A stabilization-free mixed virtual element approximation for unsteady non-Newtonian pseudoplastic Stokes flows

Zeinab Gharibi* Mehdi Dehghan[†] Ricardo Ruiz-Baier[‡]

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Abstract

In this work, we propose and investigate a mixed virtual element method within a Banach space framework to numerically study the unsteady motion of non-Newtonian pseudoplastic Stokes flows. Given the increasing focus on non-Newtonian fluids problem where stress plays a critical role, our approach introduces new unknowns such as the rate of strain and pseudostress tensors. This results in a mixed variational formulation that including velocity and these additional unknowns within a Banach space setting. We establish the well-posedness of weak solution and derive stability bounds using classical results from nonlinear monotone operator theory. The discretization of the problem in both space and time is carried out using an $\mathbb{H}(\mathbf{div})$-conforming virtual element method and the implicit Euler method, respectively. In particular, for the spatial discretization, the pseudostress is approximated using a virtual element subspace of $\mathbb{H}(\mathbf{div};\Omega)$, while piecewise polynomial subspaces of degree j are employed to approximate the velocity and rate of strain tensor. The scheme handles nonlinear terms implicitly, and its well-posedness and unconditional stability have been proven. Furthermore, a convergence analysis is performed for all variables in their natural norms, demonstrating optimal rates of convergence with respect to both spatial mesh size and time step. Finally, we conduct several numerical experiments to validate the effectiveness and accuracy of the proposed method.

1 Introduction

Flow problems are often influenced by the viscosity of the fluid, which varies from one fluid to another depending on the relationship between shear stress and shear rate. Fluids can be categorized as either Newtonian or non-Newtonian based on their response to shear stress; Newtonian fluids exhibit a linear relationship between shear stress and shear rate, whereas non-Newtonian fluids show a non-linear relationship between these variables. The latter group, which includes fluids used in the petroleum industry, chemical-pharmaceutical processes, (bio)polymer manufacturing, and food production, is often modeled using power-law and Carreau–Yasuda models. These applications have led to a growing focus among researchers on developing numerical methods for non-Newtonian fluids across a wide range of flow configurations and domains.

To the best of the authors' knowledge, Baranger and Najib [3] were among the first to analyze finite element methods (FEMs) for fluids with viscosity described by the Carreau or power law. Subsequently, Du and Gunzburger [28] investigated FEMs for non-Newtonian flows with viscosity governed by the Ladyzhenskaya law. Later, Barrett et al. [4, 5] established error bounds for velocity and pressure in appropriate quasi-norms for non-Newtonian flow models with viscosity governed by either the Carreau

*Department of Applied Mathematics, Faculty of Mathematics and Computer Sciences, Amirkabir University of Technology (Tehran Polytechnic), No. 424, Hafez Ave., 15914 Tehran, Iran. Email: z90gharibi@aut.ac.ir

[†]Department of Applied Mathematics, Faculty of Mathematics and Computer Sciences, Amirkabir University of Technology (Tehran Polytechnic), No. 424, Hafez Ave., 15914 Tehran, Iran. Email: mdehghan@aut.ac.ir

[‡]School of Mathematics, Monash University. 9 Rainforest Walk, 3800 Melbourne, Australia. Email: ricardo.ruizbaier@monash.edu.

or power law. These results were further improved in [12, 41] by proving optimal error estimates for fluids exhibiting shear-thinning behavior, characterized by a power-law exponent $r < 2$. In the context of mixed FEMs for non-Newtonian flows, Gatica et al. [31, 32] proposed a mixed FEM for nonlinear Stokes models in quasi-Newtonian fluids, introducing flux and the tensor gradient of velocity as unknowns, and provided a *priori* and a *posteriori* error analyses for the associated lowest-order Galerkin scheme. Howell [42] introduced a finite element technique for a dual-mixed formulation of Stokes and nonlinear Stokes problems, incorporating the velocity gradient, velocity, and pseudostress, and also conducted an error analysis. Other significant contributions to the numerical approximation of non-Newtonian flows include [14, 21, 27, 33, 37, 40, 43].

In recent years, there has been growing interest within the scientific community in developing numerical techniques to solve partial differential equations on polygonal and polytopal meshes in two- and three-dimensional domains, respectively. One of the significant advancements in this area is the virtual element method (VEM), initially introduced to address the primal conforming Poisson problem in [6] as a more general approach compared to the traditional H^1-conforming finite element method. Later, the generalization of mixed finite elements [13] to $\mathbf{H}(\mathrm{div})$-conforming vector fields, known as the mixed virtual element method, was introduced in [17] and further developed in subsequent works [7, 10]. Due to its strong theoretical foundation and ease of implementation, the mixed VEM has been applied to solve various linear and nonlinear problems, particularly in fluid mechanics, using velocity-pressure formulations [1, 2, 8, 9] and pseudostress-velocity-based approaches [18, 20, 30, 36]. Specifically, focusing on the latter approach, the authors in [18] introduced a mixed VEM for the Stokes problem using a pseudostress-velocity formulation, extending the mixed formulation from [34] into the virtual element framework. Similarly, in [20], this method was applied to the Brinkman equation using a pseudostress-based formulation. Further studies have extended this approach to nonlinear fluid mechanics models, such as quasi-Newtonian Stokes [19], Navier–Stokes [36], and Boussinesq equations [30, 38, 39].

According to the above bibliographic discussion, the aim of this paper is to develop and analyse a mixed virtual method within the Banach space framework for the unsteady motion of the non-Newtonian Stokes equation. To achieve this, we introduce symmetric stress and rate of strain tensors as auxiliary unknowns in the non-Newtonian Stokes equation. Subsequently, we eliminate the pressure variable by applying the incompressibility condition and then compute it using a post-processing technique. The discretization for the spatial variables type of additional and primal is based on the virtual and non-virtual spaces in [30], respectively, whereas to discretize of time variable we employ the backward Euler approach. It is worth mentioning that the discrete formulation is entirely independent of the stabilization term, with only the discrete form being related to the inf-sup term. We also prove the solvability of both the continuous and discrete schemes using classical results on nonlinear monotone operators and establish stability estimates for all unknowns without making any assumptions about the data. Unlike in [1], where the VEM is used for the steady motion of the non-Newtonian Stokes equation, our analysis relies solely on the proof of inf-sup condition, which significantly simplifies the model's analysis. Furthermore, we obtain optimal error estimates in natural norms for all unknowns of the same order, with particular attention to the case where $\delta = 0$, corresponding to the power-law equation, which is typically the most challenging regime. Combining the current findings with methods from references [4, 5] would allow the results to be generalized to cases where $\delta > 0$. To the best of the authors' knowledge, this is the first arbitrary-order polytopal approximation method in the mixed framework for non-Newtonian flow models in fluid mechanics.

The rest of the paper is organized as follows. Sec. 2 is devoted to introducing the non-Newtonian Stokes model, providing the variational formulation. The solvability is analyzed using classical results from nonlinear monotone operator theory in Sec. 3. Then, we introduce the mixed VEM in Sec. 4 by following Ref. [36]. This has four main parts, starting with the basic assumption on the mesh, defining the local and global mixed VE spaces, projection operators, and deriving the discrete version of bilinear forms. In Sec. 5, we study the unique solvability of the proposed VEM using a discrete version of nonlinear monotone operator theory developed in Sec. 3 for the continuous case, and then establish the stability bounds. In order to accomplish this, we derive the common estimates about the bilinear forms, as well as the discrete inf-sup condition. In Sec. 6, we study a *priori* error analysis. Finally, the method's performance is illustrated in Sec. 7 through several numerical examples in 2D, both with and without manufactured solutions, confirming the accuracy and flexibility of the proposed mixed VEM.

2 Continuous problem

In this section we introduce the model problem and derive its corresponding weak formulation.

2.1 The model problem

Let $\Omega \subset \mathbb{R}^d$, where $d \in 2,3$, represents a bounded connected open polytopal domain (i.e., polygonal if $d = 2$ and polyhedral if $d = 3$) with boundary Γ which split as $\Gamma = \Gamma_D \cup \Gamma_N$ such that $\Gamma_D \cap \Gamma_N = \emptyset$. Consider given function $\mu : \Omega \to \mathbb{R}$. Throughout the subsequent discussion, we assume the existence of real numbers $\underline{\mu}, \overline{\mu}$ such that, almost everywhere in Ω

$$0 < \underline{\mu} \leq \mu \leq \overline{\mu}\,.$$

Let t_F represent the final time, $\mathbf{f} : \Omega \times (0, t_F] \to \mathbb{R}^d$ denotes volumetric source term. The unsteady non-linear Stokes problem reads [1]: Find the velocity $\mathbf{u} : \Omega \times (0, t_F] \to \mathbb{R}^d$ and the pressure $p : \Omega \times (0, t_F] \to \mathbb{R}$ such that

$$\begin{aligned}
\partial_t \mathbf{u} - \mathbf{div}\big(\mu\, (\delta^\alpha + |\epsilon(\mathbf{u})|^\alpha)^{\frac{r-2}{\alpha}}\, \epsilon(\mathbf{u})\big) + \nabla p \;&=\; \mathbf{f} && \text{in} \quad \Omega_t\,, && (2.1a)\\
\mathrm{div}\,(\mathbf{u}) \;&=\; 0 && \text{in} \quad \Omega_t\,, && (2.1b)\\
\boldsymbol{\sigma}\mathbf{n} \;=\; \mathbf{0} \quad \text{on} \quad \Gamma_{N,t} \quad \text{and} \quad \mathbf{u} \;&=\; \mathbf{0} && \text{on} \quad \Gamma_{D,t}\,, && (2.1c)\\
\mathbf{u}(\cdot, 0) \;&=\; \mathbf{u}_0 && \text{in} \quad \Omega\,, && (2.1d)
\end{aligned}$$

where $\Omega_t := \Omega \times (0, t_F]$, $\Gamma_{D,t} = \Gamma_D \times (0, t_F]$, $\Gamma_{N,t} = \Gamma_N \times (0, t_F]$, $\alpha, r \in [1, \infty)$, and $\delta \geq 0$ and $\boldsymbol{\varepsilon}(\mathbf{u})$ is the symmetric part of the velocity gradient tensor, which defined as $\boldsymbol{\varepsilon}(\mathbf{u}) = \big(\nabla\mathbf{u} + (\nabla\mathbf{u})^\top\big)/2$. Note that the nonlinear shear stress-strain rate defined in the first term on the left-hand side of (2.1a) represents the Carreau–Yasuda law, which generalizes the Carreau model for the case where $\alpha = 2$. In addition, the classical power-law model can be derived from (2.1a) by considering $\delta = 0$. Shear-thinning behavior, which is observed in most real fluids described by a constitutive relation, corresponds to the case where $r < 2$. It is clear that for $r = 2$, problem (2.1) simplifies to the standard Stokes system for Newtonian fluids. For brevity, we will focus on the pseudoplastic case where $r < 2$, which is most commonly encountered in practical applications and presents greater challenges for theoretical analysis. Even so, similar results for the dilating case $r > 2$ can be established by applying the arguments that follow.

Next, to attain a mixed formulation for (2.1), we follow the approach outlined in [35]. Specifically, we introduce the strain rate and stress tensors defined as follows, respectively

$$\mathbf{t} \;:=\; \nabla\mathbf{u}\,, \quad \text{and} \quad \boldsymbol{\sigma} \;:=\; \mu\, \left(\delta^\alpha + |\mathbf{t}_{\mathtt{sym}}|^\alpha\right)^{\frac{r-2}{\alpha}}\, \mathbf{t}_{\mathtt{sym}} - p\mathbb{I} \quad \text{in} \quad \Omega_t\,. \tag{2.2}$$

Thus, applying the trace operator to the tensor $\boldsymbol{\sigma}$, and using the condition (2.1b), we obtain

$$p \;=\; -\frac{1}{2}\,\mathrm{tr}(\boldsymbol{\sigma}) \quad \text{in} \quad \Omega_t\,. \tag{2.3}$$

Thus, substituting (2.3) back into (2.2) and performing some straightforward calculations, we find out that the problem (2.1) can be equivalently rewritten as follows: Find $(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u})$, belong to the appropriate spaces specified in the following section, such that

$$\mathbf{t} = \nabla \mathbf{u} \quad \text{in} \quad \Omega_t \,, \tag{2.4a}$$

$$\boldsymbol{\sigma}^{\mathrm{d}} = \mu \, \left(\delta^{\alpha} + |\mathbf{t}_{\mathtt{sym}}|^{\alpha}\right)^{\frac{r-2}{\alpha}} \mathbf{t}_{\mathtt{sym}} \quad \text{in} \quad \Omega_t \,, \tag{2.4b}$$

$$\partial_t \mathbf{u} - \mathbf{div}\,(\boldsymbol{\sigma}) = \mathbf{f} \quad \text{in} \quad \Omega_t \,, \tag{2.4c}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma_{D,t} \quad \boldsymbol{\sigma}\mathbf{n} = \mathbf{0} \quad \text{on} \quad \Gamma_{N,t} \,, \tag{2.4d}$$

$$\mathbf{u}(\cdot, 0) = \mathbf{u}_0 \quad \text{in} \quad \Omega \quad \text{and} \quad \int_{\Omega} \mathrm{tr}\big(\boldsymbol{\sigma}\big) = 0 \,. \tag{2.4e}$$

We end this section by noting that the stress-strain law given by (2.4b) satisfies the following assumption (see [14, Appendix A]).

Assumption 2.1. The shear stress-strain rate law $\boldsymbol{\sigma}^{\mathrm{d}}$ presented in (2.4b) is a Caratheodory function that satisfies $\boldsymbol{\sigma}^{\mathrm{d}}(\mathbf{0}) = \mathbf{0}$, and for a fixed $r \in (1, 2]$ the following properties hold:

(i) (*Hölder continuity*). there exists a constant $\sigma_* > 0$, depending only on $\overline{\mu}$, α, r, satisfying

$$\left|\boldsymbol{\sigma}^{\mathrm{d}}(\mathbf{t}_{\mathtt{sym}}) - \boldsymbol{\sigma}^{\mathrm{d}}(\mathbf{s}_{\mathtt{sym}})\right| \leq \sigma_* \left(\delta^r + |\mathbf{t}_{\mathtt{sym}}|^r + |\mathbf{s}_{\mathtt{sym}}|^r\right)^{\frac{r-2}{r}} \left|\mathbf{t}_{\mathtt{sym}} - \mathbf{s}_{\mathtt{sym}}\right|, \tag{2.5}$$

(ii) (*Strong monotonicity*). there exists a constant $\sigma^* > 0$, depending only on $\underline{\mu}$, α, r, satisfying

$$\left(\boldsymbol{\sigma}^{\mathrm{d}}(\mathbf{t}_{\mathtt{sym}}) - \boldsymbol{\sigma}^{\mathrm{d}}(\mathbf{s}_{\mathtt{sym}})\right) : (\mathbf{t}_{\mathtt{sym}} - \mathbf{s}_{\mathtt{sym}}) \geq \sigma^* \left(\delta^r + |\mathbf{t}_{\mathtt{sym}}|^r + |\mathbf{s}_{\mathtt{sym}}|^r\right)^{\frac{r-2}{r}} \left|\mathbf{t}_{\mathtt{sym}} - \mathbf{s}_{\mathtt{sym}}\right|^2 . \tag{2.6}$$

2.2 The variational formulation

In this section, we focus on deriving our mixed variational formulation for the system. Specifically, by multiplying (2.4a), (2.4b) and (2.4c) by appropriate test functions $\boldsymbol{\tau}$, $\mathbf{s}$ and $\mathbf{v}$, respectively, we get

$$\int_{\Omega} \mathbf{t} : \boldsymbol{\tau} = \int_{\Omega} \nabla \mathbf{u} : \boldsymbol{\tau} \quad \text{for a.e } t \in (0, t_F) \,, \tag{2.7a}$$

$$-\int_{\Omega} \boldsymbol{\sigma}^{\mathrm{d}} : \mathbf{s} + \int_{\Omega} \mu \, \left(\delta^{\alpha} + |\mathbf{t}_{\mathtt{sym}}|^{\alpha}\right)^{\frac{r-2}{\alpha}} \mathbf{t}_{\mathtt{sym}} : \mathbf{s} = 0 \quad \text{for a.e } t \in (0, t_F) \,, \tag{2.7b}$$

$$\int_{\Omega} \partial_t \mathbf{u} \cdot \mathbf{v} - \int_{\Omega} \mathbf{div}(\boldsymbol{\sigma}) \cdot \mathbf{v} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \quad \text{for a.e } t \in (0, t_F) \,, \tag{2.7c}$$

We begin by (2.7b) as it involves a key expression. More precisely, by applying the Hölder inequality and the continuity property given in (2.5) with $\mathbf{s} = \mathbf{0}$, and denoting s as the conjugate of r defined by $s := \frac{r}{r-1}$, we find that the second term of (2.7b) is bounded as follows

$$\begin{aligned} \left| \int_{\Omega} \mu \, \left(\delta^{\alpha} + |\mathbf{t}_{\mathtt{sym}}|^{\alpha}\right)^{\frac{r-2}{\alpha}} \mathbf{t}_{\mathtt{sym}} : \mathbf{s} \right| &\leq \left(\int_{\Omega} \left| \mu \, \left(\delta^{\alpha} + |\mathbf{t}_{\mathtt{sym}}|^{\alpha}\right)^{\frac{r-2}{\alpha}} \mathbf{t}_{\mathtt{sym}} \right|^s \right)^{1/s} \|\mathbf{s}\|_{0,r;\Omega} \\ &\leq \sigma_* \left(\int_{\Omega} \left(\delta^r + |\mathbf{t}_{\mathtt{sym}}|^r\right)^{\frac{s(r-2)}{r}} \left|\mathbf{t}_{\mathtt{sym}}\right|^s \right)^{1/s} \|\mathbf{s}\|_{0,r;\Omega} \,. \end{aligned} \tag{2.8}$$

We observe that, since δ is non-negative and $r < 2$, the following inequality holds

$$\left(\delta^r + |\mathbf{t}_{\mathtt{sym}}|^r\right)^{\frac{r-2}{r}} \leq \left(2^{1-r} |\mathbf{t}_{\mathtt{sym}}|^r\right)^{\frac{r-2}{r}} .$$

Then, applying the above result to bound the term on the right-hand side of (2.8), yields

$$\left| \int_\Omega \mu \, \left(\delta^\alpha + |\mathbf{t}_{\mathtt{sym}}|^\alpha\right)^{\frac{r-2}{\alpha}} \mathbf{t}_{\mathtt{sym}} : \mathbf{s} \right| \,\leq\, \sigma_* 2^{s(r-2)} \, \|\mathbf{t}_{\mathtt{sym}}\|_{0,r;\Omega}^{r-1} \, \|\mathbf{s}\|_{0,r;\Omega} \,, \tag{2.9}$$

so we deduce that for $t \in \mathrm{J}$ the second term in (2.7b) makes sense for $\mathbf{t}(t), \mathbf{s} \in \mathbb{L}^r(\Omega)$, and consequently, the first terms in (2.7a) and (2.7b) are well-defined provided that $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}(t)$ belong to $\mathbb{L}^s(\Omega)$. Additionally, we deduce from (2.4a) that $\mathbf{u}$ should be initially sought in $\mathbf{W}^{1,r}(\Omega)$. Now, we introduce the Banach space

$$\mathbb{H}^s(\mathbf{div};\Omega) \,:=\, \Big\{ \boldsymbol{\tau} \in \mathbb{L}^s(\Omega) : \quad \mathbf{div}\,\boldsymbol{\tau} \in \mathbf{L}^2(\Omega) \Big\} \,,$$

which is endowed with the natural norm defined by

$$\|\boldsymbol{\tau}\|_{s;\mathbf{div};\Omega}^2 \,:=\, \|\boldsymbol{\tau}\|_{0,s;\Omega}^2 + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,\Omega}^2 \,.$$

Then, proceeding as in [23, Theorem 2.2], it is easy to show that, given r, there holds

$$\langle \boldsymbol{\tau}\mathbf{n}, \mathbf{v} \rangle_\Gamma \,=\, \int_\Omega \Big\{ \boldsymbol{\tau} : \nabla \mathbf{v} + \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\tau}) \Big\} \qquad \forall\, (\boldsymbol{\tau}, \mathbf{v}) \in \mathbb{H}^s(\mathbf{div};\Omega) \times \mathbf{W}^{1,r}(\Omega) \,, \tag{2.10}$$

where$\langle \cdot, \cdot \rangle$ stands for the duality pairing between $\mathbf{W}^{-1/s,s}(\Gamma)$ and $\mathbf{W}^{1/s,r}(\Gamma)$.

In this way, defining the subspace of $\mathbb{H}^s(\mathbf{div};\Omega)$ by

$$\mathbb{H}^s_{\Gamma_N}(\mathbf{div};\Omega) \,:=\, \Big\{ \boldsymbol{\tau} \in \mathbb{H}^s(\mathbf{div};\Omega) : \quad \langle \boldsymbol{\tau}\mathbf{n}, \mathbf{v} \rangle_\Gamma = \mathbf{0} \quad \forall\, \mathbf{v} \in \mathbf{W}^{1,r}_{0,\Gamma_D}(\Omega) \Big\} \,,$$

and applying (2.10) for the given $\boldsymbol{\tau} \in \mathbb{H}^s_{\Gamma_N}(\mathbf{div};\Omega)$ and $\mathbf{u} \in \mathbf{W}^{1,r}(\Omega)$, while employing the Dirichlet boundary condition on $\mathbf{u}$, (2.7a) can be rewritten as

$$\int_\Omega \mathbf{t} : \boldsymbol{\tau} + \int_\Omega \mathbf{u} \cdot \mathbf{div}(\boldsymbol{\tau}) \,=\, 0 \,, \tag{2.11}$$

It is easy to notice that, the first term is well-defined, whereas the second term makes sense for $\mathbf{u} \in \mathbf{L}^2(\Omega)$, thanks to the Sobolev embeddings $\mathbf{W}^{1,r}(\Omega) \hookrightarrow \mathbf{L}^r(\Omega)$ when $r < d$ and $\mathbf{L}^2(\Omega) \hookrightarrow \mathbf{L}^r(\Omega)$. In addition, due to the second column of (2.4e), it follows that we should look for $\boldsymbol{\sigma}$ in $\mathbb{H}^s_{0,\Gamma_N}(\mathbf{div};\Omega)$, where

$$\mathbb{H}^s_{0,\Gamma_N}(\mathbf{div};\Omega) \,:=\, \Big\{ \boldsymbol{\tau} \in \mathbb{H}^s_{\Gamma_N}(\mathbf{div};\Omega) : \quad \int_\Omega \mathrm{tr}(\boldsymbol{\sigma}) \,=\, 0 \Big\} \,.$$

In turn, considering the free trace property of $\mathbf{t}$, we look for these unknowns in $\mathbb{L}^r_{\mathtt{tr}}(\Omega)$, where

$$\mathbb{L}^r_{\mathtt{tr}}(\Omega) \,:=\, \Big\{ \mathbf{t} \in \mathbb{L}^r(\Omega) : \quad \mathrm{tr}(\mathbf{t}) = 0 \Big\} \,,$$

This implies that (2.7b) can be equivalently rewritten as

$$- \int_\Omega \boldsymbol{\sigma} : \mathbf{s} + \int_\Omega \mu \, \left(\delta^\alpha + |\mathbf{t}_{\mathtt{sym}}|^\alpha\right)^{\frac{r-2}{\alpha}} \mathbf{t}_{\mathtt{sym}} : \mathbf{s} \,=\, 0 \qquad \forall\, \mathbf{s} \in \mathbb{L}^r_{\mathtt{tr}}(\Omega) \,.$$

Consequently, the weak formulation (2.7) is well defined if we choose the spaces $\mathbb{Q} := \mathbb{L}^r_{\mathtt{tr}}(\Omega)$, $\mathbb{X} := \mathbb{H}^s_{0,\Gamma_N}(\mathbf{div};\Omega)$, $\mathbf{Y} := \mathbf{L}^2(\Omega)$, with their respective norms $\|\cdot\|_{0,r;\Omega}$, $\|\cdot\|_{s;\mathbf{div};\Omega}$, $\|\cdot\|_{0,\Omega}$.

According to the above, we arrive at the variational problem: For a.e $t \in \mathrm{J}$, find $\mathbf{t}(t) \in \mathbb{L}^r_{\mathtt{tr}}(\Omega)$, $\boldsymbol{\sigma}(t) \in \mathbb{H}^s_{0,\Gamma_N}(\mathbf{div};\Omega)$ and $\mathbf{u}(t) \in \mathbf{L}^2(\Omega)$, such that $\mathbf{u}(0) = \mathbf{u}_0$ and

$$\begin{aligned}
&[a(\mathbf{t}), \mathbf{s}] + [b_1(\mathbf{s}), \boldsymbol{\sigma}] &&= 0 && \forall\, \mathbf{s} \in \mathbb{L}^r_{\mathtt{tr}}(\Omega)\,, && (2.12a)\\
&[b_1(\mathbf{t}), \boldsymbol{\tau}] + [b_2(\mathbf{u}), \boldsymbol{\tau}] &&= 0 && \forall\, \boldsymbol{\tau} \in \mathbb{H}^s_{0,\Gamma_N}(\mathbf{div};\Omega), && (2.12b)\\
&\frac{\partial}{\partial t}[c(\mathbf{u}), \mathbf{v}] + [b_2(\mathbf{v}), \boldsymbol{\sigma}] &&= [F, \mathbf{v}] && \forall\, \mathbf{v} \in \mathbf{L}^2(\Omega)\,, && (2.12c)
\end{aligned}$$

where the operators $a : \mathbb{Q} \to \mathbb{Q}'$, $b_1 : \mathbb{Q} \to \mathbb{X}'$, $b_2 : \mathbf{Y} \to \mathbb{X}'$, $c : \mathbf{Y} \to \mathbf{Y}'$, are defined, respectively, as

$$[a(\mathbf{t}), \boldsymbol{\tau}] \ := \ \int_\Omega \mu \, \left(\delta^\alpha + |\mathbf{t}_{\mathtt{sym}}|^\alpha\right)^{\frac{r-2}{\alpha}} \mathbf{t}_{\mathtt{sym}} : \mathbf{s}\,, \tag{2.13a}$$

$$[b_1(\mathbf{s}), \boldsymbol{\tau}] \ := \ -\int_\Omega \boldsymbol{\tau} : \mathbf{s}\,, \tag{2.13b}$$

$$[b_2(\mathbf{v}), \boldsymbol{\tau}] \ := \ -\int_\Omega \mathbf{div}(\boldsymbol{\tau}) \cdot \mathbf{v}\,, \tag{2.13c}$$

$$[c(\mathbf{u}), \mathbf{v}] \ := \ \int_\Omega \mathbf{u} \cdot \mathbf{v}\,, \tag{2.13d}$$

for all $(\mathbf{s}, \boldsymbol{\tau}, \mathbf{v}) \in \mathbb{Q} \times \mathbb{X} \times \mathbf{Y}$.

In turn, $F \in \mathbf{Y}'$ is the bounded linear functional defined by

$$[F, \mathbf{v}] \ := \ \int_\Omega \mathbf{f} \cdot \mathbf{v}\,. \tag{2.14}$$

In all the terms above, $[\cdot,\cdot]$ denotes the duality pairing induced by the corresponding operators. Let us define the global unknown and space:

$$\vec{\mathbf{u}} \ := \ (\mathbf{u}, \mathbf{t}) \in \mathbf{V} \ := \ \mathbf{L}^2(\Omega) \times \mathbb{L}^r_{\mathtt{tr}}(\Omega)\,, \tag{2.15}$$

where $\mathbf{V}$ is endowed with the norm

$$\|\vec{\mathbf{u}}\|^2_{\mathbf{V}} \ = \ \|(\mathbf{v}, \mathbf{s})\|^2_{\mathbf{V}} \ = \ \|\mathbf{v}\|^2_{0,\Omega} + \|\mathbf{s}\|^2_{0,r;\Omega} \qquad \forall\, \vec{\mathbf{u}} := (\mathbf{v}, \mathbf{s}) \in \mathbf{V}\,.$$

Now, thanks to the above notation, it is easy to see that (2.12) can be rewritten equivalently as: Find $(\vec{\mathbf{u}}, \boldsymbol{\sigma}) : [0, t_F] \to \mathbf{V} \times \mathbb{H}^s_{0,\Gamma_N}(\mathbf{div};\Omega)$ such that

$$\frac{\partial}{\partial t}[\mathcal{C}(\vec{\mathbf{u}}(t)), \vec{\mathbf{v}}] + [\mathcal{A}(\vec{\mathbf{u}}(t)), \vec{\mathbf{v}}] + [\mathcal{B}(\vec{\mathbf{v}}), \boldsymbol{\sigma}(t)] \ = \ [\mathcal{F}(t), \vec{\mathbf{v}}] \qquad \forall\, \vec{\mathbf{v}} \in \mathbf{V}\,, \tag{2.16a}$$

$$[\mathcal{B}(\vec{\mathbf{u}}(t)), \boldsymbol{\tau}] \ = \ 0 \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}^s_{0,\Gamma_N}(\mathbf{div};\Omega)\,, \tag{2.16b}$$

where, the operators $\mathcal{A} : \mathbf{V} \to \mathbf{V}'$, $\mathcal{B} : \mathbf{V} \to \mathbb{X}'$ and $\mathcal{C} : \mathbf{V} \to \mathbf{V}'$ are defined by

$$[\mathcal{A}(\vec{\mathbf{w}}), \vec{\mathbf{v}}] \ := \ [a(\mathbf{r}, \mathbf{s})] \qquad \forall\, \vec{\mathbf{w}} = (\mathbf{w}, \mathbf{r}), \vec{\mathbf{v}} = (\mathbf{v}, \mathbf{s}) \in \mathbf{V}\,, \tag{2.17a}$$

$$[\mathcal{B}(\vec{\mathbf{v}}), \boldsymbol{\tau}] \ := \ [b_1(\mathbf{s}), \boldsymbol{\tau}] + [b_2(\mathbf{v}), \boldsymbol{\tau}] \qquad \forall\, (\vec{\mathbf{v}}, \boldsymbol{\tau}) = ((\mathbf{v}, \mathbf{s}), \boldsymbol{\tau}) \in \mathbf{V} \times \mathbb{X}\,, \tag{2.17b}$$

$$[\mathcal{C}(\vec{\mathbf{w}}), \vec{\mathbf{v}}] \ := \ [c(\mathbf{w}), \mathbf{v}] \qquad \forall\, \vec{\mathbf{w}},\, \vec{\mathbf{v}} \in \mathbf{V}\,, \tag{2.17c}$$

whereas the functional $\mathcal{F} \in \mathbf{V}'$ is set as

$$[\mathcal{F}, \vec{\mathbf{v}}] \ := \ [F, \mathbf{v}]\,. \tag{2.18}$$

3 Well-posedness of the continuous problem

In this section we proceed similarly to [24] to establish existence of a solution to problem (2.16). More precisely, we will recall several results that will be utilized in the upcoming analysis.

3.1 Some abstract results

We start by recalling the important result from [45, Theorem IV.6.1(b)], which will help us demonstrate the existence of a solution to problem (2.16).

Theorem 3.1. *Given V_n a seminorm space, which obtained from a symmetric and non-negative bilinear form $n(\cdot,\cdot)$, we let $\mathcal{N} : V_n \to V_n'$ be the bounded linear operator induced by n, which is defined by*

$$\mathcal{N}x(y) \;=\; n(x,y) \qquad \forall\, x, y \in V_n\,.$$

In addition, let D be a dense subspace of V_n, $\mathcal{M} : D \to V_n'$ be linear operator and $N(\mathcal{N})$ and $N(\mathcal{M})$ be the respective null spaces of operators $\mathcal{N}$ and $\mathcal{M}$. Assume that:

i) *$\mathcal{M}$ is monotone, that is,*

$$[\mathcal{M}(x) - \mathcal{M}(y), x - y] \;\geq\; 0 \qquad \forall\, x, y \in D\,.$$

ii) *$N(\mathcal{N}) \cap D \subset N(\mathcal{M})$ and $\mathcal{N} + \mathcal{M} : D \to V_m'$ is onto.*

Then for every $f \in W^{1,1}(\mathrm{J}; V_n')$ and $u_0 \in D$ there exists a solution of $\partial_t(\mathcal{N}u)(t) + \mathcal{M}u(t) = f(t)$, $t > 0$, with $\big(\mathcal{N}u\big)(0) = \mathcal{N}u_0$.

Furthermore, to establish hypothesis ii) in Theorem 3.1, we will need the following abstract result from [22, Theorem 3.1].

Theorem 3.2. *Let X_1, X_2 and Y be separable and reflexive Banach spaces, X_1 and X_2 being uniformly convex, and set $X = X_1 \times X_2$. Let $\mathcal{A} : X \to X'$ be a nonlinear operator, $\mathcal{B} \in \mathcal{L}(X, Y')$, and let K be the kernel of $\mathcal{B}$, that is,*

$$K \;:=\; \Big\{ v \in X : \quad [\mathcal{B}(v), q] \;=\; 0 \quad \forall\, q \in Y \Big\}\,.$$

Assume that

(i) *$\mathcal{A}$ is hemi-continuous, that is, for each $u, v \in X$*

$$J : \mathrm{R} \to \mathrm{R}, \quad t \to J(t) = [\mathcal{A}(u + tv), v] \quad \textit{is continuous}.$$

(ii) *there exist constants $L > 0$ and $p_1, p_2 \geq 1$, such that*

$$\|\mathcal{A}(u) - \mathcal{A}(v)\|_{X'} \;\leq\; L \sum_{j=1}^{2} \Big\{ \|u_j - v_j\|_{X_j} + \Big(\|u_j\|_{X_j} + \|v_j\|_{X_j} \Big)^{p_j - 2} \; \|u_j - v_j\|_{X_j} \Big\}\,,$$

for all $u = (u_1, u_2)$, $v = (v_1, v_2) \in X$.

(iii) *the family of operators $\Big\{ \mathcal{A}(\cdot + t) : V \to V' : \quad t \in X \Big\}$ is uniformly strictly monotone, that is there exist $\gamma > 0$ and $p_1, p_2 \geq 1$, such that*

$$[\mathcal{A}(u + t) - \mathcal{A}(v + t), u - v] \;\geq\; \gamma \Big\{ \|u_1 - v_1\|_{X_1}^{p_1} + \|u_2 - v_2\|_{X_2}^{p_2} \Big\}\,,$$

for all $t \in X$, and for all $u = (u_1, u_2)$, $v = (v_1, v_2) \in V$.

(iv) *there exist $\beta > 0$ such that*

$$\sup_{v \in X} \frac{[\mathcal{B}(v), q]}{\|v\|_X} \;\geq\; \beta \quad \forall\, q \in Y\,.$$

Then, for each $(\mathcal{F}, \mathcal{G}) \in X' \times Y'$ there exists a unique $(u, p) \in X \times Y$ such that

$$\begin{aligned} [\mathcal{A}(u), v] \;+\; [\mathcal{B}(v), p] \;&=\; \mathcal{F}(v) \qquad \forall\, v \in X\,, \\ [\mathcal{B}(u), q] \;&=\; \mathcal{G}(q) \qquad \forall\, q \in Y\,. \end{aligned}$$

Then, we rewrite the problem (2.16) in notations of Theorem 3.1. Specifically, we define the operators

$$\mathcal{N} := \begin{pmatrix} \mathcal{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} : V_n \to V_n' \quad \text{and} \quad \mathcal{M} := \begin{pmatrix} \mathcal{A} & \mathcal{B}' \\ -\mathcal{B} & \mathbf{0} \end{pmatrix} : D \to V_n', \tag{3.1}$$

with

$$V_n := \mathbf{V} \times \mathbb{X}, \quad u := \begin{pmatrix} \vec{\mathbf{u}} \\ \boldsymbol{\sigma} \end{pmatrix}, \quad V_n' := \left(\mathbf{L}^2(\Omega) \times \{\mathbf{0}\}\right) \times \{\mathbf{0}\}, \tag{3.2}$$

and

$$D := \left\{ (\vec{\mathbf{u}}, \boldsymbol{\sigma}) \in V_n : \quad \mathcal{M}(\vec{\mathbf{u}}, \boldsymbol{\sigma}) \in V_n' \right\}. \tag{3.3}$$

Finally, we derive the stability properties of the operators $\mathcal{N}$ and $\mathcal{M}$. First, we observe that the operators $\mathcal{B}$, $\mathcal{C}$ and functional $\mathcal{F}$ are linear. Additionally, employing Hölder and Cauchy–Schwarz inequalities, we obtain

$$\left|[\mathcal{B}(\vec{\mathbf{v}}), \boldsymbol{\tau}]\right| \leq \|\vec{\mathbf{v}}\|_{\mathbf{V}} \, \|\boldsymbol{\tau}\|_{s,\mathbf{div};\Omega} \qquad \forall\, (\vec{\mathbf{v}}, \boldsymbol{\tau}) \in \mathbf{V} \times \mathbb{X}, \tag{3.4a}$$

$$\left|[\mathcal{C}(\vec{\mathbf{w}}), \vec{\mathbf{v}}]\right| \leq \|\mathbf{w}\|_{0,\Omega} \, \|\mathbf{v}\|_{0,\Omega} \leq \|\vec{\mathbf{w}}\|_{\mathbf{V}} \, \|\vec{\mathbf{v}}\|_{\mathbf{V}}, \quad [\mathcal{C}(\vec{\mathbf{v}}), \vec{\mathbf{v}}] = \|\mathbf{v}\|_{0,\Omega}^2 \qquad \forall\, \vec{\mathbf{w}}, \vec{\mathbf{v}} \in \mathbf{V}, \tag{3.4b}$$

$$\left|[\mathcal{F}, \vec{\mathbf{v}}]\right| \leq \|\mathbf{f}\|_{0,\Omega} \, \|\vec{\mathbf{v}}\|_{\mathbf{V}} \qquad \forall\, \vec{\mathbf{v}} \in \mathbf{V}. \tag{3.4c}$$

This means that $\mathcal{B}$ and $\mathcal{F}$ are bounded and continuous, whereas $\mathcal{N}$ is bounded, continuous, and monotone. Additionally, by applying the estimate (2.9), it is clear that, the nonlinear operator $\mathcal{A}$ is bounded with upper bound $C_{\mathcal{A}} := \sigma^* \, 2^{s(r-1)}$, that is

$$\left|[\mathcal{A}(\vec{\mathbf{w}}), \vec{\mathbf{v}}]\right| \leq C_{\mathcal{A}} \, \|\mathbf{t}\|_{0,r;\Omega}^{r-1} \, \|\mathbf{s}\|_{0,r;\Omega} \leq C_{\mathcal{A}} \, \|\mathbf{t}\|_{0,r;\Omega}^{r-1} \, \|\vec{\mathbf{v}}\|_{\mathbf{V}}. \tag{3.5}$$

This result, along with (3.4a), implies that $\mathcal{M}$ is bounded and continuous.

Next, we will verify the hypotheses *ii)* of Theorem 3.1 to establish the well-posedness of (2.16). To this end,let us consider the resolvent system associated with (2.16): Find $(\vec{\mathbf{u}}, \boldsymbol{\sigma}) \in \mathbf{V} \times \mathbb{X}$ such that

$$[(\mathcal{C} + \mathcal{A})(\vec{\mathbf{u}}), \vec{\mathbf{v}}] + [\mathcal{B}(\vec{\mathbf{v}}), \boldsymbol{\sigma}] = [\mathcal{F}, \vec{\mathbf{v}}] \qquad \forall\, \vec{\mathbf{v}} \in \mathbf{V}, \tag{3.6a}$$

$$[\mathcal{B}(\vec{\mathbf{u}}), \boldsymbol{\tau}] = 0 \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}^s_{0,\Gamma_N}(\mathbf{div}; \Omega). \tag{3.6b}$$

3.2 The well-posedness of (3.6)

In this section, we use Theorem 3.2 to show the existence of a unique solution to problem (3.6). First, we note that, due to the uniform convexity and separability of $\mathbb{L}^r(\Omega)$ for $r \in (1, 2]$, the spaces $\mathbf{V}$ and $\mathbb{X}$ also exhibit uniform convexity and separability.

Next, we proceed to verify hypothesis ii) of Theorem 3.2, which states a continuity bound for the nonlinear operator $\mathcal{C} + \mathcal{A}$.

Lemma 3.3. *Let $r \in (1, 2]$. Then, there exists L_{nN}, depending on σ_*, s, r, such that*

$$\left\|(\mathcal{C} + \mathcal{A})(\vec{\mathbf{u}}) - (\mathcal{C} + \mathcal{A})(\vec{\mathbf{v}})\right\| \leq L_{\mathrm{nN}} \left\{ \|\mathbf{u} - \mathbf{v}\|_{0,\Omega} + \left(\|\mathbf{t}\|_{0,r;\Omega} + \|\mathbf{s}\|_{0,r;\Omega}\right)^{r-2} \|\mathbf{t} - \mathbf{s}\|_{0,r;\Omega} \right\}. \tag{3.7}$$

Proof. Let $\vec{\mathbf{u}} = (\mathbf{u}, \mathbf{t})$, $\vec{\mathbf{v}} = (\mathbf{v}, \mathbf{s}) \in \mathbf{V} = \mathbf{Y} \times \mathbb{Q}$. Then, due to the linearity of $\mathcal{C}$, and by using the definition of $\mathcal{A}$ (cf. (2.17a)) along with the boundedness estimate (3.4b), we obtain

$$\begin{aligned} \left|[(\mathcal{C} + \mathcal{A})(\vec{\mathbf{u}}), \vec{\mathbf{w}}]\right| &\leq \left|[\mathcal{C}(\vec{\mathbf{u}} - \vec{\mathbf{v}}), \vec{\mathbf{w}}]\right| + \left|[\mathcal{A}(\vec{\mathbf{u}}) - \mathcal{A}(\vec{\mathbf{v}}), \vec{\mathbf{w}}]\right| \\ &\leq \|\mathbf{u} - \mathbf{v}\|_{0,\Omega} \, \|\mathbf{w}\|_{0,\Omega} + \left|[a(\mathbf{t}) - a(\mathbf{s}), \mathbf{r}]\right|. \end{aligned} \tag{3.8}$$

To bound the second term, we now apply the definition of a given by (2.13a) along with assumption 2.1, which gives

$$
\begin{aligned}
\big|[a(\mathbf{t}) - a(\mathbf{s}), \mathbf{r}]\big| &\le \Big| \int_\Omega \mu \left(\left(\delta^\alpha + |\mathbf{t}_{\mathtt{sym}}|^\alpha\right)^{\frac{r-2}{\alpha}} \mathbf{t}_{\mathtt{sym}} - \left(\delta^\alpha + |\mathbf{s}_{\mathtt{sym}}|^\alpha\right)^{\frac{r-2}{\alpha}} \mathbf{s}_{\mathtt{sym}} \right) : \mathbf{r} \Big| \\
&\le \left(\int_\Omega \mu \Big| \left(\delta^\alpha + |\mathbf{t}_{\mathtt{sym}}|^\alpha\right)^{\frac{r-2}{\alpha}} \mathbf{t}_{\mathtt{sym}} - \left(\delta^\alpha + |\mathbf{s}_{\mathtt{sym}}|^\alpha\right)^{\frac{r-2}{\alpha}} \mathbf{s}_{\mathtt{sym}} \Big|^s \right)^{1/s} \|\mathbf{r}\|_{0,r;\Omega} \\
&\le \sigma_* \left(\int_\Omega \left(\delta^r + |\mathbf{t}_{\mathtt{sym}} - \mathbf{s}_{\mathtt{sym}}|^r\right)^{\frac{s(r-2)}{r}} \big|\mathbf{t}_{\mathtt{sym}} - \mathbf{s}_{\mathtt{sym}}\big|^s \right)^{1/s} \|\mathbf{r}\|_{0,r;\Omega} \,,
\end{aligned}
$$

form which, proceeding in a manner similar to the derivation of (2.9), we find that

$$
\begin{aligned}
\big|[a(\mathbf{t}) - a(\mathbf{s}), \mathbf{r}]\big| &\le \sigma_* \, 2^{s(r-2)} \, \|\mathbf{t} - \mathbf{s}\|_{0,r;\Omega}^{r-1} \|\mathbf{r}\|_{0,r;\Omega} \\
&\le \sigma_* \, 2^{s(r-2)} \left(\|\mathbf{t}\|_{0,r;\Omega} + \|\mathbf{s}\|_{0,r;\Omega} \right)^{r-2} \|\mathbf{t} - \mathbf{s}\|_{0,r;\Omega} \, \|\mathbf{r}\|_{0,r;\Omega} \,.
\end{aligned}
\tag{3.9}
$$

Thus, replacing back (3.9) into (3.8), we obtain (3.7) with $L_{\mathtt{nN}} := \max\left\{1, \sigma_* \, 2^{s(r-2)}\right\}$. □

Next, before examining hypothesis iii) of Theorem 3.2 for the operator $\mathcal{C} + \mathcal{A}$, let us note that, proceeding similarly to [25], the kernel of $\mathcal{B}$ by $\mathbf{K}$ can be characterized as

$$
\mathbf{K} := \left\{ \vec{\mathbf{v}} \in \mathbf{V} : \quad \boldsymbol{\nabla}\mathbf{u} = \mathbf{s} \quad \text{and} \quad \mathbf{v} \in \mathbf{W}_0^{1,r}(\Omega) \right\}. \tag{3.10}
$$

Lemma 3.4. *Let $r \in (1,2]$. The family of operators $\left\{ (\mathcal{C} + \mathcal{A})(\cdot + \vec{\mathbf{z}}) : \mathbf{K} \to \mathbf{K}' : \quad \vec{\mathbf{z}} \in \mathbf{V} \right\}$ is uniformly strictly monotone, that is, there exists a constant $\alpha_{\mathtt{nN}} > 0$, depending on σ^*, r and $|\Omega|$, such that*

$$
\begin{aligned}
[(\mathcal{C} + \mathcal{A})(\vec{\mathbf{u}} + \vec{\mathbf{z}}) - (\mathcal{C} + \mathcal{A})(\vec{\mathbf{v}} + \vec{\mathbf{z}}), \vec{\mathbf{u}} - \vec{\mathbf{v}}] \;\ge\; \alpha_{\mathtt{nN}} \Big\{ &\|\mathbf{u} - \mathbf{v}\|_{0,\Omega}^2 \\
&+ \left(\delta^r + \|\mathbf{t}\|_{0,r;\Omega}^r + \|\mathbf{s}\|_{0,r;\Omega}^r \right)^{\frac{r-2}{r}} \|\mathbf{t} - \mathbf{s}\|_{0,r;\Omega}^2 \Big\},
\end{aligned}
\tag{3.11}
$$

for all $\vec{\mathbf{z}} = (\mathbf{z}, \mathbf{r}) \in \mathbf{V}$ and $\vec{\mathbf{u}} = (\mathbf{u}, \mathbf{t}), \vec{\mathbf{v}} = (\mathbf{v}, \mathbf{s}) \in \mathbf{K}$.

Proof. Given $\vec{\mathbf{z}} = (\mathbf{z}, \mathbf{r}) \in \mathbf{V}$ and $\vec{\mathbf{u}} = (\mathbf{u}, \mathbf{t}), \vec{\mathbf{v}} = (\mathbf{v}, \mathbf{s}) \in \mathbf{K}$, from the definitions of $\mathcal{A}$, $\mathcal{C}$ (cf. (2.17a), (2.17c)), we have

$$
[(\mathcal{C} + \mathcal{A})(\vec{\mathbf{u}} + \vec{\mathbf{z}}) - (\mathcal{C} + \mathcal{A})(\vec{\mathbf{v}} + \vec{\mathbf{z}}), \vec{\mathbf{u}} - \vec{\mathbf{v}}] \;=\; \|\mathbf{u} - \mathbf{v}\|_{0,\Omega}^2 + [a(\mathbf{t} + \mathbf{r}) - a(\mathbf{s} + \mathbf{r}), \mathbf{t} - \mathbf{s}] \,.
$$

In order to find a lower bound for the second term on the right-hand side of the above expression, we use Assumption 2.1 (cf. eq (2.6)) and the Hölder inequality with exponents $(\frac{2}{2-r}, \frac{2}{r})$ to obtain

$$
\begin{aligned}
\sigma^* \, \|\mathbf{t}_{\mathtt{sym}} - \mathbf{s}_{\mathtt{sym}}\|_{0,r;\Omega}^2 &\le \sigma^* \left(\int_\Omega \left(\big|\mathbf{t}_{\mathtt{sym}} - \mathbf{s}_{\mathtt{sym}}\big|^2 \right)^{r/2} \right)^{2/r} \\
&\le \left(\int_\Omega \left(\delta^r + |\mathbf{t}_{\mathtt{sym}}|^r + |\mathbf{s}_{\mathtt{sym}}|^r\right)^{\frac{2-r}{2}} \left(\boldsymbol{\sigma}^{\mathrm{d}}(\mathbf{t}_{\mathtt{sym}}) - \boldsymbol{\sigma}^{\mathrm{d}}(\mathbf{s}_{\mathtt{sym}}) : \mathbf{t}_{\mathtt{sym}} - \mathbf{s}_{\mathtt{sym}} \right)^{r/2} \right)^{2/r} \\
&\le \left(\int_\Omega \left(\delta^r + |\mathbf{t}_{\mathtt{sym}}|^r + |\mathbf{s}_{\mathtt{sym}}|^r\right) \right)^{\frac{2-r}{r}} \left(\int_\Omega \boldsymbol{\sigma}^{\mathrm{d}}(\mathbf{t}_{\mathtt{sym}}) - \boldsymbol{\sigma}^{\mathrm{d}}(\mathbf{s}_{\mathtt{sym}}) : \mathbf{t}_{\mathtt{sym}} - \mathbf{s}_{\mathtt{sym}} \right) \\
&\le \left(|\Omega|\delta^r + \|\mathbf{t}_{\mathtt{sym}}\|_{0,r;\Omega}^r + \|\mathbf{s}_{\mathtt{sym}}\|_{0,r;\Omega}^r \right)^{\frac{2-r}{r}} [a(\mathbf{t}) - a(\mathbf{s}), \mathbf{t} - \mathbf{s}] \,,
\end{aligned}
$$

which in turn gives

$$
[a(\mathbf{t}) - a(\mathbf{s}), \mathbf{t} - \mathbf{s}] \;\ge\; \left(|\Omega|\delta^r + \|\mathbf{t}_{\mathtt{sym}}\|_{0,r;\Omega}^r + \|\mathbf{s}_{\mathtt{sym}}\|_{0,r;\Omega}^r \right)^{\frac{r-2}{r}} \sigma^* \, \|\mathbf{t}_{\mathtt{sym}} - \mathbf{s}_{\mathtt{sym}}\|_{0,r;\Omega}^2 . \tag{3.12}
$$

On the other hand, we know from (3.10) that $\nabla\mathbf{v} = \mathbf{s}$, $\nabla\mathbf{u} = \mathbf{t}$ and $\mathbf{v} \in \mathbf{W}^{1,r}_0(\Omega)$. Hence, by applying Körn's inequality and the fact that $r < 2$ we obtain

$$\begin{aligned}
&\left(|\Omega|\delta^r + \|\mathbf{t}_{\mathtt{sym}}\|^r_{0,r;\Omega} + \|\mathbf{s}_{\mathtt{sym}}\|^r_{0,r;\Omega}\right)^{\frac{r-2}{r}} \sigma^* \|\mathbf{t}_{\mathtt{sym}} - \mathbf{s}_{\mathtt{sym}}\|^2_{0,r;\Omega} \\
&\quad \ge \left(|\Omega|\delta^r + \|\mathbf{t}\|^r_{0,r;\Omega} + \|\mathbf{s}\|^r_{0,r;\Omega}\right)^{\frac{r-2}{r}} \sigma^* \|\mathbf{t}_{\mathtt{sym}} - \mathbf{s}_{\mathtt{sym}}\|^2_{0,r;\Omega} \\
&\quad \ge \left(|\Omega|\delta^r + \|\mathbf{t}\|^r_{0,r;\Omega} + \|\mathbf{s}\|^r_{0,r;\Omega}\right)^{\frac{r-2}{r}} \sigma^* \|\epsilon(\mathbf{u}-\mathbf{v})\|^2_{0,r;\Omega} \\
&\quad \ge \left(|\Omega|\delta^r + \|\mathbf{t}\|^r_{0,r;\Omega} + \|\mathbf{s}\|^r_{0,r;\Omega}\right)^{\frac{r-2}{r}} \frac{\sigma^*}{2} |\mathbf{u}-\mathbf{v}|^2_{1,r;\Omega} \\
&\quad \ge \left(|\Omega|\delta^r + \|\mathbf{t}\|^r_{0,\Omega} + \|\mathbf{s}\|^r_{0,\Omega}\right)^{\frac{r-2}{r}} \frac{\sigma^*}{2} \|\mathbf{t}-\mathbf{s}\|^2_{0,r;\Omega}.
\end{aligned} \tag{3.13}$$

Thus, replacing (3.13) back into (3.12), implies (3.11) with $\alpha_{\mathtt{nN}} := \min\left\{\frac{\sigma^*}{2}|\Omega|^{(r-2)/r},\, 1\right\}$. □

Furthermore, the following lemma states that $\mathcal{B}$ satisfies the hypothesis *iv)* Theorem 3.2.

Lemma 3.5. *There exists a constant $\beta > 0$, depending only $|\Omega|$, such that*

$$\sup_{\mathbf{0}\neq\vec{\mathbf{v}}\in\mathbf{V}} \frac{[\mathcal{B}(\vec{\mathbf{v}}),\boldsymbol{\tau}]}{\|\vec{\mathbf{v}}\|_{\mathbf{V}}} \ \ge\ \beta\,\|\boldsymbol{\tau}\|_{s,\mathbf{div};\Omega} \qquad \forall\,\boldsymbol{\tau} \in \mathbb{H}^s(\mathbf{div};\Omega)\,. \tag{3.14}$$

Proof. The proof is a modification of [25, Lemma 3.3]. More precisely, given $\boldsymbol{\tau}^{\mathrm{d}} \neq \mathbf{0}$ we set $\widetilde{\mathbf{s}} := |\boldsymbol{\tau}^{\mathrm{d}}|^{s-2}\boldsymbol{\tau}^{\mathrm{d}}$, and notice that $\|\widetilde{\mathbf{s}}\|^r_{0,r;\Omega} = \|\boldsymbol{\tau}^{\mathrm{d}}\|^s_{0,s;\Omega}$ and $\mathrm{tr}(\widetilde{\mathbf{s}}) = |\boldsymbol{\tau}^{\mathrm{d}}|^{s-2}\mathrm{tr}(\boldsymbol{\tau}^{\mathrm{d}}) = 0$, which says that $\widetilde{\mathbf{s}} \in \mathbf{L}^r_{\mathtt{tr}}(\Omega)$, and additionally there holds

$$\int_\Omega \boldsymbol{\tau} : \widetilde{\mathbf{s}} \ =\ \|\boldsymbol{\tau}^{\mathrm{d}}\|^s_{0,s} \ =\ \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,s;\Omega}\,\|\widetilde{\mathbf{s}}\|_{0,r;\Omega}\,. \tag{3.15}$$

Then, employing (3.15) we find that

$$\sup_{\mathbf{0}\neq\vec{\mathbf{v}}\in\mathbf{V}} \frac{[\mathcal{B}(\vec{\mathbf{v}}),\boldsymbol{\tau}]}{\|\vec{\mathbf{v}}\|_{\mathbf{V}}} \ \ge\ \frac{[\mathcal{B}((\mathbf{0},\widetilde{\mathbf{s}})),\boldsymbol{\tau}]}{\|\widetilde{\mathbf{s}}\|_{0,r;\Omega}} \ =\ \frac{\displaystyle\int_\Omega \widetilde{\mathbf{s}} : \boldsymbol{\tau}}{\|\widetilde{\mathbf{s}}\|_{0,r;\Omega}} \ =\ \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,s;\Omega}\,. \tag{3.16}$$

In turn, denoting by $\boldsymbol{\tau}_j$ the j-th row of $\boldsymbol{\tau}$ for $j = 1,\ldots,n$, we now set $\vec{\widetilde{\mathbf{v}}} = (\widetilde{\mathbf{v}},\mathbf{0}) \in \mathbf{V}$, with $\widetilde{\mathbf{v}} := (\mathrm{div}(\boldsymbol{\tau}_j))_{j=1,\cdots,n} \in \mathbf{L}^2(\Omega)$. Then, it follows that

$$\sup_{\mathbf{0}\neq\vec{\mathbf{v}}\in\mathbf{V}} \frac{[\mathcal{B}(\vec{\mathbf{v}}),\boldsymbol{\tau}]}{\|\vec{\mathbf{v}}\|_{\mathbf{V}}} \ \ge\ \frac{[\mathcal{B}((\widetilde{\mathbf{v}},\mathbf{0})),\boldsymbol{\tau}]}{\|\widetilde{\mathbf{v}}\|_{0,\Omega}} \ =\ \|\mathbf{div}(\boldsymbol{\tau})\|_{0,\Omega},$$

which, together with (3.16) implies

$$\sup_{\mathbf{0}\neq\vec{\mathbf{v}}\in\mathbf{V}} \frac{[\mathcal{B}(\vec{\mathbf{v}}),\boldsymbol{\tau}]}{\|\vec{\mathbf{v}}\|_{\mathbf{V}}} \ \ge\ \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,s;\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,\Omega}. \tag{3.17}$$

Additionally, by appropriately adjusting the proof of [29, Lemma 2.3], it can be demonstrated that there exists a positive constant c_Ω, depending only on Ω, such that

$$c_\Omega\|\boldsymbol{\tau}\|_{0,s;\Omega} \ \le\ \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,s;\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,\Omega}\,. \tag{3.18}$$

Finally, by combining (3.18) and (3.17) we arrive at (3.14) with $\beta = \min\{\frac{1}{2}, \frac{c_\Omega}{2}\}$. □

We are now ready to establish the main result of this section.

Lemma 3.6. *Let $r \in (1,2]$. Then, for each $\mathbf{f} \in \mathbf{L}^2(\Omega)$, there exists a unique solution $(\vec{\mathbf{u}}, \boldsymbol{\sigma}) = ((\mathbf{u}, \mathbf{t}), \boldsymbol{\sigma}) \in \mathbf{V} \times \mathbb{X}$ to* (3.6).

Proof. As a consequence of Lemmas 3.3, 3.4 and 3.5, we conclude that the operators $\mathcal{A} + \mathcal{C}$ and $\mathcal{B}$ satisfy the hypotheses of Theorem 3.2. Therefore, through a straightforward application of this abstract result, we arrive at the desired conclusion. □

We end this section by presenting an appropriate initial condition result, which is essential for applying Theorem 3.1.

Lemma 3.7. *Let $\mathbf{M}$ be the subspace of $\mathbf{W}_0^{1,r}(\Omega)$*

$$\mathbf{M} := \left\{ \mathbf{v} \in \mathbf{W}_0^{1,r}(\Omega) : \ \mathbf{div}\big(\mu\ (\delta^\alpha + |\epsilon(\mathbf{v})|^\alpha)^{\frac{r-2}{\alpha}}\, \epsilon(\mathbf{v})\big) \in \mathbf{L}^2(\Omega) \quad \textit{and} \quad \mathrm{div}(\mathbf{v}) = 0 \ \textit{in}\ \Omega \right\}, \tag{3.19}$$

and assume that $\mathbf{u}_0 \in \mathbf{Y} \cap \mathbf{M}$. Then, there exists $(\mathbf{t}_0, \boldsymbol{\sigma}_0) \in \mathbb{Q} \times \mathbb{X}$ such that $\vec{\mathbf{u}}_0 := (\mathbf{u}_0, \mathbf{t}_0)$ and $\boldsymbol{\sigma}_0$ satisfy

$$\begin{pmatrix} \mathcal{A} & \mathcal{B}' \\ -\mathcal{B} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \vec{\mathbf{u}}_0 \\ \boldsymbol{\sigma}_0 \end{pmatrix} \in \Big(\mathbf{L}^2(\Omega) \times \{\mathbf{0}\} \Big) \times \{\mathbf{0}\}. \tag{3.20}$$

Proof. Given $\mathbf{u}_0 \in \mathbf{Y} \cap \mathbf{M}$, we define

$$\mathbf{t}_0 := \boldsymbol{\nabla}\mathbf{u}_0 \quad \text{and} \quad \boldsymbol{\sigma}_0 := \mu\ (\delta^\alpha + |\epsilon(\mathbf{u}_0)|^\alpha)^{\frac{r-2}{\alpha}}\, \epsilon(\mathbf{u}_0)\,, \tag{3.21}$$

which satisfy

$$\mathrm{tr}(\mathbf{t}_0) = 0\,, \quad \mathbf{div}(\boldsymbol{\sigma}_0) = \mathbf{div}\Big(\mu\ (\delta^\alpha + |\epsilon(\mathbf{u}_0)|^\alpha)^{\frac{r-2}{\alpha}}\, \epsilon(\mathbf{u}_0)\Big)\,. \tag{3.22}$$

Notice that $\mathbf{t}_0 \in \mathbb{Q}$, and $\boldsymbol{\sigma}_0 \in \mathbb{X}$. Next, by applying integration by parts to the identity $\mathbf{t}_0 = \boldsymbol{\nabla}\mathbf{u}_0$ and following a similar approach to that used in (2.7a), we get

$$[\mathcal{B}(\vec{\mathbf{u}}_0), \boldsymbol{\tau}] = 0 \quad \forall\, \boldsymbol{\tau} \in \mathbb{X}\,.$$

Therefore, given $\mathbf{u}_0 \in \mathbf{V} \cap \mathbf{M}$, where $\mathbf{M}$ is defined in (3.19), by multiplying the second rows in (3.21) and (3.22) by the respective test functions $\mathbf{s} \in \mathbb{Q}$ and $\mathbf{v} \in \mathbf{Y}$, we find that

$$\begin{pmatrix} \mathcal{A} & \mathcal{B}' \\ -\mathcal{B} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \vec{\mathbf{u}}_0 \\ \boldsymbol{\sigma}_0 \end{pmatrix} = \begin{pmatrix} \vec{F}_0 \\ 0 \end{pmatrix}, \tag{3.23}$$

where $\vec{F}_0 = (f_0, 0)$ and

$$(f_0, \mathbf{v}) := -\int_\Omega \mathbf{div}\Big(\mu\ (\delta^\alpha + |\epsilon(\mathbf{u}_0)|^\alpha)^{\frac{r-2}{\alpha}}\, \epsilon(\mathbf{u}_0)\Big) \cdot \mathbf{v}\,. \tag{3.24}$$

Also, we have

$$\big|(f_0, \mathbf{v})\big| \le \Big\| \mathbf{div}\Big(\mu\ (\delta^\alpha + |\epsilon(\mathbf{u}_0)|^\alpha)^{\frac{r-2}{\alpha}}\, \epsilon(\mathbf{u}_0)\Big) \Big\|_{0,\Omega} \|\mathbf{v}\|_{0,\Omega}\,.$$

Thus, $\vec{F}_0 \in \mathbf{L}^2(\Omega) \times \{\mathbf{0}\}$, which implies (3.20), completing the proof. □

3.3 Main result

In this section, we establish the well-posedness of problem (2.16).

Theorem 3.8. *For each $\mathbf{f} \in W^{1,1}(\mathrm{J}; \mathbf{L}^2(\Omega))$ and every compatible initial data $(\vec{\mathbf{u}}_0, \boldsymbol{\sigma}_0) = ((\mathbf{u}_0, \mathbf{t}_0), \boldsymbol{\sigma}_0)$ derived in Lemma 3.7, there exists a unique $(\vec{\mathbf{u}}, \boldsymbol{\sigma}) = ((\mathbf{u}, \mathbf{t}), \boldsymbol{\sigma}) : [0, t_F] \to \mathbf{V} \times \mathbb{X}$ solution to* (2.16)*, such that $\mathbf{u} \in W^{1,\infty}(\mathrm{J}; \mathbf{Y})$ and $((\mathbf{u}(0), \mathbf{t}(0)), \boldsymbol{\sigma}(0)) = ((\mathbf{u}_0, \mathbf{t}_0), \boldsymbol{\sigma}_0)$.*

Proof. We note that the structure of problem (3.6) is the same as in Theorem 3.1, based on the definitions given in (3.1)-(3.3). Additionally, the operator $\mathcal{N}$ is linear, symmetric, and monotone due to the definition of C (cf. (3.4b)), while $\mathcal{M}$ is monotone thanks to the strictly monotonicity of $\mathcal{A}$ (cf. Lemma 3.4). Alternatively, Lemma 3.6 allows us to deduce that for the given $(\hat{F}, \mathbf{0}) \in V_n'$ with $\hat{F} = (\hat{\mathbf{f}}, \mathbf{0})$, there exists a unique $(\vec{\mathbf{u}}, \boldsymbol{\sigma}) = ((\mathbf{u}, \mathbf{t}), \boldsymbol{\sigma}) \in \mathbf{V} \times \mathbb{X}$ such that

$$(\hat{F}, \mathbf{0}) \;=\; (\mathcal{N} + \mathcal{M})(\vec{\mathbf{u}}, \boldsymbol{\sigma}),$$

which implies $\mathcal{N} + \mathcal{M} : D \to V_n'$ is onto, where D is defined in (3.3). Lastly, by taking $\mathbf{u}_0 \in \mathbf{Y} \cap \mathbf{M}$, where $\mathbf{M}$ is defined in (3.19), a direct application of Lemma 3.7 allows us to determine $(\mathbf{t}_0, \boldsymbol{\sigma}_0) \in \mathbb{Q} \times \mathbb{X}$ such that $(\vec{\mathbf{u}}_0, \boldsymbol{\sigma}_0) \in D$. Consequently, by utilizing Theorem 3.1, we can conclude that for each $t \in \mathrm{J}$ there exists a solution $(\vec{\mathbf{u}}(t), \boldsymbol{\sigma}(t)) = ((\mathbf{u}(t), \mathbf{t}(t)), \boldsymbol{\sigma}(t)) \in \mathbf{V} \times \mathbb{X}$ to (2.16) such that $\mathbf{u} \in W^{1,\infty}(\mathrm{J}; \mathbf{Y})$, with initial value $\mathbf{u}(0) = \mathbf{u}_0$.

We will now establish the uniqueness of the solution to (2.16). To do this, consider $(\vec{\mathbf{u}}_i, \boldsymbol{\sigma}_i)$ for $i \in \{1, 2\}$, which are two solutions that correspond to the same input data. Then, considering test functions $(\vec{\mathbf{v}}, \boldsymbol{\tau}) = (\vec{\mathbf{u}}_1 - \vec{\mathbf{u}}_2, \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)$ to (2.16), we find that

$$\frac{1}{2}\partial_t \|\mathbf{u}_1 - \mathbf{u}_2\|_{0,\Omega}^2 + [\mathcal{A}(\vec{\mathbf{u}}_1) - \mathcal{A}(\vec{\mathbf{u}}_2), \vec{\mathbf{u}}_1 - \vec{\mathbf{u}}_2] \;=\; 0,$$

from which using the fact that $\vec{\mathbf{u}}_1 - \vec{\mathbf{u}}_2 \in \mathbf{K}$ and the strict monotonicity of $\mathcal{A}$ (cf. (3.11)), we can conclude that

$$\frac{1}{2}\partial_t \|\mathbf{u}_1 - \mathbf{u}_2\|_{0,\Omega}^2 + \frac{\sigma^*}{2}\left(\delta^r + \|\mathbf{t}_1 - \mathbf{t}_2\|_{0,r;\Omega}^r\right)^{(r-2)/r} \|\mathbf{t}_1 - \mathbf{t}_2\|_{0,r;\Omega} \;\le\; 0. \tag{3.25}$$

Next, we will consider two possible cases as follows

$$\|\mathbf{t}_1 - \mathbf{t}_2\|_{0,r;\Omega} \;\ge\; \delta \quad \text{or} \quad \|\mathbf{t}_1 - \mathbf{t}_2\|_{0,r;\Omega} \;<\; \delta\,.$$

In the first case, (3.25) yields

$$\partial_t \|\mathbf{u}_1 - \mathbf{u}_2\|_{0,\Omega}^2 \;+\; \sigma^* 2^{(r-2)/r} \|\mathbf{t}_1 - \mathbf{t}_2\|_{0,r;\Omega}^r \;\le\; 0\,, \tag{3.26}$$

whereas for the second case, we get

$$\partial_t \|\mathbf{u}_1 - \mathbf{u}_2\|_{0,\Omega}^2 \;+\; \frac{\sigma^* 2^{(r-2)/2}}{\delta^{2-r}} \|\mathbf{t}_1 - \mathbf{t}_2\|_{0,r;\Omega}^2 \;\le\; 0\,. \tag{3.27}$$

Thus, by integrating (3.26) and (3.27) over time from 0 to $t \in (0, t_F]$, combining them, and utilizing the condition $\mathbf{u}_1(0) = \mathbf{u}_2(0)$, we obtain:

$$\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_{0,\Omega}^2 + \int_0^t \left(\|\mathbf{t}_1(s) - \mathbf{t}_2(s)\|_{0,r;\Omega}^r + \|\mathbf{t}_1(s) - \mathbf{t}_2(s)\|_{0,r;\Omega}^2\right) ds \;\le\; 0. \tag{3.28}$$

Thus, it can be concluded from (3.28) that $\mathbf{u}_1(t) = \mathbf{u}_2(t)$ and $\mathbf{t}_1(t) = \mathbf{t}_2(t)$ for all $t \in (0, t_F]$. Next, utilizing the inf-sup condition of the operator $\mathcal{B}$ (cf. (3.14)) along with the first equation of (2.16) yields

$$\begin{aligned}
\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\|_{s,\mathbf{div};\Omega} \;&\le\; \frac{1}{\beta} \sup_{\mathbf{0} \neq \vec{\mathbf{v}} \in \mathbf{V}} \frac{[\mathcal{B}(\vec{\mathbf{v}}), \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2]}{\|\vec{\mathbf{v}}\|_{\mathbf{V}}} \\
&\le\; \frac{1}{\beta} \sup_{\mathbf{0} \neq \vec{\mathbf{v}} \in \mathbf{V}} \frac{-\dfrac{\partial}{\partial t}[C(\vec{\mathbf{u}}_1 - \vec{\mathbf{u}}_2), \vec{\mathbf{v}}] - [\mathcal{A}(\vec{\mathbf{u}}_1) - \mathcal{A}(\vec{\mathbf{u}}_2), \vec{\mathbf{v}}]}{\|\vec{\mathbf{v}}\|_{\mathbf{V}}} \;=\; 0\,.
\end{aligned}$$

This implies that $\boldsymbol{\sigma}_1(t) = \boldsymbol{\sigma}_2(t)$ for all $t \in (0, t_F]$, and consequently, (2.16) has a unique solution. □

The following result provides the stability bound for solution of (2.16).

Theorem 3.9. *Let $r \in (1,2]$. Assume that $\mathbf{u}_0 \in \mathbf{Y} \cap \mathbf{M}$ satisfies* (3.20)*. Then, there exists a constant $C_{\texttt{stab}}$, depending on σ^*, r, δ, β, the norm of the continuous injections $\mathbf{i}_{dr/(d-r)} : \mathbf{W}^{1,r}(\Omega) \to \mathbf{L}^{dr/(d-r)}$ and $\mathbf{i}_2 : \mathbf{L}^{dr/(d-r)} \to \mathbf{L}^2(\Omega)$, such that*

$$\begin{aligned} \|\mathbf{u}(t)\|_{0,\Omega}^2 + \|\mathbf{t}(t)\|_{0,r;\Omega}^r + \int_0^t \Big(\|\mathbf{t}(s)\|_{0,r;\Omega}^r + \|\mathbf{u}(s)\|_{0,\Omega}^r + \|\boldsymbol{\sigma}(s)\|_{s,\mathbf{div};\Omega}^2 \Big)\, ds \\ \le C_{\texttt{stab}} \Big\{ \|\mathbf{u}_0\|_{0,\Omega}^2 + \Big(\delta^r + \|\mathbf{u}_0\|_{1,r;\Omega}^r \Big)^{\frac{r-2}{r}} \|\mathbf{u}_0\|_{1,r;\Omega}^2 \\ + \int_0^t \Big(\|\mathbf{f}(s)\|_{0,\Omega}^{r/(r-1)} + \|\mathbf{f}(s)\|_{0,\Omega}^2 \Big)\, ds \Big\} := \mathcal{N}(\mathbf{f}, \mathbf{u}_0)\,, \end{aligned} \tag{3.29}$$

Proof. We start by selecting $(\vec{\mathbf{v}}, \boldsymbol{\tau}) = (\vec{\mathbf{u}}, \boldsymbol{\sigma})$ in (2.16), to obtain

$$\frac{1}{2}\partial_t \|\mathbf{u}\|_{0,\Omega}^2 + \big|[\mathcal{A}(\vec{\mathbf{u}}), \vec{\mathbf{u}}]\big| \le \big|[\mathcal{F}, \vec{\mathbf{u}}]\big|.$$

As a result of the monotonicity of $\mathcal{A}$ (cf. (3.11)) and the boundedness of $\mathcal{F}$ (cf. (3.4c)), it is clear that

$$\frac{1}{2}\partial_t \|\mathbf{u}\|_{0,\Omega}^2 + \frac{\sigma^*}{2} \Big(\delta^r + \|\mathbf{t}\|_{0,r;\Omega}^r \Big)^{\frac{r-2}{r}} \|\mathbf{t}\|_{0,r;\Omega}^2 \le \|\mathbf{f}\|_{0,\Omega} \|\mathbf{u}\|_{0,\Omega}\,. \tag{3.30}$$

We notice that for the term $\|\mathbf{t}\|_{0,r}$ there are two possibilities, namely,

$$\|\mathbf{t}\|_{0,r;\Omega} \ge \delta \quad \text{or} \quad \|\mathbf{t}\|_{0,r;\Omega} < \delta\,.$$

If the first case occurs, from (3.30), we conclude that

$$\partial_t \|\mathbf{u}\|_{0,\Omega}^2 + \sigma^* \, 2^{\frac{r-2}{r}} \|\mathbf{t}\|_{0,r;\Omega}^r \le 2\|\mathbf{f}\|_{0,\Omega} \|\mathbf{u}\|_{0,\Omega}\,. \tag{3.31}$$

In turn, observing from the second row of (2.16) that $\vec{\mathbf{u}}$ belongs to $\mathbf{K}$ (cf. (3.10)), we understand that $\mathbf{t} = \boldsymbol{\nabla}\mathbf{u}$ and $\mathbf{u} \in \mathbf{W}_0^{1,r}(\Omega)$, which invoking the continuous injections $\mathbf{i}_{dr/(d-r)} : \mathbf{W}^{1,r}(\Omega) \to \mathbf{L}^{dr/(d-r)}$ and $\mathbf{i}_2 : \mathbf{L}^{dr/(d-r)} \to \mathbf{L}^2(\Omega)$, imply

$$\begin{aligned} \frac{1}{2}\sigma^* \, 2^{\frac{r-2}{r}} \|\mathbf{t}\|_{0,r;\Omega}^r &= \frac{1}{2}\sigma^* \, 2^{\frac{r-2}{r}} \|\boldsymbol{\nabla}\mathbf{u}\|_{0,r;\Omega}^r \\ \ge \frac{1}{2\|\mathbf{i}_{dr/(d-r)}\|^r}\sigma^* \, 2^{\frac{r-2}{r}} \|\mathbf{u}\|_{0,dr/(d-r);\Omega}^r &\ge \frac{1}{2\|\mathbf{i}_{dr/(d-r)}\|^r \|\mathbf{i}_2\|^r}\sigma^* \, 2^{\frac{r-2}{r}} \|\mathbf{u}\|_{0,\Omega}^r. \end{aligned} \tag{3.32}$$

It then suffices to combine (3.32) with (3.31) and Young's inequality

$$ab \le \frac{\alpha}{p} a^p + \frac{\alpha^{-q/p}}{q} b^q \qquad \forall\, p, q \ge 0 \quad \text{and} \quad \frac{1}{p} + \frac{1}{q} = 1\,, \tag{3.33}$$

to arrive at

$$\partial_t \|\mathbf{u}\|_{0,\Omega}^2 + \frac{1}{2\|\mathbf{i}_{dr/(d-r)}\|^r \|\mathbf{i}_2\|^r}\sigma^* \, 2^{\frac{r-2}{r}} \|\mathbf{u}\|_{0,\Omega}^r + \frac{1}{2}\sigma^* \, 2^{\frac{r-2}{r}} \|\mathbf{t}\|_{0,r;\Omega}^r \le \frac{\alpha_1}{r} \|\mathbf{u}\|_{0,\Omega}^r + 4\frac{\alpha_1^{-1/(r-1)}}{r/(r-1)} \|\mathbf{f}\|_{0,\Omega}^{r/(r-1)}\,.$$

From the above bound, by choosing α_1 such that $\alpha_1 \le \dfrac{1}{4\|\mathbf{i}_{dr/(d-r)}\|^r \|\mathbf{i}_2\|^r}\sigma^* \, 2^{\frac{r-2}{r}}$ and integrating over time from 0 to $t \in (0, t_F]$, we obtain

$$\|\mathbf{u}(t)\|_{0,\Omega}^2 + \int_0^t \Big(\|\mathbf{t}(s)\|_{0,r;\Omega}^r + \|\mathbf{u}(s)\|_{0,\Omega}^r \Big)\, ds \le C_1 \Big\{ \|\mathbf{u}_0\|_{0,\Omega}^2 + \int_0^t \|\mathbf{f}(s)\|_{0,\Omega}^{r/(r-1)}\, ds \Big\}\,. \tag{3.34}$$

On the other hand, for the second case according to (3.30), and utilizing (3.32) with $r = 2$ along with the Young inequality again, we conclude

$$\begin{aligned} \partial_t \|\mathbf{u}\|_{0,\Omega}^2 + \frac{1}{2\|\mathbf{i}_{dr/(d-r)}\|^2 \|\mathbf{i}_2\|^2}\sigma^* \, 2^{\frac{r-2}{r}} \delta^{r-2} \|\mathbf{u}\|_{0,\Omega}^2 + \frac{1}{2}\sigma^* \, 2^{\frac{r-2}{2}} \delta^{r-2} \|\mathbf{t}\|_{0,r;\Omega}^2 \\ \le \alpha_2 \|\mathbf{u}\|_{0,\Omega}^2 + \frac{1}{\alpha_2} \|\mathbf{f}\|_{0,\Omega}^2\,, \end{aligned}$$

where, by considering α_2 to be sufficiently small so that $\alpha_2 \le \dfrac{1}{4\|\mathbf{i}_{dr/(d-r)}\|^2\|\mathbf{i}_2\|^2}\sigma^*\, 2^{\frac{r-2}{r}}\delta^{r-2}$ and integrating from 0 to $t \in (0, t_F]$, readily gives

$$\|\mathbf{u}(t)\|_{0,\Omega}^2 + \int_0^t \left(\|\mathbf{t}(s)\|_{0,r;\Omega}^2 + \|\mathbf{u}(s)\|_{0,\Omega}^2\right) ds \;\le\; C_2\left\{\|\mathbf{u}_0\|_{0,\Omega}^2 + \int_0^t \|\mathbf{f}(s)\|_{0,\Omega}^2\, ds\right\}. \tag{3.35}$$

Thus, by combining estimates (3.34) and (3.35), and using the fact that $r < 2$, imply

$$\begin{aligned}\|\mathbf{u}(t)\|_{0,\Omega}^2 + \int_0^t \left(\|\mathbf{t}(s)\|_{0,r;\Omega}^r + \|\mathbf{u}(s)\|_{0,\Omega}^r\right) ds \;\le\; \widetilde{C}\Big\{\|\mathbf{u}_0\|_{0,\Omega}^2 \\ + \int_0^t \left(\|\mathbf{f}(s)\|_{0,\Omega}^{r/(r-1)} + \|\mathbf{f}(s)\|_{0,\Omega}^2\right) ds\Big\}.\end{aligned} \tag{3.36}$$

Lastly, to establish the a priori estimate of pseudostress, we employ the inf-sup condition of $\mathcal{B}$ provided by (3.14), along with the first row of (2.16) and the stability bounds of $\mathcal{F}$, C and $\mathcal{A}$ (cf. (3.4c), (3.4b), (3.5)), which leads to

$$\begin{aligned}\beta\,\|\boldsymbol{\sigma}\|_{s,\mathbf{div};\Omega} \;\le\; \sup_{\mathbf{0}\neq\vec{\mathbf{v}}\in\mathbf{V}} \frac{[\mathcal{B}(\vec{\mathbf{v}}),\boldsymbol{\sigma}]}{\|\vec{\mathbf{v}}\|_{\mathbf{V}}} \;=\; \sup_{\mathbf{0}\neq\vec{\mathbf{v}}\in\mathbf{V}} \frac{[\mathcal{F},\vec{\mathbf{v}}] - \frac{\partial}{\partial t}[C(\vec{\mathbf{u}}),\vec{\mathbf{v}}] - [\mathcal{A}(\vec{\mathbf{u}}),\vec{\mathbf{v}}]}{\|\vec{\mathbf{v}}\|_{\mathbf{V}}} \\ \le\; C_0\left(\|\mathbf{f}\|_{0,\Omega} + \|\partial_t\mathbf{u}\|_{0,\Omega} + \|\mathbf{t}\|_{0,r;\Omega}^{r-1}\right).\end{aligned}$$

Then, taking squares, integrating from 0 to $t \in (0, t_F]$, and (3.36), we get

$$\int_0^t \|\boldsymbol{\sigma}(s)\|_{t,\mathbf{div};\Omega}^2\, ds \;\le\; C_0^2 \int_0^t \left(\|\mathbf{f}(s)\|_{0,\Omega}^2 + \|\partial_t\mathbf{u}(s)\|_{0,\Omega}^2 + \|\mathbf{t}(s)\|_{0,r;\Omega}^{2(r-1)}\right) ds. \tag{3.37}$$

Next, to find an upper bound for the second term in (3.37), we differentiate the second equation of (2.16) with respect to time and take $(\vec{\mathbf{v}}, \boldsymbol{\sigma}) = ((\partial_t\mathbf{u}, \partial_t\mathbf{t}), \boldsymbol{\sigma})$, which implies

$$\frac{\partial}{\partial t}[C(\vec{\mathbf{u}}), \partial_t\vec{\mathbf{u}}] \;+\; [\mathcal{A}(\vec{\mathbf{u}}), \partial_t\vec{\mathbf{u}}] = [\mathcal{F}, \partial_t\vec{\mathbf{u}}],$$

and this result, combined with the identity

$$[\mathcal{A}(\vec{\mathbf{u}}), \partial_t\vec{\mathbf{u}}] \;=\; \frac{1}{2}\partial_t\,[\mathcal{A}(\vec{\mathbf{u}}), \vec{\mathbf{u}}]\,,$$

and the monotonicity of $\mathcal{A}$ given by (3.11), yields

$$\|\partial_t\mathbf{u}\|_{0,\Omega}^2 + \frac{1}{4}\partial_t\left(\sigma^*\left(\delta^r + \|\mathbf{t}\|_{0,r;\Omega}^r\right)^{\frac{r-2}{r}}\|\mathbf{t}\|_{0,r;\Omega}^2\right) \;\le\; \frac{1}{2}\left(\|\mathbf{f}\|_{0,\Omega}^2 + \|\partial_t\mathbf{u}\|_{0,\Omega}^2\right).$$

Integrating from 0 to $t \in (0, t_F]$ and proceeding similar with derivation (3.36), we get

$$\int_0^t \|\partial_t\mathbf{u}(s)\|_{0,\Omega}^2\, ds + \|\mathbf{t}(t)\|_{0,r;\Omega}^r \;\le\; C_3\left\{\int_0^t \|\mathbf{f}(s)\|_{0,\Omega}^2 + \left(\delta^r + \|\mathbf{t}(0)\|_{0,r;\Omega}^r\right)^{\frac{r-2}{r}}\|\mathbf{t}(0)\|_{0,r;\Omega}^2\right\}. \tag{3.38}$$

Then, substituting (3.38) back into (3.37) and employing (3.36) yields

$$\begin{aligned}\int_0^t \|\boldsymbol{\sigma}(s)\|_{s,\mathbf{div};\Omega}^2\, ds \;\le\; C_4\Big\{\|\mathbf{u}_0\|_{0,\Omega}^2 + \left(\delta^r + \|\mathbf{t}(0)\|_{0,r;\Omega}^r\right)^{\frac{r-2}{r}}\|\mathbf{t}(0)\|_{0,r;\Omega}^2 \\ + \int_0^t \left(\|\mathbf{f}(s)\|_{0,\Omega}^{r/(r-1)} + \|\mathbf{f}(s)\|_{0,\Omega}^2\right) ds\Big\}\end{aligned}$$

which, combined with the estimate in (3.36) and the fact that $(\mathbf{u}(0), \mathbf{t}(0)) = (\mathbf{u}_0, \mathbf{t}_0)$, where $\mathbf{t}_0 = \boldsymbol{\nabla}\mathbf{u}_0$, implies (3.29). $\square$

4 The discrete setting

In this section, we present a $\mathbb{H}(\mathbf{div})$-conforming virtual element method to approximate the mixed problem (2.4). To this purpose, we first specify the concept of polygonal decomposition of Ω in Section 4.1. Subsequently, in Sections 4.2 and 4.3, we define a set of discrete spaces, projectors, and discrete bilinear forms. With these foundational elements established, we proceed to propose our discrete formulation in Section 4.5.

4.1 Polygonal decomposition and notation

We begin by considering a sequence $\{\Omega_h\}_{h>0}$ of partitions of Ω into general polygons T, where each polygon T has a diameter denoted by h_T and a number of edges denoted by d_T, respectively. As usual, we set $h := \max_{T\in\Omega_h} h_T$, and let $N_h^{\mathtt{ed}}$ and $N_h^{\mathtt{el}}$ be the number of edges and elements, respectively, and $\mathbf{n}_e^T$ be the unit outward normal on edge $e \subset \partial T$. Also, we denote the edges of ∂T by e, its length by $h_e := |e|$ and the set of edges e of Ω_h by Γ_h. For any $l \in N$ and any mesh object $\varpi \in \Omega_h \cup \Gamma_h$, let $\mathrm{P}_l(\varpi)$, $\mathbf{P}_l(\varpi)$, $\mathbb{P}_l(\varpi)$ be the space of scalar, vectorial and matrix polynomials defined on ϖ of degree less than or equal to l, respectively (with the extended notation $\mathrm{P}_{-1}(\varpi) = \{0\}$). The dimension of such spaces, for each $T \in \Omega_h$ and $e \in \Gamma_h$, are

$$\dim\big(\mathrm{P}_l(T)\big) = \pi_l^{\mathtt{el}} := \frac{(l+1)(l+2)}{2}, \quad \dim\big(\mathbf{P}_l(T)\big) = 2\pi_l^{\mathtt{el}}, \quad \dim\big(\mathbb{P}_l(T)\big) = 4\pi_l^{\mathtt{el}},$$

and

$$\dim\big(\mathrm{P}_l(e)\big) = \pi_l^{\mathtt{ed}} := l+1, \quad \dim\big(\mathbf{P}_l(e)\big) = 2\pi_l^{\mathtt{ed}}, \quad \dim\big(\mathbb{P}_l(e)\big) = 4\pi_l^{\mathtt{ed}}.$$

Also, for any $l \in N$ we introduce the broken space

$$\mathrm{P}_\ell(\Omega_h) := \Big\{ v \in L^2(\Omega): \quad v|_E \in \mathrm{P}_\ell(E), \quad \forall\, E \in \Omega_h \Big\}.$$

In addition, we suppose that $\{\Omega_h\}_h$ satisfies the following mesh-regularity assumptions:

Assumption 4.1. There exists a positive constant ρ such that for any $T \in \{\Omega_h\}_h$:

- T is star-shaped with respect to every point of a disk with radius $\geq \rho h_T$;
- every edge $e \subset \partial T$ of cell T has length $\geq \rho\, h_T$.

We note that the above assumptions, while generally not too restrictive in many practical scenarios, could potentially be further relaxed by combining the current analysis with the research presented in [11, 15, 16].

4.2 Projection operators

In this section, we follow very closely [30, Section 3.1] to introduce the polynomial projection, which is a key ingredient in the set up of VEM. We start with introducing L^1-projection operator $\mathcal{P}_\ell^T : \mathrm{L}^1(T) \to \mathrm{P}_\ell(T)$, which satisfies the following variational problem for any function $v \in \mathrm{L}^1(T)$:

$$\int_T \big(\mathcal{P}_\ell^T(v) - v\big)\, q = 0 \qquad \forall\, q \in \mathrm{P}_\ell(T)\,. \tag{4.1}$$

Thanks to Assumption 4.1, the boundedness and approximation properties of P_ℓ^T are stated as follows [36, Lemma 3.1].

Lemma 4.2. *Let $p > 1$, and ℓ, s, m be integers such that $\ell \geq 0$ and $0 \leq m \leq s \leq \ell + 1$. It holds, for all $v \in \mathrm{W}^{s,p}(T)$, and $T \in \Omega_h$:*

- *Boundedness. there exists a constant M_ℓ, depending only on ℓ and ρ, such that there holds*

$$|\mathcal{P}_\ell^T(v)|_{s,p;T} \leq M_\ell\, |v|_{s,p;T}\,. \tag{4.2}$$

- *Approximation. there exists a constant C_ℓ, depending only on ℓ and ρ, and hence independent of T, such that*
$$|v - \mathcal{P}_\ell^T(v)|_{m,p;T} \quad \le \quad C_\ell\, h_K^{s-m}\, |v|_{s,p;T}\,. \tag{4.3}$$

We remark that scaled projector P_ℓ^T can be generalized for vector and tensor versions, and denoted by $\boldsymbol{\mathcal{P}}_\ell^T : \mathbf{L}^1(T) \to \mathbf{P}_\ell(T)$ and $\boldsymbol{\mathscr{P}}_\ell^T : \mathbb{L}^1(T) \to \mathbb{P}_\ell(T)$, respectively. In addition, the estimates (4.2) and (4.3) remain valid for $\boldsymbol{\mathcal{P}}_\ell^T$ and $\boldsymbol{\mathscr{P}}_\ell^T$.

Finally, for any element $T \in \Omega_h$ and functions $v \in \mathrm{L}^1(\Omega)$, $\boldsymbol{\eta} \in \mathbf{L}^1(\Omega)$, $\boldsymbol{\tau} \in \mathbb{L}^1(\Omega)$, the global projection operators $\mathcal{P}_\ell^h$, $\boldsymbol{\mathcal{P}}_\ell^h$ and $\boldsymbol{\mathscr{P}}_\ell^h$ are defined by
$$\mathcal{P}_\ell^h(v)|_T = \mathcal{P}_\ell^T(v|_T)\,, \quad \boldsymbol{\mathcal{P}}_\ell^h(\boldsymbol{\eta})|_T = \boldsymbol{\mathcal{P}}_\ell^T(\boldsymbol{\eta}|_T) \quad \text{and} \quad \boldsymbol{\mathscr{P}}_\ell^h(\boldsymbol{\tau})|_T = \boldsymbol{\mathscr{P}}_\ell^T(\boldsymbol{\tau}|_T)\,.$$

4.3 Discrete spaces

We now present the $\mathbb{H}(\mathbf{div})$-conforming virtual element subspace discussed in [36]. In this regard, we first recall the following notations:

- $\mathbf{rot}(\boldsymbol{\tau}) := \left(\partial_{x_1}\tau_{12} - \partial_{x_2}\tau_{11}, \partial_{x_1}\tau_{22} - \partial_{x_2}\tau_{21}\right)^\top$.
- $\mathcal{G}_\ell(T) := \nabla \mathbf{P}_{\ell+1}(T) \subset \mathbb{P}_\ell(T)$.
- $\mathbb{P}_\ell(T) = \mathcal{G}_\ell(T) \oplus \mathcal{G}_\ell(T)^\perp$, where $\mathcal{G}_\ell(T)^\perp$ is the L^2 orthogonal of $\mathcal{G}_\ell(T)$ in $\mathbb{P}_\ell(T)$.

For integer $k \ge 0$, we define
$$\mathbb{X}_k^T := \Big\{\boldsymbol{\tau} \in \mathbb{H}^s(\mathbf{div};T) \cap \mathbb{H}(\mathbf{rot};T) : \quad (\boldsymbol{\tau}\mathbf{n}_e^T)|_e \in \mathbf{P}_k(e) \quad \text{for each edge e of } \partial T$$
$$\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{P}_k(T) \quad \text{and} \quad \mathbf{rot}(\boldsymbol{\tau}) \in \mathbf{P}_{k-1}(T)\Big\}\,.$$

Next, for $\boldsymbol{\tau} \in \mathbb{X}_k^T$, we introduce the following local degrees of freedom:

- the edge moments
$$\mathbf{D1}(\boldsymbol{\tau}) := \int_e \boldsymbol{\tau}\mathbf{n}_e^T \cdot \mathbf{q} \qquad \forall\, \mathbf{q} \in \mathbf{P}_k(e)\,, \tag{4.4a}$$
- the element moments of the gradient
$$\mathbf{D2}(\boldsymbol{\tau}) := \int_T \boldsymbol{\tau} : \boldsymbol{\xi} \qquad \forall\, \boldsymbol{\xi} \in \mathcal{G}_{k-1}(T)\,, \tag{4.4b}$$
- the element moments
$$\mathbf{D3}(\boldsymbol{\tau}) := \int_T \boldsymbol{\tau} : \boldsymbol{\xi} \qquad \forall\, \boldsymbol{\xi} \in \mathcal{G}_k(T)^\perp\,. \tag{4.4c}$$

As shown in [18], the degrees of freedom **D1**-**D3** given by (4.4a)-(4.4c) guarantee unisolvency for every function in $\mathbb{X}_k^T$, and quantities
$$\boldsymbol{\mathscr{P}}_k^T(\boldsymbol{\tau}) \quad \text{and} \quad \mathbf{div}(\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in \mathbb{X}_k^T\,,$$
are computable thanks to the degrees of freedom **D1**-**D3**. The global virtual element subspace of $\mathbb{X}$ is defined as
$$\mathbb{X}_h := \Big\{\boldsymbol{\tau} \in \mathbb{X} : \quad \boldsymbol{\tau}|_T \in \mathbb{X}_k^T \quad \forall\, T \in \Omega_h\Big\}\,.$$
Focusing on approximating strain rate tensor, and velocity, we use piecewise polynomial spaces as follow
$$\mathbb{Q}_h := \mathbb{P}_k(\Omega_h) \cap \mathbb{L}^r_{\mathtt{tr}}(\Omega)\,, \quad \text{and} \quad \mathbf{Y}_h = \mathbf{P}_k(\Omega_h)\,.$$
We end this section by upgrading of notations provided by (2.15) in the discrete type as
$$\vec{\mathbf{v}}_h := (\mathbf{u}_h, \mathbf{t}_h) \in \mathbf{V}_h := \mathbf{Y}_h \times \mathbb{Q}_h\,.$$

4.4 Interpolation estimates

In order to define an interpolation operator in the local space $\mathbb{X}_k(T)$, for each element $T \in \Omega_h$ we denote by χ_i^T the operator associated to the i-th local degree of freedom, $i = 1, \cdots, n_k^T$. From the definition of this space, it is easily seen that for every smooth enough function $\boldsymbol{\tau} \in \mathbb{W}^{1,1}(T)$ there exists a unique operator $\boldsymbol{\Pi}_k^T(\boldsymbol{\tau}) \in \mathcal{H}_k(T)$ such that

$$\chi_i^T\big(\boldsymbol{\tau} - \boldsymbol{\Pi}_k^T(\boldsymbol{\tau})\big) = 0 \quad \forall\, i = 1, \cdots, n_k^T . \tag{4.5}$$

In addition, following the discussion of [18] (also see [36]), we have the standard interpolation estimate

$$\|\boldsymbol{\tau} - \boldsymbol{\Pi}_k^T(\boldsymbol{\tau})\|_{0,T} \precsim h_T^j\, |\boldsymbol{\tau}|_{j,T} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}^j(T) . \tag{4.6}$$

We also recall the following commutative property

$$\mathbf{div}(\boldsymbol{\Pi}_k^T(\boldsymbol{\tau})) = \boldsymbol{\mathcal{P}}_k^T(\mathbf{div}(\boldsymbol{\tau})) \qquad \forall\, \boldsymbol{\tau} \in \mathbb{W}^{1,1}(T) . \tag{4.7}$$

Now, from Lemma 4.2 (cf. (4.3)) and (4.7) we deduce, for each $\boldsymbol{\tau} \in \mathbb{W}^{1,1}(E)$ such that $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{H}^j(T)$, with $j \in [0, k+1]$, there holds (see, e.g., [36, eq. (3.14)])

$$\|\mathbf{div}\big(\boldsymbol{\tau} - \boldsymbol{\Pi}_k^T(\boldsymbol{\tau})\big)\|_{0,T} \precsim h_T^j\, |\mathbf{div}(\boldsymbol{\tau})|_{j,T} . \tag{4.8}$$

As a consequence of the local approximation properties stated by (4.6), (4.8), and Lemma 4.2, we easily derive for each integer $j \in [1, k+1]$ the following global ones:

($\mathbf{AP}_h^{\mathbf{t}}$) for any $\mathbf{t} \in \mathbb{L}_{\mathrm{tr}}^r \cap \mathbb{W}^{j,r}(\Omega)$ there hold

$$\|\mathbf{t} - \boldsymbol{\mathcal{P}}_k^h(\mathbf{t})\|_{0,r;\Omega} \precsim h^j\, |\mathbf{t}|_{j,r;\Omega} ,$$

($\mathbf{AP}_h^{\boldsymbol{\sigma}}$) for any $\boldsymbol{\tau} \in \mathbb{X}_h \cap \mathbb{W}^{j,s}(\Omega)$ such that $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{H}^j(\Omega)$, there hold

$$\|\boldsymbol{\tau} - \boldsymbol{\Pi}_k(\boldsymbol{\tau})\|_{s,\mathbf{div};\Omega} \precsim h^j \left\{ |\boldsymbol{\tau}|_{j,s;\Omega} + |\mathbf{div}(\boldsymbol{\tau})|_{j,\Omega} \right\} ,$$

($\mathbf{AP}_h^{\mathbf{u}}$) for any $\mathbf{v} \in \mathbf{L}^2(\Omega) \cap \mathbf{H}^j(\Omega)$ there hold

$$\|\mathbf{v} - \mathcal{P}_k^h(\mathbf{v})\|_{0,\Omega} \precsim h^j\, |\mathbf{v}|_{j,\Omega} ,$$

4.5 The fully-discrete scheme

To formulate the discrete scheme for problem (2.12), we proceed by introducing computable discrete versions of the operators involving the virtual space, as needed. In particular, we observe initially from the definitions of the discrete spaces and the linear (and nonlinear) operators (cf. (2.13)) and functionals (cf. (2.14)) that it is necessary to define only the discrete version of the inf-sup term. This term is discretized by the operator $\mathcal{B}_h^T : \mathbf{V}_h \to \mathbb{X}_h'$ such that

$$\big[\mathcal{B}_h^T(\vec{\mathbf{v}}_h), \boldsymbol{\tau}_h\big] := -\int_T \mathbf{div}(\boldsymbol{\tau}_h) \cdot \mathbf{v}_h - \int_T \boldsymbol{\mathcal{P}}_k^T \boldsymbol{\tau}_h : \mathbf{s}_h . \tag{4.9}$$

In addition, as usual we define the global operator by

$$\big[\mathcal{B}_h(\mathbf{s}_h), \boldsymbol{\tau}_h\big] := \sum_{T \in \Omega_h} \big[\mathcal{B}_h^T(\mathbf{s}_h), \boldsymbol{\tau}_h\big] .$$

Finally, by discretizing in time using the backward Euler method, which include introducing a sequence of time steps $t_n = n\Delta t$, $n = 1, \cdots, N$ with constant step-size $\Delta t = t_F/N$ and denoting $f^n := f(\cdot, t_n)$, $\delta_t f^n := (f^n - f^{n-1})/\Delta t$ for a generic function f, combined to mixed VEM with considering the above discrete form, we construct the following fully-discrete mixed VE scheme: Find $(\vec{\mathbf{u}}_h^n, \boldsymbol{\sigma}_h^n,) \in \mathbf{V}_h \times \mathbb{X}_h$, for each $n = 1, \cdots, N$, such that

$$[C(\delta_t \vec{\mathbf{u}}_h^n), \vec{\mathbf{v}}_h] + [\mathcal{A}(\vec{\mathbf{u}}_h^n), \vec{\mathbf{v}}_h] + [\mathcal{B}_h(\vec{\mathbf{v}}_h), \boldsymbol{\sigma}_h^n] = [\mathcal{F}^n, \vec{\mathbf{v}}_h] \qquad \forall\, \vec{\mathbf{v}}_h \in \mathbf{V}_h\,, \tag{4.10a}$$

$$[\mathcal{B}_h(\vec{\mathbf{u}}_h^n), \boldsymbol{\tau}_h] = 0 \qquad \forall\, \boldsymbol{\tau}_h \in \mathbb{X}_h\,. \tag{4.10b}$$

We set the initial condition by taking $(\vec{\mathbf{u}}_h^0, \boldsymbol{\sigma}_h^0) = ((\mathbf{u}_h^0, \mathbf{t}_h^0), \boldsymbol{\sigma}_h^0) \in \mathbf{V}_h \times \mathbb{X}_h$ as an appropriate approximation of $(\vec{\mathbf{u}}_0\,, \boldsymbol{\sigma}_0)$ that satisfying

$$\begin{aligned} [\mathcal{A}(\vec{\mathbf{u}}_h^0), \vec{\mathbf{v}}_h] + [\mathcal{B}_h(\vec{\mathbf{v}}_h), \boldsymbol{\sigma}_h^0] &= [F^0, \vec{\mathbf{v}}_h] \qquad \forall\, \vec{\mathbf{v}}_h \in \mathbf{V}_h\,, \\ [\mathcal{B}_h(\vec{\mathbf{u}}_h^0), \boldsymbol{\tau}_h] &= 0 \qquad \forall\, \boldsymbol{\tau}_h \in \mathbb{X}_h\,, \end{aligned} \tag{4.11}$$

with $F^0 \in \mathbf{L}^2(\Omega) \times \{\mathbf{0}\}$ defined in (3.24). The purpose of this choice is to ensure that the discrete initial datum is compatible with Lemma 3.7, in order to apply Theorem 3.1.

5 Discrete solvability analysis

In this section, we proceed similarly to Section 3 and establish the well-posedness of the fully-discrete scheme (4.10) by employing the discrete versions of Theorems 3.1 and 3.2. In this regard, since $\mathcal{B}_h$ is the only discrete operator in (4.10), we will first discuss the stability properties of this discrete operator.

5.1 Discrete inf-sup condition

Here, we focus on deriving the discrete inf-sup condition for $\mathcal{B}_h$. To achieve this, we first recall the abstract result established in [25, Lemmas 5.1 and 5.2], which will serve as an essential tool for the aforementioned purpose.

Lemma 5.1. *Let U, V, V_1, V_2 and W be reflexive Banach spaces with V_1 and V_2 being closed subspaces of V such that $V = V_1 \oplus V_2$, and assume that the norm of V can be redefined, equivalently, but with constants independent of V_1 and V_2, as $\|v\| := \|v_1\| + \|v_2\|$ for any $v \in V$, with $v_i \in V_i$ for $i \in \{1, 2\}$. In addition, let $\mathcal{B} : \mathcal{L}(U \times V, W')$ be a linear operator, and define the following subspaces:*

$$\begin{aligned} Z &:= \Big\{(u, v) \in U \times V : \quad [\mathcal{B}(u, v), w] = 0 \quad \forall\, w \in W\Big\}, \quad \textit{and} \\ W_0 &:= \Big\{w \in W : \quad [\mathcal{B}(u, v_2), w] = 0 \quad \forall\, (u, v_2) \in U \times V_2\Big\}. \end{aligned} \tag{5.1}$$

The the following statements are equivalents

(*i*) *there exists positive constants β_1, β_2 such that*

$$\sup_{\mathbf{0} \neq (u,v) \in U \times V} \frac{[\mathcal{B}(u, v), w]}{\|(u, v)\|} \geq \beta_1 \|w\| \qquad \forall\, w \in W\,,$$

and

$$\|u_1\| \geq \beta_2 \|(u, v_2)\| \qquad \forall\, (u, v) \in Z\,.$$

(*ii*) *there exist positive constants β_3, β_4 such that*

$$\begin{aligned} \sup_{0 \neq w \in W} \frac{[\mathcal{B}(u, v_2), w]}{\|w\|} &\geq \beta_3 \|(u, v_2)\| \qquad \forall\, (u, v_2) \in U \times V_2\,, \quad \textit{and} \\ \sup_{0 \neq v_1 \in V_1} \frac{[\mathcal{B}(0, v_1), w]}{\|v_1\|} &\geq \beta_4 \|w\| \qquad \forall\, w \in W_0\,. \end{aligned} \tag{5.2}$$

The following lemma provides sufficient conditions for the inf-sup condition of $\mathcal{B} \in \mathcal{L}(U \times V_2, W')$ (cf. the first row of (5.2)).

Lemma 5.2. *In addition to the hypotheses and notations of Lemma 5.1, we introduce the subspace*

$$W_1 \;:=\; \Big\{ w \in W : \quad [\mathcal{B}(u,0), w] \;=\; 0 \quad \forall\, u \in U \Big\}\,,$$

and assume that there exist positive constants β_5, β_6 such that

$$\begin{aligned} \sup_{0 \neq w \in W} \frac{[\mathcal{B}(u,0), w]}{\|w\|} \;&\geq\; \beta_5\,\|u\| \qquad \forall\, u \in U\,, \quad \textit{and} \\ \sup_{0 \neq w \in W_1} \frac{[\mathcal{B}(0,v_2), w]}{\|w\|} \;&\geq\; \beta_6\,\|v_2\| \qquad \forall\, v_2 \in V_2\,. \end{aligned} \tag{5.3}$$

Then, there holds the inf-sup condition given in the first row of (5.2).

Now, we are ready to utilize the two results above for establishing discrete inf-sup condition for $\mathcal{B}_h$. We begin with defining spaces U, V, and W in Lemmas 5.1 and 5.2 by

$$U \;:=\; \mathbf{Y}_h\,, \qquad V \;:=\; \mathbb{Q}_h, \quad \text{and} \quad W \;:=\; \mathbb{X}_h\,, \tag{5.4}$$

and letting

$$\mathbb{Q}_{h,\mathtt{sym}} \;:=\; \Big\{ \mathbf{s} \in \mathbb{Q}_h : \quad \mathbf{s}^{\perp} - \mathbf{s} = 0 \Big\} \quad \text{and} \quad \mathbb{Q}_{h,\mathtt{skw}} \;:=\; \Big\{ \mathbf{s} \in \mathbb{Q}_h : \quad \mathbf{s}^{\perp} + \mathbf{s} = \mathbf{0} \Big\}\,,$$

we proceed to split the space $\mathbb{Q}_h$ as $\mathbb{Q}_h \;=\; \mathbb{Q}_{h,\mathtt{sym}} + \mathbb{Q}_{h,\mathtt{skw}}$, and realize from the orthogonality of spaces $\mathbb{Q}_{h,\mathtt{sym}}$ and $\mathbb{Q}_{h,\mathtt{skw}}$ that for any $\mathbf{s} = \mathbf{s}_{\mathtt{sym}} + \mathbf{s}_{\mathtt{skw}} \in \mathbb{Q}_h$, there holds

$$\|\mathbf{s}\|_{0,r;\Omega} \;=\; \|\mathbf{s}_{\mathtt{sym}}\|_{0,r;\Omega} \;+\; \|\mathbf{s}_{\mathtt{skw}}\|_{0,r;\Omega},$$

from which, we deduce that the space V – as well as the other spaces U and W given by (5.4) – satisfy the hypotheses of Lemmas 5.1 and 5.2. In this way, considering the product spaces and setting notations

$$\begin{aligned} \vec{\mathbf{v}}_1 \;&:=\; (\mathbf{v}, \mathbf{s}_{\mathtt{skw}}) \in \mathbf{V}_{1,h} \;:=\; \mathbf{Y}_h \times \mathbb{Q}_{h,\mathtt{skw}} \quad \text{and} \\ \vec{\mathbf{v}}_2 \;&:=\; (\mathbf{0}, \mathbf{s}_{\mathtt{sym}}) \in \mathbf{V}_{2,h} \;:=\; \{\mathbf{0}\} \times \mathbb{Q}_{h,\mathtt{sym}}\,, \end{aligned} \tag{5.5}$$

respectively, according to Lemma 5.1, in order to establish the discrete inf-sup condition for $\mathcal{B}_h : \mathcal{L}(\mathbf{V}_h, \mathbb{X}_h^{'})$ we need to show the discrete inf-sup conditions of $\mathcal{B}_h : \mathcal{L}(\mathbf{V}_{1,h}, \mathbb{X}_h^{'})$ and $\mathcal{B}_h : \mathcal{L}(\mathbf{V}_{2,h}, W_0^{'})$ (cf. (5.2)), where W_0 is the kernel of $\mathcal{B}_h : \mathcal{L}(\mathbf{V}_{1,h}, \mathbb{X}_h^{'})$ and using (5.1), (5.4)-(5.5) redefine it as:

$$\begin{aligned} W_0 \;&:=\; \Big\{ \boldsymbol{\tau} \in \mathbb{X}_h : \quad [\mathcal{B}_h(\vec{\mathbf{v}}_1), \boldsymbol{\tau}] \;=\; 0 \quad \forall\, \vec{\mathbf{v}}_1 \in \mathbf{V}_{1,h} \Big\} \\ &=\; \Big\{ \boldsymbol{\tau} \in \mathbb{X}_h : \int_{\Omega_h} \boldsymbol{\mathcal{P}}_r(\boldsymbol{\tau}) : \mathbf{s}_{\mathtt{skw}} = 0 \quad \text{and} \quad \int_{\Omega_h} \mathbf{v} \cdot \mathbf{div}\boldsymbol{\tau} \;=\; 0 \quad \forall\, \vec{\mathbf{v}}_1 = (\mathbf{v}, \mathbf{s}_{\mathtt{skw}}) \in \mathbf{V}_{1,h} \Big\}. \end{aligned} \tag{5.6}$$

On the other hand, invoking Lemma 5.2 with notations given by

$$\vec{\mathbf{v}}_3 \;:=\; (\mathbf{v}, \mathbf{0}) \in \mathbf{V}_{3,h} \;:=\; \mathbf{Y}_h \times \{\mathbf{0}\} \quad \text{and} \quad \vec{\mathbf{v}}_4 \;:=\; (\mathbf{0}, \mathbf{s}_{\mathtt{skw}}) \in \mathbf{V}_{4,h} \;:=\; \{\mathbf{0}\} \times \mathbb{Q}_{h,\mathtt{skw}}, \tag{5.7}$$

we conclude that to obtain the discrete inf-sup condition $\mathcal{B}_h : \mathcal{L}(\mathbf{V}_{1,h}, \mathbb{X}_h^{'})$, we just need to prove inf-sup conditions of $\mathcal{B}_h : \mathcal{L}(\mathbf{V}_{3,h}, \mathbb{X}_h^{'})$ and $\mathcal{B}_h : \mathcal{L}(\mathbf{V}_{1,h}, W_1^{'})$ provided by (5.3), where

$$W_1 \;:=\; \Big\{ \boldsymbol{\tau} \in \mathbb{X}_h : \quad [\mathcal{B}_h(\vec{\mathbf{v}}_3), \boldsymbol{\tau}] \;=\; 0 \quad \forall\, \vec{\mathbf{v}}_3 = (\mathbf{v}, \mathbf{0}) \in \mathbf{V}_{3,h} \Big\}\,.$$

It follows from (5.4) and notations (5.7) that W_1 can be redefined as

$$W_1 \;=\; \Big\{ \boldsymbol{\tau} \in \mathbb{X}_h : \quad \int_{\Omega_h} \mathbf{v} \cdot \mathbf{div}\boldsymbol{\tau} \;=\; 0 \quad \forall\, \mathbf{v} \in \mathbf{Y} \Big\}. \tag{5.8}$$

We note that as a consequence of $\mathbf{div}(\mathbb{X}_h) \subseteq \mathbf{V}_h$ and the orthogonality of $\boldsymbol{\mathcal{P}}_r$ (cf. tensor version of (4.1)) we easily deduce from (5.6) and (5.8) that

$$\begin{aligned} W_0 &= \left\{ \boldsymbol{\tau} \in \mathbb{X}_h : \quad \int_{\Omega_h} \boldsymbol{\mathcal{P}}_r^K \boldsymbol{\tau} : \mathbf{s}_{\mathtt{skw}} = 0 \quad \forall\, \mathbf{s}_{\mathtt{skw}} \in \mathbb{Q}_{h,\mathtt{skw}}, \quad \text{and} \quad \mathbf{div}\,\boldsymbol{\tau} = \mathbf{0} \quad \text{in}\ \Omega \right\}, \\ W_1 &= \left\{ \boldsymbol{\tau} \in \mathbb{X}_h : \quad \mathbf{div}\,\boldsymbol{\tau} = \mathbf{0} \quad \text{in}\ \Omega \right\}. \end{aligned} \tag{5.9}$$

As a summation of the above discussion, to prove discrete inf-sup condition $\mathcal{B}_h \in \mathcal{L}(\mathbf{V}_h, \mathbb{X}_h')$ it is sufficient to obtain the inf-sup conditions $\mathcal{B}_h \in \mathcal{L}(\mathbf{V}_{3,h}, \mathbb{X}_h')$, $\mathcal{B}_h \in \mathcal{L}(\mathbf{V}_{2,h}, W_0')$ and $\mathcal{B}_h \in \mathcal{L}(\mathbf{V}_{4,h}, W_1')$. We begin with the following lemma establishing the first case, for which we recall preliminary result, which recently established in [36, Lemma 4.8].

Lemma 5.3. *For $r < 2$, there exist a constant $C_{\mathtt{sta}}$ such that*

$$\|\mathbf{\Pi}_k(\boldsymbol{\tau})\|_{0,\Omega} \,\leq\, C_{\mathtt{sta}}\, \|\boldsymbol{\tau}\|_{1,r;\Omega} \quad \forall \boldsymbol{\tau} \in \mathbb{W}^{1,r}(\Omega)\,. \tag{5.10}$$

We are now in position to establish discrete inf-sup condition for $\mathcal{B}_h : \mathbf{V}_{3,h} \to \mathbb{X}_h'$.

Lemma 5.4. *There exists a positive constant $\beta_{5,\mathrm{d}}$, independent of h, such that*

$$\sup_{\boldsymbol{\tau}_h \in \mathbb{X}_h} \frac{[\mathcal{B}_h(\mathbf{v}_h, \mathbf{0}), \boldsymbol{\tau}_h]}{\|\boldsymbol{\tau}_h\|_{s,\mathbf{div};\Omega}} \,\geq\, \beta_{5,h}\, \|\mathbf{v}_h\|_{0,\Omega} \quad \forall\, \vec{\mathbf{v}}_{3,h} = (\mathbf{v}_h, \mathbf{0}) \in \mathbf{V}_{3,h}\,, \tag{5.11}$$

Proof. Due to (3.14), it is sufficient to show the existence of a Fortin operator. More precisely, we need to construct a $\boldsymbol{\tau}_h \in \mathbb{X}_h$ such that

$$[\mathcal{B}_h(\mathbf{v}_h, \mathbf{0}), \boldsymbol{\tau}_h] \,=\, [\mathcal{B}(\mathbf{v}_h, \mathbf{0}), \boldsymbol{\tau}] \qquad \forall\, \mathbf{v}_h \in \mathbf{Y}_h\,,$$

with

$$\|\boldsymbol{\tau}_h\|_{s,\mathbf{div};\Omega} \,\leq\, C\, \|\boldsymbol{\tau}\|_{s,\mathbf{div};\Omega}\,,$$

for some constant $C > 0$, independent of h. To do that, we proceeds analogously to the proof of [29, Lemma 4.4]. Given $\boldsymbol{\tau} \in \mathbb{H}_0^s(\mathbf{div};\Omega)$, we set

$$\mathbf{h}_{\boldsymbol{\tau}} \,:=\, \begin{cases} \mathbf{div}(\boldsymbol{\tau}) & \text{in} \quad \Omega\,, \\ \mathbf{0} & \text{in} \quad \mathscr{D} \setminus \Omega\,, \end{cases}$$

where $\mathscr{D}$ is an open ball containing $\bar{\Omega}$. Since $\mathbf{h}_{\boldsymbol{\tau}} \in \mathbf{L}^2(\Omega)$, a well-known result on regularity of elliptic problems implies that there exists a unique weak solution $\mathbf{w} \in \mathbf{H}_0^1(\mathscr{D}) \cap \mathbf{H}^2(\mathscr{D})$ of the boundary value problem

$$\boldsymbol{\Delta}\mathbf{w} \,=\, \mathbf{h}_{\boldsymbol{\tau}} \quad \text{in} \quad \mathscr{D}\,, \quad \mathbf{w} \,=\, \mathbf{0} \quad \text{on} \quad \partial\mathscr{D}\,,$$

that satisfies

$$\|\mathbf{w}\|_{2,\mathscr{D}} \,\leq\, C\, \|\mathbf{h}_{\boldsymbol{\tau}}\|_{0,\mathscr{D}}.$$

Then, setting $\widehat{\boldsymbol{\tau}} = -\nabla \mathbf{w}$, implies

$$\mathbf{div}(\widehat{\boldsymbol{\tau}}) \,=\, \mathbf{div}(\boldsymbol{\tau}) \quad \text{in} \quad \Omega \quad \text{and} \quad \|\widehat{\boldsymbol{\tau}}\|_{1,\Omega} \,\leq\, C\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,\Omega}\,. \tag{5.12}$$

We can now define the Fortin operator $\mathbf{\Pi}^{\mathscr{F}} : \mathbb{H}^1(\Omega) \to \mathbb{X}_h$ as

$$\mathbf{\Pi}^{\mathscr{F}}(\boldsymbol{\tau}) \,:=\, \mathbf{\Pi}_k(\widehat{\boldsymbol{\tau}}) - \Big(\frac{1}{2\Omega|} \int_{\Omega} \mathbf{\Pi}_k(\widehat{\boldsymbol{\tau}})\Big)\mathbb{I}\,,$$

where $\mathbb{I}$ is the identity matrix in $\mathbb{R}^d$. In turn, employing the estimate (5.10) given by Lemma 5.3, the inequality from (5.12), along with the Sobolev embeddings of $\mathbb{L}^s(\Omega) \hookrightarrow \mathbb{L}^2(\Omega)$ and $\mathbb{H}^1(\Omega) \hookrightarrow \mathbb{W}^{1,r}(\Omega)$, where $\frac{1}{r} + \frac{1}{s} = 1$, we obtain

$$\begin{aligned} \left\|\mathbf{\Pi}^{\mathcal{F}}(\boldsymbol{\tau})\right\|_{0,s;\Omega} \,&\leq\, \left\|\mathbf{\Pi}_k(\widehat{\boldsymbol{\tau}})\right\|_{0,s;\Omega} \,\leq\, |\Omega|^{\frac{1}{s}-\frac{1}{2}} \left\|\mathbf{\Pi}_k(\widehat{\boldsymbol{\tau}})\right\|_{0,\Omega} \\ &\leq\, |\Omega|^{\frac{1}{s}-\frac{1}{2}} C_{\mathtt{sta}} \|\widehat{\boldsymbol{\tau}}\|_{1,r,\Omega} \,\leq\, |\Omega|^{\frac{1}{s}+\frac{1}{r}-1} C_{\mathtt{sta}} \|\widehat{\boldsymbol{\tau}}\|_{1,\Omega} \\ &=\, C_{\mathtt{sta}} \|\widehat{\boldsymbol{\tau}}\|_{1,\Omega} \,\leq\, \widetilde{C}\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,\Omega}. \end{aligned} \tag{5.13}$$

On the other hand, an application of identities (4.7) and (5.12) gives

$$\mathbf{div}\big(\mathbf{\Pi}^{\mathcal{F}}(\boldsymbol{\tau})\big) \,=\, \mathbf{div}\big(\mathbf{\Pi}_k(\widehat{\boldsymbol{\tau}})\big) \,=\, \mathcal{P}_k\big(\mathbf{div}(\widehat{\boldsymbol{\tau}})\big) \,=\, \mathcal{P}_k\big(\mathbf{div}(\boldsymbol{\tau})\big)\,, \tag{5.14}$$

which thanks to (4.2) also gives

$$\left\|\mathbf{div}\big(\mathbf{\Pi}^{\mathcal{F}}(\boldsymbol{\tau})\big)\right\|_{0,\Omega} \,\leq\, M_k \,\|\mathbf{div}(\boldsymbol{\tau})\|_{0,\Omega}\,. \tag{5.15}$$

Now, taking square in (5.13) and (5.15), then adding two resulting inequalities, we arrive at

$$\|\mathbf{\Pi}^{\mathcal{F}}(\boldsymbol{\tau})\|_{s,\mathbf{div};\Omega} \,\leq\, \left(\widetilde{C} + M_k\right) \|\boldsymbol{\tau}\|_{s,\mathbf{div};\Omega}\,. \tag{5.16}$$

Finally, from (5.14) and (5.16), we find

$$\begin{aligned} \sup_{\boldsymbol{\tau}_h\in\mathbb{X}_h} \frac{[\mathcal{B}_h(\mathbf{v}_h,\mathbf{0}),\boldsymbol{\tau}_h]}{\|\boldsymbol{\tau}_h\|_{s,\mathbf{div};\Omega}} \,&\geq\, \sup_{\boldsymbol{\tau}\in\mathbb{X}} \frac{[\mathcal{B}_h(\mathbf{v}_h,\mathbf{0}),\mathbf{\Pi}^{\mathcal{F}}(\boldsymbol{\tau})]}{\|\mathbf{\Pi}^{\mathcal{F}}(\boldsymbol{\tau})\|_{s,\mathbf{div};\Omega}} \\ &\geq\, \sup_{\boldsymbol{\tau}\in\mathbb{X}} \frac{1}{\widehat{C}+M_k} \frac{\displaystyle\int_\Omega \mathcal{P}_k\big(\mathbf{div}(\boldsymbol{\tau})\big)\cdot\mathbf{v}_h}{\|\boldsymbol{\tau}\|_{s,\mathbf{div};\Omega}} \,=\, \sup_{\boldsymbol{\tau}\in\mathbb{X}} \frac{1}{\widehat{C}+M_k} \frac{[\mathcal{B}(\mathbf{v}_h,\mathbf{0}),\boldsymbol{\tau}]}{\|\boldsymbol{\tau}\|_{s,\mathbf{div};\Omega}}, \end{aligned}$$

which gives (5.11) with $\beta_{5,\mathrm{d}} := \beta/\big(\widehat{C}+M_k\big)$. □

We are now in position of establishing the next result, that is, the discrete inf-sup condition of $\mathcal{B}_h \in \mathcal{L}(\mathbf{V}_{2,h}, W_0')$, where $\mathbf{V}_{2,h}$ and W_0' are given by the first and second rows of (5.5) and (5.9), respectively.

Lemma 5.5. *There exists a positive constant $\beta_{4,\mathrm{d}}$, independent of h, such that*

$$\sup_{\mathbf{0}\neq\mathbf{s}_{h,\mathtt{sym}}\in\mathbb{Q}_{h,\mathtt{sym}}} \frac{[\mathcal{B}_h(\mathbf{0},\mathbf{s}_{h,\mathtt{sym}}),\boldsymbol{\tau}_h]}{\|\mathbf{s}_{h,\mathtt{sym}}\|_{0,r;\Omega}} \,\geq\, \beta_{4,\mathrm{d}}\,\|\boldsymbol{\tau}_h\|_{s,\mathbf{div};\Omega} \qquad \forall\,\boldsymbol{\tau}_h \in W_0\,. \tag{5.17}$$

Proof. First, we observe from the first identity given in (5.9), that is $\mathbf{div}(\boldsymbol{\tau}_h) = \mathbf{0}$ in Ω_h, implies the existence function $\boldsymbol{\phi} \in \mathbf{H}^1(T)$ for each $T \in \Omega_h$, such that there holds

$$\boldsymbol{\tau}_h \,=\, \mathbf{curl}(\boldsymbol{\phi}) \quad \text{in} \quad T\,.$$

In turn, applying the operator **rot** to the above equation we find that

$$\mathbf{rot}(\boldsymbol{\tau}_h) \,=\, \mathbf{rot}(\mathbf{curl}(\boldsymbol{\phi})) \,=\, \Delta\boldsymbol{\phi} \quad \text{in} \quad T\,. \tag{5.18}$$

Consequently, noting that $\boldsymbol{\tau}_h \in \mathbb{X}_k(T)$, so necessarily $\mathbf{rot}(\boldsymbol{\tau}_h) \in \mathbf{P}_{k-1}(T)$ which implies that $\Delta\boldsymbol{\phi} \in \mathbf{P}_{k-1}(T)$. On the other hand, by recalling from [7, third proposition of eq. (2.10)] that the rotation operator **rot** is an isomorphism from $\mathcal{G}_r^\perp$ to the whole $\mathbf{P}_{k-1}$, where

$$\mathcal{G}_r \,=\, \Big\{\mathbf{z}_k \,:=\, \boldsymbol{\nabla}(v_{k+1}) \;\text{ with } v_{k+1} \in \mathrm{P}_{k+1}\Big\}\,, \qquad \mathbf{P}_k \,=\, \mathcal{G}_k \oplus \mathcal{G}_k^\perp\,.$$

Then, for given $\Delta\boldsymbol{\phi} \in \mathbf{P}_{k-1}(K)$ there exists a unique $\mathbf{z}_k \in \mathcal{G}_k^{\perp}$ such that

$$\mathbf{rot}(\mathbf{z}_k) \;=\; \Delta\boldsymbol{\phi} \quad \text{in} \quad T\,. \tag{5.19}$$

Therefore, from (5.18) and (5.19) we deduce that $\boldsymbol{\tau}_h \in \mathbb{P}_k(T)$. This result combined with the first equation given in (5.9), that is $\int_{\Omega_h} \boldsymbol{\tau}_h^{\mathrm{d}} : \mathbf{s}_{h,\mathtt{skw}} = 0$, allow us set $\widetilde{\mathbf{s}}_{h,\mathtt{sym}} := |\boldsymbol{\tau}_h^{\mathrm{d}}|^{s-2}\boldsymbol{\tau}_h^{\mathrm{d}}$. It is easy to see that

$$\|\widetilde{\mathbf{s}}_{h,\mathtt{sym}}\|_{0,r;\Omega}^{r} = \|\boldsymbol{\tau}_h^{\mathrm{d}}\|_{0,s;\Omega}^{s}\,, \quad \mathrm{tr}(\widetilde{\mathbf{s}}) = |\boldsymbol{\tau}_h^{\mathrm{d}}|^{s-2}\mathrm{tr}(\boldsymbol{\tau}_h^{\mathrm{d}}) = 0 \quad \text{and} \quad (\widetilde{\mathbf{s}}_{h,\mathtt{sym}})^{\perp} \;=\; \widetilde{\mathbf{s}}_{h,\mathtt{sym}}\,,$$

which says that $\widetilde{\mathbf{s}}_{h,\mathtt{sym}} \in \mathbb{Q}_{h,\mathtt{sym}}$, and additionally there holds

$$\int_{\Omega} \boldsymbol{\tau}_h : \widetilde{\mathbf{s}}_{h,\mathtt{sym}} \;=\; \|\boldsymbol{\tau}_h^{\mathrm{d}}\|_{0,s}^{s} \;=\; \|\boldsymbol{\tau}_h^{\mathrm{d}}\|_{0,s;\Omega}\,\|\widetilde{\mathbf{s}}_{h,\mathtt{sym}}\|_{0,r;\Omega}\,. \tag{5.20}$$

Then, bounding from below the supremum in (5.17) with $\widetilde{\mathbf{s}}_{h,\mathtt{sym}}$, and employing (5.20), we obtain

$$\sup_{\mathbf{0}\neq\mathbf{s}_{h,\mathtt{sym}}\in\mathbb{Q}_{h,\mathtt{sym}}} \frac{[\mathcal{B}_h(\mathbf{0},\mathbf{s}_{h,\mathtt{sym}}),\boldsymbol{\tau}_h]}{\|\mathbf{s}_{h,\mathtt{sym}}\|_{0,r;\Omega}} \;\geq\; \frac{\displaystyle\int_{\Omega} \widetilde{\mathbf{s}}_{h,\mathtt{sym}} : \boldsymbol{\tau}_h}{\|\widetilde{\mathbf{s}}_{h,\mathtt{sym}}\|_{0,r;\Omega}} \;=\; \|\boldsymbol{\tau}_h^{\mathrm{d}}\|_{0,s;\Omega}\,,$$

from which, using (3.18) and the fact that $\mathbf{div}(\boldsymbol{\tau}) = \mathbf{0}$, it follows (5.17) with $\beta_{4,\mathrm{d}} = c_{\Omega}$. □

The following result, which is a slight modification of the proof of [39, Lemma 5.6], state the discrete inf-sup condition for $\mathcal{B}_h \in \mathcal{L}(\mathbf{V}_{4,h}, W_1')$, where $\mathbf{V}_{4,h}$ and W_1 are given by (5.7) and (5.9), respectively.

Lemma 5.6. *There exists a constant $\beta_{3,\mathrm{d}}$ such that*

$$\sup_{\boldsymbol{\tau}_h\in W_1} \frac{[\mathcal{B}_h(\mathbf{0},\mathbf{s}_{h,\mathtt{skw}}),\boldsymbol{\tau}_h]}{\|\boldsymbol{\tau}_h\|_{s,\mathbf{div};\Omega}} \;\geq\; \beta_{3,\mathrm{d}}\,\|\mathbf{s}_{h,\mathtt{skw}}\|_{0,r;\Omega} \quad \forall\,\vec{\mathbf{v}}_{4,h} = (0,\mathbf{s}_{h,\mathtt{skw}}) \in \mathbf{V}_{4,h}\,.$$

Proof. It reduces to a minor variation of the proof of [39, Lemma 5.6]. □

We are now in position to establish the main result of this section. More precisely, we have the following lemma.

Lemma 5.7. *Let $\mathbf{K}_h$ be the kernel of $\mathcal{B}_h$, that is,*

$$\mathbf{K}_h \;:=\; \Big\{ \boldsymbol{\tau}_h \in \mathbb{X}_h : \quad [\mathcal{B}_h(\vec{\mathbf{v}}_h),\boldsymbol{\tau}_h] \;=\; 0 \quad \forall\,\vec{\mathbf{v}}_h \in \mathbf{V}_h \Big\}\,.$$

Then there exist positive constants $\beta_{1,\mathrm{d}},\beta_{2,\mathrm{d}}$, independent of h, satisfying

$$\sup_{\mathbf{0}\neq\vec{\mathbf{v}}\in\mathbf{V}_h} \frac{[\mathcal{B}_h(\vec{\mathbf{v}}_h),\boldsymbol{\tau}_h]}{\|\vec{\mathbf{v}}_h\|} \;\geq\; \beta_{1,\mathrm{d}}\,\|\boldsymbol{\tau}_h\|_{s,\mathbf{div};\Omega} \qquad \forall\,\boldsymbol{\tau}_h \in \mathbb{X}_h\,, \tag{5.21a}$$

and

$$\|\mathbf{s}_{h,\mathtt{sym}}\|_{0,r;\Omega} \;\geq\; \beta_{2,\mathrm{d}}\,\|(\mathbf{v}_h,\mathbf{s}_{h,\mathtt{skw}})\|_{\mathbf{V}} \qquad \forall\,\vec{\mathbf{v}}_h \in \mathbf{K}_h\,. \tag{5.21b}$$

Proof. It is a direct consequence of Lemmas 5.1-5.6. □

5.2 The main result

We begin by observing from (5.21a) that discrete operator $\mathcal{B}_h$ verifies the hypothesis $iv)$ of Theorem 3.2. In addition, the Lipschitz-continuity of $C+\mathcal{A}$ (cf. Lemma 3.3), is also valid on $\mathbf{V}_h \times \mathbb{X}_h$, which means that, with the same constant L_{nN}, there hold

$$\big\|(C+\mathcal{A})(\vec{\mathbf{u}}_h)-(C+\mathcal{A})(\vec{\mathbf{v}}_h)\big\| \;\leq\; L_{\mathrm{nN}}\Big\{\|\mathbf{u}_h-\mathbf{v}_h\|_{0,\Omega}+\big(\|\mathbf{t}_h\|_{0,r;\Omega}+\|\mathbf{s}_h\|_{0,r;\Omega}\big)^{r-2}\,\|\mathbf{t}_h-\mathbf{s}_h\|_{0,r;\Omega}\Big\}\,, \tag{5.22}$$

for all $\vec{\mathbf{u}}_h, \vec{\mathbf{v}}_h \in \mathbf{V}_h$.

Next, we address the discrete counterpart of Lemma 3.4.

Lemma 5.8. *Let $r < 2$ and $\mathbf{K}_h$ be defined as in Lemma 5.7. The family of operators $\Big\{(\mathcal{C}+\mathcal{A})(\cdot+\vec{\mathbf{z}}_h) : \mathbf{K}_h \to \mathbf{K}_h' : \quad \vec{\mathbf{z}}_h \in \mathbf{V}_h\Big\}$ is uniformly strictly monotone, i.e., there exists $\alpha_{\mathtt{nN,d}}$, independent of h, such that*

$$
\begin{aligned}
&[(\mathcal{C}+\mathcal{A})(\vec{\mathbf{u}}_h+\vec{\mathbf{z}}_h) - (\mathcal{C}+\mathcal{A})(\vec{\mathbf{v}}_h+\vec{\mathbf{z}}_h), \vec{\mathbf{u}}_h - \vec{\mathbf{v}}_h] \\
&\quad \geq \; \alpha_{\mathtt{nN,d}}\Big\{\|\mathbf{u}_h - \mathbf{v}_h\|_{0,\Omega}^2 \; + \; \Big(\delta^r + \|\mathbf{t}_h\|_{0,r;\Omega}^r + \|\mathbf{s}_h\|_{0,r;\Omega}^r\Big)^{\frac{r-2}{r}} \; \|\mathbf{t}_h - \mathbf{s}_h\|_{0,r;\Omega}^2\Big\},
\end{aligned} \tag{5.23}
$$

for all $\vec{\mathbf{z}}_h = (\mathbf{z}_h, \mathbf{r}_h) \in \mathbf{V}_h$ and $\vec{\mathbf{u}}_h = (\mathbf{u}_h, \mathbf{t}_h), \vec{\mathbf{v}}_h = (\mathbf{v}_h, \mathbf{s}_h) \in \mathbf{K}_h$.

Proof. Let $\vec{\mathbf{z}}_h = (\mathbf{z}_h, \mathbf{r}_h) \in \mathbf{V}_h$ and $\vec{\mathbf{u}}_h = (\mathbf{u}_h, \mathbf{t}_h), \vec{\mathbf{v}}_h = (\mathbf{v}_h, \mathbf{s}_h) \in \mathbf{K}_h$. Similar to derivation of (3.12) in Lemma 3.4, we have

$$
\begin{aligned}
&[(\mathcal{C}+\mathcal{A})(\vec{\mathbf{u}}_h+\vec{\mathbf{z}}_h) - (\mathcal{C}+\mathcal{A})(\vec{\mathbf{v}}_h+\vec{\mathbf{z}}_h), \vec{\mathbf{u}}_h - \vec{\mathbf{v}}_h] \\
&\quad \geq \; \|\mathbf{u}_h - \mathbf{v}_h\|_{0,\Omega}^2 \; + \; \Big(|\Omega|\delta^r + \|\mathbf{t}_{h,\mathtt{sym}}\|_{0,r;\Omega}^r + \|\mathbf{s}_{h,\mathtt{sym}}\|_{0,r;\Omega}^r\Big)^{\frac{r-2}{r}} \sigma^* \, \|\mathbf{t}_{h,\mathtt{sym}} - \mathbf{s}_{h,\mathtt{sym}}\|_{0,r;\Omega}^2 \, .
\end{aligned} \tag{5.24}
$$

To find the lower bound of the second term, we first use the fact that $r < 2$ to conclude that

$$
\Big(|\Omega|\delta^r + \|\mathbf{t}_{h,\mathtt{sym}}\|_{0,r;\Omega}^r + \|\mathbf{s}_{h,\mathtt{sym}}\|_{0,r;\Omega}^r\Big)^{\frac{r-2}{r}} \;\leq\; \Big(|\Omega|\delta^r + \|\mathbf{t}_h\|_{0,r;\Omega}^r + \|\mathbf{s}_h\|_{0,r;\Omega}^r\Big)^{\frac{r-2}{r}}, \tag{5.25}
$$

and then apply inequality (5.21b) to arrive at

$$
\begin{aligned}
\sigma^* \, \|\mathbf{t}_{h,\mathtt{sym}} - \mathbf{s}_{h,\mathtt{sym}}\|_{0,r;\Omega}^2 \;&\geq\; \frac{\sigma^*}{2} \, \|\mathbf{t}_{h,\mathtt{sym}} - \mathbf{s}_{h,\mathtt{sym}}\|_{0,r;\Omega}^2 \\
&\quad + \; \frac{\sigma^*}{2} \, \beta_{2,\mathtt{d}}^2 \, \Big(\|\mathbf{t}_{h,\mathtt{skw}} - \mathbf{t}_{h,\mathtt{skw}}\|_{0,r;\Omega}^2 + \|\mathbf{u}_h - \mathbf{v}_h\|_{0,\Omega}^2\Big) \\
&\geq\; \frac{\sigma^*}{2} \min\big\{1, \beta_{2,\mathtt{d}}^2\big\} \|\vec{\mathbf{u}}_h - \vec{\mathbf{v}}_h\|^2 \, .
\end{aligned} \tag{5.26}
$$

Therefore, replacing back (5.25) and (5.26) into (5.24) implies (5.23) with the specification $\alpha_{\mathtt{nN,d}} := \min\big\{1, \frac{\sigma^*}{2} \min\big\{1, \beta_{2,\mathtt{d}}^2\big\}\big\}$, thus completing the proof. □

We are now ready to provide the fully discrete counterpart of Theorems 3.8 and 3.9.

Theorem 5.9. *Let $r < 2$. For every $\mathbf{f}^n \in \mathbf{L}^2(\Omega)$, for $n = 1, \cdots, N$ and each $(\vec{\mathbf{u}}_h^0, \boldsymbol{\sigma}_h^0) = ((\mathbf{u}_{h,0}, \mathbf{t}_{h,0}), \boldsymbol{\sigma}_{h,0})$ satisfying* (4.11)*, there exists a unique solution $(\vec{\mathbf{u}}_h^n, \boldsymbol{\sigma}_h^n) = ((\mathbf{u}_h^n, \mathbf{t}_h^n), \boldsymbol{\sigma}_h^n) \in \mathbf{V}_h \times \mathbb{X}_h$ to the virtual scheme* (4.10)*. Moreover, under a suitable extra regularity assumption on the data, there exists a constant $C_{\mathtt{d.stab}}$, independent of h, such that*

$$
\begin{aligned}
&\|\mathbf{u}_h\|_{L^\infty(\mathrm{J};\mathrm{Y})}^2 \; + \; \|\mathbf{t}_h\|_{L^\infty(\mathrm{J};\mathbb{Q})}^2 \; + \; \Delta t \sum_{m=1}^{n} \|\mathbf{u}_h^m\|_{0,\Omega}^2 \; + \; \Delta t \sum_{m=1}^{n} \|\mathbf{t}_h^m\|_{0,r;\Omega}^r \; + \; \Delta t \sum_{m=1}^{n} \|\boldsymbol{\sigma}_h^m\|_{s,\mathbf{div};\Omega}^2 \\
&\quad \leq \; C_{\mathtt{d.stab}} \Big\{\|\mathbf{u}_{h,0}\|_{\mathbf{H}^1(\Omega)} \; + \; \Delta t \textstyle\sum_{m=1}^{n} \big(\|\mathbf{f}^m\|_{0,\Omega}^{r/(r-1)} + \|\mathbf{f}^m\|_{0,\Omega}^2\big)\Big\} \; := \; \mathcal{N}_{\mathtt{dis}}(\mathbf{f}^n, \mathbf{u}_0).
\end{aligned} \tag{5.27}
$$

Proof. The well-posedness of the fully discrete scheme (4.10) at every time step, for $n = 1, \cdots, N$, can be established using arguments analogue to those in the proof of Lemma 3.6. Furthermore, to prove the second part, we start by taking $(\vec{\mathbf{v}}_h, \boldsymbol{\tau}_h) = (\vec{\mathbf{u}}_h^n, \boldsymbol{\sigma}_h^n)$ in (4.10), using the identity

$$
\int_\Omega \delta_t \mathbf{u}_h^n \cdot \mathbf{u}_h^n \;=\; \frac{1}{2}\delta_t \|\mathbf{u}_h^n\|_{0,\Omega}^2 \; + \; \frac{1}{2}\Delta t \|\delta_t \mathbf{u}_h^n\|_{0,\Omega}^2 \, ,
$$

and the discrete strict monotonicity of $\mathcal{A}$ ((5.23)), to obtain

$$
\frac{1}{2}\delta_t \|\mathbf{u}_h^n\|_{0,\Omega}^2 \; + \; \frac{1}{2}\Delta t \|\delta_t \mathbf{u}_h^n\|_{0,\Omega}^2 \; + \; \alpha_{\mathtt{nN,d}} \Big(\delta^r + \|\mathbf{t}_h^n\|_{0,r;\Omega}^r\Big)^{\frac{r-2}{r}} \; \|\mathbf{t}_h^n\|_{0,r;\Omega}^2 \;\leq\; \|\mathbf{f}^n\|_{0,\Omega} \|\mathbf{u}_h^n\|_{0,\Omega} \, . \tag{5.28}
$$

Similar to the proof of Theorem 3.9, if $\|\mathbf{t}_h^n\|_{0,r;\Omega} \geq \delta$ occurs, we conclude from (5.28) that

$$\frac{1}{2}\delta_t\|\mathbf{u}_h^n\|_{0,\Omega}^2 + \frac{1}{2}\Delta t\|\delta_t\mathbf{u}_h^n\|_{0,\Omega}^2 + \alpha_{\mathtt{nN}}\, 2^{(r-2)/r}\|\mathbf{t}_h^n\|_{0,r;\Omega}^r \leq \|\mathbf{f}^n\|_{0,\Omega}\|\mathbf{u}_h^n\|_{0,\Omega}\,. \tag{5.29}$$

In turn, an application of (5.21b) implies

$$\frac{1}{2}\alpha_{\mathtt{nN}}\, 2^{(r-2)/r}\|\mathbf{t}_h^n\|_{0,r;\Omega}^r \geq \frac{1}{2}\alpha_{\mathtt{nN}}\, 2^{(r-2)/r}\|\mathbf{t}_{h,\mathtt{sym}}^n\|_{0,r;\Omega}^r \geq \frac{1}{2}\alpha_{\mathtt{nN}}\, 2^{(r-2)/r}\,\beta_{2,\mathtt{d}}^r\,\|\mathbf{u}_h^n\|_{0,\Omega}^r\,,$$

which, combined with (5.29) and employ the Young inequality (3.33) gives

$$\begin{aligned}\frac{1}{2}\delta_t\|\mathbf{u}_h^n\|_{0,\Omega}^2 &+ \frac{1}{2}\Delta t\|\delta_t\mathbf{u}_h^n\|_{0,\Omega}^2\\ + \frac{1}{2}\alpha_{\mathtt{nN}}\, 2^{(r-2)/r}\left(\|\mathbf{t}_h^n\|_{0,r;\Omega}^r + \beta_{2,\mathtt{d}}^r\,\|\mathbf{u}_h^n\|_{0,\Omega}^r\right) &\leq \frac{\alpha_1}{r}\|\mathbf{u}_h^n\|_{0,\Omega}^r + C(\alpha_1)\|\mathbf{f}^n\|_{0,\Omega}^{r/(r-1)}\,.\end{aligned} \tag{5.30}$$

Now, by the suitable choosing α_1 in (5.30), summing up over the time index $n = 1,\cdots,m$, with $m = 1,\cdots,N$, and multiplying by Δt, we get

$$\begin{aligned}\|\mathbf{u}_h^m\|_{0,\Omega}^2 &+ (\Delta t)^2\sum_{n=1}^m\|\delta_t\mathbf{u}_h^n\|_{0,\Omega}^2 + \sum_{n=1}^m\Delta t\left(\|\mathbf{t}_h^n\|_{0,r;\Omega}^r + \|\mathbf{u}_h^n\|_{0,\Omega}^r\right)\\ &\leq C_1\left(\|\mathbf{u}_h^0\|_{0,\Omega}^2 + \sum_{n=1}^m\|\mathbf{f}^n\|_{0,\Omega}^{r/(r-1)}\right).\end{aligned} \tag{5.31}$$

Whereas in the case $\|\mathbf{t}_h^n\|_{0,r;\Omega} < \delta$, by proceeding analogously, one can deduce that

$$\begin{aligned}\|\mathbf{u}_h^m\|_{0,\Omega}^2 &+ (\Delta t)^2\sum_{n=1}^m\|\delta_t\mathbf{u}_h^n\|_{0,\Omega}^2 + \sum_{n=1}^m\Delta t\left(\|\mathbf{t}_h^n\|_{0,r;\Omega}^2 + \|\mathbf{u}_h^n\|_{0,\Omega}^2\right)\\ &\leq C_2\left(\|\mathbf{u}_h^0\|_{0,\Omega}^2 + \sum_{n=1}^m\|\mathbf{f}^n\|_{0,\Omega}^2\right).\end{aligned}$$

Therefore, from (5.31) and (5.25), and the fact that $r < 2$ we infer that

$$\begin{aligned}\|\mathbf{u}_h^m\|_{0,\Omega}^2 &+ (\Delta t)^2\sum_{n=1}^m\|\delta_t\mathbf{u}_h^n\|_{0,\Omega}^2 + \sum_{n=1}^m\Delta t\left(\|\mathbf{t}_h^n\|_{0,r;\Omega}^r + \|\mathbf{u}_h^n\|_{0,\Omega}^r\right)\\ &\leq C\left(\|\mathbf{u}_h^0\|_{0,\Omega}^2 + \sum_{n=1}^m\left(\|\mathbf{f}^n\|_{0,\Omega}^{r/(r-1)} + \|\mathbf{f}^n\|_{0,\Omega}^2\right)\right).\end{aligned} \tag{5.32}$$

On the other hand, employing the discrete inf-sup condition of $\mathcal{B}_h$ (cf. (5.21a)) and the first row of (4.10), we conclude that

$$\|\boldsymbol{\sigma}_h^n\|_{s,\mathbf{div};\Omega}^2 \leq C\left\{\|\mathbf{f}^n\|_{0,\Omega}^2 + \|\delta_t\mathbf{u}_h^n\|_{0,\Omega}^2 + \|\mathbf{t}_h^n\|_{0,r;\Omega}^{2(r-1)}\right\}, \tag{5.33}$$

In turn, using Young's inequality (cf. (3.33)) and fact that $r - 1 < 1$, we readily obtain

$$\|\mathbf{t}_h^n\|_{0,r;\Omega}^{2(r-1)} \leq \frac{r-1}{r}\|\mathbf{t}_h^n\|_{0,r;\Omega}^r + \frac{1}{r}\|\mathbf{t}_h^n\|_{0,r;\Omega}^{r(r-1)} \leq \|\mathbf{t}_h^n\|_{0,r;\Omega}^r,$$

which, combined with (5.33), and then summing over n and employing (5.32), yields

$$\begin{aligned}\Delta t\sum_{n=1}^m\|\boldsymbol{\sigma}_h^n\|_{s,\mathbf{div};\Omega}^2 &\leq C\Delta t\left\{\sum_{n=1}^m\|\mathbf{f}^n\|_{0,\Omega}^2 + \sum_{n=1}^m\|\delta_t\mathbf{u}_h^n\|_{0,\Omega}^2 + \sum_{n=1}^m\|\mathbf{t}_h^n\|_{0,r;\Omega}^r\right\}\\ &\leq C\left\{\Delta t\sum_{n=1}^m\|\delta_t\mathbf{u}_h^n\|_{0,\Omega}^2 + \|\mathbf{u}_h^0\|_{0,\Omega}^2 + \Delta t\sum_{n=1}^m\left(\|\mathbf{f}^n\|_{0,\Omega}^{r/(r-1)} + \|\mathbf{f}^n\|_{0,\Omega}^2\right)\right\}.\end{aligned} \tag{5.34}$$

Next, in order to bound the first term in (5.34), we choose $(\vec{\mathbf{v}}_h, \boldsymbol{\tau}_h) = ((\delta_t \mathbf{u}_h^n, \delta_t \mathbf{t}_h^n), \boldsymbol{\sigma}_h^n)$ in (4.10), perform some algebraic manipulations, and use the Cauchy–Schwarz inequality to obtain

$$\|\delta_t \mathbf{u}_h^n\|_{0,\Omega}^2 + \frac{\alpha_{\mathtt{nN,d}}}{2} \delta_t \left((\delta^r + \|\mathbf{t}_h^n\|_{0,r;\Omega}^r)^{(r-2)/r} \|\mathbf{t}_h^n\|_{0,r;\Omega}^2 \right) \leq \frac{1}{2} \left(\|\mathbf{f}^n\|_{0,\Omega}^2 + \|\delta_t \mathbf{u}_h^n\|_{0,\Omega}^2 \right).$$

Therefore, by summing over the time index $n = 1, \cdots, m$ with $m = 1, \cdots, N$ and multiplying by Δt, we obtain

$$\Delta t \sum_{n=1}^{m} \|\delta_t \mathbf{u}_h^n\|_{0,\Omega}^2 + \|\mathbf{t}_h^n\|_{0,r;\Omega}^r \leq C \Big((\delta^r + \|\mathbf{t}_h^0\|_{0,r;\Omega}^r)^{(r-2)/r} \|\mathbf{t}_h^0\|_{0,r;\Omega}^2 + \Delta t \sum_{n=1}^{m} \|\mathbf{f}^n\|_{0,\Omega}^2 \Big).$$

Combining this with (5.34), the fact that $\mathbf{t}_h^0 = \mathbf{t}_{h,0}$, and (5.32), leads to the desired result (5.27). □

6 A priori error analysis

In this section, we now focus on proving an optimal a *priori* error estimates for $\vec{\mathbf{u}}$, $\boldsymbol{\sigma}$ in the norms $\mathbf{V}$ and $\mathbb{H}^s(\mathbf{div};\Omega)$, respectively. To this end, given the Sobolev exponent $r \in (1,2]$ and $a > 0$, we define the convex function φ_a by

$$\varphi_a(t) := \int_0^t \big(a+s\big)^{r-2} s\, ds\,.$$

The following Lemma provides important properties of the shifted function φ_a.

Lemma 6.1. *(Young-type inequality). For all $\epsilon > 0$ there exists $C(\epsilon) > 0$ only depending on r and δ such that for all $s, t, a, \delta \geq 0$ there holds*

$$s\varphi_a'(t) + t\varphi_a'(s) \leq \epsilon\varphi_a(s) + C(\epsilon)\varphi_a(t)\,. \tag{6.1}$$

Proof. See [26, Lemmata 28-32]. □

The following result, which demonstrates the equivalence of various quantities, is closely connected to the continuity and monotonicity assumptions outlined in Assumption 2.1.

Lemma 6.2. *Let $\boldsymbol{\sigma}$ satisfy* (2.5) *and* (2.6) *for $r \in (1,2]$ and $\delta \geq 0$. Then, uniformly for all $\boldsymbol{\tau}, \boldsymbol{\eta} \in \mathbb{R}^{d\times d}$ there hold*

$$\big|\boldsymbol{\sigma}^{\mathrm{d}}(\boldsymbol{\tau}) - \boldsymbol{\sigma}^{\mathrm{d}}(\boldsymbol{\eta})\big| \lesssim \big(\delta + |\boldsymbol{\tau}| + |\boldsymbol{\eta}|\big)^{r-2} |\boldsymbol{\tau} - \boldsymbol{\eta}| \simeq \varphi_{\delta+|\boldsymbol{\tau}|}'\big(|\boldsymbol{\tau} - \boldsymbol{\eta}|\big)\,, \tag{6.2a}$$

$$\big(\boldsymbol{\sigma}^{\mathrm{d}}(\boldsymbol{\tau}) - \boldsymbol{\sigma}^{\mathrm{d}}(\boldsymbol{\eta})\big) : (\boldsymbol{\tau} - \boldsymbol{\eta}) \lesssim \big(\delta + |\boldsymbol{\tau}| + |\boldsymbol{\eta}|\big)^{r-2} |\boldsymbol{\tau} - \boldsymbol{\eta}|^2 \simeq \varphi_{\delta+|\boldsymbol{\tau}|}\big(|\boldsymbol{\tau} - \boldsymbol{\eta}|\big)\,. \tag{6.2b}$$

Proof. See [41, Section 2.3]. □

We continue the analysis with defining

$$\mathbf{e}_{\mathbf{t}}^n := \mathbf{t}^n - \mathbf{t}_h^n\,, \quad \mathbf{e}_{\boldsymbol{\sigma}}^n := \boldsymbol{\sigma}^n - \boldsymbol{\sigma}_h^n\,, \quad \text{and} \quad \mathbf{e}_{\mathbf{u}}^n := \mathbf{u}^n - \mathbf{u}_h^n\,,$$

and write the above errors as follows

$$\begin{aligned}
\mathbf{e}_{\mathbf{t}}^n &:= \big(\mathbf{t}^n - \boldsymbol{\mathcal{P}}_k^h(\mathbf{t}^n)\big) + \big(\boldsymbol{\mathcal{P}}_k^h(\mathbf{t}^n) - \mathbf{t}_h^n\big) =: \boldsymbol{\vartheta}_{\mathbf{t}}^n + \boldsymbol{\theta}_{\mathbf{t}}^n\,,\\
\mathbf{e}_{\boldsymbol{\sigma}}^n &:= \big(\boldsymbol{\sigma}^n - \mathbf{\Pi}_h(\boldsymbol{\sigma}^n)\big) + \big(\mathbf{\Pi}_h(\boldsymbol{\sigma}^n) - \boldsymbol{\sigma}_h^n\big) =: \boldsymbol{\vartheta}_{\boldsymbol{\sigma}}^n + \boldsymbol{\theta}_{\boldsymbol{\sigma}}^n\,,\\
\mathbf{e}_{\mathbf{u}}^n &:= \big(\mathbf{u}^n - \mathcal{P}_k^h(\mathbf{u}^n)\big) + \big(\mathcal{P}_k^h(\mathbf{u}^n) - \vec{\mathbf{u}}_h^n\big) =: \boldsymbol{\vartheta}_{\mathbf{u}}^n + \boldsymbol{\theta}_{\mathbf{u}}^n\,,
\end{aligned}$$

where $\boldsymbol{\vartheta}^n_{\mathbf{t}}$, $\boldsymbol{\vartheta}^n_{\sigma}$ and $\boldsymbol{\vartheta}^n_{\mathbf{u}}$ are estimated according to properties $(\mathbf{AP}^{\mathbf{t}}_h)$, $(\mathbf{AP}^{\sigma}_h)$ and $(\mathbf{AP}^{\mathbf{u}}_h)$, respectively. Next, we proceed to estimate $\boldsymbol{\theta}^n_{\mathbf{t}}$, $\boldsymbol{\theta}^n_{\sigma}$ and $\boldsymbol{\theta}^n_{\mathbf{u}}$. By making use of the definition of $\mathcal{B}_h$ (cf. (4.9)), commuting diagram property given by (4.7) and the identity (4.5), we get

$$
\begin{aligned}
[\mathcal{B}_h(\vec{\mathbf{v}}_h), \boldsymbol{\Pi}^h_k\boldsymbol{\sigma}] &= -\int_{\Omega_h} \mathbf{div}(\boldsymbol{\Pi}^h_k\boldsymbol{\sigma})\cdot\mathbf{v}_h - \int_{\Omega_h} \boldsymbol{\mathcal{P}}^h_k(\boldsymbol{\Pi}^h_k\boldsymbol{\sigma}) : \mathbf{s}_h \\
&= -\int_{\Omega_h} \boldsymbol{\mathcal{P}}^h_k(\mathbf{div}(\boldsymbol{\sigma}))\cdot\mathbf{v}_h - \int_{\Omega_h} \boldsymbol{\Pi}^h_k\boldsymbol{\sigma} : \boldsymbol{\mathcal{P}}^h_k\mathbf{s}_h \\
&= -\int_{\Omega_h} \mathbf{div}(\boldsymbol{\sigma})\cdot\mathbf{v}_h - \int_{\Omega_h} \boldsymbol{\sigma} : \boldsymbol{\mathcal{P}}^h_k\mathbf{s}_h \\
&= [\mathcal{B}(\vec{\mathbf{v}}_h), \boldsymbol{\sigma}]\,.
\end{aligned}
\tag{6.3}
$$

On the other hand, a straightforward application of the discrete and continuous problems (4.10) and (2.12), along with identities (6.3), (3.21) gives

$$
\begin{aligned}
&\delta_t[C(\boldsymbol{\theta}^n_{\vec{\mathbf{u}}}), \vec{\mathbf{v}}_h] + [(\mathcal{A}(\vec{\mathbf{u}}^n_h) - \mathcal{A}(\mathcal{P}^h_k\vec{\mathbf{u}}^n)), \vec{\mathbf{v}}_h] + [\mathcal{B}_h(\vec{\mathbf{v}}_h), \boldsymbol{\theta}^n_{\sigma}] \\
&\qquad = -\int_{\Omega}(\partial_t\mathbf{u}^n - \delta_t\mathbf{u}^n)\cdot\mathbf{v}_h - \delta_t[C(\boldsymbol{\vartheta}^n_{\vec{\mathbf{u}}}), \vec{\mathbf{v}}_h] + [(\mathcal{A}(\vec{\mathbf{u}}^n) - \mathcal{A}(\mathcal{P}^h_k\vec{\mathbf{u}}^n)), \vec{\mathbf{v}}_h]\,,
\end{aligned}
\tag{6.4a}
$$

$$
[\mathcal{B}_h(\boldsymbol{\theta}^n_{\vec{\mathbf{u}}}), \boldsymbol{\tau}_h] = 0\,. \tag{6.4b}
$$

Consequently, thanks to (6.4) we are in position to establish the rates of convergence.

Theorem 6.3. *Let* $(\mathbf{t}, \boldsymbol{\sigma}, \vec{\mathbf{u}}) \in \mathbb{L}^2_{\mathtt{tr}}(\Omega)\times\mathbb{H}^s_{0,\Gamma_N}(\mathbf{div};\Omega)\times\mathbf{V}$ *and* $(\mathbf{t}_h, \boldsymbol{\sigma}_h, \vec{\mathbf{u}}_h) \in \mathbb{Q}_h\times\mathbb{X}_h\times\mathbf{V}_h$ *be the unique solutions of* (2.12) *and* (4.10), *respectively, whose existences are guaranteed by Theorems 3.9 and 5.9, respectively. Furthermore, given an integer* $k \geq 0$, *assume that there exist* $j \in [0, k+1]$ *and* $l \in [1, k+1]$ *such that* $\mathbf{t} \in \mathbb{W}^{j,r}(\Omega)$, $\boldsymbol{\sigma} \in \mathbb{W}^{l,s}(\Omega)$, $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{H}^l(\Omega)$, $\mathbf{u} \in \mathbf{H}^j(\Omega)$. *Then, there exists a positive constant* $C_{\mathtt{opt}}$, *independent of h, such that*

$$
\|\boldsymbol{\theta}^n_{\mathbf{u}}\|^2_{0,\Omega} + \Delta t\sum_{n=1}^{m}\left(\|\boldsymbol{\theta}^n_{\mathbf{t}}\|^2_{0,r;\Omega} + \|\boldsymbol{\theta}^n_{\mathbf{u}}\|^2_{0,\Omega} + \|\boldsymbol{\theta}^n_{\sigma}\|^2_{s,\mathbf{div};\Omega}\right) \leq C_{\mathtt{opt}}\left(\Delta t^2 + h^{\min\{j,l\}r}\right). \tag{6.5}
$$

Proof. We begin by considering $(\vec{\mathbf{v}}_h, \boldsymbol{\tau}_h) := ((\boldsymbol{\theta}^n_{\mathbf{u}}, \boldsymbol{\theta}^n_{\mathbf{t}}), \boldsymbol{\theta}^n_{\sigma}) \in \mathbb{V}_h\times\mathbb{X}_h$, and adding two resulting equations to arrive at

$$
\begin{aligned}
&\frac{1}{2}\delta_t\|\boldsymbol{\theta}^n_{\mathbf{u}}\|^2_{0,\Omega} + [(\mathcal{A}(\vec{\mathbf{u}}^n_h) - \mathcal{A}(\mathcal{P}^h_k\vec{\mathbf{u}}^n)), \boldsymbol{\theta}^n_{\vec{\mathbf{u}}}] \\
&\quad \leq \left|\int_{\Omega}(\partial_t\mathbf{u}^n - \delta_t\mathbf{u}^n)\cdot\boldsymbol{\theta}^n_{\mathbf{u}} - \delta_t[C(\boldsymbol{\vartheta}^n_{\vec{\mathbf{u}}}), \boldsymbol{\theta}^n_{\vec{\mathbf{u}}}]\right| + \left|[(\mathcal{A}(\vec{\mathbf{u}}^n) - \mathcal{A}(\mathcal{P}^h_k\vec{\mathbf{u}}^n)), \boldsymbol{\theta}^n_{\vec{\mathbf{u}}}]\right| \\
&\quad =: \sum_{i=1}^{3} E_i\,.
\end{aligned}
\tag{6.6}
$$

We now aim to bound each one of the terms appearing on the right-side of (6.6). The first and second terms can be estimated exactly as for standard finite elements, see for instance [48]:

$$
\begin{aligned}
|E_1| &= \left|\int_{\Omega}(\partial_t\mathbf{u}^n - \delta_t\mathbf{u}^n)\cdot\boldsymbol{\theta}^n_{\mathbf{u}}\right| \leq \|\partial_t\mathbf{u}^n - \delta_t\mathbf{u}^n\|_{0,\Omega}\|\boldsymbol{\theta}^n_{\mathbf{u}}\|_{0,\Omega} \\
&\leq \left(\int_{t_{n-1}}^{t_n}\|\partial_{tt}\mathbf{u}(s)\|_{0,\Omega}\,ds\right)\|\boldsymbol{\theta}^n_{\mathbf{u}}\|_{0,\Omega} \leq \Delta t^{1/2}\left(\int_{t_{n-1}}^{t_n}\|\partial_{tt}\mathbf{u}(s)\|^2_{0,\Omega}\,ds\right)^{1/2}\|\boldsymbol{\theta}^n_{\mathbf{u}}\|_{0,\Omega}\,,
\end{aligned}
$$

and

$$
\begin{aligned}
|E_2| &= \left|\delta_t[C(\boldsymbol{\vartheta}^n_{\vec{\mathbf{u}}}), \boldsymbol{\theta}^n_{\vec{\mathbf{u}}}]\right| \leq \|\delta_t\boldsymbol{\vartheta}^n_{\mathbf{u}}\|_{0,\Omega}\|\boldsymbol{\theta}^n_{\mathbf{u}}\|_{0,\Omega} \\
&\leq \frac{1}{\Delta t}\left\|\int_{t_{n-1}}^{t_n}\partial_s\boldsymbol{\vartheta}_{\mathbf{u}}(s)\,ds\right\|_{0,\Omega}\|\boldsymbol{\theta}^n_{\mathbf{u}}\|_{0,\Omega} \leq \Delta t^{-1/2}\|\partial_t\boldsymbol{\vartheta}_{\mathbf{u}}\|_{L^2(\mathrm{J}_n,\mathbf{L}^2(\Omega))}\|\boldsymbol{\theta}^n_{\mathbf{u}}\|_{0,\Omega}\,.
\end{aligned}
$$

In addition, by employing the definition of $\mathcal{A}$ given by (2.17a) and inequalities (6.1), (6.2a), and (6.2b) with $\delta = 0$, we easily obtain

$$
\begin{aligned}
\left|E_3\right| \;\leq\; & \int_\Omega \Big|\mu\left(\left(\delta^\alpha + |\mathbf{t}^n_{\mathtt{sym}}|^\alpha\right)^{\frac{r-2}{\alpha}}\mathbf{t}^n_{\mathtt{sym}} - \left(\delta^\alpha + |\boldsymbol{\mathcal{P}}^h_k\mathbf{t}^n_{\mathtt{sym}}|^\alpha\right)^{\frac{r-2}{\alpha}}\boldsymbol{\mathcal{P}}^h_k\mathbf{t}^n_{\mathtt{sym}}\right)\Big|\,|\boldsymbol{\theta}^n_{\mathbf{t}}| \\
\precsim\; & \int_\Omega \varphi'_{|\boldsymbol{\mathcal{P}}^h_k\mathbf{t}^n_{\mathtt{sym}}|}\left(|\boldsymbol{\mathcal{P}}^h_k\mathbf{t}^n_{\mathtt{sym}} - \mathbf{t}^n_{\mathtt{sym}}|\right)|\boldsymbol{\theta}^n_{\mathbf{t}}| \\
\precsim\; & \epsilon\int_\Omega \varphi_{|\boldsymbol{\mathcal{P}}^h_k\mathbf{t}^n_{\mathtt{sym}}|}(\boldsymbol{\theta}^n_{\mathbf{t}}) \;+\; C(\epsilon)\int_\Omega \varphi_{|\boldsymbol{\mathcal{P}}^h_k\mathbf{t}^n_{\mathtt{sym}}|}\left(|\boldsymbol{\mathcal{P}}^h_k\mathbf{t}^n_{\mathtt{sym}} - \mathbf{t}^n_{\mathtt{sym}}|\right) \\
\precsim\; & \epsilon\left(\boldsymbol{\sigma}^{\mathrm{d}}(\mathbf{t}^n_{h,\mathtt{sym}}) - \boldsymbol{\sigma}^{\mathrm{d}}(\boldsymbol{\mathcal{P}}^h_k\mathbf{t}^n_{\mathtt{sym}}),\boldsymbol{\theta}^n_{\mathbf{t}}\right)_{0,\Omega} \;+\; C(\epsilon)\int_\Omega \left(|\boldsymbol{\mathcal{P}}^h_k\mathbf{t}^n_{\mathtt{sym}}| + |\mathbf{t}^n_{\mathtt{sym}}|\right)^{r-2}\left|\boldsymbol{\mathcal{P}}^h_k\mathbf{t}^n_{\mathtt{sym}} - \mathbf{t}^n_{\mathtt{sym}}\right|^2 \\
\precsim\; & \epsilon\left[\left(\mathcal{A}(\vec{\mathbf{u}}^n_h) - \mathcal{A}(\mathcal{P}^h_k\vec{\mathbf{u}}^n)\right),\boldsymbol{\theta}^n_{\vec{\mathbf{u}}}\right] \;+\; C(\epsilon)\,\|\boldsymbol{\vartheta}^n_{\mathbf{t},\mathtt{sym}}\|^r_{0,r;\Omega}\,,
\end{aligned}
\tag{6.7}
$$

where the fact $\left(|\boldsymbol{\mathcal{P}}^h_k\mathbf{t}^n_{\mathtt{sym}}| + |\mathbf{t}^n_{\mathtt{sym}}|\right)^{r-2} \leq \left|\boldsymbol{\mathcal{P}}^h_k\mathbf{t}^n_{\mathtt{sym}} - \mathbf{t}^n_{\mathtt{sym}}\right|^{r-2}$, for $r < 2$, was used in the last step.

Consequently, considering $\epsilon = \frac{1}{2}$ in (6.7), and then replacing the estimates from (6.7) up to (3.22) back into (6.6), implies that

$$
\begin{aligned}
&\frac{1}{2}\delta_t\|\boldsymbol{\theta}^n_{\mathbf{u}}\|^2_{0,\Omega} + \frac{1}{2}\left[\left(\mathcal{A}(\vec{\mathbf{u}}^n_h) - \mathcal{A}(\mathcal{P}^h_k\vec{\mathbf{u}}^n)\right),\boldsymbol{\theta}^n_{\vec{\mathbf{u}}}\right] \\
&\quad\leq C\,\|\boldsymbol{\vartheta}_{\mathbf{t}}\|^r_{0,r;\Omega} + \left(\Delta t^{1/2}\|\partial_{tt}\mathbf{u}\|_{L^2(\mathrm{J}_n;\mathbf{Y})} + \Delta t^{-1/2}\|\partial_t\boldsymbol{\vartheta}_{\mathbf{u}}\|_{L^2(\mathrm{J}_n;\mathrm{Y})}\right)\|\boldsymbol{\theta}^n_{\mathbf{u}}\|_{0,\Omega}\,.
\end{aligned}
\tag{6.8}
$$

Now, to derive a lower bound for the second term on the right-hand side of (6.8), we employ the monotonicity of $\mathcal{A}$ (cf. (5.23)) and triangle inequality, which yields

$$
\begin{aligned}
\left[\left(\mathcal{A}(\vec{\mathbf{u}}^n_h) - \mathcal{A}(\mathcal{P}^h_k\vec{\mathbf{u}}^n)\right),\boldsymbol{\theta}^n_{\vec{\mathbf{u}}}\right] &\geq \alpha_{\mathtt{nN,d}}\left(\|\mathbf{t}_h\|^r_{0,r;\Omega} + \|\boldsymbol{\mathcal{P}}^h_k\mathbf{t}^n\|^r_{0,r;\Omega}\right)^{(r-2)/r}\|\boldsymbol{\theta}^n_{\mathbf{t}}\|^2_{0,r;\Omega} \\
&\geq \alpha_{\mathtt{nN,d}}\left(\|\mathbf{t}^n_h\|^r_{0,r;\Omega} + \|\mathbf{t}^n\|^r_{0,r;\Omega} + \|\mathbf{t}^n - \boldsymbol{\mathcal{P}}^h_k\mathbf{t}^n\|^r_{0,r;\Omega}\right)^{(r-2)/r}\|\boldsymbol{\theta}^n_{\mathbf{t}}\|^2_{0,r;\Omega} \\
&\geq \alpha_{\mathtt{nN,d}}\left(\|\mathbf{t}^n_h\|^r_{0,r;\Omega} + \|\mathbf{t}^n\|^r_{0,r;\Omega}\right)^{(r-2)/r}\|\boldsymbol{\theta}^n_{\mathbf{t}}\|^2_{0,r;\Omega}\,.
\end{aligned}
\tag{6.9}
$$

Substituting back into (6.8), gives

$$
\begin{aligned}
&\frac{1}{2}\delta_t\|\boldsymbol{\theta}^n_{\mathbf{u}}\|^2_{0,\Omega} + \frac{1}{2}\alpha_{\mathtt{nN,d}}\left(\|\mathbf{t}^n_h\|^r_{0,r;\Omega} + \|\mathbf{t}^n\|^r_{0,r;\Omega}\right)^{(r-2)/r}\|\boldsymbol{\theta}^n_{\mathbf{t}}\|^2_{0,r;\Omega} \\
&\quad\leq C\,\|\boldsymbol{\vartheta}_{\mathbf{t}}\|^r_{0,r;\Omega} + \left(\Delta t^{1/2}\|\partial_{tt}\mathbf{u}\|_{L^2(\mathrm{J}_n;\mathbf{Y})} + \Delta t^{-1/2}\|\partial_t\boldsymbol{\vartheta}_{\mathbf{u}}\|_{L^2(\mathrm{J}_n;\mathrm{Y})}\right)\|\boldsymbol{\theta}^n_{\mathbf{u}}\|_{0,\Omega}\,.
\end{aligned}
\tag{6.10}
$$

In turn, from (5.21b) we have

$$
\frac{1}{2}\|\boldsymbol{\theta}^n_{\mathbf{t}}\|^2_{0,r;\Omega} \;\geq\; \frac{1}{2}\|\boldsymbol{\theta}^n_{\mathbf{t},\mathtt{sym}}\|^2_{0,r;\Omega} \;\geq\; \frac{1}{2}\beta^2_{2,\mathtt{d}}\|\boldsymbol{\theta}^n_{\mathbf{u}}\|^2_{0,\Omega},
$$

which combined with (6.10), then summing up over the time index $n = 1,\cdots,m$, with $m = 1,\cdots,N$, and multiplying by Δt, we obtain

$$
\begin{aligned}
&\|\boldsymbol{\theta}^n_{\mathbf{u}}\|^2_{0,\Omega} + \Delta t\sum_{n=1}^m\left(\|\mathbf{t}^n_h\|^r_{0,r;\Omega} + \|\mathbf{t}^n\|^r_{0,r;\Omega}\right)^{(r-2)/r}\left(\|\boldsymbol{\theta}^n_{\mathbf{t}}\|^2_{0,r;\Omega} + \|\boldsymbol{\theta}^n_{\mathbf{u}}\|^2_{0,\Omega}\right) \\
&\leq C\Big\{\|\boldsymbol{\theta}^0_{\mathbf{u}}\|^2_{0,\Omega} + \Delta t\sum_{n=1}^m\|\boldsymbol{\vartheta}_{\mathbf{t}}\|^r_{0,r;\Omega} + \Delta t\sum_{n=1}^m\left(\Delta t\|\partial_{tt}\mathbf{u}\|^2_{L^2(\mathrm{J}_n;\mathbf{Y})} + \Delta t^{-1}\|\partial_t\boldsymbol{\vartheta}_{\mathbf{u}}\|^2_{L^2(\mathrm{J}_n;\mathrm{Y})}\right) + \epsilon\Delta t\sum_{n=1}^m\|\boldsymbol{\theta}^n_{\mathbf{u}}\|^2_{0,\Omega}\Big\}.
\end{aligned}
$$

Then, after bounding the terms $\|\mathbf{t}\|_{L^\infty(\mathrm{J}_n,\mathbb{Q})}$ and $\|\mathbf{t}_h\|_{L^\infty(\mathrm{J}_n,\mathbb{Q})}$ by estimates (3.29) and (5.27), respectively, and choosing $\epsilon < \frac{1}{2}\left(\mathcal{N}(\mathbf{f}^n,\mathbf{u}_0),\mathcal{N}_{\mathtt{dis}}(\mathbf{f}^n,\mathbf{u}_0)\right)^{(r-2)/r}$ we deduce the existence of a positive constant C_1, depending on r, σ_*, $\beta_{1,\mathtt{d}}$, $\beta_{2,\mathtt{d}}$, δ, $\mathcal{N}(\mathbf{f}^n, \mathbf{u}_0)$, $\mathcal{N}_{\mathtt{dis}}(\mathbf{f}^n, \mathbf{u}_0)$, such that there holds

$$\begin{aligned}\|\boldsymbol{\theta}_{\mathbf{u}}^n\|_{0,\Omega}^2 + \Delta t \sum_{n=1}^m \left(\|\boldsymbol{\theta}_{\mathbf{t}}^n\|_{0,r;\Omega}^2 + \|\boldsymbol{\theta}_{\mathbf{u}}^n\|_{0,\Omega}^2\right) \;\leq\; C \Big\{ &\|\boldsymbol{\theta}_{\mathbf{u}}^0\|_{0,\Omega}^2 + \Delta t \sum_{n=1}^m \|\boldsymbol{\vartheta}_{\mathbf{t}}\|_{0,r;\Omega}^r \\ &+ \Delta t \sum_{n=1}^m \left(\Delta t \|\partial_{tt}\mathbf{u}\|_{L^2(\mathrm{J}_n;\mathbf{Y})}^2 + \Delta t^{-1}\|\partial_t \boldsymbol{\vartheta}_{\mathbf{u}}\|_{L^2(\mathrm{J}_n;\mathrm{Y})}^2\right) \Big\}\,. \end{aligned} \tag{6.11}$$

Next, in order to bound the first term in (6.11), we subtract the continuous and discrete initial condition problems (3.23) and (4.11), to obtain the error system:

$$\begin{aligned} [\big(\mathcal{A}(\vec{\mathbf{u}}_{h,0}) - \mathcal{A}(\vec{\mathbf{u}}_0)\big), \vec{\mathbf{v}}_h] + [\mathcal{B}_h(\vec{\mathbf{v}}_h), \boldsymbol{\sigma}_{h,0} - \boldsymbol{\sigma}_0] &= 0 \qquad \forall\, \vec{\mathbf{v}}_h \in \mathbf{V}_h\,, \\ [\mathcal{B}_h(\vec{\mathbf{u}}_{h,0} - \vec{\mathbf{u}}_0), \boldsymbol{\tau}_h] &= 0 \qquad \forall\, \boldsymbol{\tau}_h \in \mathbb{X}_h. \end{aligned}$$

Then, proceeding as in (6.10), recalling from Theorems 3.8 and 5.9 that $(\mathbf{u}(0), \mathbf{t}(0)) = (\mathbf{u}_0, \mathbf{t}_0)$ and $(\mathbf{u}_h(0), \mathbf{t}_h(0)) = (\mathbf{u}_{h,0}, \mathbf{t}_{h,0})$, respectively, we get

$$\|\boldsymbol{\theta}_{\mathbf{t}}(0)\|_{0,r;\Omega}^2 \;\leq\; C\,\|\boldsymbol{\vartheta}_{\mathbf{t}}(0)\|_{0,r;\Omega}^r \left(\|\mathbf{t}_h(0)\|_{0,r;\Omega}^r + \|\mathbf{t}(0)\|_{0,r;\Omega}^r\right)^{(2-r)/r}\,. \tag{6.12}$$

Replacing back (6.12) into (6.11) and involving the approximation projections from (4.3), we arrive at

$$\|\boldsymbol{\theta}_{\mathbf{u}}^n\|_{0,\Omega}^2 + \Delta t \sum_{n=1}^m \left(\|\boldsymbol{\theta}_{\mathbf{t}}^n\|_{0,r;\Omega}^2 + \|\boldsymbol{\theta}_{\mathbf{u}}^n\|_{0,\Omega}^2\right) \;\leq\; C\,\left(\Delta t^2 + h^{jr}\right)\,.$$

On the other hand, to get the pseudostress estimate, we observe that from the discrete inf-sup condition of $\mathcal{B}_h$ (cf. (5.21a)), the first equation of (6.4), and the continuity of $\mathcal{A}$ (cf. (5.22)), there holds

$$\begin{aligned} \beta_{1,\mathtt{d}}\|\boldsymbol{\theta}_{\boldsymbol{\sigma}}^n\|_{s,\mathbf{div};\Omega} \;&\leq\; \sup_{\mathbf{0}\neq\vec{\mathbf{v}}_h\in\mathbf{V}_h} \frac{[\mathcal{B}_h(\vec{\mathbf{v}}_h), \boldsymbol{\theta}_{\boldsymbol{\sigma}}^n]}{\|\vec{\mathbf{v}}\|_{\mathbf{V}}} \\ = \sup_{\mathbf{0}\neq\vec{\mathbf{v}}_h\in\mathbf{V}_h} & \frac{-\delta_t[C(\boldsymbol{\theta}_{\vec{\mathbf{u}}}^n), \vec{\mathbf{v}}_h] - [\big(\mathcal{A}(\vec{\mathbf{u}}_h^n) - \mathcal{A}(\vec{\mathbf{u}}^n)\big), \vec{\mathbf{v}}_h] - \displaystyle\int_\Omega (\partial_t\mathbf{u}^n - \delta_t\mathbf{u}^n)\cdot\mathbf{v}_h - \delta_t[C(\boldsymbol{\vartheta}_{\vec{\mathbf{u}}}^n), \vec{\mathbf{v}}_h]}{\|\vec{\mathbf{v}}\|_{\mathbf{V}}} \\ \leq\; C\,\Big(&\|\delta_t\boldsymbol{\theta}_{\mathbf{u}}^n\|_{0,\Omega} + \left(\|\mathbf{t}_h^n\|_{0,r;\Omega} + \|\mathbf{t}^n\|_{0,r;\Omega}\right)^{r-2}\|\mathbf{e}_{\mathbf{t}}^n\|_{0,r;\Omega} \\ &+ \Delta t^{1/2}\|\partial_{tt}\mathbf{u}\|_{L^2(\mathrm{J}_n;\mathbf{Y})} + \Delta t^{-1/2}\|\partial_t\boldsymbol{\vartheta}_{\mathbf{u}}\|_{L^2(\mathrm{J}_n;\mathrm{Y})}\Big). \end{aligned}$$

Then, taking square in the above inequality, summing up over the time index$n = 1, \cdots, m$, with $m = 1, \cdots, N$, multiplying by Δt, we deduce that

$$\begin{aligned} \Delta t \sum_{n=1}^m \|\boldsymbol{\theta}_{\boldsymbol{\sigma}}^n\|_{s,\mathbf{div};\Omega}^2 \;\leq\; C\Big\{&\Delta t \sum_{n=1}^m \|\delta_t\boldsymbol{\theta}_{\mathbf{u}}^n\|_{0,\Omega}^2 + \left(\|\mathbf{t}_h\|_{L^\infty(\mathrm{J}_n;\mathbb{Q})} + \|\mathbf{t}\|_{L^\infty(\mathrm{J}_n;\mathbb{Q})}\right)^{2(r-2)}\Delta t \sum_{n=1}^m \|\mathbf{e}_{\mathbf{t}}^n\|_{0,r;\Omega}^2 \\ &+ \Delta t \sum_{n=1}^m \left(\Delta t\|\partial_{tt}\mathbf{u}\|_{L^2(\mathrm{J}_n;\mathbf{Y})}^2 + \Delta t^{-1}\|\partial_t\boldsymbol{\vartheta}_{\mathbf{u}}\|_{L^2(\mathrm{J}_n;\mathbf{Y})}^2\right)\Big\}. \end{aligned} \tag{6.13}$$

Next, in order to bound the first term in the right-hand side, we differentiate in time the second equation of (6.4), then choose $(\vec{\mathbf{v}}_h, \boldsymbol{\tau}_h) = ((\delta_t\boldsymbol{\theta}_{\mathbf{u}}^n, \delta_t\boldsymbol{\theta}_{\mathbf{t}}^n), \boldsymbol{\theta}_{\boldsymbol{\sigma}}^n)$, to find that

$$\|\delta_t\boldsymbol{\theta}_{\mathbf{u}}^n\|_{0,\Omega}^2 + [\big(\mathcal{A}(\vec{\mathbf{u}}_h^n) - \mathcal{A}(\mathcal{P}_k^h\vec{\mathbf{u}}^n)\big), \delta_t\boldsymbol{\theta}_{\vec{\mathbf{u}}}^n] \;\leq\; \|\boldsymbol{\vartheta}_{\mathbf{t}}^n\|_{0,r;\Omega}^r \;+\; \Delta t\|\partial_{tt}\mathbf{u}\|_{L^2(\mathrm{J}_n;\mathbf{Y})}^2 + \Delta t^{-1}\|\partial_t\boldsymbol{\vartheta}_{\mathbf{u}}\|_{L^2(\mathrm{J}_n;\mathrm{Y})}^2,$$

which, using the identity

$$[\big(\mathcal{A}(\vec{\mathbf{u}}_h^n) - \mathcal{A}(\mathcal{P}_k^h\vec{\mathbf{u}}^n)\big), \delta_t\boldsymbol{\theta}_{\vec{\mathbf{u}}}^n] \;=\; \frac{1}{2}\delta_t[\big(\mathcal{A}(\vec{\mathbf{u}}_h^n) - \mathcal{A}(\mathcal{P}_k^h\vec{\mathbf{u}}^n)\big), \boldsymbol{\theta}_{\vec{\mathbf{u}}}^n],$$

and the estimate (6.9), yields

$$\|\delta_t \boldsymbol{\theta}_{\mathbf{u}}^n\|_{0,\Omega}^2 + \alpha_{\mathtt{nN,d}}\, \delta_t \Big(\big(\|\mathbf{t}_h^n\|_{0,r;\Omega}^r + \|\mathbf{t}^n\|_{0,r;\Omega}^r\big)^{(r-2)/r} \|\boldsymbol{\theta}_{\mathbf{t}}^n\|_{0,r;\Omega}^2 \Big)$$
$$\leq\ \|\boldsymbol{\vartheta}_{\mathbf{t}}^n\|_{0,r;\Omega}^r \ +\ \Delta t \|\partial_{tt}\mathbf{u}\|_{L^2(\mathrm{J}_n;\mathbf{Y})}^2 + \Delta t^{-1} \|\partial_t \boldsymbol{\vartheta}_{\mathbf{u}}\|_{L^2(\mathrm{J}_n;\mathbf{Y})}^2 .$$

Then, summing up on n and employing (6.12) gives

$$\begin{aligned}
\Delta t \sum_{n=1}^m \|\delta_t \boldsymbol{\theta}_{\mathbf{u}}^n\|_{0,\Omega}^2 &+ \alpha_{\mathtt{nN,d}} \Big(\big(\|\mathbf{t}_h^n\|_{0,r;\Omega}^r + \|\mathbf{t}^n\|_{0,r;\Omega}^r\big)^{(r-2)/r} \|\boldsymbol{\theta}_{\mathbf{t}}^n\|_{0,r;\Omega}^2 \Big) \\
&\leq\ \Delta t \sum_{n=1}^m \|\boldsymbol{\vartheta}_{\mathbf{t}}^n\|_{0,r;\Omega}^r \ +\ \alpha_{\mathtt{nN,d}} \Big(\big(\|\mathbf{t}_h^0\|_{0,r;\Omega}^r + \|\mathbf{t}^0\|_{0,r;\Omega}^r\big)^{(r-2)/r} \|\boldsymbol{\theta}_{\mathbf{t}}^0\|_{0,r;\Omega}^2 \Big) \\
&\quad +\ \Delta t^2 \sum_{n=1}^m \|\partial_{tt}\mathbf{u}\|_{L^2(\mathrm{J}_n;\mathbf{Y})}^2 + \Delta t \sum_{n=1}^m \|\partial_t \boldsymbol{\vartheta}_{\mathbf{u}}\|_{L^2(\mathrm{J}_n;\mathbf{Y})}^2 \\
&\leq\ \Delta t \sum_{n=1}^m \|\boldsymbol{\vartheta}_{\mathbf{t}}^n\|_{0,r;\Omega}^r \ +\ \|\boldsymbol{\vartheta}_{\mathbf{t}}(0)\|_{0,r;\Omega}^r \ +\ \Delta t^2 \sum_{n=1}^m \|\partial_{tt}\mathbf{u}\|_{L^2(\mathrm{J}_n;\mathbf{Y})}^2 + \Delta t \sum_{n=1}^m \|\partial_t \boldsymbol{\vartheta}_{\mathbf{u}}\|_{L^2(\mathrm{J}_n;\mathbf{Y})}^2 ,
\end{aligned}$$

which, replacing back into (6.13) implies

$$\begin{aligned}
\Delta t \sum_{n=1}^m \|\boldsymbol{\theta}_{\boldsymbol{\sigma}}^n\|_{s,\mathbf{div};\Omega}^2 \ &\leq\ C\Big\{ \Delta t \sum_{n=1}^m \|\boldsymbol{\vartheta}_{\mathbf{t}}^n\|_{0,r;\Omega}^r \ +\ \|\boldsymbol{\vartheta}_{\mathbf{t}}(0)\|_{0,r;\Omega}^r \\
&+ \big(\|\mathbf{t}_h\|_{L^\infty(\mathrm{J}_n;\mathbb{Q})} + \|\mathbf{t}\|_{L^\infty(\mathrm{J}_n;\mathbb{Q})}\big)^{2(r-2)} \Delta t \sum_{n=1}^m \Big(\|\boldsymbol{\theta}_{\mathbf{t}}^n\|_{0,r;\Omega}^2 + \|\boldsymbol{\vartheta}_{\mathbf{t}}^n\|_{0,r;\Omega}^2 \Big) \\
&+ \Delta t \sum_{n=1}^m \Big(\Delta t \|\partial_{tt}\mathbf{u}\|_{L^2(\mathrm{J}_n;\mathbf{Y})}^2 + \Delta t^{-1} \|\partial_t \boldsymbol{\vartheta}_{\mathbf{u}}\|_{L^2(\mathrm{J}_n;\mathbf{Y})}^2 \Big) \Big\}.
\end{aligned}$$

Then, bounding the terms $\|\mathbf{t}\|_{L^\infty(\mathrm{J}_n,\mathbb{Q})}$ and $\|\mathbf{t}_h\|_{L^\infty(\mathrm{J}_n,\mathbb{Q})}$ by estimates (3.29) and (5.27), respectively, and employing (6.11) to bound $\Delta t \sum_{n=1}^m \|\boldsymbol{\theta}_{\mathbf{t}}^n\|_{0,r;\Omega}^2$ we arrive at (6.5), thus completing the proof. □

7 Numerical results

In this section, we conduct several numerical tests using the publicly available software `MATLAB` R2024a to validate the theoretical analysis and demonstrate the scheme's effectiveness. In all tests, the Picard method is employed, and its iterations are halted when either the absolute or relative ℓ^2-norm of the residuals falls below 1e-6. Absolute errors for each variable are computed in the following way

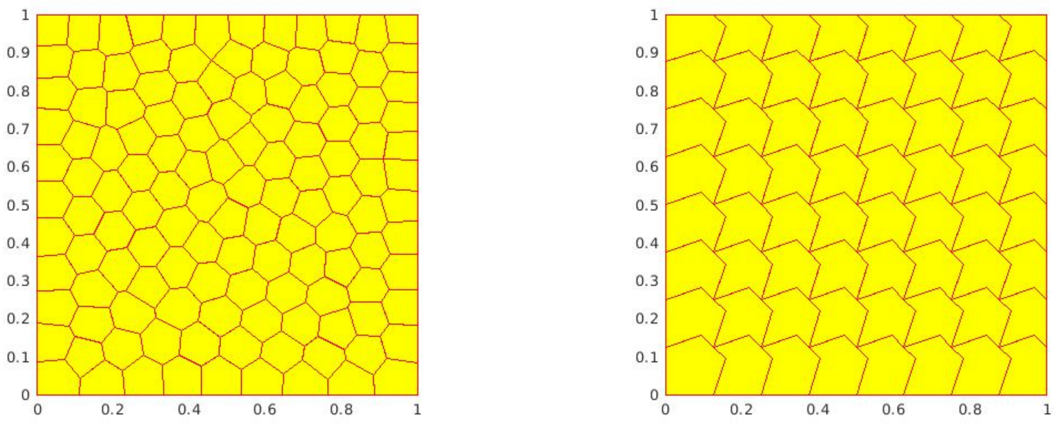
$$e(\mathbf{t}^n) \ :=\ \|\mathbf{t}^n - \mathbf{t}_h^n\|_{0,r;\Omega} , \quad e(\boldsymbol{\sigma}^n) \ :=\ \|\boldsymbol{\sigma}^n - \boldsymbol{\sigma}_h^n\|_{s,\mathbf{div};\Omega} , \quad e(\mathbf{u}^n) \ :=\ \|\mathbf{u}^n - \mathbf{u}_h^n\|_{0,\Omega} \quad \text{and}$$
$$e(p^n) \ :=\ \|p^n - p_h^n\|_{0,\Omega} .$$

The examples discussed in this section are as follows: In the first example, we solve a two-dimensional problem with manufactured exact solutions to validate the theoretical error estimates for the strain of rate, pseudostress, velocity, vorticity, and pressure presented in this study. Examples 2 and 3 are used to assess the effectiveness of the discrete scheme by simulating practical problems for which no analytical solutions are available.

7.1 Example 1: Accuracy assessment

In the present example we consider problem (2.1) with parameters $\mu = 1$, $\alpha = 1$ and the following exact solution, velocity field and pressure term

$$\mathbf{u}(x_1,x_2,t) = \cos(t) \begin{pmatrix} \sin(\frac{\pi}{2}x_1)\cos(\frac{\pi}{2}x_2) \\ -\sin(\frac{\pi}{2}x_2)\cos(\frac{\pi}{2}x_1) \end{pmatrix} \qquad p(x_1,x_2,t) = \cos(t) \Big(-\sin(\tfrac{\pi}{2}x_1)\ \sin(\tfrac{\pi}{2}x_2) + \tfrac{4}{\pi^2} \Big) ,$$



(a) HEXAGON (b) NONCONVEX

Figure 7.1: Example of the adopted polygonal meshes.

for all $(x_1, x_2)^\top \in \Omega := (0,1)^2$ and $t \in [0, t_F] := [0, 0.3]$. In addition, the external force and non-homogeneous Dirichlet boundary condition are taken in accordance with the above manufactured solution. The domain Ω is partitioned with the following sequences of polygonal meshes: `HEXAGON` and `NONCONVEX` meshes (cf. Fig. 7.1), which are generated by `PolyMesher` package [47]. For each family of meshes, we take the sequence with diameter $h = 1/4\,,\ 1/8\,,\ 1/6\,,\ 1/32$. In Figs. 7.2 and 7.3, we display the errors $e(\boldsymbol{\sigma})$, $e(\mathbf{u})$, $e(\boldsymbol{\gamma})$, $e(p)$, considering parameter $r \in \{1.1, 1.15, 1.25, 1.5, 1.75, 2\}$, the degree $j = 1$ and the time step $\Delta t = h$, for two values $\delta = 1$ and $\delta = 0$, respectively. In addition, the errors considering the degree $j = 2$ and $\Delta t = h^2$, for two values $\delta = 1$ and $\delta = 0$ are displayed in Figs. 7.4 and 7.5, respectively. It can be seen from Figs. 7.2 and 7.4 that all the unknowns converge to the discrete solutions with optimal order for all $r \in \{1.1, 1.15, 1.25, 1.5, 1.75, 2\}$, whereas Figs. 7.3 and 7.5 show that optimal convergence occurs as r increases, which agree with the theoretical result proposed in Theorem 6.3. All results indicate the optimal convergence of order $O(h^{jr})$ for all the unknowns and for each one of the utilized decompositions of Ω with considering $\Delta t = h^j$, which agree with the theoretical result proposed in Theorem 6.3.

7.2 Example 2: Lid driven cavity flow

This classic problem is a key benchmark for evaluating the numerical algorithms performance across various flow problems and has been studied within the Navier-Stokes framework in [46, 49]. The problem is set in a unit square domain $\Omega = (0,1)^2$, where a unit tangential velocity $\mathbf{g} = (1, 0)$ is applied along the top edge (i.e. $x_2 = 1$), while wall boundary conditions are applied along the remaining edges. In addition, we consider the initial value $\mathbf{u}_0 = [0,0]^{\mathtt{t}}$, time step $\Delta t = $ 1e-2 and the final time $t_F = 0.5$. In the first row of Figure 7.6, we display the stream function, while in the second row, we plot the horizontal component u_{1h} of the velocity along the vertical centreline $x_1 = 0.5$ and the vertical component u_{2h} along the horizontal centreline $x_2 = 0.5$.

7.3 Example 3: Flow past a cylinder

This well-known test problem has been studied extensively by various researchers [27, 44, 46]. The geometry is illustrated in Fig. 7.7, focusing on the fluid dynamics around a cylinder. The cylinder is placed in an incompressible flow, with its center located at $(0.25, 0.2)$ and a diameter of 0.1. The boundary conditions for inflow and outflow are applied at the left and right edges, respectively, as follow

$$\mathbf{g} = \begin{cases} \left(\frac{0.3}{0.41^2}\, 4x_2(0.41 - x_2),\ 0\right)^{\mathtt{t}} & \text{on} \quad \Gamma_{\mathtt{in}} \cup \Gamma_{\mathtt{out}}\,, \\ \mathbf{0} & \text{on} \quad \Gamma \setminus (\Gamma_{\mathtt{in}} \cup \Gamma_{\mathtt{out}})\,. \end{cases}$$

For this test, we consider the physical and discretization parameters $\mu = 0.1$, $\delta = 1$, $\alpha = 1$, $j = 1$,

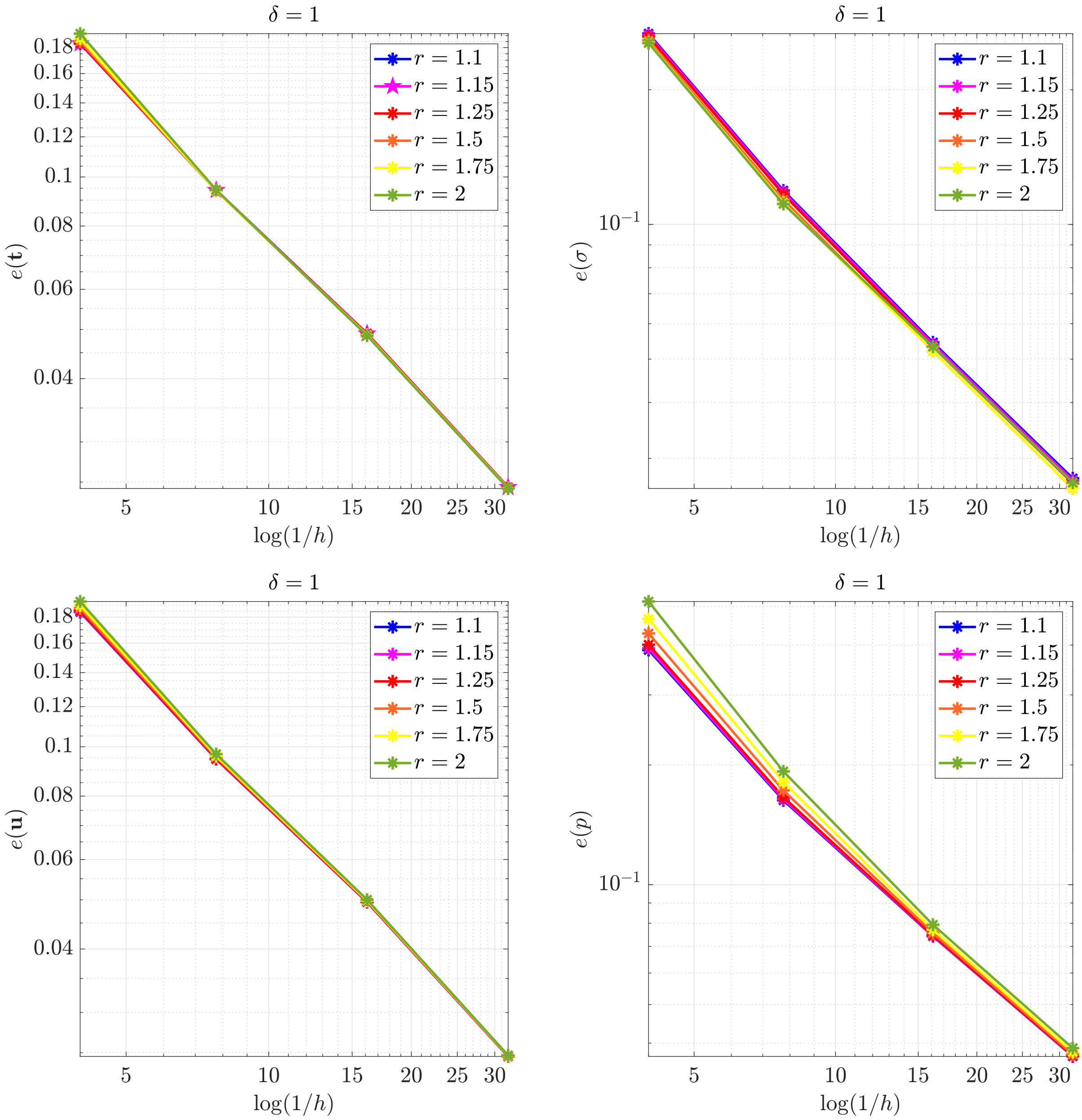


Figure 7.2: Convergence results with `HEXAGON` mesh, $\delta = 1$, $j = 1$ and $\Delta t = h$.

Δt = 3e-2, $t_F = 0.5$. In Figs. 7.8 and 7.9 we have portrayed the approximate solutions generated with the first-order mixed virtual element family for two values $r = 1.25$ and $r = 2.25$, respectively. All plots are consistent with those obtained in [27] and what is expected to be observed from the physical point of view, in accordance to [46].

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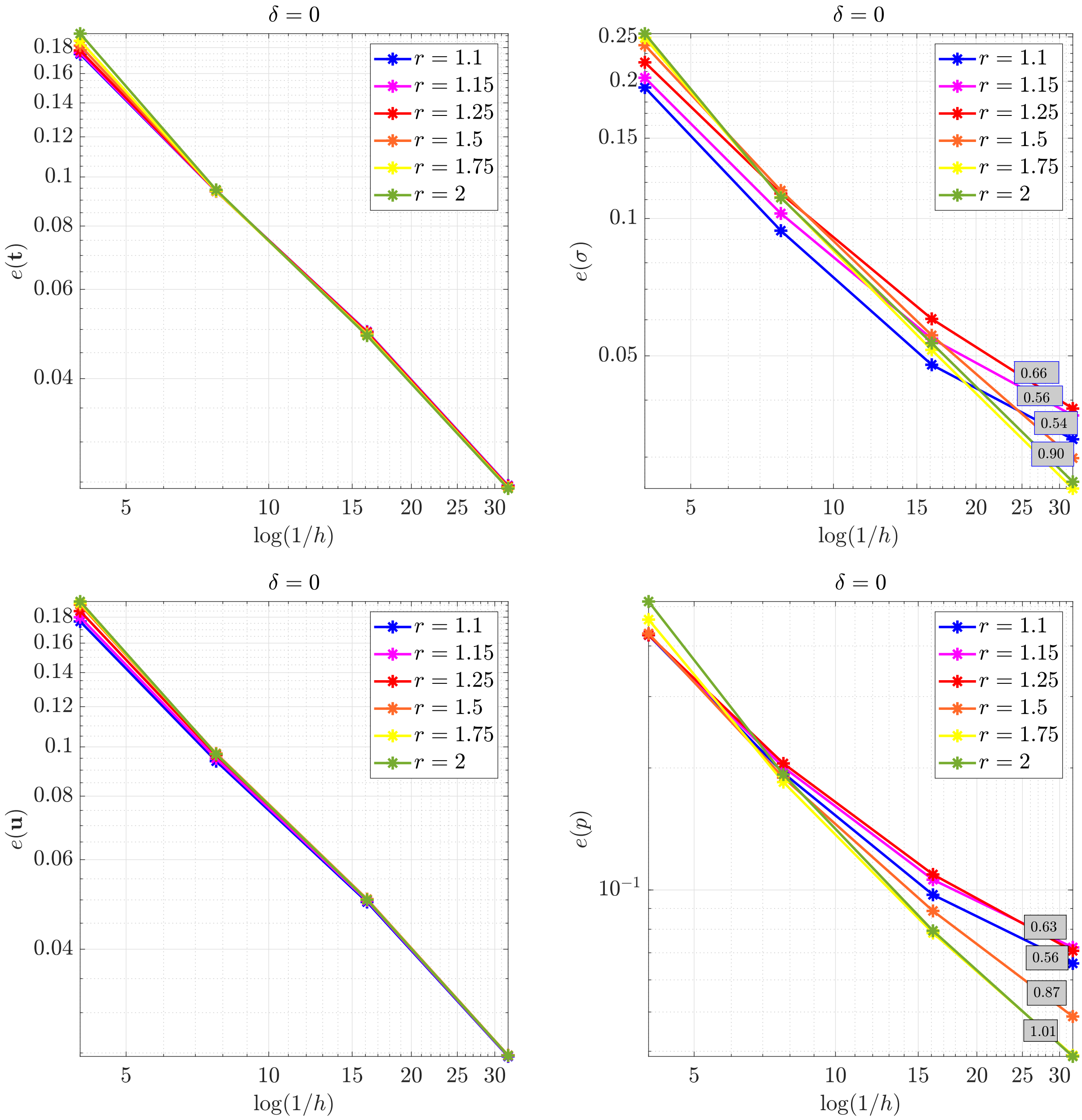

Figure 7.3: Convergence results with `HEXAGON` mesh, $\delta = 0$, $j = 1$ and $\Delta t = h$.

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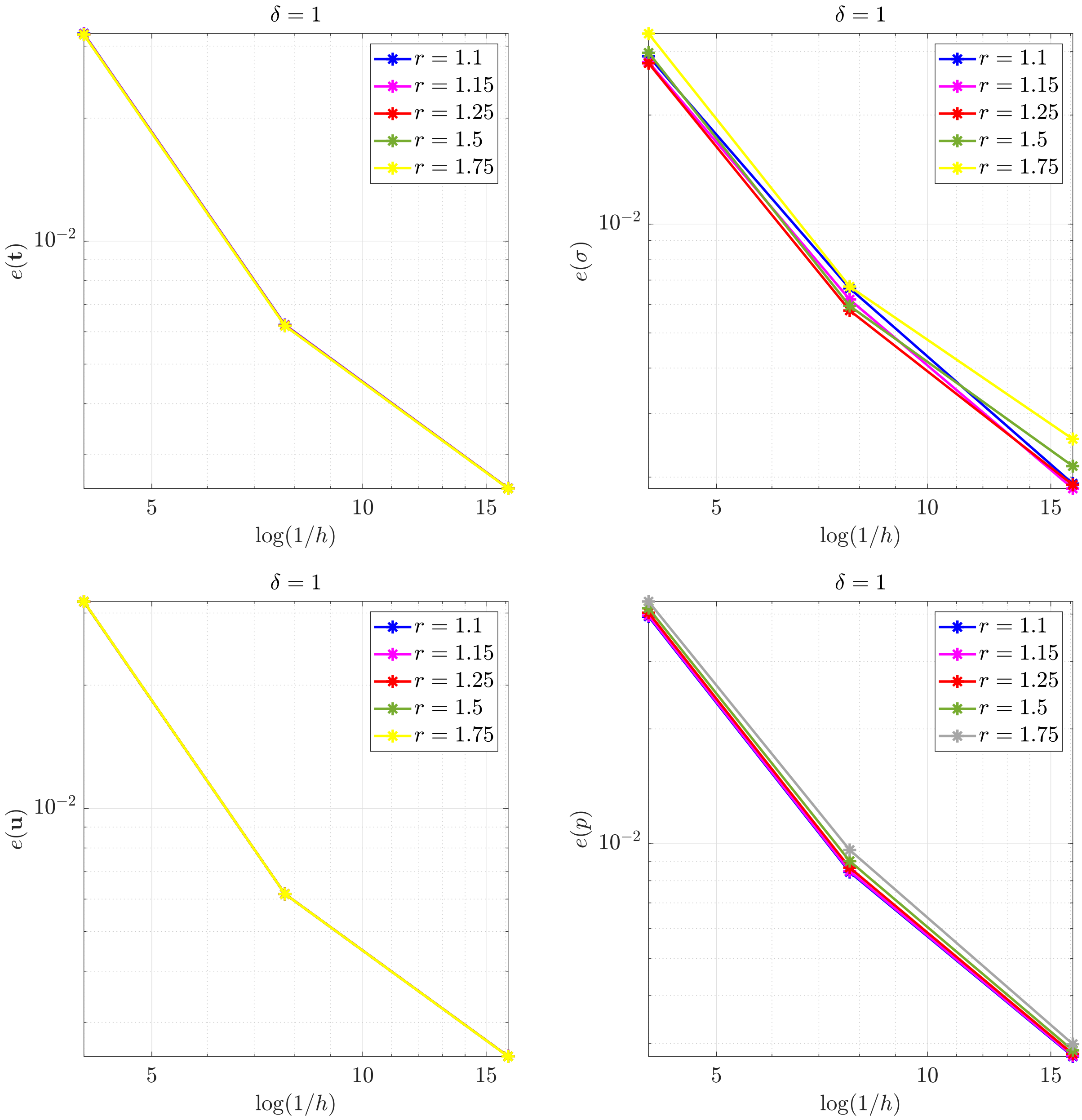

Figure 7.4: Convergence results with `HEXAGON` mesh, $\delta = 1$, $j = 2$ and $\Delta t = h^2$.

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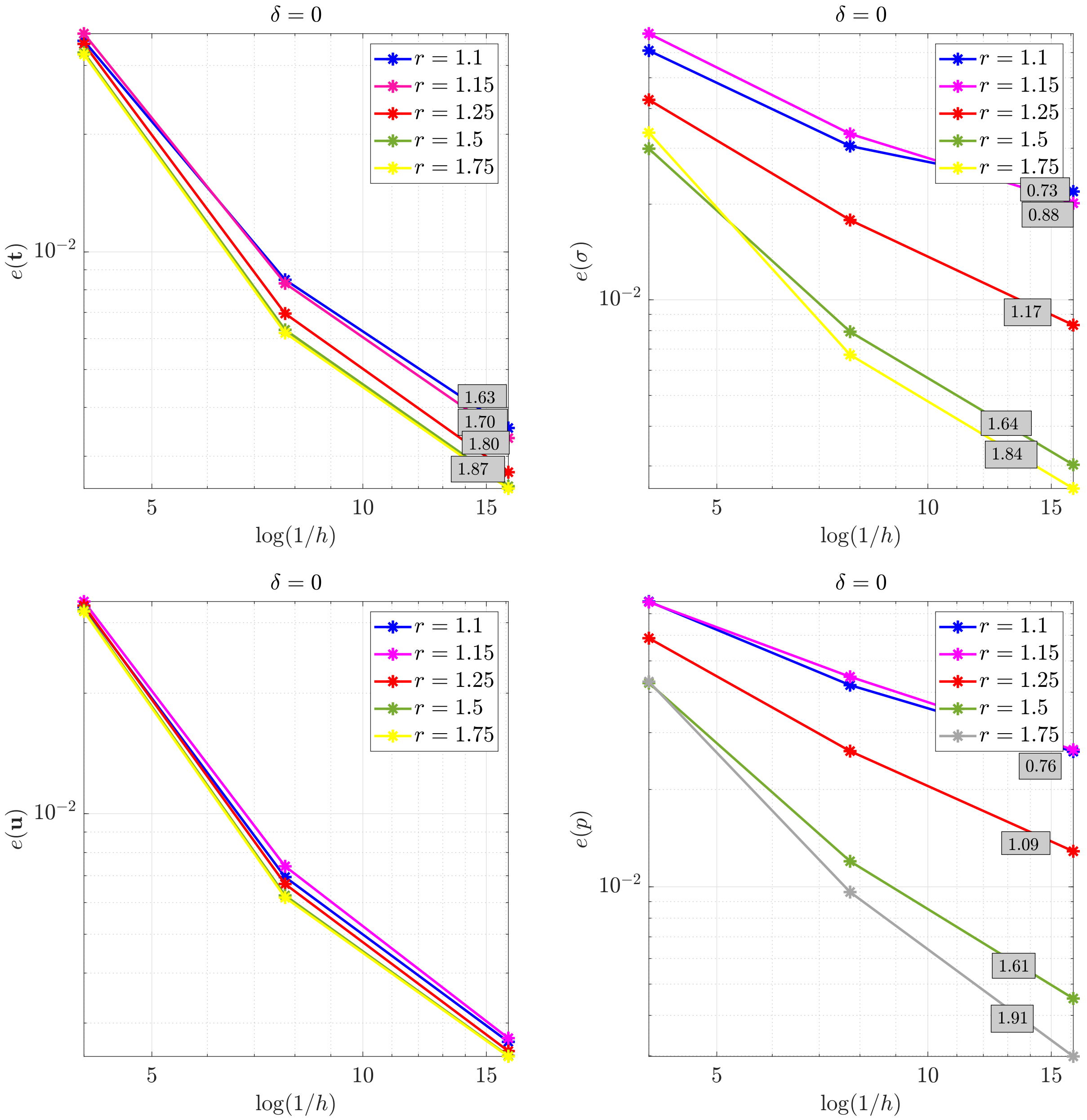

Figure 7.5: Convergence results with `HEXAGON` mesh, $\delta = 0$, $j = 2$ and $\Delta t = h^2$.

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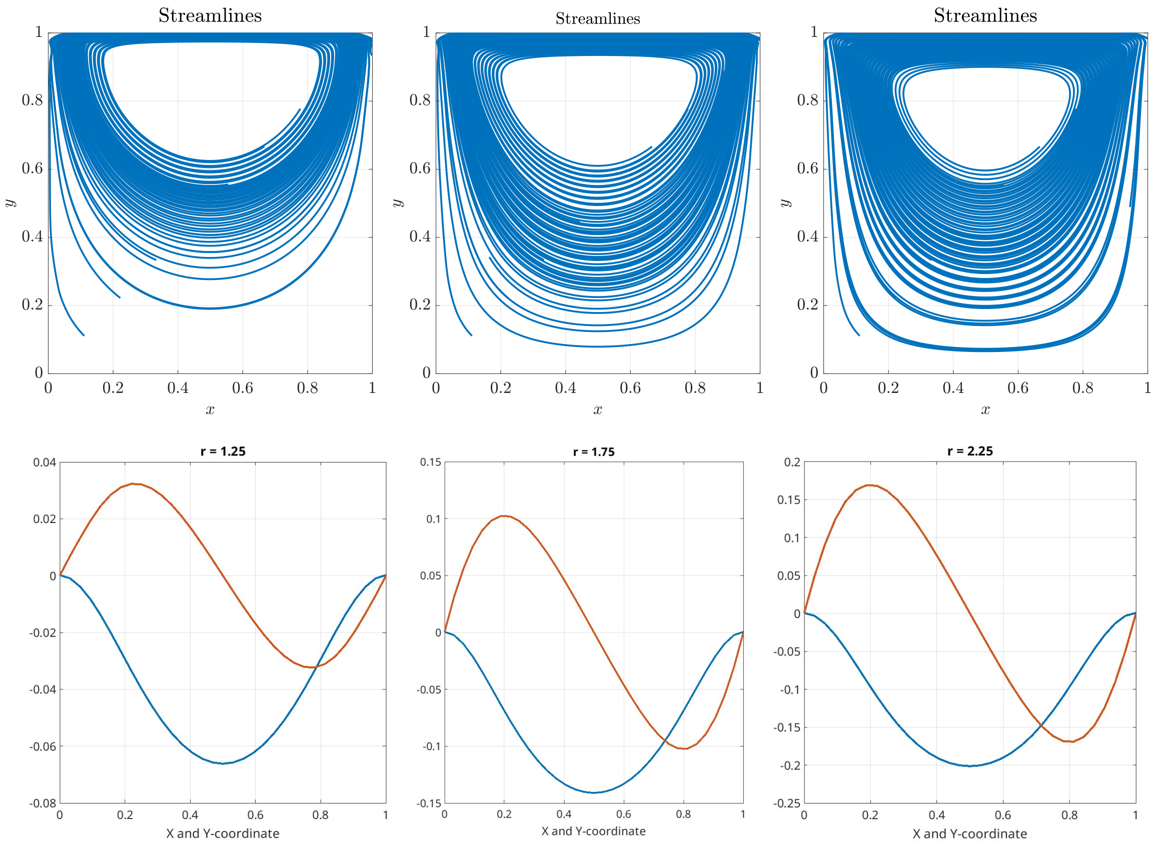

Figure 7.6: Numerical results for the test case of Section 5.2. Top: velocity magnitude contours computed on a Cartesian mesh of size 32×32 with $j = 2$. Bottom: horizontal component u_{1h} of the velocity along the vertical centreline $x_1 = 0.5$ and vertical component $u_{2,h}$ of the velocity along the horizontal centreline $x_2 = 0.5$.

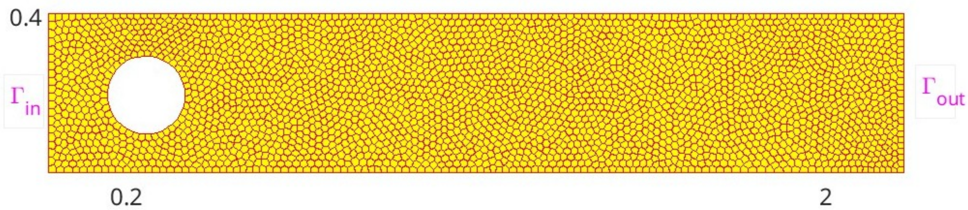

Figure 7.7: Example 3. An illustration of the mesh.

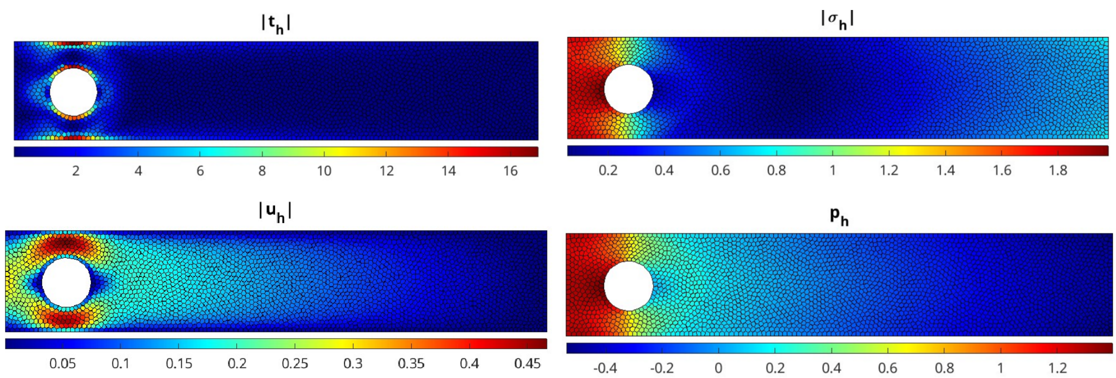

Figure 7.8: Example 3. Snapshots of the numerical solution for $r = 1.25$.

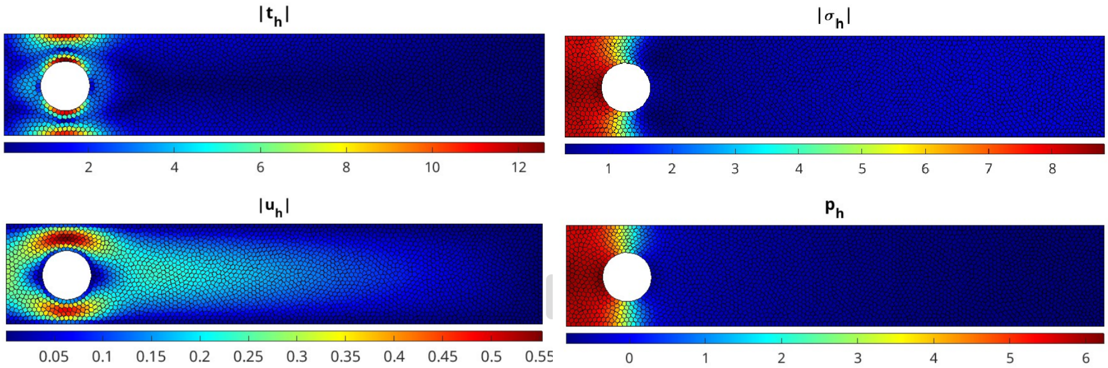

Figure 7.9: Example 3. Snapshots of the numerical solution for $r = 2.25$.

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