,\text{sym}} := |\mathbf{dev}(\boldsymbol{\tau}_h)|^{\rho-2} \mathbf{dev}(\boldsymbol{\tau}_h)$. It is straightforward to verify that

$$\|\widetilde{\mathbf{s}}_{h,\text{sym}}\|_{\mathbf{X}}^r = \|\mathbf{dev}(\boldsymbol{\tau}_h)\|_{0,\rho;\Omega}^\rho, \quad \text{tr}(\widetilde{\mathbf{s}}) = |\mathbf{dev}(\boldsymbol{\tau}_h)|^{\rho-2} \text{tr}(\mathbf{dev}(\boldsymbol{\tau}_h)) = 0 \quad \text{and} \quad (\widetilde{\mathbf{s}}_{h,\text{sym}})^T = \widetilde{\mathbf{s}}_{h,\text{sym}},$$

which implies that $\widetilde{\mathbf{s}}_{h,\text{sym}} \in \mathbf{X}_{h,\text{sym}}$, and furthermore, the following relation holds:

$$\int_{\Omega} \boldsymbol{\tau}_h : \widetilde{\mathbf{s}}_{h,\text{sym}} = \|\mathbf{dev}(\boldsymbol{\tau}_h)\|_{0,\rho;\Omega}^\rho = \|\mathbf{dev}(\boldsymbol{\tau}_h)\|_{0,\rho;\Omega} \|\widetilde{\mathbf{s}}_{h,\text{sym}}\|_{\mathbf{X}}. \quad (5.20)$$

By bounding the supremum in (5.17) from below using $\widetilde{\mathbf{s}}_{h,\text{sym}}$ and applying (5.20), we deduce

$$\sup_{\mathbf{0} \neq \mathbf{s}_{h,\text{sym}} \in \mathbf{X}_{h,\text{sym}}} \frac{[\mathcal{B}_h(\mathbf{0}, \mathbf{s}_{h,\text{sym}}), \boldsymbol{\tau}_h]}{\|\mathbf{s}_{h,\text{sym}}\|_{\mathbf{X}}} \geq \frac{\int_{\Omega} \widetilde{\mathbf{s}}_{h,\text{sym}} : \boldsymbol{\tau}_h}{\|\widetilde{\mathbf{s}}_{h,\text{sym}}\|_{\mathbf{X}}} = \|\mathbf{dev}(\boldsymbol{\tau}_h)\|_{0,\rho;\Omega},$$

from which, noting that $\mathbf{div}(\boldsymbol{\tau}_h) = \mathbf{0}$, it follows that (5.17) holds with constant $\beta_{4,d} = \widehat{C}_\rho$. \square

The following result, which constitutes a slight modification of the proof of Lemma 5.6 in Ref. 45, establishes the discrete inf-sup condition for the operator $\mathcal{B}_h \in \mathcal{L}(\mathbf{V}_{4,h}, \mathbf{W}'_1)$, where the spaces $\mathbf{V}_{4,h}$ and \mathbf{W}_1 are defined in (5.7) and (5.9), respectively.

Lemma 5.5. *There exists a constant $\beta_{3,d} > 0$, independent of h , such that*

$$\sup_{\boldsymbol{\tau}_h \in \mathbf{W}_1} \frac{[\mathcal{B}_h(\mathbf{0}, \mathbf{s}_{h,\text{skw}}), \boldsymbol{\tau}_h]}{\|\boldsymbol{\tau}_h\|_{\mathbf{Y}}} \geq \beta_{3,d} \|\mathbf{s}_{h,\text{skw}}\|_{\mathbf{X}} \quad \forall \vec{\mathbf{v}}_{4,h} = (\mathbf{0}, \mathbf{s}_{h,\text{skw}}) \in \mathbf{V}_{4,h}.$$

Proof. It reduces to a minor adaptation of the proof presented in Lemma 5.6 of Ref. 45. \square

We are now in a position to state and prove the main result of this section. Specifically, we present the following lemma.

Lemma 5.6. *Let \mathbf{K}_h be the kernel of \mathcal{B}_h , that is,*

$$\mathbf{K}_h := \left\{ \vec{\mathbf{v}}_h \in \mathbf{V}_h : [\mathcal{B}_h(\vec{\mathbf{v}}_h), \boldsymbol{\tau}_h] = 0 \quad \forall \boldsymbol{\tau}_h \in \mathbf{Y}_h \right\}.$$

Then there exist positive constants $\beta_{1,d}, \beta_{2,d}$, independent of h , satisfying

$$\sup_{\mathbf{0} \neq \vec{\mathbf{v}}_h \in \mathbf{V}_h} \frac{[\mathcal{B}_h(\vec{\mathbf{v}}_h), \boldsymbol{\tau}_h]}{\|\vec{\mathbf{v}}_h\|_{\mathbf{V}}} \geq \beta_{1,d} \|\boldsymbol{\tau}_h\|_{\mathbf{Y}} \quad \forall \boldsymbol{\tau}_h \in \mathbf{Y}_h, \quad (5.21a)$$

and

$$\|\mathbf{s}_{h,\text{sym}}\|_{\mathbf{X}} \geq \beta_{2,d} \|(\mathbf{v}_h, \mathbf{s}_{h,\text{skw}})\|_{\mathbf{V}} \quad \forall \vec{\mathbf{v}}_h \in \mathbf{K}_h. \quad (5.21b)$$

Proof. It is a direct consequence of Lemmas 5.1-5.5. \square

5.2. The main result

We begin by noting from (5.21a) that the discrete operator \mathcal{B}_h satisfies hypothesis *iv*) of Theorem 3.2. Moreover, the Lipschitz continuity of the operator $C + \mathcal{A}$, established in Lemma 3.1, remains valid on the discrete space $\mathbf{V}_h \times \mathbf{V}_h$. Consequently, with the same Lipschitz constant L_{nN} , the following estimate holds

$$\|(C + \mathcal{A})(\bar{\mathbf{u}}_h) - (C + \mathcal{A})(\bar{\mathbf{v}}_h)\|_{V'} \leq L_{\text{nN}} \left\{ \|\mathbf{u}_h - \mathbf{v}_h\|_{\mathbf{Q}} + \|\mathbf{t}_h - \mathbf{s}_h\|_{\mathbf{X}}^{r-1} \right\}, \quad (5.22)$$

for all $\bar{\mathbf{u}}_h, \bar{\mathbf{v}}_h \in \mathbf{V}_h$.

We now turn to the discrete analogue of Lemma 3.2.

Lemma 5.7. *Let $r \in (1, 2]$, and let \mathbf{K}_h be defined as in Lemma 5.6. The family of operators $\{(C + \mathcal{A})(\cdot + \bar{\mathbf{z}}_h) : \mathbf{K}_h \rightarrow \mathbf{K}'_h : \bar{\mathbf{z}}_h \in \mathbf{V}_h\}$ is uniformly strictly monotone; that is, there exists a constant $\alpha_{\text{nN,d}} > 0$, independent of h , such that*

$$\begin{aligned} & [(C + \mathcal{A})(\bar{\mathbf{u}}_h + \bar{\mathbf{z}}_h) - (C + \mathcal{A})(\bar{\mathbf{v}}_h + \bar{\mathbf{z}}_h), \bar{\mathbf{u}}_h - \bar{\mathbf{v}}_h] \\ & \geq \alpha_{\text{nN,d}} \left\{ \|\mathbf{u}_h - \mathbf{v}_h\|_{\mathbf{Q}}^2 + \frac{\|\mathbf{t}_h - \mathbf{s}_h\|_{\mathbf{X}}^2}{1 + (\|\mathbf{t}_h\|_{\mathbf{X}} + \|\mathbf{s}_h\|_{\mathbf{X}})^{2-r}} \right\}, \end{aligned} \quad (5.23)$$

for all $\bar{\mathbf{z}}_h = (\mathbf{z}_h, \mathbf{r}_h) \in \mathbf{V}_h$ and $\bar{\mathbf{u}}_h = (\mathbf{u}_h, \mathbf{t}_h), \bar{\mathbf{v}}_h = (\mathbf{v}_h, \mathbf{s}_h) \in \mathbf{K}_h$.

Proof. Let $\bar{\mathbf{z}}_h = (\mathbf{z}_h, \mathbf{r}_h) \in \mathbf{V}_h$ and $\bar{\mathbf{u}}_h = (\mathbf{u}_h, \mathbf{t}_h), \bar{\mathbf{v}}_h = (\mathbf{v}_h, \mathbf{s}_h) \in \mathbf{K}_h$. Proceeding analogously to the derivation of (3.12) in Lemma 3.2, we obtain

$$\begin{aligned} & [(C + \mathcal{A})(\bar{\mathbf{u}}_h + \bar{\mathbf{z}}_h) - (C + \mathcal{A})(\bar{\mathbf{v}}_h + \bar{\mathbf{z}}_h), \bar{\mathbf{u}}_h - \bar{\mathbf{v}}_h] \\ & \geq \|\mathbf{u}_h - \mathbf{v}_h\|_{0,\Omega}^2 + \frac{\alpha_{\mathcal{G}}}{\max\{|\Omega|^{\frac{2-r}{r}} \delta^{2-r}, 1\}} \frac{\|\mathbf{t}_{h,\text{sym}} - \mathbf{s}_{h,\text{sym}}\|_{\mathbf{X}}^2}{1 + (\|\mathbf{t}_{\text{sym}}\|_{\mathbf{X}} + \|\mathbf{s}_{\text{sym}}\|_{\mathbf{X}})^{2-r}}. \end{aligned} \quad (5.24)$$

To derive a lower bound for the second term, we first exploit the assumption $r \leq 2$ to deduce that

$$1 + (\|\mathbf{t}_{\text{sym}}\|_{\mathbf{X}} + \|\mathbf{s}_{\text{sym}}\|_{\mathbf{X}})^{2-r} \leq 1 + (\|\mathbf{t}\|_{\mathbf{X}} + \|\mathbf{s}\|_{\mathbf{X}})^{2-r}, \quad (5.25)$$

and subsequently apply inequality (5.21b) to obtain

$$\begin{aligned} \|\mathbf{t}_{h,\text{sym}} - \mathbf{s}_{h,\text{sym}}\|_{\mathbf{X}}^2 & \geq \frac{1}{2} \|\mathbf{t}_{h,\text{sym}} - \mathbf{s}_{h,\text{sym}}\|_{\mathbf{X}}^2 + \frac{\beta_{2,\text{d}}^2}{2} \left(\|\mathbf{t}_{h,\text{skw}} - \mathbf{s}_{h,\text{skw}}\|_{\mathbf{X}}^2 + \|\mathbf{u}_h - \mathbf{v}_h\|_{\mathbf{Q}}^2 \right) \\ & \geq \frac{\min\{1, \beta_{2,\text{d}}^2\}}{2} \|\bar{\mathbf{u}}_h - \bar{\mathbf{v}}_h\|_{\mathbf{V}}^2. \end{aligned} \quad (5.26)$$

Therefore, substituting (5.25) and (5.26) into (5.24) yields (5.23), with the constant $\alpha_{\text{nN,d}}$ defined as $\alpha_{\text{nN,d}} := \min\{1, \frac{\alpha_{\mathcal{G}}}{2} \min\{1, \beta_{2,\text{d}}^2\}\}$. This concludes the proof. \square

We are now ready to present the fully discrete analogue of Theorem 3.4.

Theorem 5.1. *Let $r < 2$. For every $\mathbf{f}^n \in \mathbf{L}^2(\Omega)$, with $n = 1, \dots, N$, and for each initial condition $(\bar{\mathbf{u}}_h^0, \sigma_h^0) = ((\mathbf{u}_{h,0}, \mathbf{t}_{h,0}), \sigma_{h,0})$ satisfying (4.11), there exists a unique solution*

$(\bar{\mathbf{u}}_h^n, \boldsymbol{\sigma}_h^n) = ((\mathbf{u}_h^n, \mathbf{t}_h^n), \boldsymbol{\sigma}_h^n) \in \mathbf{V}_h \times \mathbf{Y}_h$ to the fully discrete scheme (4.10). Moreover, under the assumption that $\mathbf{u}_0 \in \mathbf{Q} \cap \mathbf{M}$, there exists a constant $C_{d, \text{stab}} > 0$, independent of h , such that

$$\begin{aligned} & \|\mathbf{u}_h\|_{L^\infty(J; \mathbf{Q})}^2 + \Delta t \sum_{m=1}^n \|\mathbf{u}_h^m\|_{\mathbf{Q}}^2 + \Delta t \sum_{m=1}^n \|\mathbf{t}_h^m\|_{\mathbf{X}}^r + \Delta t \sum_{m=1}^n \|\boldsymbol{\sigma}_h^m\|_{\mathbf{Y}}^2 \\ & \leq C_{d, \text{stab}} \left\{ \left\| \operatorname{div}(\boldsymbol{\vartheta}(\epsilon(\mathbf{u}_0))) \right\|_{0, \Omega}^{r/(r-1)} + \left\| \operatorname{div}(\boldsymbol{\vartheta}(\epsilon(\mathbf{u}_0))) \right\|_{0, \Omega}^2 \right. \\ & \quad \left. + \Delta t \sum_{m=1}^n (\|\mathbf{f}^m\|_{0, \Omega}^{r/(r-1)} + \|\mathbf{f}^m\|_{0, \Omega}^2) \right\} := \mathcal{N}_{\text{dis}}(\mathbf{f}^n, \mathbf{u}_0). \end{aligned} \quad (5.27)$$

Proof. The well-posedness of the fully discrete scheme (4.10) at each time step, for $n = 1, \dots, N$, can be established by arguments analogous to those used in the proof of Lemma 3.5. Furthermore, to establish the second part of the result, we begin by choosing the test functions $(\bar{\mathbf{v}}_h, \boldsymbol{\tau}_h) = (\bar{\mathbf{u}}_h^n, \boldsymbol{\sigma}_h^n)$ in (4.10), and applying the identity

$$\int_{\Omega} \delta_t \mathbf{u}_h^n \cdot \mathbf{u}_h^n = \frac{1}{2} \delta_t \|\mathbf{u}_h^n\|_{\mathbf{Q}}^2 + \frac{1}{2} \Delta t \|\delta_t \mathbf{u}_h^n\|_{\mathbf{Q}}^2,$$

together with the discrete strict monotonicity of the operator \mathcal{A} (cf. (5.23)) to obtain:

$$\frac{1}{2} \delta_t \|\mathbf{u}_h^n\|_{\mathbf{Q}}^2 + \frac{1}{2} \Delta t \|\delta_t \mathbf{u}_h^n\|_{\mathbf{Q}}^2 + \alpha_{\text{N,N,d}} \frac{\|\mathbf{t}_h^n\|_{\mathbf{X}}^2}{1 + \|\mathbf{t}_h^n\|_{\mathbf{X}}^{2-r}} \leq \|\mathbf{f}^n\|_{0, \Omega} \|\mathbf{u}_h^n\|_{\mathbf{Q}}. \quad (5.28)$$

Reasoning as in the proof of Theorem 3.4, if the condition $\|\mathbf{t}_h^n\|_{\mathbf{X}} \geq 1$ holds, then it follows from (5.28) that

$$\frac{1}{2} \delta_t \|\mathbf{u}_h^n\|_{\mathbf{Q}}^2 + \frac{1}{2} \Delta t \|\delta_t \mathbf{u}_h^n\|_{\mathbf{Q}}^2 + \frac{\alpha_{\text{N,N,d}}}{2} \|\mathbf{t}_h^n\|_{\mathbf{X}}^r \leq \|\mathbf{f}^n\|_{0, \Omega} \|\mathbf{u}_h^n\|_{\mathbf{Q}}. \quad (5.29)$$

In turn, applying inequality (5.21b) yields

$$\|\mathbf{t}_h^n\|_{\mathbf{X}}^r \geq \|\mathbf{t}_{h, \text{sym}}^n\|_{\mathbf{X}}^r \geq \beta_{2, \text{d}}^r \|\mathbf{u}_h^n\|_{\mathbf{Q}}^r,$$

which, together with (5.29) and the use of Young's inequality (3.37), leads to

$$\begin{aligned} & \frac{1}{2} \delta_t \|\mathbf{u}_h^n\|_{\mathbf{Q}}^2 + \frac{1}{2} \Delta t \|\delta_t \mathbf{u}_h^n\|_{\mathbf{Q}}^2 + \frac{\alpha_{\text{N,N,d}}}{4} \left(\|\mathbf{t}_h^n\|_{\mathbf{X}}^r + \beta_{2, \text{d}}^r \|\mathbf{u}_h^n\|_{\mathbf{Q}}^r \right) \\ & \leq \frac{\alpha_1}{r} \|\mathbf{u}_h^n\|_{\mathbf{Q}}^r + C(\alpha_1) \|\mathbf{f}^n\|_{0, \Omega}^{r/(r-1)}. \end{aligned} \quad (5.30)$$

Now, by suitably choosing α_1 in (5.30), summing over the time index $n = 1, \dots, m$ for $m = 1, \dots, N$, and multiplying the resulting inequality by Δt , we obtain

$$\begin{aligned} & \|\mathbf{u}_h^m\|_{\mathbf{Q}}^2 + (\Delta t)^2 \sum_{n=1}^m \|\delta_t \mathbf{u}_h^n\|_{\mathbf{Q}}^2 + \sum_{n=1}^m \Delta t \left(\|\mathbf{t}_h^n\|_{\mathbf{X}}^r + \|\mathbf{u}_h^n\|_{\mathbf{Q}}^r \right) \\ & \leq C_1 \left(\|\mathbf{u}_h^0\|_{\mathbf{Q}}^2 + \sum_{n=1}^m \|\mathbf{f}^n\|_{0, \Omega}^{r/(r-1)} \right). \end{aligned} \quad (5.31)$$

On the other hand, if $\|\mathbf{t}_h^n\|_{\mathbf{X}} < 1$, then by proceeding analogously, one obtains

$$\begin{aligned} & \|\mathbf{u}_h^m\|_{\mathbf{Q}}^2 + (\Delta t)^2 \sum_{n=1}^m \|\delta_t \mathbf{u}_h^n\|_{\mathbf{Q}}^2 + \sum_{n=1}^m \Delta t \left(\|\mathbf{t}_h^n\|_{\mathbf{X}}^2 + \|\mathbf{u}_h^n\|_{\mathbf{Q}}^2 \right) \\ & \leq C_2 \left(\|\mathbf{u}_h^0\|_{\mathbf{Q}}^2 + \sum_{n=1}^m \|\mathbf{f}^n\|_{0, \Omega}^2 \right). \end{aligned} \quad (5.32)$$

Therefore, combining (5.31) and (5.32) with the assumption $r < 2$, we infer that

$$\begin{aligned} & \|\mathbf{u}_h^m\|_{\mathbf{Q}}^2 + (\Delta t)^2 \sum_{n=1}^m \|\delta_t \mathbf{u}_h^n\|_{\mathbf{Q}}^2 + \sum_{n=1}^m \Delta t \left(\|\mathbf{t}_h^n\|_{\mathbf{X}}^r + \|\mathbf{u}_h^n\|_{\mathbf{Q}}^r \right) \\ & \leq C \left(\|\mathbf{u}_h^0\|_{\mathbf{Q}}^2 + \sum_{n=1}^m (\|\mathbf{f}^n\|_{0,\Omega}^{r/(r-1)} + \|\mathbf{f}^n\|_{0,\Omega}^2) \right). \end{aligned} \quad (5.33)$$

We now turn to bounding $\|\mathbf{u}_h^0\|_{\mathbf{Q}}$. To this end, by choosing $(\vec{\mathbf{v}}_h, \boldsymbol{\tau}_h) = (\vec{\mathbf{u}}_h^0, \boldsymbol{\sigma}_h^0)$ in (4.11), and invoking the monotonicity of \mathcal{A} (cf. (5.23)) along with the continuity estimate for F^0 (cf. (3.31)) and proceeding in a manner analogous to (5.29)-(5.33), we deduce the existence of a constant $C_0 > 0$, depending only on $\alpha_{\text{NN},d}$, $\beta_{2,d}$, r , and thus independent of h , such that

$$\|\mathbf{t}_h^0\|_{\mathbf{X}}^r + \|\mathbf{u}_h^0\|_{\mathbf{Q}}^2 \leq C_0 \left\{ \left\| \operatorname{div}(\boldsymbol{\vartheta}(\boldsymbol{\epsilon}(\mathbf{u}_0))) \right\|_{0,\Omega}^{r/(r-1)} + \left\| \operatorname{div}(\boldsymbol{\vartheta}(\boldsymbol{\epsilon}(\mathbf{u}_0))) \right\|_{0,\Omega}^2 \right\}. \quad (5.34)$$

Moreover, invoking the discrete inf-sup condition for \mathcal{B}_h (cf. (5.21a)) together with the first equation in (4.10), we conclude that

$$\|\boldsymbol{\sigma}_h^n\|_{\mathbf{X}}^2 \leq \tilde{C} \left\{ \|\mathbf{f}^n\|_{0,\Omega}^2 + \|\delta_t \mathbf{u}_h^n\|_{\mathbf{Q}}^2 + \|\mathbf{t}_h^n\|_{\mathbf{X}}^{2(r-1)} \right\}. \quad (5.35)$$

Since $2(r-1) < r$ for all $r \leq 2$, we have $\|\mathbf{t}_h^n\|_{\mathbf{X}}^{2(r-1)} \leq \|\mathbf{t}_h^n\|_{\mathbf{X}}^r$, which, combined with (5.35), and upon summing over n and applying (5.33), yields

$$\begin{aligned} & \Delta t \sum_{n=1}^m \|\boldsymbol{\sigma}_h^n\|_{\mathbf{X}}^2 \leq \tilde{C} \Delta t \left\{ \sum_{n=1}^m \|\mathbf{f}^n\|_{0,\Omega}^2 + \sum_{n=1}^m \|\delta_t \mathbf{u}_h^n\|_{\mathbf{Q}}^2 + \sum_{n=1}^m \|\mathbf{t}_h^n\|_{\mathbf{X}}^r \right\} \\ & \leq \tilde{C} \left\{ \Delta t \sum_{n=1}^m \|\delta_t \mathbf{u}_h^n\|_{\mathbf{Q}}^2 + \|\mathbf{u}_h^0\|_{\mathbf{Q}}^2 + \Delta t \sum_{n=1}^m (\|\mathbf{f}^n\|_{0,\Omega}^{r/(r-1)} + \|\mathbf{f}^n\|_{0,\Omega}^2) \right\}. \end{aligned} \quad (5.36)$$

Next, to estimate the first term in (5.36), we select the test functions $(\vec{\mathbf{v}}_h, \boldsymbol{\tau}_h) = ((\delta_t \mathbf{u}_h^n, \delta_t \mathbf{t}_h^n), \boldsymbol{\sigma}_h^n)$ in (4.10), employ (5.22) and apply the Young inequality to obtain

$$\|\delta_t \mathbf{u}_h^n\|_{\mathbf{Q}}^2 \leq \|\mathbf{f}^n\|_{0,\Omega}^2 + L_{\text{NN}}^2 \|\mathbf{t}_h^n\|_{\mathbf{X}}^{2(r-1)} + \frac{1}{2} \|\delta_t \mathbf{u}_h^n\|_{\mathbf{Q}}^2.$$

Accordingly, by summing over $n = 1, \dots, m$ with $m = 1, \dots, N$, and multiplying through by Δt , substituting into (5.36), and making use of (5.33) and (5.34), we obtain the desired result (5.27). \square

6. A priori error analysis

In this section, we focus on establishing optimal *a priori* error estimates for $\vec{\mathbf{u}}$ and $\boldsymbol{\sigma}$ in the norms of \mathbf{V} and \mathbf{Y} , respectively. To this end, given the Sobolev exponent $r \in (1, 2]$ and a parameter $a > 0$, we define the convex function φ_a by

$$\varphi_a(t) := \int_0^t (a+s)^{r-2} s \, ds.$$

The following lemma establishes key property of the shifted function φ_a

Lemma 6.1. (*Young-type inequality*). *For every $\epsilon > 0$, there exists a constant $C(\epsilon) > 0$, depending only on r and δ , such that for all $s, t, a \geq 0$, the following inequality holds:*

$$s\varphi_a'(t) + t\varphi_a'(s) \leq \epsilon\varphi_a(s) + C(\epsilon)\varphi_a(t). \quad (6.1)$$

Proof. See Lemmata 28-32 in Ref. 33. \square

The following result, which establishes the equivalence among several quantities, is intimately related to the continuity and monotonicity conditions specified in Assumption 2.1.

Lemma 6.2. *Let $\boldsymbol{\vartheta}$ satisfy (2.6) and (2.7) for $r \in (1, 2]$ and $\delta \geq 0$. Then, uniformly for all $\boldsymbol{\tau}, \boldsymbol{\eta} \in \mathbb{R}^{n \times n}$, the following inequalities hold*

$$|\boldsymbol{\vartheta}(\boldsymbol{\tau}) - \boldsymbol{\vartheta}(\boldsymbol{\eta})| \lesssim (\delta + |\boldsymbol{\tau}| + |\boldsymbol{\eta}|)^{r-2} |\boldsymbol{\tau} - \boldsymbol{\eta}| \simeq \varphi'_{\delta+|\boldsymbol{\tau}|}(|\boldsymbol{\tau} - \boldsymbol{\eta}|), \quad (6.2a)$$

$$(\boldsymbol{\vartheta}(\boldsymbol{\tau}) - \boldsymbol{\vartheta}(\boldsymbol{\eta})) : (\boldsymbol{\tau} - \boldsymbol{\eta}) \lesssim (\delta + |\boldsymbol{\tau}| + |\boldsymbol{\eta}|)^{r-2} |\boldsymbol{\tau} - \boldsymbol{\eta}|^2 \simeq \varphi_{\delta+|\boldsymbol{\tau}|}(|\boldsymbol{\tau} - \boldsymbol{\eta}|). \quad (6.2b)$$

Proof. See Section 2.3 of Ref. 49. \square

Continuing the analysis, we define

$$\mathbf{e}_t^n := \mathbf{t}^n - \mathbf{t}_h^n, \quad \mathbf{e}_\sigma^n := \boldsymbol{\sigma}^n - \boldsymbol{\sigma}_h^n, \quad \text{and} \quad \mathbf{e}_u^n := \mathbf{u}^n - \mathbf{u}_h^n,$$

and represent the aforementioned errors as follows

$$\begin{aligned} \mathbf{e}_t^n &:= (\mathbf{t}^n - \mathcal{P}_k^h(\mathbf{t}^n)) + (\mathcal{P}_k^h(\mathbf{t}^n) - \mathbf{t}_h^n) =: \boldsymbol{\varrho}_t^n + \boldsymbol{\theta}_t^n, \\ \mathbf{e}_\sigma^n &:= (\boldsymbol{\sigma}^n - \boldsymbol{\Pi}_k^h(\boldsymbol{\sigma}^n)) + (\boldsymbol{\Pi}_k^h(\boldsymbol{\sigma}^n) - \boldsymbol{\sigma}_h^n) =: \boldsymbol{\varrho}_\sigma^n + \boldsymbol{\theta}_\sigma^n, \\ \mathbf{e}_u^n &:= (\mathbf{u}^n - \mathcal{P}_k^h(\mathbf{u}^n)) + (\mathcal{P}_k^h(\mathbf{u}^n) - \tilde{\mathbf{u}}_h^n) =: \boldsymbol{\varrho}_u^n + \boldsymbol{\theta}_u^n. \end{aligned} \quad (6.3)$$

Here, $\boldsymbol{\varrho}_t^n$, $\boldsymbol{\varrho}_\sigma^n$, and $\boldsymbol{\varrho}_u^n$ are estimated in accordance with the properties (\mathbf{AP}_h^t) , (\mathbf{AP}_h^σ) , and (\mathbf{AP}_h^u) , respectively. We now proceed to estimate the corresponding terms $\boldsymbol{\theta}_t^n$, $\boldsymbol{\theta}_\sigma^n$, and $\boldsymbol{\theta}_u^n$. By employing the definition of \mathcal{B}_h (cf. (4.9)), the commuting diagram property stated in (4.7), and the identity (4.5), we obtain

$$\begin{aligned} [\mathcal{B}_h(\tilde{\mathbf{v}}_h), \boldsymbol{\Pi}_k^h \boldsymbol{\sigma}] &= - \int_{\Omega_h} \operatorname{div}(\boldsymbol{\Pi}_k^h \boldsymbol{\sigma}) \cdot \mathbf{v}_h - \int_{\Omega_h} \mathcal{P}_k^h(\boldsymbol{\Pi}_k^h \boldsymbol{\sigma}) : \mathbf{s}_h \\ &= - \int_{\Omega_h} \mathcal{P}_k^h(\operatorname{div}(\boldsymbol{\sigma})) \cdot \mathbf{v}_h - \int_{\Omega_h} \boldsymbol{\Pi}_k^h \boldsymbol{\sigma} : \mathcal{P}_k^h \mathbf{s}_h \\ &= - \int_{\Omega_h} \operatorname{div}(\boldsymbol{\sigma}) \cdot \mathbf{v}_h - \int_{\Omega_h} \boldsymbol{\sigma} : \mathcal{P}_k^h \mathbf{s}_h \\ &= [\mathcal{B}(\tilde{\mathbf{v}}_h), \boldsymbol{\sigma}]. \end{aligned} \quad (6.4)$$

On the other hand, a direct application of the discrete and continuous formulations (4.10) and (2.16), together with the identities (6.4) and (3.27), yields

$$\begin{aligned} &[\delta_t \mathcal{C}(\boldsymbol{\theta}_u^n), \tilde{\mathbf{v}}_h] + [(\mathcal{A}(\tilde{\mathbf{u}}_h^n) - \mathcal{A}(\mathcal{P}_k^h \tilde{\mathbf{u}}_h^n)), \tilde{\mathbf{v}}_h] + [\mathcal{B}_h(\tilde{\mathbf{v}}_h), \boldsymbol{\theta}_\sigma^n] \\ &= - \int_{\Omega} (\partial_t \mathbf{u}^n - \delta_t \mathbf{u}^n) \cdot \mathbf{v}_h - [\delta_t \mathcal{C}(\boldsymbol{\varrho}_u^n), \tilde{\mathbf{v}}_h] + [(\mathcal{A}(\tilde{\mathbf{u}}^n) - \mathcal{A}(\mathcal{P}_k^h \tilde{\mathbf{u}}^n)), \tilde{\mathbf{v}}_h], \end{aligned} \quad (6.5a)$$

$$[\mathcal{B}_h(\boldsymbol{\theta}_u^n), \boldsymbol{\tau}_h] = 0. \quad (6.5b)$$

Consequently, thanks to (6.5) we are in position to establish the rates of convergence.

Theorem 6.1. *Let $(\tilde{\mathbf{u}}, \boldsymbol{\sigma}) \in L^2(\mathbf{J}; \mathbf{V}) \times L^2(\mathbf{J}; \mathbf{Y})$ and $(\tilde{\mathbf{u}}_h^n, \boldsymbol{\sigma}_h^n) \in \mathbf{V}_h \times \mathbf{Y}_h$ for any $n = 1, \dots, N$ be the unique solutions of (2.16) and (4.10), respectively, whose existences are guaranteed by Theorems 3.4 and 5.1, respectively. Furthermore, given an integer $k \geq 0$, assume that there*

exist $j \in [0, k+1]$ and $l \in [1, k+1]$ such that $\mathbf{u}_0 \in \mathbf{W}^{1,r}(\Omega) \cap \mathbf{W}^{j+1,r}(\Omega)$, $\partial_{tt}\mathbf{u} \in L^2(J; \mathbf{Q})$, $\partial_t\mathbf{u} \in L^2(J; \mathbf{H}^j(\Omega))$, $\mathbf{u}^n \in \mathbf{H}^j(\Omega)$, $\mathbf{t}^n \in \mathbb{W}^{j,r}(\Omega)$, $\boldsymbol{\sigma}^n \in \mathbb{W}^{l,\rho}(\Omega)$ and $\operatorname{div}(\boldsymbol{\sigma}^n) \in \mathbf{H}^l(\Omega)$ for any $n = 1, \dots, N$. Then, there exists a positive constant C_{opt} , independent of h and Δt , such that

$$\|\mathbf{e}_\mathbf{u}^n\|_{\mathbf{Q}}^2 + \Delta t \sum_{n=1}^m (\|\mathbf{e}_\mathbf{t}^n\|_{\mathbf{X}}^2 + \|\mathbf{e}_\mathbf{u}^n\|_{\mathbf{Q}}^2 + \|\mathbf{e}_\boldsymbol{\sigma}^n\|_{\mathbf{Y}}^2) \leq C_{\text{opt}} \left(\Delta t^2 + h^{\min\{rj, 2(r-1)j, 2l\}} \right). \quad (6.6)$$

Proof. We begin by choosing the test functions $(\vec{\mathbf{v}}_h, \boldsymbol{\tau}_h) := ((\boldsymbol{\theta}\mathbf{u}^n, \boldsymbol{\theta}_\mathbf{t}^n), \boldsymbol{\theta}_\boldsymbol{\sigma}^n) \in \mathbf{V}_h \times \mathbf{Y}_h$, and adding the two corresponding equations to deduce:

$$\begin{aligned} & \frac{1}{2} \delta_t \|\boldsymbol{\theta}_\mathbf{u}^n\|_{\mathbf{Q}}^2 + [(\mathcal{A}(\vec{\mathbf{u}}_h^n) - \mathcal{A}(\mathcal{P}_k^h \vec{\mathbf{u}}^n)), \boldsymbol{\theta}_\mathbf{u}^n] \\ & \leq \left| \int_{\Omega} (\partial_t \mathbf{u}^n - \delta_t \mathbf{u}^n) \cdot \boldsymbol{\theta}_\mathbf{u}^n - [\delta_t \mathbf{C}(\boldsymbol{\varrho}_\mathbf{u}^n), \boldsymbol{\theta}_\mathbf{u}^n] \right| + [(\mathcal{A}(\vec{\mathbf{u}}^n) - \mathcal{A}(\mathcal{P}_k^h \vec{\mathbf{u}}^n)), \boldsymbol{\theta}_\mathbf{u}^n] \\ & =: \sum_{i=1}^3 E_i. \end{aligned} \quad (6.7)$$

We now proceed to estimate each term appearing on the right-hand side of (6.7). The first and second terms can be bounded following the standard finite element arguments; for further details, see, for example, Ref. 61:

$$\begin{aligned} |E_1| &= \left| \int_{\Omega} (\partial_t \mathbf{u}^n - \delta_t \mathbf{u}^n) \cdot \boldsymbol{\theta}_\mathbf{u}^n \right| \leq \|\partial_t \mathbf{u}^n - \delta_t \mathbf{u}^n\|_{0,\Omega} \|\boldsymbol{\theta}_\mathbf{u}^n\|_{\mathbf{Q}} \\ &\leq \left(\int_{t_{n-1}}^{t_n} \|\partial_{tt}\mathbf{u}(s)\|_{0,\Omega} \, ds \right) \|\boldsymbol{\theta}_\mathbf{u}^n\|_{\mathbf{Q}} \leq \Delta t^{1/2} \left(\int_{t_{n-1}}^{t_n} \|\partial_{tt}\mathbf{u}(s)\|_{0,\Omega}^2 \, ds \right)^{1/2} \|\boldsymbol{\theta}_\mathbf{u}^n\|_{\mathbf{Q}}, \end{aligned}$$

and

$$\begin{aligned} |E_2| &= |[\delta_t \mathbf{C}(\boldsymbol{\varrho}_\mathbf{u}^n), \boldsymbol{\theta}_\mathbf{u}^n]| \leq \|\delta_t \boldsymbol{\varrho}_\mathbf{u}^n\|_{0,\Omega} \|\boldsymbol{\theta}_\mathbf{u}^n\|_{\mathbf{Q}} \\ &\leq \frac{1}{\Delta t} \left\| \int_{t_{n-1}}^{t_n} \partial_s \boldsymbol{\varrho}_\mathbf{u}(s) \, ds \right\|_{0,\Omega} \|\boldsymbol{\theta}_\mathbf{u}^n\|_{\mathbf{Q}} \leq \Delta t^{-1/2} \|\partial_t \boldsymbol{\varrho}_\mathbf{u}\|_{L^2(J_n, L^2(\Omega))} \|\boldsymbol{\theta}_\mathbf{u}^n\|_{\mathbf{Q}}. \end{aligned}$$

Furthermore, by utilizing the definition of \mathcal{A} as given in (2.21a), together with inequalities (6.1), (6.2a), and (6.2b) for the case $\delta = 0$, we readily obtain

$$\begin{aligned} |E_3| &\leq \int_{\Omega} \left| \boldsymbol{\vartheta}(\mathbf{t}_{\text{sym}}^n) - \boldsymbol{\vartheta}(\mathcal{P}_k^h(\mathbf{t}_{\text{sym}}^n)) \right| |\boldsymbol{\theta}_\mathbf{t}^n| \lesssim \int_{\Omega} \varphi'_{|\mathcal{P}_k^h(\mathbf{t}_{\text{sym}}^n)|} (|\mathcal{P}_k^h(\mathbf{t}_{\text{sym}}^n) - \mathbf{t}_{\text{sym}}^n|) |\boldsymbol{\theta}_\mathbf{t}^n| \\ &\lesssim \epsilon \int_{\Omega} \varphi_{|\mathcal{P}_k^h(\mathbf{t}_{\text{sym}}^n)|} (|\boldsymbol{\theta}_\mathbf{t}^n|) + C(\epsilon) \int_{\Omega} \varphi_{|\mathcal{P}_k^h(\mathbf{t}_{\text{sym}}^n)|} (|\mathcal{P}_k^h(\mathbf{t}_{\text{sym}}^n) - \mathbf{t}_{\text{sym}}^n|) \\ &\lesssim \epsilon \left(\boldsymbol{\vartheta}(\mathbf{t}_{h,\text{sym}}^n) - \boldsymbol{\vartheta}(\mathcal{P}_k^h(\mathbf{t}_{\text{sym}}^n)), \boldsymbol{\theta}_\mathbf{t}^n \right)_{0,\Omega} \\ &+ C(\epsilon) \int_{\Omega} \left(|\mathcal{P}_k^h(\mathbf{t}_{\text{sym}}^n)| + |\mathbf{t}_{\text{sym}}^n| \right)^{r-2} |\mathcal{P}_k^h(\mathbf{t}_{\text{sym}}^n) - \mathbf{t}_{\text{sym}}^n|^2 \\ &\lesssim \epsilon [(\mathcal{A}(\vec{\mathbf{u}}_h^n) - \mathcal{A}(\mathcal{P}_k^h \vec{\mathbf{u}}^n)), \boldsymbol{\theta}_\mathbf{u}^n] + C(\epsilon) \|\boldsymbol{\varrho}_{\mathbf{t},\text{sym}}^n\|_{\mathbf{X}}^r, \end{aligned} \quad (6.8)$$

where, in the final step, we used the fact that $(|\mathcal{P}_k^h(\mathbf{t}_{\text{sym}}^n)| + |\mathbf{t}_{\text{sym}}^n|)^{r-2} \leq |\mathcal{P}_k^h(\mathbf{t}_{\text{sym}}^n) - \mathbf{t}_{\text{sym}}^n|^{r-2}$, which holds for $r < 2$.

Accordingly, by setting $\epsilon = \frac{1}{2}$ in (6.8) and subsequently substituting the estimates from (6.8) up (3.28) into (6.7), we obtain that

$$\begin{aligned} & \frac{1}{2} \delta_t \|\boldsymbol{\theta}_u^n\|_{\mathbf{Q}}^2 + \frac{1}{2} [(\mathcal{A}(\bar{\mathbf{u}}_h^n) - \mathcal{A}(\mathcal{P}_k^h \bar{\mathbf{u}}^n)), \boldsymbol{\theta}_u^n] \\ & \leq C \|\boldsymbol{\varrho}_t^n\|_{\mathbf{X}}^r + \left(\Delta t^{1/2} \|\partial_{tt} \mathbf{u}\|_{L^2(J_n; \mathbf{Q})} + \Delta t^{-1/2} \|\partial_t \boldsymbol{\varrho}_u\|_{L^2(J_n; \mathbf{L}^2)} \right) \|\boldsymbol{\theta}_u^n\|_{\mathbf{Q}}. \end{aligned} \quad (6.9)$$

To derive a lower bound for the second term on the right-hand side of (6.9), we invoke the monotonicity of \mathcal{A} (cf. (5.23)), together with the stability of the projection and the fact that $r - 2 < 0$, which yields

$$\begin{aligned} [(\mathcal{A}(\bar{\mathbf{u}}_h^n) - \mathcal{A}(\mathcal{P}_k^h \bar{\mathbf{u}}^n)), \boldsymbol{\theta}_u^n] & \geq \alpha_{\text{N},d} \left(\|\mathbf{t}_h\|_{\mathbf{X}}^r + \|\mathcal{P}_k^h \mathbf{t}^n\|_{\mathbf{X}}^r \right)^{(r-2)/r} \|\boldsymbol{\theta}_t^n\|_{\mathbf{X}}^2 \\ & \geq \alpha_{\text{N},d} \left(\|\mathbf{t}_h^n\|_{\mathbf{X}}^r + \|\mathbf{t}^n\|_{\mathbf{X}}^r \right)^{(r-2)/r} \|\boldsymbol{\theta}_t^n\|_{\mathbf{X}}^2. \end{aligned} \quad (6.10)$$

Substituting this expression back into (6.9) yields

$$\begin{aligned} & \frac{1}{2} \delta_t \|\boldsymbol{\theta}_u^n\|_{\mathbf{Q}}^2 + \frac{1}{2} \alpha_{\text{N},d} \left(\|\mathbf{t}_h^n\|_{\mathbf{X}}^r + \|\mathbf{t}^n\|_{\mathbf{X}}^r \right)^{(r-2)/r} \|\boldsymbol{\theta}_t^n\|_{\mathbf{X}}^2 \\ & \leq C \|\boldsymbol{\varrho}_t^n\|_{\mathbf{X}}^r + \left(\Delta t^{1/2} \|\partial_{tt} \mathbf{u}\|_{L^2(J_n; \mathbf{Q})} + \Delta t^{-1/2} \|\partial_t \boldsymbol{\varrho}_u\|_{L^2(J_n; \mathbf{L}^2)} \right) \|\boldsymbol{\theta}_u^n\|_{\mathbf{Q}}. \end{aligned} \quad (6.11)$$

In turn, it follows from (5.21b) that

$$\frac{1}{2} \|\boldsymbol{\theta}_t^n\|_{\mathbf{X}}^2 \geq \frac{1}{2} \|\boldsymbol{\theta}_{t, \text{sym}}^n\|_{\mathbf{X}}^2 \geq \frac{\beta_{2,d}^2}{2} \|\boldsymbol{\theta}_u^n\|_{\mathbf{Q}}^2,$$

upon combining with (6.11), summing over $n = 1, \dots, m$ for each $m = 1, \dots, N$, and multiplying by Δt , we arrive at

$$\begin{aligned} & \|\boldsymbol{\theta}_u^m\|_{\mathbf{Q}}^2 + \Delta t \sum_{n=1}^m \left(\|\mathbf{t}_h^n\|_{\mathbf{X}}^r + \|\mathbf{t}^n\|_{\mathbf{X}}^r \right)^{(r-2)/r} (\|\boldsymbol{\theta}_t^n\|_{\mathbf{X}}^2 + \|\boldsymbol{\theta}_u^n\|_{\mathbf{Q}}^2) \\ & \leq C \left\{ \|\boldsymbol{\theta}_u^0\|_{\mathbf{Q}}^2 + \Delta t \sum_{n=1}^m \|\boldsymbol{\varrho}_t^n\|_{\mathbf{X}}^r + \Delta t \sum_{n=1}^m \left(\Delta t \|\partial_{tt} \mathbf{u}\|_{L^2(J_n; \mathbf{Q})}^2 + \Delta t^{-1} \|\partial_t \boldsymbol{\varrho}_u\|_{L^2(J_n; \mathbf{L}^2)}^2 \right) + \epsilon \Delta t \sum_{n=1}^m \|\boldsymbol{\theta}_u^n\|_{\mathbf{Q}}^2 \right\}. \end{aligned}$$

Next, by bounding the terms $\|\mathbf{t}\|_{L^\infty(J_n; \mathbf{X})}$ and $\|\mathbf{t}_h\|_{L^\infty(J_n; \mathbf{X})}$ using estimates (3.33) and (5.27), respectively, and selecting $\epsilon < \frac{1}{2} (\mathcal{N}(\mathbf{f}^n, \mathbf{u}_0), \mathcal{N}_{\text{dis}}(\mathbf{f}^n, \mathbf{u}_0))^{(r-2)/r}$, we deduce the existence of a positive constant C_1 , depending on r , $\alpha_{\text{N},d}$, $\beta_{1,d}$, $\beta_{2,d}$, $\mathcal{N}(\mathbf{f}^n, \mathbf{u}_0)$, $\mathcal{N}_{\text{dis}}(\mathbf{f}^n, \mathbf{u}_0)$, such that there holds

$$\begin{aligned} \|\boldsymbol{\theta}_u^m\|_{\mathbf{Q}}^2 + \Delta t \sum_{n=1}^m \left(\|\boldsymbol{\theta}_t^n\|_{\mathbf{X}}^2 + \|\boldsymbol{\theta}_u^n\|_{\mathbf{Q}}^2 \right) & \leq C \left\{ \|\boldsymbol{\theta}_u^0\|_{\mathbf{Q}}^2 + \Delta t \sum_{n=1}^m \|\boldsymbol{\varrho}_t^n\|_{\mathbf{X}}^r \right. \\ & \quad \left. + \Delta t \sum_{n=1}^m \left(\Delta t \|\partial_{tt} \mathbf{u}\|_{L^2(J_n; \mathbf{Q})}^2 + \Delta t^{-1} \|\partial_t \boldsymbol{\varrho}_u\|_{L^2(J_n; \mathbf{L}^2)}^2 \right) \right\}. \end{aligned} \quad (6.12)$$

To bound the first term in (6.12), we next subtract the continuous and discrete initial condition problems (3.29) and (4.11), respectively, thereby obtaining the following error system:

$$\begin{aligned} [(\mathcal{A}(\bar{\mathbf{u}}_{h,0}) - \mathcal{A}(\bar{\mathbf{u}}_0)), \bar{\mathbf{v}}_h] + [\mathcal{B}_h(\bar{\mathbf{v}}_h), \boldsymbol{\sigma}_{h,0} - \boldsymbol{\sigma}_0] & = 0 \quad \forall \bar{\mathbf{v}}_h \in \mathbf{V}_h, \\ [\mathcal{B}_h(\bar{\mathbf{u}}_{h,0} - \bar{\mathbf{u}}_0), \boldsymbol{\tau}_h] & = 0 \quad \forall \boldsymbol{\tau}_h \in \mathbf{Y}_h. \end{aligned}$$

Then, following the same approach as in the derivation of (5.34), and noting from Theorems 3.3 and 5.1 that $(\mathbf{u}(0), \mathbf{t}(0)) = (\mathbf{u}_0, \mathbf{t}_0)$ and $(\mathbf{u}_h(0), \mathbf{t}_h(0)) = (\mathbf{u}_{h,0}, \mathbf{t}_{h,0})$, respectively, we consider the test pair $(\vec{\mathbf{v}}_h, \boldsymbol{\tau}_h) = (\boldsymbol{\theta}_{\mathbf{u}}^0, \boldsymbol{\theta}_{\boldsymbol{\sigma}}^0)$ in (6.10), which yields

$$\|\boldsymbol{\theta}_{\mathbf{t}}^0\|_{\mathbf{X}} \leq C_1 \|\boldsymbol{\varrho}_{\mathbf{t}}(0)\|_{\mathbf{X}}^{r-1} \left(\|\mathbf{t}_h(0)\|_{\mathbf{X}}^r + \|\mathbf{t}(0)\|_{\mathbf{X}}^r \right)^{(2-r)/r}. \quad (6.13)$$

Combining (6.13) and (5.21b), and subsequently substituting the resulting inequality into (6.12), while using the error decomposition (6.3), we arrive at

$$\begin{aligned} \|\mathbf{e}_{\mathbf{u}}^n\|_{\mathbf{Q}}^2 + \Delta t \sum_{n=1}^m \left(\|\mathbf{e}_{\mathbf{t}}^n\|_{\mathbf{X}}^2 + \|\mathbf{e}_{\mathbf{u}}^n\|_{\mathbf{Q}}^2 \right) &\leq \widehat{C} \left\{ \|\boldsymbol{\varrho}_{\mathbf{t}}(0)\|_{\mathbf{X}}^{2(r-1)} \left(\|\mathbf{t}_h(0)\|_{\mathbf{X}}^r + \|\mathbf{t}(0)\|_{\mathbf{X}}^r \right)^{2(2-r)/r} \right. \\ &\quad \left. + \|\boldsymbol{\varrho}_{\mathbf{u}}^n\|_{\mathbf{Q}}^2 + \Delta t \sum_{n=1}^m \left(\|\boldsymbol{\varrho}_{\mathbf{t}}^n\|_{\mathbf{X}}^2 + \|\boldsymbol{\varrho}_{\mathbf{t}}^n\|_{\mathbf{X}}^r + \|\boldsymbol{\varrho}_{\mathbf{u}}^n\|_{\mathbf{Q}}^2 \right) \right. \\ &\quad \left. + \Delta t \sum_{n=1}^m \left(\Delta t \|\partial_{tt} \mathbf{u}\|_{L^2(J_n; \mathbf{Q})}^2 + \Delta t^{-1} \|\partial_t \boldsymbol{\varrho}_{\mathbf{u}}\|_{L^2(J_n; \mathbf{L}^2)}^2 \right) \right\}. \end{aligned} \quad (6.14)$$

On the other hand, to derive the estimate for the pseudostress, we invoke the discrete inf-sup condition for \mathcal{B}_h (cf. (5.21a)), the first equation in (6.5), and the continuity of \mathcal{A} (cf. (5.22)), which together yield

$$\begin{aligned} \beta_{1,d} \|\boldsymbol{\theta}_{\boldsymbol{\sigma}}^n\|_{\mathbf{Y}} &\leq \sup_{\mathbf{0} \neq \vec{\mathbf{v}}_h \in \mathbf{V}_h} \frac{[\mathcal{B}_h(\vec{\mathbf{v}}_h), \boldsymbol{\theta}_{\boldsymbol{\sigma}}^n]}{\|\vec{\mathbf{v}}\|_{\mathbf{V}}} \\ &= \sup_{\mathbf{0} \neq \vec{\mathbf{v}}_h \in \mathbf{V}_h} \frac{-[\delta_t C(\boldsymbol{\theta}_{\mathbf{u}}^n), \vec{\mathbf{v}}_h] - [(\mathcal{A}(\vec{\mathbf{u}}_h^n) - \mathcal{A}(\vec{\mathbf{u}}^n)), \vec{\mathbf{v}}_h] - \int_{\Omega} (\partial_t \mathbf{u}^n - \delta_t \mathbf{u}^n) \cdot \mathbf{v}_h - [\delta_t C(\boldsymbol{\varrho}_{\mathbf{u}}^n), \vec{\mathbf{v}}_h]}{\|\vec{\mathbf{v}}\|_{\mathbf{V}}} \\ &\leq C \left(\|\delta_t \boldsymbol{\theta}_{\mathbf{u}}^n\|_{\mathbf{Q}} + \|\boldsymbol{\theta}_{\mathbf{t}}^n\|_{\mathbf{X}}^{r-1} + \|\boldsymbol{\varrho}_{\mathbf{t}}^n\|_{\mathbf{X}}^{r-1} \right. \\ &\quad \left. + \Delta t^{1/2} \|\partial_{tt} \mathbf{u}\|_{L^2(J_n; \mathbf{Q})} + \Delta t^{-1/2} \|\partial_t \boldsymbol{\varrho}_{\mathbf{u}}\|_{L^2(J_n; \mathbf{Q})} \right). \end{aligned}$$

Then, taking square in the above inequality, summing up over the time index $n = 1, \dots, m$, with $m = 1, \dots, N$, multiplying by Δt , we deduce that

$$\begin{aligned} \Delta t \sum_{n=1}^m \|\boldsymbol{\theta}_{\boldsymbol{\sigma}}^n\|_{\mathbf{Y}}^2 &\leq C \left\{ \Delta t \sum_{n=1}^m \|\delta_t \boldsymbol{\theta}_{\mathbf{u}}^n\|_{0,\Omega}^2 + \Delta t \sum_{n=1}^m \left(\|\boldsymbol{\theta}_{\mathbf{t}}^n\|_{\mathbf{X}}^{2(r-1)} + \|\boldsymbol{\varrho}_{\mathbf{t}}^n\|_{\mathbf{X}}^{2(r-1)} \right) \right. \\ &\quad \left. + \Delta t \sum_{n=1}^m \left(\Delta t \|\partial_{tt} \mathbf{u}\|_{L^2(J_n; \mathbf{Q})}^2 + \Delta t^{-1} \|\partial_t \boldsymbol{\varrho}_{\mathbf{u}}\|_{L^2(J_n; \mathbf{Q})}^2 \right) \right\}. \end{aligned} \quad (6.15)$$

Next, to bound the first term on the right-hand side, we differentiate the second equation of (6.5) with respect to time, and then choose the test functions as $(\vec{\mathbf{v}}_h, \boldsymbol{\tau}_h) = ((\delta_t \boldsymbol{\theta}_{\mathbf{u}}^n, \delta_t \boldsymbol{\theta}_{\mathbf{t}}^n), \boldsymbol{\theta}_{\boldsymbol{\sigma}}^n)$, which leads to

$$\begin{aligned} \|\delta_t \boldsymbol{\theta}_{\mathbf{u}}^n\|_{\mathbf{Q}}^2 &= -[(\mathcal{A}(\vec{\mathbf{u}}_h^n) - \mathcal{A}(\mathcal{P}_k^h \vec{\mathbf{u}}^n)), \delta_t \boldsymbol{\theta}_{\mathbf{u}}^n] - \int_{\Omega} (\partial_t \mathbf{u}^n - \delta_t \mathbf{u}^n) \cdot \delta_t \boldsymbol{\theta}_{\mathbf{u}}^n \\ &\quad - [\delta_t C(\boldsymbol{\varrho}_{\mathbf{u}}^n), \delta_t \boldsymbol{\theta}_{\mathbf{u}}^n] + [(\mathcal{A}(\vec{\mathbf{u}}^n) - \mathcal{A}(\mathcal{P}_k^h \vec{\mathbf{u}}^n)), \delta_t \boldsymbol{\theta}_{\mathbf{u}}^n], \end{aligned}$$

which, by the continuity of \mathcal{A} (cf. (5.22)) and the boundedness of C together with the Young inequality, implies

$$\|\delta_t \boldsymbol{\theta}_{\mathbf{u}}^n\|_{\mathbf{Q}}^2 \leq C_1 \left\{ \|\boldsymbol{\varrho}_{\mathbf{t}}^n\|_{\mathbf{X}}^{2(r-1)} + \|\boldsymbol{\theta}_{\mathbf{t}}^n\|_{\mathbf{X}}^{2(r-1)} + \Delta t \|\partial_{tt} \mathbf{u}\|_{L^2(J_n; \mathbf{Q})}^2 + \Delta t^{-1} \|\partial_t \boldsymbol{\varrho}_{\mathbf{u}}\|_{L^2(J_n; \mathbf{Q})}^2 \right\}.$$

Summing over n and substituting the result back into (6.15) yields

$$\begin{aligned} \Delta t \sum_{n=1}^m \|\boldsymbol{\theta}_\sigma^n\|_Y^2 &\leq C \left\{ \Delta t \sum_{n=1}^m (\|\boldsymbol{\theta}_t^n\|_X^{2(r-1)} + \|\boldsymbol{\varrho}_t^n\|_X^{2(r-1)}) \right. \\ &\quad \left. + \Delta t \sum_{n=1}^m \left(\Delta t \|\partial_{tt} \mathbf{u}\|_{L^2(J_n; \mathbf{Q})}^2 + \Delta t^{-1} \|\partial_t \boldsymbol{\varrho}_u\|_{L^2(J_n; \mathbf{Q})}^2 \right) \right\}. \end{aligned} \quad (6.16)$$

By noting that $2(r-1) < 2$ for $r \in (1, 2]$ and combining (6.16) with the estimate (6.14), it follows that

$$\begin{aligned} \|\mathbf{e}_u^n\|_Q^2 + \Delta t \sum_{n=1}^m \left(\|\mathbf{e}_t^n\|_X^2 + \|\mathbf{e}_u^n\|_Q^2 + \|\mathbf{e}_\sigma^n\|_Y^2 \right) &\leq \widehat{C} \left\{ \|\boldsymbol{\varrho}_t(0)\|_X^{2(r-1)} \left(\|\mathbf{t}_h(0)\|_X^r + \|\mathbf{t}(0)\|_X^r \right)^{2(2-r)/r} \right. \\ &\quad + \|\boldsymbol{\varrho}_u^n\|_Q^2 + \Delta t \sum_{n=1}^m \left(\|\boldsymbol{\varrho}_t^n\|_X^2 + \|\boldsymbol{\varrho}_t^n\|_X^r + \|\boldsymbol{\varrho}_t^n\|_X^{2(r-1)} + \|\boldsymbol{\varrho}_u^n\|_Q^2 + \|\boldsymbol{\varrho}_\sigma^n\|_Y^2 \right) \\ &\quad \left. + \Delta t \sum_{n=1}^m \left(\Delta t \|\partial_{tt} \mathbf{u}\|_{L^2(J_n; \mathbf{Q})}^2 + \Delta t^{-1} \|\partial_t \boldsymbol{\varrho}_u\|_{L^2(J_n; \mathbb{L}^2)}^2 \right) \right\}. \end{aligned}$$

Then, by bounding the term $\|\mathbf{t}_h^0\|_X$ using Estimate (5.34), and invoking the identity $\mathbf{t}_0 = \nabla \mathbf{u}_0$ along with the approximation properties (\mathbf{AP}_h^t) , (\mathbf{AP}_h^σ) , and (\mathbf{AP}_h^u) , we arrive at (6.6), thereby completing the proof. \square

We end this section by remarking that (2.4) suggests the following postprocessed approximation for the pressure

$$p_h^n := -\frac{1}{2} \operatorname{tr}(\mathcal{P}_k^h(\boldsymbol{\sigma}_h^n)) \quad \text{in } \Omega, \quad n = 1, \dots, N,$$

so that, it is easy to show that (e.g., cf. Eq. (5.14) in Ref. 44)

$$\|p^n - p_h^n\|_{0,\Omega} \leq C \left(\|\boldsymbol{\sigma}^n - \mathcal{P}_k^h(\boldsymbol{\sigma}^n)\|_{0,\Omega} + \|\boldsymbol{\sigma}^n - \boldsymbol{\sigma}_h^n\|_{0,\Omega} \right).$$

Hence, by virtue of Theorem 6.1 and the approximation property of \mathcal{P}_k^h (cf. (4.3)), we obtain

$$\|\mathbf{e}_u^n\|_Q^2 + \|\mathbf{e}_p^n\|_{0,\Omega}^2 + \Delta t \sum_{n=1}^m \left(\|\mathbf{e}_t^n\|_X^2 + \|\mathbf{e}_u^n\|_Q^2 + \|\mathbf{e}_\sigma^n\|_Y^2 \right) \leq \widetilde{C}_{\text{opt}} \left(\Delta t^2 + h^{\min\{rj, 2(r-1)j, 2l\}} \right). \quad (6.17)$$

7. Numerical results

In this section, we present several numerical experiments using the publicly available software MATLAB R2024a to validate the theoretical analysis and demonstrate the effectiveness of the proposed scheme. In all tests, the Picard iteration method is employed, with iterations terminated when either the absolute or relative ℓ^2 -norm of the residual falls below 10^{-6} . The absolute errors for each variable are computed as follows:

$$\begin{aligned} e(\mathbf{t}^n) &:= \|\mathbf{t}^n - \mathbf{t}_h^n\|_X, \quad e(\boldsymbol{\sigma}^n) := \|\boldsymbol{\sigma}^n - \boldsymbol{\sigma}_h^n\|_Y, \quad e(\mathbf{u}^n) := \|\mathbf{u}^n - \mathbf{u}_h^n\|_Q \quad \text{and} \\ e(p^n) &:= \|p^n - p_h^n\|_{0,\Omega}. \end{aligned}$$

The numerical examples considered in this section are structured as follows: the first example addresses a two-dimensional problem with manufactured exact solutions, aimed at validating the theoretical error estimates for the strain rate, pseudostress, velocity, vorticity, and pressure derived in this study. Examples 2 and 3 are designed to evaluate the performance of the proposed discrete scheme in practical scenarios where analytical solutions are not available.

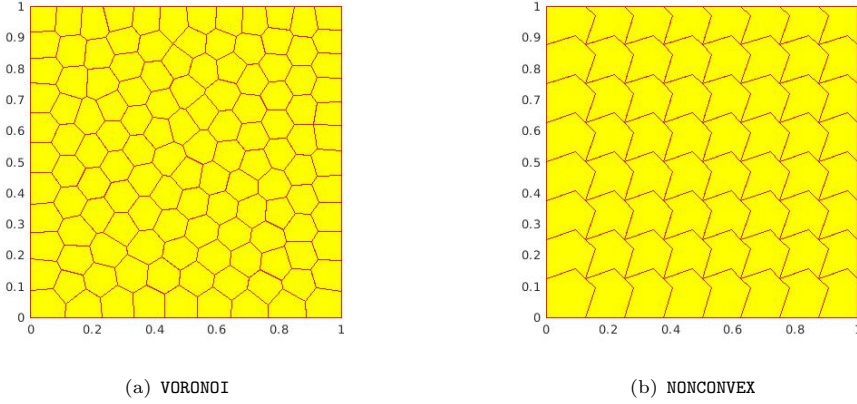


Fig. 7.1. Example of the adopted polygonal meshes.

7.1. Example 1: Accuracy assessment

In this example, we consider problem (2.1) with parameters $\mu = 1$ and $\alpha = 1$, along with the following exact solution for the velocity field and pressure term:

$$\mathbf{u}(x_1, x_2, t) = \cos(t) \begin{pmatrix} \sin(\frac{\pi}{2}x_1) \cos(\frac{\pi}{2}x_2) \\ -\sin(\frac{\pi}{2}x_2) \cos(\frac{\pi}{2}x_1) \end{pmatrix} \quad p(x_1, x_2, t) = \cos(t) \left(-\sin(\frac{\pi}{2}x_1) \sin(\frac{\pi}{2}x_2) + \frac{4}{\pi^2} \right),$$

for all $(x_1, x_2)^\top \in \Omega := (0, 1)^2$ and $t \in [0, t_F] := [0, 0.3]$, the external force and non-homogeneous Dirichlet boundary condition are defined in accordance with the manufactured solution described above. The domain Ω is discretized using two families of polygonal meshes: VORONOI and NONCONVEX meshes (cf. Fig.7.1), both generated by the `PolyMesher` package⁶⁰. For each mesh type, we consider a sequence of meshes with diameters $h = 1/4, 1/8, 1/16, \text{ and } 1/32$. In Figs.7.2 and 7.3, we report the errors $e(\mathbf{t})$, $e(\mathbf{u})$, $e(\boldsymbol{\sigma})$, and $e(p)$ for various values of the parameter $r \in \{1.1, 1.15, 1.25, 1.5, 1.75, 2\}$, using polynomial degree $j = l = 1$ and time step $\Delta t = h$, for two different cases: $\delta = 1$ and $\delta = 0$, respectively. Furthermore, the errors obtained using polynomial degree $j = l = 2$ and time step $\Delta t = h^2$, for both $\delta = 1$ and $\delta = 0$, are presented in Figs.7.4 and 7.5, respectively. As observed in Figs.7.2 and 7.4, all unknowns exhibit optimal-order convergence toward the discrete solutions for all values of $r \in \{1.1, 1.15, 1.25, 1.5, 1.75, 2\}$. Table 7.1 reports the computed errors and the corresponding convergence rates, further illustrating the observed behavior. In contrast, Figs.7.3 and 7.5 illustrate that optimal convergence is attained as r increases, in agreement with the theoretical result established in Theorem6.1.

7.2. Example 2: Lid driven cavity flow

This classical problem serves as a key benchmark for assessing the performance of numerical algorithms across a range of flow scenarios and has been extensively studied within the Navier–Stokes framework (cf. Refs. 63, 59). More precisely, we consider the nonlinear stress-strain law given by (2.2), with fixed parameters $\mu = 0.1$, $\delta = 0$, $\alpha = 1$, and $r \in \{1.25, 1.75, 2.25\}$. The problem is defined on the unit square domain $\Omega = (0, 1)^2$, where a unit tangential velocity $\mathbf{g} = (1, 0)$ is prescribed along the top boundary (i.e., $x_2 = 1$), while no-slip (wall) boundary conditions are imposed along the remaining edges. The initial velocity is taken as $\mathbf{u}_0 = [0, 0]^\top$, with a time step $\Delta t = 10^{-2}$, final time $t_F = 0.5$, and a Cartesian

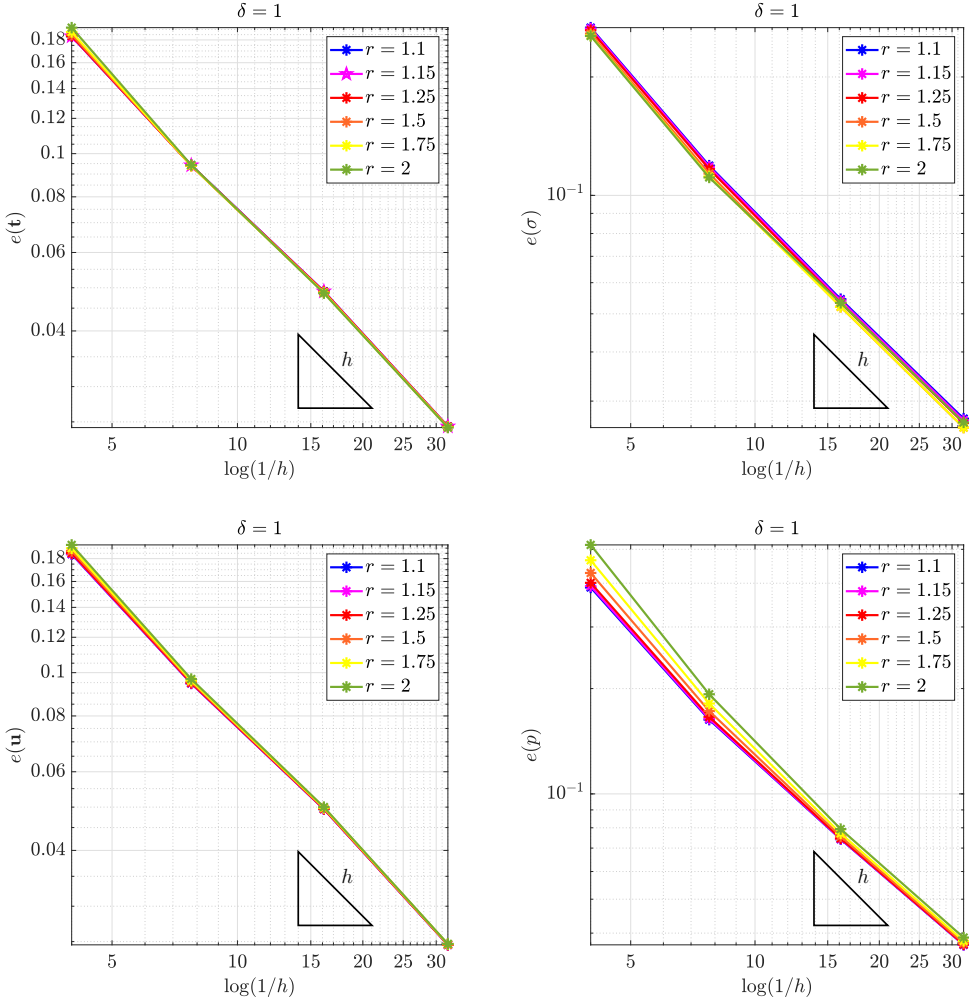


Fig. 7.2. Convergence results with VORONOI mesh, $\delta = 1$, $j = l = 1$ and $\Delta t = h$.

mesh of size 32×32 . In the first row of Figure 7.6, the stream function is depicted, while the second row presents the horizontal velocity component u_{1h} along the vertical centerline $x_1 = 0.5$, and the vertical component u_{2h} along the horizontal centerline $x_2 = 0.5$. More specifically, Figure 7.6 illustrates the influence of the parameter r on the flow structure. As r increases, the solution becomes smoother, with a more regular vortex structure and less pronounced shear near the lid, reflecting the transition toward Newtonian-like behavior. The centerline velocity profiles confirm this trend, showing reduced gradients and increased symmetry for larger values of r . We remark that the choice $r = 2.25$ lies outside the range explicitly covered by our theoretical analysis. This value was selected to explore the method's behavior beyond the aforementioned range, and to assess its robustness in more general settings. As shown in Figure 7.6, the numerical results for $r = 2.25$ remain consistent with the trends observed for admissible values of r . This suggests that the method retains its stability and accuracy even for values of r slightly outside the theoretical range, although a rigorous justification would require further analysis.

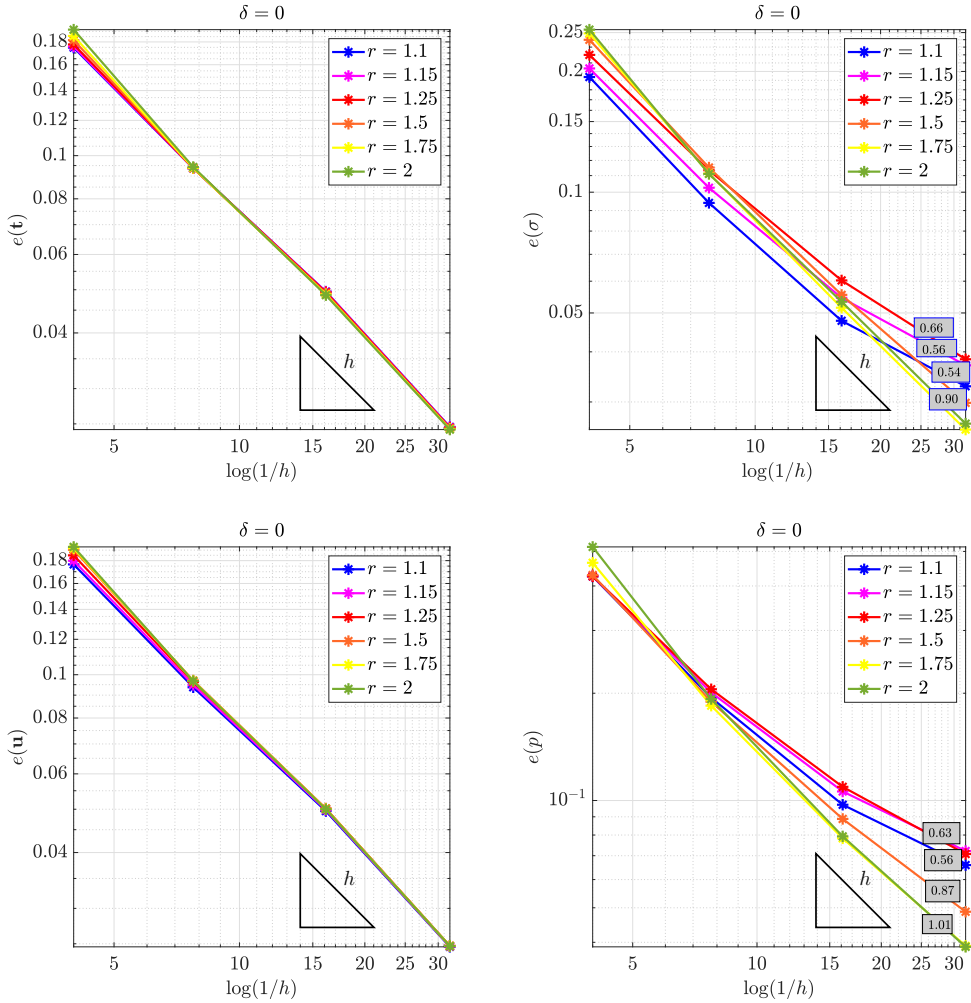


Fig. 7.3. Convergence results with VORONOI mesh, $\delta = 0$, $j = l = 1$ and $\Delta t = h$.

7.3. Example 3: Flow past a cylinder

This well-known benchmark problem has been extensively studied by numerous researchers^{59,57,34}. The geometry, depicted in Fig. 7.7, focuses on fluid flow around a cylinder. The cylinder is positioned in an incompressible flow with its center located at $(0.25, 0.2)$ and diameter 0.1. Inflow and outflow boundary conditions are imposed on the left and right edges, respectively, as detailed below:

$$\mathbf{g} = \begin{cases} \left(\frac{0.3}{0.41^2} 4x_2(0.41 - x_2), 0 \right)^t & \text{on } \Gamma_{\text{in}} \cup \Gamma_{\text{out}}, \\ \mathbf{0} & \text{on } \Gamma \setminus (\Gamma_{\text{in}} \cup \Gamma_{\text{out}}). \end{cases}$$

For this test, we consider the physical and discretization parameters $\mu = 0.1$, $\delta = 1$, $\alpha = 1$, polynomial degree $j = l = 1$, time step $\Delta t = 3 \times 10^{-2}$, final time $t_F = 0.5$, and mesh size $h = 0.018$. Figures 7.8 and 7.9 present the approximate solutions obtained using the first-order mixed virtual element method for two values of r , namely $r = 1.25$ and $r = 2.25$,

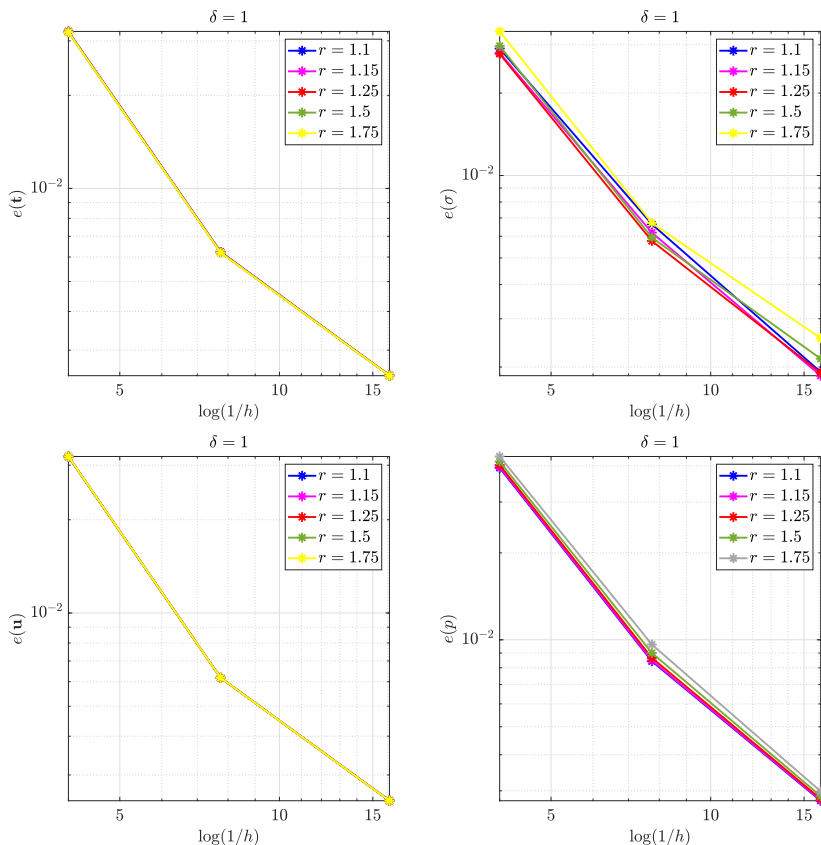


Fig. 7.4. Convergence results with VORONOI mesh, $\delta = 1$, $j = l = 2$ and $\Delta t = h^2$.

respectively. All results are consistent with those reported in Ref. 34 and align with the expected physical behavior as discussed in Ref. 59.

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Table 7.1. Convergence results with VORONOI mesh, $\delta = 1$, $j = l = 2$ and $\Delta t = h^2$.

r	h	$e(\mathbf{t}^n)$	$\mathbf{r}(\mathbf{t}^n)$	$e(\boldsymbol{\sigma}^n)$	$\mathbf{r}(\boldsymbol{\sigma}^n)$	$e(\mathbf{u}^n)$	$\mathbf{r}(\mathbf{u}^n)$	$e(p^n)$	$\mathbf{r}(p^n)$
1.1	2.500e-01	3.21887e-02	-	2.90597e-02	-	3.19826e-02	-	3.94007e-02	-
	1.291e-01	6.23638e-03	2.483	6.64266e-03	2.233	6.17594e-03	2.488	8.43036e-03	2.333
	6.202e-02	2.48505e-03	1.254	1.91815e-03	1.694	2.47853e-03	1.245	2.77168e-03	1.517
1.15	2.500e-01	3.21659e-02	-	2.86494e-02	-	3.19797e-02	-	3.94681e-02	-
	1.291e-01	6.23238e-03	2.483	6.47826e-03	2.249	6.17607e-03	2.488	8.44394e-03	2.333
	6.202e-02	2.48473e-03	1.254	1.89251e-03	1.678	2.47861e-03	1.245	2.77453e-03	1.518
1.25	2.500e-01	3.21248e-02	-	2.80238e-02	-	3.19748e-02	-	3.96395e-02	-
	1.291e-01	6.22537e-03	2.483	6.18768e-03	2.285	6.17621e-03	2.487	8.48178e-03	2.333
	6.202e-02	2.48416e-03	1.253	1.85946e-03	1.639	2.47872e-03	1.245	2.78196e-03	1.520
1.5	2.500e-01	3.20483e-02	-	2.78055e-02	-	3.19686e-02	-	4.03121e-02	-
	1.291e-01	6.21246e-03	2.482	5.76941e-03	2.379	6.17595e-03	2.487	8.65782e-03	2.327
	6.202e-02	2.48296e-03	1.250	1.90415e-03	1.511	2.47861e-03	1.245	2.81390e-03	1.532
1.75	2.500e-01	3.20103e-02	-	2.97155e-02	-	3.19756e-02	-	4.14191e-02	-
	1.291e-01	6.20493e-03	2.482	5.93705e-03	2.436	6.17504e-03	2.488	9.00761e-03	2.308
	6.202e-02	2.48176e-03	1.249	2.14408e-03	1.389	2.47790e-03	1.245	2.87549e-03	1.557

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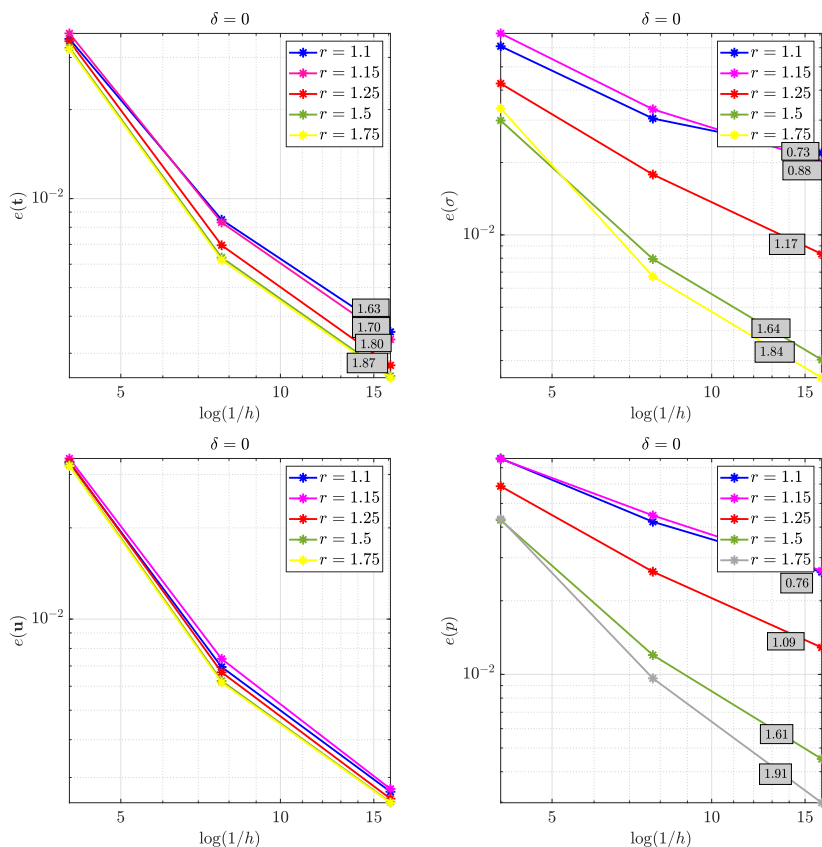


Fig. 7.5. Convergence results with VORONOI mesh, $\delta = 0$, $j = l = 2$ and $\Delta t = h^2$.

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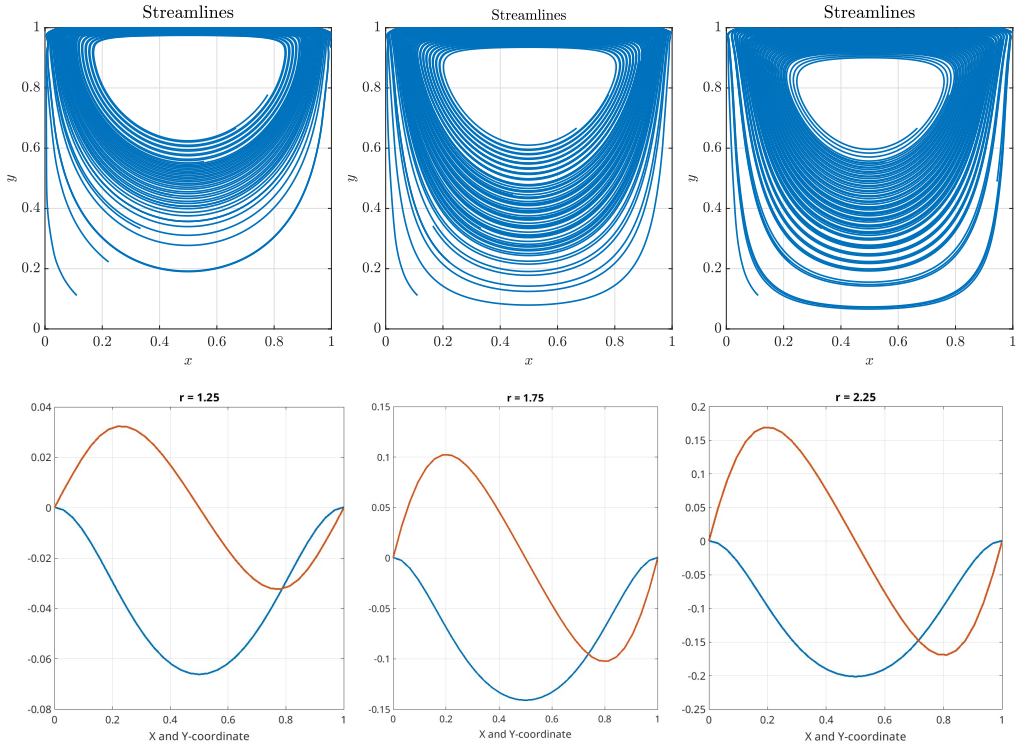


Fig. 7.6. Numerical results for the test case of Section 5.2. Top: velocity magnitude contours computed on a Cartesian mesh of size 32×32 with $j = 2$. Bottom: horizontal component $u_{1,h}$ of the velocity along the vertical centreline $x_1 = 0.5$ and vertical component $u_{2,h}$ of the velocity along the horizontal centreline $x_2 = 0.5$.

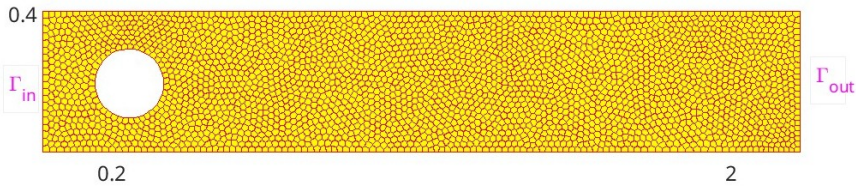


Fig. 7.7. Example 3. An illustration of the mesh.

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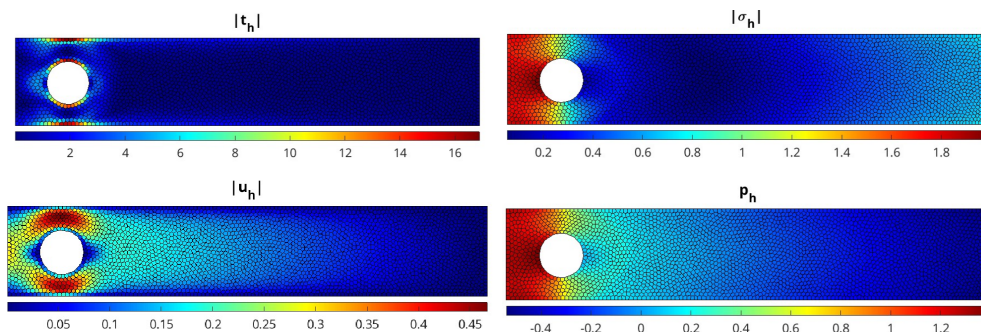


Fig. 7.8. Example 3. Snapshots of the numerical solution for $r = 1.25$.

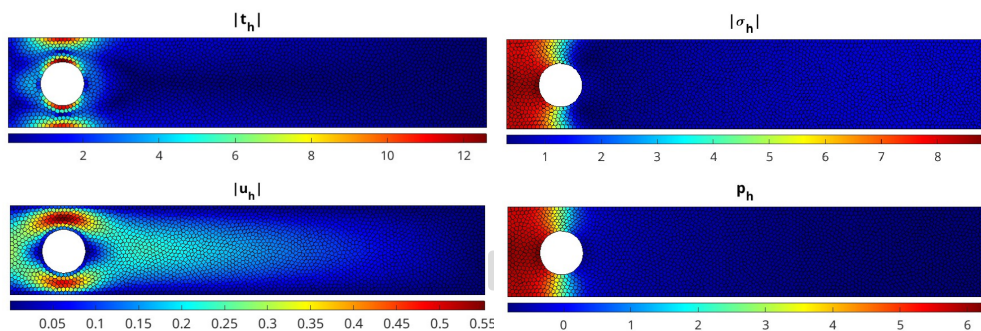


Fig. 7.9. Example 3. Snapshots of the numerical solution for $r = 2.25$.

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