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Analysis and mixed-primal finite element discretisations for stress-assisted diffusion problems

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Abstract

We analyse the solvability of a static coupled system of PDEs describing the diffusion of a solute into an elastic material, where the process is affected by the stresses exerted in the solid. The problem is formulated in terms of solid stress, rotation tensor, solid displacement, and concentration of the solute. Existence and uniqueness of weak solutions follow from adapting a fixed-point strategy decoupling linear elasticity from a generalised Poisson equation. We then construct mixed-primal and augmented mixed-primal Galerkin schemes based on adequate finite element spaces, for which we rigorously derive a priori error bounds. The convergence of these methods is confirmed through a set of computational tests in 2D and 3D.

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1. Introduction

This work is motivated by the mathematical and numerical investigation of stress-enhanced diffusion processes in deformable solids. Starting from the early works by e.g. Truesdell [1], Podstrigach [2], or Aifantis [3], a number of applicative studies and different models have been developed. Many of these contributions have focused on the modelling of hydrogen diffusion in metals [4], damage of electrodes in lithium ion batteries [5], sorption in fibre-reinforced polymeric materials [6], drying of liquid paint layers [7], gels and general-purpose solute penetration [8,9], anisotropy of cardiac dynamics [10], and several other effects. Irrespective of the specific interaction under consideration, the assumptions in these models convey that the species diffuses on the elastic medium obeying a Fickean law enriched with additional contributions arising from local effects by exerted stresses.

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Although there exist numerous advances on the modelling considerations for stress-assisted and strain-assisted diffusion problems, their counterparts from the viewpoint of mathematical and numerical analysis are still far behind. A few punctual references include the study of plane steady solutions [11], asymptotic analysis [7,12], and the very recent general well-posedness theory for static and transient problems in a primal formulation, developed in [13]. Our goal at this stage is to focus on a simple stationary problem that represents the main ingredients of diffusion–deformation interaction models where the Cauchy stress acts as a coupling variable. We will concentrate on the regime of linear elasticity, and we will further assume that there are no additional nonlinearities in the diffusion process other than the coupling through stresses. In turn, it is supposed that the diffusing species affects the motion of the solid skeleton through external forces, constituting a two-way coupled system.

Apart from stress and displacement, the elasticity equations will incorporate the tensor of solid rotations as supplementary field variable, serving to impose symmetry of the Cauchy stress. This approach has been exploited in several mixed formulations for elastostatics [14–16], and in our case has particular importance as the stress influences directly the diffusion process. In contrast, we will use a primal formulation for the diffusion equation. Then, following a similar approach to the one employed in [17] and [18], the existence and uniqueness of weak solutions to the coupled system will be established invoking the Lax–Milgram lemma, the Babuška–Brezzi theory, suitable regularity estimates, and fixed-point arguments permitting us to decouple the solid mechanics from the generalised Poisson problem. Nevertheless, while there are in fact certain similarities with [17] and [18], it is important to remark that the problems involved deal with very different models and that there are substantial differences between the respective analyses. In particular, in [17] and [18] it was needed to assume, without proof, a regularity result, whereas in the present paper the regularity estimates that are required for the analysis are either proved or available in the literature. Also, in [17] and [18] the authors were able to show existence of solution for sufficiently small data only whereas in the present paper this assumption is not required for that purpose. More specifically, Schauder's fixed-point theorem will yield existence of weak solutions, whereas Banach's fixed-point theorem (in combination with assumptions on the data) will give uniqueness of solution. Additionally, the Sobolev embedding and Rellich–Kondrachov compactness theorems will constitute essential tools in the analysis of the continuous problem. In turn, the regularity estimates needed for the uncoupled elasticity and diffusion problems will be adapted from those appearing in [19] and [20], respectively. Even if these results are valid provided one restricts the analysis to convex domains in two spatial dimensions, our computational tests indicate that this requirement may only be technical.

Regarding the numerical approximation of the problem, we propose two families of finite element discretisations: one that will follow the same mixed-primal character as in the continuous case, and a second one that utilises augmentation of the elasticity problem through redundant Galerkin contributions in order to achieve conformity and well-definiteness of appropriate terms. As a consequence, the resulting augmented scheme allows more flexibility in the choice of the finite element subspaces for the aforementioned problem. In addition, the Brouwer fixed-point theorem will be utilised to establish existence of solutions to the associated Galerkin schemes. In this context, the recent theory leading to the well-posedness of Stokes-transport coupled systems developed in [17,18] will be modified accordingly. The convergence analysis in each case will be conducted using a blend of a Strang-type argument, Céa estimates, and the approximation properties of specific finite element spaces. To the best of our knowledge, the results presented in this paper constitute the first rigorous analysis of continuous and discrete mixed formulations for stress-assisted diffusion problems. The structure of the paper is as follows. Required definitions and preliminary notation are recalled in the remainder of this section, where we also present the governing equations in strong form together with main assumptions on the model. The weak formulation stated in mixed-primal form, as well as its solvability analysis, is provided in Section 2. We then provide a mixed-primal Galerkin method and derive existence of discrete solution along with the corresponding a priori error estimates in Section 3. Section 4 is dedicated to the derivation and analysis of an augmented mixed-primal formulation in continuous form, a suitable discretisation, and the derivation of error bounds. We then present a set of numerical examples in Section 5 that illustrate the accuracy and applicability of the proposed numerical schemes, and we close with summary and concluding remarks in Section 6.

Preliminaries. Let us denote by $\Omega \subseteq \mathrm{R}^n$, $n \in \{2, 3\}$ a given bounded domain with polyhedral boundary $\Gamma = \partial\Omega$, and denote by $\boldsymbol{\nu}$ the outward unit normal vector on the boundary. We will adopt a fairly standard notation for Lebesgue and Sobolev spaces: $L^p(\Omega)$ and $\mathrm{H}^s(\Omega)$, respectively. Norms and seminorms for the latter will be written as $\|\cdot\|_{s,\Omega}$ and $|\cdot|_{s,\Omega}$. The space $\mathrm{H}^{1/2}(\Gamma)$ contains traces of functions of $\mathrm{H}^1(\Omega)$, and $\mathrm{H}^{-1/2}(\Gamma)$ denotes its dual. In general, the notation $\mathbf{M}$ and $\mathbb{M}$ will refer to vectorial and tensorial counterparts of a generic scalar functional space M. Furthermore, by

$$\|\boldsymbol{w}\|_{\infty,\Omega} := \max_{i=1,n}\{\|w_i\|_{\infty,\Omega}\}, \quad \text{and} \quad \|\psi\|_{1,\infty,\Omega} := \max_{\alpha\le 1}\left(\operatorname{ess\,sup}_{x\in\Omega}|\partial^\alpha\psi(x)|\right),$$

we will denote norms for the Banach spaces $\mathbf{L}^\infty(\Omega)$ and $\mathrm{W}^{1,\infty}(\Omega)$, respectively. Next we recall the definition of the tensorial Hilbert space and its usual norm

$$\mathbb{H}(\mathbf{div},\Omega) := \left\{\boldsymbol{\tau}\in\mathbb{L}^2(\Omega):\quad \mathbf{div}\,\boldsymbol{\tau}\in\mathbf{L}^2(\Omega)\right\},\qquad \|\boldsymbol{\tau}\|^2_{\mathbf{div},\Omega} := \|\boldsymbol{\tau}\|^2_{0,\Omega} + \|\mathbf{div}\,\boldsymbol{\tau}\|^2_{0,\Omega}\,,$$

where $\mathbf{div}\,\boldsymbol{\tau}$ indicates the divergence operator acting along the rows of the tensor field $\boldsymbol{\tau}$. As usual, $\mathbb{I}$ stands for the identity tensor in $\mathrm{R}^{n\times n}$, and $|\cdot|$ denotes both the Euclidean norm in R^n and the Frobenius norm in $\mathrm{R}^{n\times n}$. Finally, for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,n}$, and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,n}$, we recall the transpose, trace, tensor product, and deviatoric splitting operators defined respectively as

$$\boldsymbol{\tau}^{\mathrm{t}} := (\tau_{ji})_{i,j=1,n},\qquad \mathrm{tr}(\boldsymbol{\tau}) := \sum_{i=1}^n \tau_{ii}\qquad \boldsymbol{\tau}:\boldsymbol{\zeta} := \sum_{i,j=1}^n \tau_{ij}\zeta_{ij},\qquad\text{and}\qquad \boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{n}\mathrm{tr}(\boldsymbol{\tau})\mathbb{I}.$$

A model for stress-assisted diffusion in elastic solids. The following system of partial differential equations describes balance laws governing the motion of an elastic solid occupying the domain Ω and a diffusing solute interacting with it:

$$\begin{aligned}\boldsymbol{\sigma} &= \lambda\,\mathrm{tr}\boldsymbol{\varepsilon}(\boldsymbol{u})\,\mathbb{I} + 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u})\,, &\quad -\,\mathbf{div}\,\boldsymbol{\sigma} &= \boldsymbol{f}(\phi),\\ \widetilde{\boldsymbol{\sigma}} &= \widetilde{\vartheta}(\boldsymbol{\varepsilon}(\boldsymbol{u}))\nabla\phi\,, &\quad -\,\mathrm{div}\,\widetilde{\boldsymbol{\sigma}} &= g(\boldsymbol{u}),\end{aligned}\tag{1.1}$$

where ϕ represents the local concentration of species, $\boldsymbol{\sigma}$ is the Cauchy solid stress, $\boldsymbol{u}$ is the displacement field, $\boldsymbol{\varepsilon}(\boldsymbol{u}) := \frac{1}{2}\left(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^{\mathrm{t}}\right)$ is the infinitesimal strain tensor (symmetrised gradient of displacements), $\widetilde{\boldsymbol{\sigma}}$ is the diffusive flux, $\lambda,\mu > 0$ are the Lamé constants (dilation and shear moduli) characterising the properties of the material, $\widetilde{\vartheta}:\mathrm{R}^{n\times n}\to\mathrm{R}^{n\times n}$ is a tensorial diffusivity function, $\boldsymbol{f}:\mathrm{R}\to\mathrm{R}^n$ is a vector field of body loads (which will depend on the species concentration), and $g:\mathrm{R}^n\to\mathrm{R}$ denotes an additional source term depending locally on the solid displacement. Specific requirements on these functions will be given below. We note that system (1.1) describes the constitutive relations inherent to linear elastic materials, conservation of linear momentum, the constitutive description of diffusive fluxes, and the mass transport of the diffusive substance, respectively. It also assumes that diffusive time scales are much lower than those of the elastic wave propagation, justifying the static character of the system (*cf.* [13]).

Hooke's law [21, eq. (2.36)] asserts that $\mathcal{C}^{-1}\boldsymbol{\sigma} = \boldsymbol{\varepsilon}(\boldsymbol{u})$, where $\mathcal{C}^{-1}$ is the fourth order compliance tensor. This relation allows us to recast the strain-dependent diffusivity $\widetilde{\vartheta}(\boldsymbol{\varepsilon}(\boldsymbol{u}))$ as a *stress-dependent* diffusivity $\vartheta(\boldsymbol{\sigma}) := \widetilde{\vartheta}(\mathcal{C}^{-1}\boldsymbol{\sigma})$. Throughout this work we will suppose that ϑ is of class C^1 and uniformly positive definite, meaning that there exists $\vartheta_0 > 0$ such that

$$\vartheta(\boldsymbol{\tau})\boldsymbol{w}\cdot\boldsymbol{w} \ge \vartheta_0|\boldsymbol{w}|^2\quad\forall\,\boldsymbol{w}\in\mathrm{R}^n,\quad\forall\,\boldsymbol{\tau}\in\mathrm{R}^{n\times n}.\tag{1.2}$$

We will also require uniform boundedness and Lipschitz continuity: there exist positive constants ϑ_1, ϑ_2 and L_ϑ, such that

$$\vartheta_1 \le |\vartheta(\boldsymbol{\tau})| \le \vartheta_2,\quad |\vartheta(\boldsymbol{\tau}) - \vartheta(\boldsymbol{\zeta})| \le L_\vartheta|\boldsymbol{\tau}-\boldsymbol{\zeta}|\quad\forall\,\boldsymbol{\tau},\boldsymbol{\zeta}\in\mathrm{R}^{n\times n}.\tag{1.3}$$

Similar assumptions will be placed on the load and source functions $\boldsymbol{f}$ and g: we suppose that there exist positive constants f_1, f_2, L_f, g_1, g_2 and L_g, such that

$$f_1 \le |\boldsymbol{f}(s)| \le f_2,\quad |\boldsymbol{f}(s) - \boldsymbol{f}(t)| \le L_f|s-t|\quad\forall\,s,t\in\mathrm{R},\tag{1.4}$$

$$g_1 \le g(\boldsymbol{w}) \le g_2,\quad |g(\boldsymbol{v}) - g(\boldsymbol{w})| \le L_g|\boldsymbol{v}-\boldsymbol{w}|\quad\forall\,\boldsymbol{v},\boldsymbol{w}\in\mathrm{R}^n.\tag{1.5}$$

Moreover, for each $\gamma\in(0,1)$, there exists a constant $C_\gamma > 0$, such that $g(\boldsymbol{w})\in\mathrm{H}^\gamma(\Omega)$ for each $\boldsymbol{w}\in\mathrm{H}^\gamma(\Omega)$ and

$$\|g(\boldsymbol{w})\|_{\gamma,\Omega} \le C_\gamma\,\|\boldsymbol{w}\|_{\gamma,\Omega}\,.\tag{1.6}$$

An additional assumption is that for every $\phi\in\mathrm{H}^1(\Omega)$, we have $\boldsymbol{f}(\phi)\in\mathbf{H}^1(\Omega)$. Finally, given $\boldsymbol{u}_{\mathrm{D}}\in\mathbf{H}^{1/2}(\Gamma)$, the following Dirichlet boundary conditions complement (1.1): $\boldsymbol{u} = \boldsymbol{u}_{\mathrm{D}}$ and $\phi = 0$ on Γ. Thus, we arrive at the following coupled system:

$$\begin{aligned}\boldsymbol{\sigma} &= \lambda\,\mathrm{tr}\boldsymbol{\varepsilon}(\boldsymbol{u})\,\mathbb{I} + 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) &\quad\text{and}\quad -\,\mathbf{div}\,\boldsymbol{\sigma} &= \boldsymbol{f}(\phi)\ \text{ in }\ \Omega, &\quad \boldsymbol{u} &= \boldsymbol{u}_{\mathrm{D}}\ \text{ on }\ \Gamma,\\ \widetilde{\boldsymbol{\sigma}} &= \vartheta(\boldsymbol{\sigma})\nabla\phi &\quad\text{and}\quad -\,\mathrm{div}\,\widetilde{\boldsymbol{\sigma}} &= g(\boldsymbol{u})\ \text{ in }\ \Omega, &\quad \phi &= 0\ \text{ on }\ \Gamma.\end{aligned}\tag{1.7}$$

Examples of specific constitutive relations for the tensor diffusivity in terms of stress appearing in the relevant literature include exponential functions of the volumetric stress for lithiation of batteries [22], simple polynomial relationships for biological materials [10], or Carreau-type laws

$$\vartheta(\boldsymbol{\sigma}) = C_0 \exp(-\operatorname{tr}\boldsymbol{\sigma})\mathbb{I}, \quad \vartheta(\boldsymbol{\sigma}) = C_0\,\mathbb{I} + C_1\boldsymbol{\sigma} + C_2\boldsymbol{\sigma}^2, \quad \vartheta(\boldsymbol{\sigma}) = (C_0 + C_1(1-|\boldsymbol{\sigma}|^2)^{-1/2})\mathbb{I},$$

respectively. Regarding the concentration-dependent body load we cite linear dependences modelling isotropic swelling in composite materials [23], saturation-based descriptions for viscous layers [7], or concentration gradient modulations for single-cell mechanics [24], adopting the form

$$\boldsymbol{f}(\phi) = \boldsymbol{C}\phi, \quad \boldsymbol{f}(\phi) = \boldsymbol{C}(1-\phi)^{m-1}, \quad \boldsymbol{f}(\phi) = C_0\nabla\phi,$$

respectively, where $\boldsymbol{C} \in \mathrm{R}^n$, $m > 1$.

2. The mixed-primal formulation

In this section we derive a mixed-primal variational formulation for (1.7) and verify the hypotheses of Schauder's fixed-point theorem, implying existence of weak solutions. In turn, an application of Banach's fixed-point theorem will be employed to prove uniqueness of solution under the assumption of adequately small data.

2.1. The continuous setting

The present treatment follows closely those in [17,21]. First we note that Hooke's law can be recast in terms of the rotation tensor as follows

$$\mathcal{C}^{-1}\boldsymbol{\sigma} = \boldsymbol{\varepsilon}(\boldsymbol{u}) = \nabla\boldsymbol{u} - \boldsymbol{\rho}, \quad \text{where} \quad \boldsymbol{\rho} := \frac{1}{2}(\nabla\boldsymbol{u} - \nabla\boldsymbol{u}^{\mathrm{t}}),$$

and we observe that $\boldsymbol{\rho} \in \mathbb{L}^2_{\mathrm{skew}}(\Omega) := \{\boldsymbol{\eta} \in \mathbb{L}^2(\Omega) : \quad \boldsymbol{\eta} + \boldsymbol{\eta}^{\mathrm{t}} = 0\}$. The weak form associated with the first row of (1.7) eventually reads: find $(\boldsymbol{\sigma}, (\boldsymbol{u}, \boldsymbol{\rho})) \in \mathbb{H}(\mathbf{div}, \Omega) \times (\mathbf{L}^2(\Omega) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega))$ such that

$$\begin{aligned} a(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, (\boldsymbol{u}, \boldsymbol{\rho})) &= G(\boldsymbol{\tau}) && \forall\, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}, \Omega), \\ b(\boldsymbol{\sigma}, (\boldsymbol{v}, \boldsymbol{\eta})) &= F_\phi(\boldsymbol{v}, \boldsymbol{\eta}) && \forall\, (\boldsymbol{v}, \boldsymbol{\eta}) \in \mathbf{L}^2(\Omega) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega), \end{aligned} \tag{2.1}$$

where the bilinear forms $a : \mathbb{H}(\mathbf{div}, \Omega) \times \mathbb{H}(\mathbf{div}, \Omega) \to \mathbb{R}$ and $b : \mathbb{H}(\mathbf{div}, \Omega) \times (\mathbf{L}^2(\Omega) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega)) \to \mathbb{R}$ are specified as

$$a(\boldsymbol{\zeta}, \boldsymbol{\tau}) := \frac{1}{2\mu}\int_\Omega \boldsymbol{\zeta} : \boldsymbol{\tau} - \frac{\lambda}{2\mu(n\,\lambda + 2\mu)}\int_\Omega \operatorname{tr}(\boldsymbol{\zeta})\operatorname{tr}(\boldsymbol{\tau}), \tag{2.2}$$

$$b(\boldsymbol{\tau}, (\boldsymbol{v}, \boldsymbol{\eta})) := \int_\Omega \boldsymbol{v} \cdot \mathbf{div}\,\boldsymbol{\tau} + \int_\Omega \boldsymbol{\eta} : \boldsymbol{\tau}, \tag{2.3}$$

for $\boldsymbol{\zeta}, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}, \Omega)$ and $(\boldsymbol{v}, \boldsymbol{\eta}) \in \mathbf{L}^2(\Omega) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega)$. In turn, the functionals $F_\phi \in \mathbb{H}(\mathbf{div}, \Omega)'$ and $G \in (\mathbf{L}^2(\Omega) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega))'$ are given by

$$G(\boldsymbol{\tau}) := \langle \boldsymbol{\tau}\boldsymbol{\nu}, \boldsymbol{u}_{\mathrm{D}}\rangle_\Gamma \qquad \text{and} \qquad F_\phi(\boldsymbol{v}, \boldsymbol{\eta}) := -\int_\Omega \boldsymbol{f}(\phi) \cdot \boldsymbol{v}\,, \tag{2.4}$$

for $(\boldsymbol{\tau}, (\boldsymbol{v}, \boldsymbol{\eta})) \in \mathbb{H}(\mathbf{div}, \Omega) \times (\mathbf{L}^2(\Omega) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega))$, where $\boldsymbol{\nu}$ denotes from now on the unit outward normal on Γ, and $\langle \cdot, \cdot \rangle_\Gamma$ stands for the duality pairing of $\mathbf{H}^{-1/2}(\Gamma)$ and $\mathbf{H}^{1/2}(\Gamma)$ with respect to the inner product in $\mathbf{L}^2(\Gamma)$.

From (2.2) and (2.3) it follows that, for any $(\boldsymbol{\tau}, (\boldsymbol{v}, \boldsymbol{\eta})) \in \mathbb{H}(\mathbf{div}, \Omega) \times (\mathbf{L}^2(\Omega) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega))$, there holds

$$a(\mathbb{I}, \boldsymbol{\tau}) = \frac{1}{n\,\lambda + 2\mu}\int_\Omega \operatorname{tr}(\boldsymbol{\tau}) \qquad \text{and} \qquad b(\mathbb{I}, (\boldsymbol{v}, \boldsymbol{\eta})) = 0. \tag{2.5}$$

Algebraic manipulations then show that the bilinear form a can be recast as

$$a(\boldsymbol{\zeta}, \boldsymbol{\tau}) = \frac{1}{2\,\mu}\int_\Omega \boldsymbol{\zeta}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} + \frac{1}{n(n\,\lambda + 2\mu)}\int_\Omega \operatorname{tr}(\boldsymbol{\zeta})\operatorname{tr}(\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\zeta}, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}, \Omega).$$

On the other hand, we recall from [25] that $\mathbb{H}(\mathbf{div}, \Omega) = \mathbb{H}_0(\mathbf{div}, \Omega) \oplus \mathrm{R}\mathbb{I}$, where

$$\mathbb{H}_0(\mathbf{div}, \Omega) := \left\{\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}, \Omega) : \quad \int_\Omega \operatorname{tr}(\boldsymbol{\tau}) = 0\right\},$$

that is, for each $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}, \Omega)$ there exist unique

$$\boldsymbol{\tau}_0 := \boldsymbol{\tau} - \left\{ \frac{1}{n|\Omega|} \int_\Omega \operatorname{tr}(\boldsymbol{\tau}) \right\} \mathbb{I} \in \mathbb{H}_0(\mathbf{div}, \Omega) \quad \text{and} \quad d := \frac{1}{n|\Omega|} \int_\Omega \operatorname{tr}(\boldsymbol{\tau}) \in \mathrm{R},$$

such that $\boldsymbol{\tau} = \boldsymbol{\tau}_0 + d\mathbb{I}$. In particular, we obtain from the first row of (1.7) that

$$\operatorname{tr}(\boldsymbol{\sigma}) = (n\lambda + 2\mu)\, \mathbf{div}\, \boldsymbol{u},$$

which yields $\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + c\mathbb{I}$, where

$$\boldsymbol{\sigma}_0 \in \mathbb{H}_0(\mathbf{div}, \Omega) \quad \text{and} \quad c := \frac{n\lambda + 2\mu}{n|\Omega|} \int_\Gamma \boldsymbol{u}_{\mathrm{D}} \cdot \boldsymbol{\nu}.$$

Then, replacing $\boldsymbol{\sigma}$ by the expression $\boldsymbol{\sigma}_0 + c\mathbb{I}$ in (2.1), applying (2.5) and denoting from now on the remaining unknown $\boldsymbol{\sigma}_0 \in \mathbb{H}_0(\mathbf{div}, \Omega)$ simply by $\boldsymbol{\sigma}$, we find that the mixed variational formulation for the elasticity problem (*cf.* first row of (1.7)) reduces to: find $(\boldsymbol{\sigma}, (\boldsymbol{u}, \boldsymbol{\rho})) \in \mathbb{H}_0(\mathbf{div}, \Omega) \times (\mathbf{L}^2(\Omega) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega))$ such that

$$\begin{aligned} a(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, (\boldsymbol{u}, \boldsymbol{\rho})) &= G(\boldsymbol{\tau}) && \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}, \Omega), \\ b(\boldsymbol{\sigma}, (\boldsymbol{v}, \boldsymbol{\eta})) &= F_\phi(\boldsymbol{v}, \boldsymbol{\eta}) && \forall\, (\boldsymbol{v}, \boldsymbol{\eta}) \in \mathbf{L}^2(\Omega) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega). \end{aligned} \tag{2.6}$$

On the other hand, the boundary condition for ϕ indicates the appropriate trial and test space

$$\mathrm{H}^1_0(\Omega) := \left\{ \psi \in \mathrm{H}^1(\Omega) : \quad \psi = 0 \text{ on } \Gamma \right\},$$

and Poincaré's inequality implies that there exists $c_p > 0$, depending only on Ω and Γ, such that

$$\|\psi\|_{1,\Omega} \le c_p |\psi|_{1,\Omega} \qquad \forall\, \psi \in \mathrm{H}^1_0(\Omega). \tag{2.7}$$

We can then deduce a primal formulation for the diffusion equation: find $\phi \in \mathrm{H}^1_0(\Omega)$ such that

$$A_{\boldsymbol{\sigma}}(\phi, \psi) = G_{\boldsymbol{u}}(\psi) \quad \forall\, \psi \in \mathrm{H}^1_0(\Omega), \tag{2.8}$$

where

$$A_{\boldsymbol{\sigma}}(\phi, \psi) := \int_\Omega \vartheta(\boldsymbol{\sigma}) \nabla\phi \cdot \nabla\psi \qquad \forall\, \phi, \psi \in \mathrm{H}^1_0(\Omega), \tag{2.9}$$

$$G_{\boldsymbol{u}}(\psi) := \int_\Omega g(\boldsymbol{u})\, \psi \qquad \forall\, \psi \in \mathrm{H}^1_0(\Omega). \tag{2.10}$$

In this way, the mixed-primal formulation for (1.7) consists in (2.6) and (2.8), that is: find $(\boldsymbol{\sigma}, (\boldsymbol{u}, \boldsymbol{\rho}), \phi) \in \mathbb{H}_0(\mathbf{div}, \Omega) \times (\mathbf{L}^2(\Omega) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega)) \times \mathrm{H}^1_0(\Omega)$, such that

$$\begin{aligned} a(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, (\boldsymbol{u}, \boldsymbol{\rho})) &= G(\boldsymbol{\tau}) && \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}, \Omega), \\ b(\boldsymbol{\sigma}, (\boldsymbol{v}, \boldsymbol{\eta})) &= F_\phi(\boldsymbol{v}, \boldsymbol{\eta}) && \forall\, (\boldsymbol{v}, \boldsymbol{\eta}) \in \mathbf{L}^2(\Omega) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega), \\ A_{\boldsymbol{\sigma}}(\phi, \psi) &= G_{\boldsymbol{u}}(\psi) && \forall\, \psi \in \mathrm{H}^1_0(\Omega). \end{aligned} \tag{2.11}$$

2.2. *Fixed-point approach and well-posedness of the uncoupled problems*

In this section, we proceed similarly as in [17] and utilise a fixed-point strategy to prove that (2.11) is uniquely solvable. We first set $H := \mathbb{H}_0(\mathbf{div}, \Omega) \times (\mathbf{L}^2(\Omega) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega))$ and let $\mathbf{S} : \mathrm{H}^1_0(\Omega) \to H$ be the operator defined by

$$\mathbf{S}(\phi) := (\mathbf{S}_1(\phi), (\mathbf{S}_2(\phi), \mathbf{S}_3(\phi))) := (\boldsymbol{\sigma}, (\boldsymbol{u}, \boldsymbol{\rho})) \quad \forall \phi \in \mathrm{H}^1_0(\Omega),$$

where, for a given ϕ, the triple $(\boldsymbol{\sigma}, (\boldsymbol{u}, \boldsymbol{\rho}))$ is the unique solution of (2.6). In turn, let $\widetilde{\mathbf{S}} : \mathbb{H}_0(\mathbf{div}, \Omega) \times \mathbf{L}^2(\Omega) \to \mathrm{H}^1_0(\Omega)$ be the operator defined by

$$\widetilde{\mathbf{S}}(\boldsymbol{\sigma}, \boldsymbol{u}) := \phi \qquad \forall\, (\boldsymbol{\sigma}, \boldsymbol{u}) \in \mathbb{H}_0(\mathbf{div}, \Omega) \times \mathbf{L}^2(\Omega),$$

where ϕ is the unique solution of (2.8), for a given pair $(\boldsymbol{\sigma}, \boldsymbol{u})$. Then, we define the map $\mathbf{T} : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$ as

$$\mathbf{T}(\phi) := \widetilde{\mathbf{S}}(\mathbf{S}_1(\phi), \mathbf{S}_2(\phi)) \qquad \forall \phi \in \mathrm{H}^1_0(\Omega),$$

and one readily realises that solving (2.11) is equivalent to seeking a fixed point of the solution operator $\mathbf{T}$, that is: find $\phi \in \mathrm{H}_0^1(\Omega)$ such that

$$\mathbf{T}(\phi) = \phi. \tag{2.12}$$

The following technical lemma will serve to establish solvability of (2.6) for a given ϕ.

Lemma 2.1. *There exists $c_1 > 0$ such that*

$$c_1 \|\boldsymbol{\tau}\|_{0,\Omega}^2 \le \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega}^2 + \|\mathbf{div}\,\boldsymbol{\tau}\|_{0,\Omega}^2 \qquad \forall \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}, \Omega).$$

Proof. See [21, Lemma 2.3]. □

We now proceed to show that the uncoupled problems defined by $\mathbf{S}$ and $\widetilde{\mathbf{S}}$ are well-posed.

Lemma 2.2. *For each $\phi \in \mathrm{H}_0^1(\Omega)$ the problem (2.6) has a unique solution $\mathbf{S}(\phi) := (\boldsymbol{\sigma}, (\boldsymbol{u}, \boldsymbol{\rho})) \in H$. Moreover, there exists $c_{\mathbf{S}} > 0$ independent of ϕ, such that*

$$\|\mathbf{S}(\phi)\|_H = \|(\boldsymbol{\sigma}, (\boldsymbol{u}, \boldsymbol{\rho}))\|_H \le c_{\mathbf{S}} \left\{ \|\boldsymbol{u}_{\mathrm{D}}\|_{1/2,\Gamma} + f_2 |\Omega|^{1/2} \right\}. \tag{2.13}$$

Proof. Along the lines of [21, Section 2.4.3.1], we first observe that

$$|a(\boldsymbol{\zeta}, \boldsymbol{\tau})| \le \frac{1}{\mu} \|\boldsymbol{\zeta}\|_{\mathbf{div},\Omega} \|\boldsymbol{\tau}\|_{\mathbf{div},\Omega} \qquad \forall \boldsymbol{\zeta}, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}, \Omega),$$

proving that $\mathbf{A} : \mathbb{H}_0(\mathbf{div}, \Omega) \to \mathbb{H}_0(\mathbf{div}, \Omega)$, the operator induced by a, is bounded with $\|\mathbf{A}\| \le \frac{1}{\mu}$. In turn we define the operator induced by the bilinear form b as $\mathbf{B} : \mathbb{H}_0(\mathbf{div}, \Omega) \to \mathbf{L}^2(\Omega) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega)$, with

$$\mathbf{B}(\boldsymbol{\tau}) := \left(\mathbf{div}\,\boldsymbol{\tau}, \frac{1}{2}(\boldsymbol{\tau} - \boldsymbol{\tau}^{\mathrm{t}}) \right) \qquad \forall \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}, \Omega), \tag{2.14}$$

from which one readily has that $\|\mathbf{B}\| \le 1$. Next, from (2.14) we deduce that

$$V := N(\mathbf{B}) = \left\{ \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}, \Omega) : \quad \mathbf{div}\,\boldsymbol{\tau} = 0 \quad \text{in} \quad \Omega, \quad \boldsymbol{\tau} = \boldsymbol{\tau}^{\mathrm{t}} \quad \text{in} \quad \Omega \right\}.$$

Consequently, using Lemma 2.1, we find that

$$a(\boldsymbol{\tau}, \boldsymbol{\tau}) \ge \frac{1}{2\mu} \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega}^2 \ge \frac{c_1}{2\mu} \|\boldsymbol{\tau}\|_{0,\Omega}^2 = \alpha \|\boldsymbol{\tau}\|_{\mathbf{div},\Omega}^2 \qquad \forall \boldsymbol{\tau} \in V, \tag{2.15}$$

thus showing that a is V-elliptic with ellipticity constant $\alpha_1 := \frac{c_1}{2\mu}$. On the other hand, the surjectivity of $\mathbf{B}$ follows exactly as in [21, Sect. 2.4.3.1]. Finally, from (2.4), we find that the functionals G and F_ϕ are bounded with

$$\|G\| \le \|\boldsymbol{u}_{\mathrm{D}}\|_{1/2,\Gamma} \quad \text{and} \quad \left\| F_\phi \right\| \le f_2 |\Omega|^{1/2}. \tag{2.16}$$

Therefore, a straightforward application of the Babuška–Brezzi theory [21, Thm. 2.3] guarantees that, for each $\phi \in \mathrm{H}_0^1(\Omega)$, problem (2.6) has a unique solution $(\boldsymbol{\sigma}, (\boldsymbol{u}, \boldsymbol{\rho})) \in H$, and there holds

$$\|\mathbf{S}(\phi)\|_H = \|(\boldsymbol{\sigma}, (\boldsymbol{u}, \boldsymbol{\rho}))\|_H \le c_{\mathbf{S}} \left\{ \|\boldsymbol{u}_{\mathrm{D}}\|_{1/2,\Gamma} + f_2 |\Omega|^{1/2} \right\},$$

where $c_{\mathbf{S}}$ is a constant depending on α_1, μ and the inf–sup constant associated with the bilinear form b. □

The following result asserts the unique solvability of (2.8).

Lemma 2.3. *For each $(\boldsymbol{\sigma}, \boldsymbol{u}) \in \mathbb{H}_0(\mathbf{div}, \Omega) \times \mathbf{L}^2(\Omega)$, the problem (2.8) has a unique solution $\phi := \widetilde{\mathbf{S}}(\boldsymbol{\sigma}, \boldsymbol{u}) \in \mathrm{H}_0^1(\Omega)$. Moreover, there exists a constant $r > 0$ depending on c_p, ϑ_0, g_2 and Ω (cf. (2.7), (1.2), (1.5)), such that*

$$\|\widetilde{\mathbf{S}}(\boldsymbol{\sigma}, \boldsymbol{u})\|_{1,\Omega} = \|\phi\|_{1,\Omega} \le r. \tag{2.17}$$

Proof. We note from (2.9) that $A_{\boldsymbol{\sigma}}$ is a bilinear form. Next, from (1.3) and (2.9), we deduce that

$$|A_{\boldsymbol{\sigma}}(\phi, \psi)| \le \vartheta_2 \|\phi\|_{1,\Omega} \|\psi\|_{1,\Omega} \qquad \forall \phi, \psi \in \mathrm{H}_0^1(\Omega),$$

which gives $\|A_{\boldsymbol{\sigma}}\| \le \vartheta_2$, and thus $A_{\boldsymbol{\sigma}}$ is bounded independently of $\boldsymbol{\sigma}$ and $\boldsymbol{u}$. Furthermore, from (1.2) and the estimate (2.7), for each $\phi \in \mathrm{H}_0^1(\Omega)$, we find that

$$A_{\boldsymbol{\sigma}}(\phi, \phi) = \int_{\Omega} \vartheta(\boldsymbol{\sigma})\nabla\phi \cdot \nabla\phi \ge \frac{\vartheta_0}{c_p^2}\, \|\phi\|_{1,\Omega}^2 \,, \tag{2.18}$$

which proves that $A_{\boldsymbol{\sigma}}$ is $\mathrm{H}_0^1(\Omega)$-elliptic with constant $\alpha_2 := \frac{\vartheta_0}{c_p^2}$, independently of $\boldsymbol{\sigma}$ and $\boldsymbol{u}$ as well. Now, using (1.5), (2.10) and applying Cauchy–Schwarz's inequality, we deduce that

$$|G_{\boldsymbol{u}}(\psi)| \le g_2|\Omega|^{1/2}\, \|\psi\|_{0,\Omega} \qquad \forall\, \psi \in \mathrm{H}_0^1(\Omega), \tag{2.19}$$

which implies that $G_{\boldsymbol{u}} \in \mathrm{H}_0^1(\Omega)'$ and $\|G_{\boldsymbol{u}}\| \le g_2|\Omega|^{1/2}$. Thus, a straightforward application of the Lax–Milgram Lemma (see, e.g. [21], Thm. 1.1) proves that for each $(\boldsymbol{\sigma}, \boldsymbol{u}) \in \mathbb{H}_0(\mathbf{div}, \Omega) \times \mathbf{L}^2(\Omega)$, problem (2.8) has a unique solution $\phi := \widetilde{\mathbf{S}}(\boldsymbol{\sigma}, \boldsymbol{u}) \in \mathrm{H}_0^1(\Omega)$. Moreover, the corresponding continuous dependence on the data is formulated as

$$\|\phi\|_{1,\Omega} \le r,$$

where

$$r := \frac{c_p^2}{\vartheta_0} g_2|\Omega|^{1/2}. \quad \square \tag{2.20}$$

The next step consists in deriving regularity estimates for the problems defining $\mathbf{S}$ and $\widetilde{\mathbf{S}}$. The following theorem (which summarises the respective analysis in [19]) is particularly crucial in the treatment for the operator $\mathbf{S}$.

Theorem 2.4. *Given a convex polygonal domain $\Omega \subseteq \mathrm{R}^2$ and $\mathbf{F} \in \mathbf{L}^2(\Omega)$, we let $\boldsymbol{u}$ be the solution of the elasticity problem*

$$\begin{aligned} \mu\Delta\boldsymbol{u} + (\mu + \lambda)\nabla(\nabla \cdot \boldsymbol{u}) &= \mathbf{F} \quad \text{in} \quad \Omega, \\ \boldsymbol{u} &= \mathbf{0} \quad \text{on} \quad \partial\Omega, \end{aligned}$$

where the Lamé moduli are bounded as $\mu \in [\mu_1, \mu_2]$ and $\lambda \in [0, \infty)$, with fixed constants $\mu_1, \mu_2 > 0$. Then, there exists $\gamma > 0$ such that whenever $\mathbf{F} \in \mathbf{H}^{\gamma}(\Omega)$, there holds $\boldsymbol{u} \in \mathbf{H}^{2+\gamma}(\Omega)$ and

$$\|\boldsymbol{u}\|_{2+\gamma,\Omega} \le \widetilde{C}_1\, \|\mathbf{F}\|_{\gamma,\Omega} \;,$$

with a constant $\widetilde{C}_1$ independent of the Lamé coefficients.

According to Theorem 2.4, in what follows we should probably concentrate in the case where Ω is a convex polygonal domain and $n = 2$. Nevertheless, it is easy to see that, assuming the regularity provided by this theorem, the forthcoming analysis and all the associated results hold even for the non-convex or 3D cases. We then recall that $\boldsymbol{f}(\psi) \in \mathbf{H}^1(\Omega)$ for each $\psi \in \mathrm{H}_0^1(\Omega)$, and suppose from now on that $\boldsymbol{u}_{\mathrm{D}} \in \mathbf{H}^{3/2+\gamma}(\Omega)$. Then, applying the theorem or the respective assumption, and recalling from the constitutive equation that the regularities of the unknowns are connected, we immediately find that $\mathbf{S}(\psi) \in \mathbb{H}_0(\mathbf{div}, \Omega) \cap \mathbb{H}^{1+\gamma}(\Omega) \times \mathbf{H}^{2+\gamma}(\Omega) \times \mathbb{L}^2_{\text{skew}}(\Omega) \cap \mathbb{H}^{1+\gamma}(\Omega)$.

In turn, for the operator $\widetilde{\mathbf{S}}$, we invoke [26, Remark (a)] and [20, Thm. 3.12], and observe that, for a given pair $(\boldsymbol{\zeta}, \boldsymbol{w}) := (\mathbf{S}_1(\psi), \mathbf{S}_2(\psi)) \in \mathbb{H}_0(\mathbf{div}, \Omega) \cap \mathbb{H}^{1+\gamma}(\Omega) \times \mathbf{H}^{2+\gamma}(\Omega)$ (which denote the first and second components of the unique solution produced by the operator $\mathbf{S}$), the hypothesis given by relation (1.6) implies in particular that $g(\boldsymbol{w}) \in \mathrm{H}^{\gamma}(\Omega)$. If one further assumes that the coefficients $\vartheta(\boldsymbol{\zeta})_{ij}$ are in $C^{1+\gamma}(\overline{\Omega})$, then elliptic regularity results (cf. [27,28]) guarantee that $\phi := \widetilde{\mathbf{S}}(\boldsymbol{\zeta}, \boldsymbol{w}) \in \mathrm{H}_0^1(\Omega) \cap \mathrm{H}^{2+\gamma}(\Omega)$, and we conclude that there exists a constant $\widetilde{C}_2 > 0$ such that

$$\|\widetilde{\mathbf{S}}(\boldsymbol{\zeta}, \boldsymbol{w})\|_{2+\gamma,\Omega} = \|\phi\|_{2+\gamma,\Omega} \le \widetilde{C}_2\, \|g(\boldsymbol{w})\|_{\gamma,\Omega} \,. \tag{2.21}$$

On the other hand, the Sobolev embedding theorem (*cf.* [29], Thm. 4.12, [30], Thm. A.5) gives the continuous injection $i_\gamma : \mathrm{H}^{2+\gamma}(\Omega) \longrightarrow C^1(\overline{\Omega})$, with boundedness constant $\widetilde{C}_\gamma$. Then, using the aforementioned continuous injection and applying (2.21), we deduce that

$$\|\widetilde{\mathbf{S}}(\boldsymbol{\zeta}, \boldsymbol{w})\|_{1,\infty,\Omega} = \|\phi\|_{1,\infty,\Omega} \le \widetilde{C}_\gamma\, \|\phi\|_{2+\gamma,\Omega} \le \widetilde{C}_\gamma \widetilde{C}_2\, \|g(\boldsymbol{w})\|_{\gamma,\Omega} \,. \tag{2.22}$$

Finally, using (1.6) and (2.13), we find that

$$\|\widetilde{\mathbf{S}}(\boldsymbol{\zeta}, \boldsymbol{w})\|_{1,\infty,\Omega} = \|\phi\|_{1,\infty,\Omega} \leq C_\infty\, c_{\mathbf{S}} \left\{ \|\boldsymbol{u}_{\mathrm{D}}\|_{1/2,\Gamma} + f_2|\Omega|^{1/2} \right\}, \tag{2.23}$$

where C_∞ is a positive constant depending on C_γ, $\widetilde{C}_\gamma$ and $\widetilde{C}_2$ (*cf.* (1.6), (2.21), (2.22)).

2.3. Solvability of the fixed-point equation

In this section we address the solvability analysis of the fixed-point equation (2.12). To this end, we will verify the hypotheses of the Schauder fixed-point theorem (see, *e.g*. [31, Thm. 9.12-1(b)]).

Lemma 2.5. *Let $r > 0$ be the constant from (2.20) (cf. proof of Lemma 2.3). Then, for the closed ball $W := \left\{\phi \in \mathrm{H}^1_0(\Omega) : \ \|\phi\|_{1,\Omega} \leq r\right\}$, it holds that $\mathbf{T}(W) \subseteq W$.*

Proof. It suffices to recall the definition of $\mathbf{T}$ (*cf.* Section 2.2), and simply apply estimate (2.17). □

Lemma 2.6. *There exists $C_{\mathbf{S}} > 0$ depending on μ, L_f, α (cf. (1.1), (1.4), (2.15)) and the inf–sup constant of b, such that*

$$\|\mathbf{S}(\phi) - \mathbf{S}(\varphi)\|_H \leq C_{\mathbf{S}}\, \|\phi - \varphi\|_{0,\Omega} \quad \forall\, \phi, \varphi \in \mathrm{H}^1_0(\Omega). \tag{2.24}$$

Proof. Given $\phi, \varphi \in \mathrm{H}^1_0(\Omega)$, we let $(\boldsymbol{\sigma}, (\boldsymbol{u}, \boldsymbol{\rho})), (\boldsymbol{\zeta}, (\boldsymbol{w}, \boldsymbol{\chi})) \in H$ be two solutions to (2.6), corresponding to ϕ and φ, respectively. That is, $(\boldsymbol{\sigma}, (\boldsymbol{u}, \boldsymbol{\rho})) = \mathbf{S}(\phi)$ and $(\boldsymbol{\zeta}, (\boldsymbol{w}, \boldsymbol{\chi})) = \mathbf{S}(\varphi)$. We then invoke the linearity of the forms a and b to deduce (using both formulations arising from (2.6)) that

$$\begin{aligned} a(\boldsymbol{\sigma} - \boldsymbol{\zeta}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, (\boldsymbol{u}, \boldsymbol{\rho}) - (\boldsymbol{w}, \boldsymbol{\chi})) &= 0 && \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}, \Omega),\\ b(\boldsymbol{\sigma} - \boldsymbol{\zeta}, (\boldsymbol{v}, \boldsymbol{\eta})) &= (F_\phi - F_\varphi)(\boldsymbol{v}, \boldsymbol{\eta}) && \forall\, (\boldsymbol{v}, \boldsymbol{\eta}) \in \mathbf{L}^2(\Omega) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega). \end{aligned} \tag{2.25}$$

From (2.4), we readily note that $\left\|F_\phi - F_\varphi\right\| \leq L_f\, \|\phi - \varphi\|_{0,\Omega}$. Consequently, and similarly to the proof of Lemma 2.2, the Babuška–Brezzi theory implies that for each $\phi, \varphi \in \mathrm{H}^1_0(\Omega)$, problem (2.25) has a unique solution $(\boldsymbol{\sigma} - \boldsymbol{\zeta}, (\boldsymbol{u} - \boldsymbol{w}, \boldsymbol{\rho} - \boldsymbol{\chi})) \in H$, as well as the continuous dependence on the data

$$\|\mathbf{S}(\phi) - \mathbf{S}(\varphi)\|_H = \|(\boldsymbol{\sigma}, (\boldsymbol{u}, \boldsymbol{\rho})) - (\boldsymbol{\zeta}, (\boldsymbol{w}, \boldsymbol{\chi}))\|_H \leq C_{\mathbf{S}}\, \|\phi - \varphi\|_{0,\Omega}\,,$$

which gives (2.24) and concludes the proof. □

The following result is a consequence of Lemma 2.6.

Lemma 2.7. *Assume that $C_{\mathbf{S}}$ is as in Lemma 2.6. Then, for each $\phi, \varphi \in \mathrm{H}^1_0(\Omega)$, there holds*

$$\|\mathbf{T}(\phi) - \mathbf{T}(\varphi)\|_{1,\Omega} \leq \frac{1}{\alpha_2}\, C_{\mathbf{S}} \left\{ L_g + L_\vartheta\, \|\mathbf{T}(\varphi)\|_{1,\infty,\Omega} \right\} \|\phi - \varphi\|_{0,\Omega}\,. \tag{2.26}$$

Proof. Firstly we recall that $\mathbf{T}(\phi) = \widetilde{\mathbf{S}}(\mathbf{S}_1(\phi), \mathbf{S}_2(\phi))$ and $\mathbf{T}(\varphi) = \widetilde{\mathbf{S}}(\mathbf{S}_1(\varphi), \mathbf{S}_2(\varphi)) \quad \forall\, \phi, \varphi \in \mathrm{H}^1_0(\Omega)$. In view of unifying the notation throughout the paper, we apply the following renaming

$$(\boldsymbol{\sigma}, \boldsymbol{u}) := (\mathbf{S}_1(\phi), \mathbf{S}_2(\phi)) \quad \text{and} \quad (\boldsymbol{\zeta}, \boldsymbol{w}) := (\mathbf{S}_1(\varphi), \mathbf{S}_2(\varphi)),$$

where $(\boldsymbol{\sigma}, \boldsymbol{u}), (\boldsymbol{\zeta}, \boldsymbol{w}) \in \mathbb{H}_0(\mathbf{div}, \Omega) \times \mathbf{L}^2(\Omega)$. In addition, we let $\widetilde{\phi} := \widetilde{\mathbf{S}}(\boldsymbol{\sigma}, \boldsymbol{u})$ and $\widetilde{\varphi} := \widetilde{\mathbf{S}}(\boldsymbol{\zeta}, \boldsymbol{w})$, that is

$$A_{\boldsymbol{\sigma}}(\widetilde{\phi}, \widetilde{\psi}) = G_{\boldsymbol{u}}(\widetilde{\psi}) \quad \text{and} \quad A_{\boldsymbol{\zeta}}(\widetilde{\varphi}, \widetilde{\psi}) = G_{\boldsymbol{w}}(\widetilde{\psi}) \quad \forall\, \widetilde{\psi} \in \mathrm{H}^1_0(\Omega).$$

Adding and subtracting appropriate terms, and appealing to the ellipticity of $A_{\boldsymbol{\sigma}}$, we readily find that

$$\begin{aligned} \alpha_2 \|\widetilde{\phi} - \widetilde{\varphi}\|^2_{1,\Omega} &\leq A_{\boldsymbol{\sigma}}(\widetilde{\phi}, \widetilde{\phi} - \widetilde{\varphi}) - A_{\boldsymbol{\sigma}}(\widetilde{\varphi}, \widetilde{\phi} - \widetilde{\varphi})\\ &= (G_{\boldsymbol{u}} - G_{\boldsymbol{w}})(\widetilde{\phi} - \widetilde{\varphi}) + (A_{\boldsymbol{\zeta}} - A_{\boldsymbol{\sigma}})(\widetilde{\varphi}, \widetilde{\phi} - \widetilde{\varphi}). \end{aligned} \tag{2.27}$$

Next we use (2.9), (2.10), we apply Cauchy–Schwarz's inequality, and exploit the assumptions (1.3) and (1.5), to obtain the bounds

$$\begin{aligned}|(G_{\boldsymbol{u}} - G_{\boldsymbol{w}})(\widetilde{\phi} - \widetilde{\varphi})| &= \left|\int_\Omega (g(\boldsymbol{u}) - g(\boldsymbol{w}))(\widetilde{\phi} - \widetilde{\varphi})\right| \\ &\le L_g \,\|\boldsymbol{u} - \boldsymbol{w}\|_{0,\Omega}\, \|\widetilde{\phi} - \widetilde{\varphi}\|_{0,\Omega}\,,\end{aligned} \tag{2.28}$$

and

$$\begin{aligned}|(A_{\boldsymbol{\zeta}} - A_{\boldsymbol{\sigma}})(\widetilde{\varphi}, \widetilde{\phi} - \widetilde{\varphi})| &= \left|\int_\Omega (\vartheta(\boldsymbol{\zeta}) - \vartheta(\boldsymbol{\sigma}))\nabla\widetilde{\varphi} \cdot \nabla(\widetilde{\phi} - \widetilde{\varphi})\right| \\ &\le L_\vartheta \|\nabla\widetilde{\varphi}\|_{\infty,\Omega}\, \|\boldsymbol{\sigma} - \boldsymbol{\zeta}\|_{0,\Omega}\, |\widetilde{\phi} - \widetilde{\varphi}|_{1,\Omega}\,.\end{aligned} \tag{2.29}$$

We then observe that the inequalities (2.27)–(2.29) imply that

$$\|\widetilde{\phi} - \widetilde{\varphi}\|_{1,\Omega} \le \frac{1}{\alpha_2}\left\{L_g \,\|\boldsymbol{u} - \boldsymbol{w}\|_{0,\Omega} + L_\vartheta \,\|\widetilde{\varphi}\|_{1,\infty,\Omega}\, \|\boldsymbol{\sigma} - \boldsymbol{\zeta}\|_{0,\Omega}\right\}. \tag{2.30}$$

Next, according to the definitions given at the beginning of the proof, we can rewrite (2.30) as

$$\begin{aligned}&\|\widetilde{\mathbf{S}}(\mathbf{S}_1(\phi), \mathbf{S}_2(\phi)) - \widetilde{\mathbf{S}}(\mathbf{S}_1(\varphi), \mathbf{S}_2(\varphi))\|_{1,\Omega} \\ &\qquad \le \frac{1}{\alpha_2}\left\{L_g \,\|\mathbf{S}_2(\phi) - \mathbf{S}_2(\varphi)\|_{0,\Omega} + L_\vartheta \|\widetilde{\mathbf{S}}(\mathbf{S}_1(\varphi), \mathbf{S}_2(\varphi))\|_{1,\infty,\Omega}\, \|\mathbf{S}_1(\phi) - \mathbf{S}_1(\varphi)\|_{0,\Omega}\right\}.\end{aligned} \tag{2.31}$$

It is important to note here that the term $\|\widetilde{\mathbf{S}}(\mathbf{S}_1(\varphi), \mathbf{S}_2(\varphi))\|_{1,\infty,\Omega}$ is bounded for each $\varphi \in \mathrm{H}^1_0(\Omega)$, thanks to (2.23). In this way, we are in a position to prove the Lipschitz continuity of $\mathbf{T}$. In fact, from (2.24) and (2.31) we find that

$$\begin{aligned}\|\mathbf{T}(\phi) - \mathbf{T}(\varphi)\|_{1,\Omega} &= \|\widetilde{\mathbf{S}}(\mathbf{S}_1(\phi), \mathbf{S}_2(\phi)) - \widetilde{\mathbf{S}}(\mathbf{S}_1(\varphi), \mathbf{S}_2(\varphi))\|_{1,\Omega} \\ &\le \frac{1}{\alpha_2}\left\{L_g \,\|\mathbf{S}(\phi) - \mathbf{S}(\varphi)\|_H + L_\vartheta \|\mathbf{T}(\varphi)\|_{1,\infty,\Omega}\, \|\mathbf{S}(\phi) - \mathbf{S}(\varphi)\|_H\right\} \\ &\le \frac{1}{\alpha_2} C_{\mathbf{S}}\left\{L_g + L_\vartheta \,\|\mathbf{T}(\varphi)\|_{1,\infty,\Omega}\right\} \|\phi - \varphi\|_{0,\Omega}\,,\end{aligned}$$

which gives (2.26) and completes the proof. □

Lemma 2.8. *Let W be as in Lemma 2.5. Then, $\mathbf{T} : W \to W$ is continuous and $\overline{\mathbf{T}(W)}$ is compact.*

Proof. It follows analogously to the proof of [17, Lemma 3.12], and it is a consequence of the Rellich–Kondrachov compactness Theorem [29, Thm. 6.3] in combination with (2.23), and the fact that every bounded sequence in a Hilbert space has a weakly convergent subsequence. □

The main result of this section is stated next.

Theorem 2.9. *The mixed-primal problem (2.11) has at least one solution $(\boldsymbol{\sigma}, (\boldsymbol{u}, \boldsymbol{\rho}), \phi) \in H \times \mathrm{H}^1_0(\Omega)$ satisfying the bounds*

$$\|\phi\|_{1,\Omega} \le r \tag{2.32}$$

and

$$\|(\boldsymbol{\sigma}, (\boldsymbol{u}, \boldsymbol{\rho}))\|_H \le c_{\mathbf{S}}\left\{\|\boldsymbol{u}_{\mathrm{D}}\|_{1/2,\Gamma} + f_2|\Omega|^{1/2}\right\}. \tag{2.33}$$

Moreover, if the data is such that

$$\frac{1}{\alpha_2} C_{\mathbf{S}}\left\{L_g + L_\vartheta C_\infty c_{\mathbf{S}}\left(\|\boldsymbol{u}_{\mathrm{D}}\|_{1/2,\Gamma} + f_2|\Omega|^{1/2}\right)\right\} < 1, \tag{2.34}$$

then the solution ϕ is unique in W.

Proof. Thanks to Lemmas 2.5 and 2.8, the existence of solution is merely an application of the Schauder fixed-point theorem. In turn, the estimates (2.32) and (2.33) follow from Lemmas 2.3 and 2.2, respectively. Furthermore, given another solution $\varphi \in W$ of (2.12), the estimate in (2.23) confirms (2.34) as a sufficient condition for concluding, together with (2.26), that $\phi = \varphi$. □

As announced in the Introduction, we notice here that, differently from the analysis in [17] and [18], the existence result provided by Theorem 2.9 does not require the data to be sufficiently small. However, we point out that the existence of the fourth component ϕ of the solution is restricted to the ball $W := \{\phi \in \mathrm{H}_0^1(\Omega) : \|\phi\|_{1,\Omega} \leq r\}$, whose radius r depends on the data ϑ_0 and g_2 (cf. (2.20)).

3. A mixed-primal Galerkin scheme

In this section we define a first numerical approximation associated with (2.11). We derive general hypotheses on the finite-dimensional subspaces defining the Galerkin finite element method, and ensuring that the discrete problem is indeed well-posed. Existence of solutions will follow by means of Brouwer's fixed-point theorem, and we will derive adequate a priori error estimates.

3.1. The mixed-primal discrete formulation

Let $\mathcal{T}_h$ be a regular partition of $\overline{\Omega}$ into triangles K of diameter h_K, where $h := \max\{h_K : K \in \mathcal{T}_h\}$ is the meshsize. Let us also consider arbitrary finite-dimensional subspaces

$$\mathbb{H}_h^{\boldsymbol{\sigma}} \subseteq \mathbb{H}_0(\mathbf{div}, \Omega), \quad \mathbf{H}_h^{\boldsymbol{u}} \subseteq \mathbf{L}^2(\Omega), \quad \mathbb{H}_h^{\boldsymbol{\rho}} \subseteq \mathbb{L}^2_{\mathrm{skew}}(\Omega) \quad \text{and} \quad \mathrm{H}_h^{\phi} \subseteq \mathrm{H}_0^1(\Omega),$$

whose specification will be made clear later on, in Section 3.4. The corresponding Galerkin scheme can be already defined as: find $(\boldsymbol{\sigma}_h, (\boldsymbol{u}_h, \boldsymbol{\rho}_h), \phi_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times (\mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}}) \times \mathrm{H}_h^{\phi}$ such that

$$\begin{aligned} a(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h, (\boldsymbol{u}_h, \boldsymbol{\rho}_h)) &= G(\boldsymbol{\tau}_h) && \forall\, \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}, \\ b(\boldsymbol{\sigma}_h, (\boldsymbol{v}_h, \boldsymbol{\eta}_h)) &= F_{\phi_h}(\boldsymbol{v}_h, \boldsymbol{\eta}_h) && \forall\, (\boldsymbol{v}_h, \boldsymbol{\eta}_h) \in \mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}}, \\ A_{\boldsymbol{\sigma}_h}(\phi_h, \psi_h) &= G_{\boldsymbol{u}_h}(\psi_h) && \forall\, \psi_h \in \mathrm{H}_h^{\phi}. \end{aligned} \tag{3.1}$$

A discrete analogue to the fixed-point strategy from Section 2.2 will be presented in what follows.

3.2. Discrete fixed-point approach

Let us introduce the operator $\mathbf{S}_h : \mathrm{H}_h^{\phi} \to \mathbb{H}_h^{\boldsymbol{\sigma}} \times (\mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}})$ defined by

$$\mathbf{S}_h(\phi_h) := (\mathbf{S}_{1,h}(\phi_h), (\mathbf{S}_{2,h}(\phi_h), \mathbf{S}_{3,h}(\phi_h))) := (\boldsymbol{\sigma}_h, (\boldsymbol{u}_h, \boldsymbol{\rho}_h)) \quad \forall \phi_h \in \mathrm{H}_h^{\phi},$$

where $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \boldsymbol{\rho}_h)$ solves uniquely the problem

$$\begin{aligned} a(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h, (\boldsymbol{u}_h, \boldsymbol{\rho}_h)) &= G(\boldsymbol{\tau}_h) && \forall\, \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}, \\ b(\boldsymbol{\sigma}_h, (\boldsymbol{v}_h, \boldsymbol{\eta}_h)) &= F_{\phi_h}(\boldsymbol{v}_h, \boldsymbol{\eta}_h) && \forall\, (\boldsymbol{v}_h, \boldsymbol{\eta}_h) \in \mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}}, \end{aligned} \tag{3.2}$$

with F_{ϕ_h} defined in (2.4) with $\phi = \phi_h$. On the other hand, we define $\widetilde{\mathbf{S}}_h : \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}} \to \mathrm{H}_h^{\phi}$ as

$$\widetilde{\mathbf{S}}_h(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) := \phi_h \qquad \forall\, (\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}},$$

where ϕ_h is the unique solution of

$$A_{\boldsymbol{\sigma}_h}(\phi_h, \psi_h) = G_{\boldsymbol{u}_h}(\psi_h) \qquad \forall\, \psi_h \in \mathrm{H}_h^{\phi}, \tag{3.3}$$

with $A_{\boldsymbol{\sigma}_h}$ and $G_{\boldsymbol{u}_h}$ being defined by (2.9) with $\boldsymbol{\sigma} = \boldsymbol{\sigma}_h$ and (2.10) with $\boldsymbol{u} = \boldsymbol{u}_h$, respectively. Therefore, solving (3.1) is equivalent to find $\phi_h \in \mathrm{H}_h^{\phi}$ such that

$$\mathbf{T}_h(\phi_h) = \phi_h,$$

where the fixed-point operator is characterised by

$$\mathbf{T}_h : \mathrm{H}_h^{\phi} \to \mathrm{H}_h^{\phi}, \qquad \mathbf{T}_h(\phi_h) := \widetilde{\mathbf{S}}_h(\mathbf{S}_{1,h}(\phi_h), \mathbf{S}_{2,h}(\phi_h)) \quad \forall \phi_h \in \mathrm{H}_h^{\phi}.$$

The well-definition of $\mathbf{T}_h$ then hinges on the well-posedness of $\widetilde{\mathbf{S}}_h$ and $\mathbf{S}_h$. For the latter, we anticipate that further hypotheses on the discrete spaces $\mathbb{H}_h^{\boldsymbol{\sigma}}$, $\mathbf{H}_h^{\boldsymbol{u}}$ and $\mathbb{H}_h^{\boldsymbol{\rho}}$ will be required. To this end, we now let V_h be the discrete kernel of b, that is

$$V_h := \left\{\boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}} \;:\; b(\boldsymbol{\tau}_h, (\boldsymbol{v}_h, \boldsymbol{\eta}_h)) = 0 \quad \forall\, (\boldsymbol{v}_h, \boldsymbol{\eta}_h) \in \mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}}\right\},$$

and assume the following discrete inf–sup conditions (which do hold for some finite element spaces, as those listed in Section 3.4):

[H.0] There exists a constant $\widehat{\alpha} > 0$, independent of h, such that

$$\sup_{\substack{\boldsymbol{\tau}_h \in V_h \\ \boldsymbol{\tau}_h \neq 0}} \frac{a(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbf{div},\Omega}} \geq \widehat{\alpha}\, \|\boldsymbol{\sigma}_h\|_{\mathbf{div},\Omega} \quad \forall\, \boldsymbol{\sigma}_h \in V_h. \tag{3.4}$$

[H.1] There exists a constant $\widehat{\beta} > 0$, independent of h, such that

$$\sup_{\substack{\boldsymbol{\tau}_h \in \mathbf{H}_h^{\boldsymbol{\sigma}} \\ \boldsymbol{\tau}_h \neq 0}} \frac{b(\boldsymbol{\tau}_h, (\boldsymbol{v}_h, \boldsymbol{\eta}_h))}{\|\boldsymbol{\tau}_h\|_{\mathbf{div},\Omega}} \geq \widehat{\beta}\, \left\|(\boldsymbol{v}_h, \boldsymbol{\eta}_h)\right\|_{\mathbf{L}^2(\Omega)\times\mathbb{L}^2_{\text{skew}}(\Omega)} \quad \forall\, (\boldsymbol{v}_h, \boldsymbol{\eta}_h) \in \mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}}. \tag{3.5}$$

Lemma 3.1. *For each $\phi_h \in \mathrm{H}_h^{\phi}$ the problem (3.2) has a unique solution $\mathbf{S}_h(\phi_h) := (\boldsymbol{\sigma}_h, (\boldsymbol{u}_h, \boldsymbol{\rho}_h)) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times (\mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}})$. Moreover, there exists $\widetilde{C} > 0$, depending on $\mu, \widehat{\alpha}, \widehat{\beta}$, but independent of ϕ_h, such that*

$$\|\mathbf{S}_h(\phi_h)\|_H = \left\|(\boldsymbol{\sigma}_h, (\boldsymbol{u}_h, \boldsymbol{\rho}_h))\right\|_H \leq \widetilde{C}\left\{\|\boldsymbol{u}_{\mathrm{D}}\|_{1/2,\Gamma} + f_2|\Omega|^{1/2}\right\}.$$

Proof. It follows directly from the discrete Babuška–Brezzi theory [21, Thm. 2.4]. Indeed, the induced operators for the forms a and b are bounded on subspaces of the corresponding continuous spaces. Furthermore, the linear functional G restricted to $\mathbb{H}_h^{\boldsymbol{\sigma}}$ is bounded as indicated in (2.16), and for each $\phi_h \in \mathrm{H}_h^{\phi}$, the functional F_{ϕ_h} restricted to $\mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}}$ is bounded as well. The remaining hypotheses are precisely **[H.0]** and **[H.1]**, and hence the proof is finished. □

Lemma 3.2. *Let $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}}$. Then, there exists a unique $\phi_h := \widetilde{\mathbf{S}}_h(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in \mathrm{H}_h^{\phi}$ solution of (3.3). Moreover, with the same constant r provided by Lemma 2.3, there holds*

$$\|\widetilde{\mathbf{S}}_h(\boldsymbol{\sigma}_h, \boldsymbol{u}_h)\|_{1,\Omega} = \|\phi_h\|_{1,\Omega} \leq r.$$

Proof. It suffices to note that for each $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}}$, the operator $A_{\boldsymbol{\sigma}_h}$ is elliptic on H_h^{ϕ} with the same constant α_2 from the proof of Lemma 2.3, and that $G_{\boldsymbol{u}_h}$ restricted to H_h^{ϕ} is bounded as in (2.19). Hence, the result is a direct application of the Lax–Milgram Lemma. □

3.3. Solvability of the discrete fixed-point equation

The following steps verify the hypotheses of the Brouwer fixed-point theorem (see, *e.g.* [31, Thm. 9.9-2]).

Lemma 3.3. *For the closed ball $W_h := \left\{\phi_h \in \mathrm{H}_h^{\phi} : \|\phi_h\|_{1,\Omega} \leq r\right\}$, we have that $\mathbf{T}_h(W_h) \subseteq W_h$.*

Proof. It is a straightforward consequence of Lemma 3.2. □

Lemma 3.4. *There exists $C > 0$ depending on μ, L_f, $\widehat{\alpha}$ and $\widehat{\beta}$ (cf. (1.1), (1.4), (3.4), (3.5)) such that*

$$\|\mathbf{S}_h(\phi_h) - \mathbf{S}_h(\varphi_h)\|_H \leq C\, \|\phi_h - \varphi_h\|_{0,\Omega} \quad \forall\, \phi_h, \varphi_h \in \mathrm{H}_h^{\phi}.$$

Proof. It follows analogously to the proof of Lemma 2.6. □

Lemma 3.5. *For each $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h), (\boldsymbol{\zeta}_h, \boldsymbol{w}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}}$, there holds*

$$\|\widetilde{\mathbf{S}}_h(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) - \widetilde{\mathbf{S}}_h(\boldsymbol{\zeta}_h, \boldsymbol{w}_h)\|_{1,\Omega} \leq \frac{1}{\alpha_2}\left\{L_g\, \|\boldsymbol{u}_h - \boldsymbol{w}_h\|_{0,\Omega} + L_\vartheta \|\nabla \widetilde{\mathbf{S}}_h(\boldsymbol{\zeta}_h, \boldsymbol{w}_h)\|_{\infty,\Omega} \left\|\boldsymbol{\sigma}_h - \boldsymbol{\zeta}_h\right\|_{0,\Omega}\right\}. \tag{3.6}$$

Proof. Given $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h), (\boldsymbol{\zeta}_h, \boldsymbol{w}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}}$, we let $\phi_h := \widetilde{\mathbf{S}}_h(\boldsymbol{\sigma}_h, \boldsymbol{u}_h)$ and $\varphi_h := \widetilde{\mathbf{S}}_h(\boldsymbol{\zeta}_h, \boldsymbol{w}_h)$. We then proceed similarly to the proof of Lemma 2.7 to obtain

$$\alpha_2\, \|\phi_h - \varphi_h\|_{1,\Omega}^2 \leq \left\{L_g\, \|\boldsymbol{u}_h - \boldsymbol{w}_h\|_{0,\Omega} + L_\vartheta \|\nabla \varphi_h\|_{\infty,\Omega} \left\|\boldsymbol{\sigma}_h - \boldsymbol{\zeta}_h\right\|_{0,\Omega}\right\} \|\phi_h - \varphi_h\|_{1,\Omega},$$

and realise that H_h^{ϕ} consists of piecewise polynomials (see Section 3.4) to conclude that $\|\nabla\varphi_h\|_{\infty,\Omega} < +\infty$, and hence (3.6) holds. □

The following result is a consequence of Lemmas 3.3–3.5.

Lemma 3.6. *Let C be as in Lemma 3.4. Then, for all $\phi_h, \varphi_h \in \mathrm{H}_h^{\phi}$, there holds*

$$\|\mathbf{T}_h(\phi_h) - \mathbf{T}_h(\varphi_h)\|_{1,\Omega} \le \frac{1}{\alpha_2} C \left(L_g + L_\vartheta \|\nabla \mathbf{T}_h(\varphi_h)\|_{\infty,\Omega}\right) \|\phi_h - \varphi_h\|_{0,\Omega}\,.$$

Proof. It follows after recalling that $\mathbf{T}_h(\phi_h) = \widetilde{\mathbf{S}}_h(\mathbf{S}_{1,h}(\phi_h), \mathbf{S}_{2,h}(\phi_h))$ for all $\phi_h \in \mathrm{H}_h^{\phi}$, and applying Lemmas 3.3–3.5. □

Finally, thanks to Lemmas 3.3 and 3.6, a straightforward application of the aforementioned Brouwer fixed-point theorem implies the main result of this section, stated as follows.

Theorem 3.7. *The Galerkin scheme (3.1) has at least one solution $(\boldsymbol{\sigma}_h, (\boldsymbol{u}_h, \boldsymbol{\rho}_h), \phi_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times (\mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}}) \times \mathrm{H}_h^{\phi}$. Furthermore, there exists a positive constant $\widetilde{C}$, independent of the discretisation parameters, such that*

$$\|\phi\|_{1,\Omega} \le r \quad \text{and} \quad \left\|(\boldsymbol{\sigma}_h, (\boldsymbol{u}_h, \boldsymbol{\rho}_h))\right\|_H \le \widetilde{C}\left\{\|\boldsymbol{u}_{\mathrm{D}}\|_{1/2,\Gamma} + f_2|\Omega|^{1/2}\right\}.$$

3.4. Specific finite element subspaces

Given an integer $k \ge 0$, for each $K \in \mathcal{T}_h$ we let $\mathrm{P}_k(K)$ be the space of polynomial functions on K of degree $\le k$ and recall the definition of the local Raviart–Thomas space of order k as $\mathbf{RT}_k(K) := \mathbf{P}_k(K) \oplus \mathrm{P}_k(K)\boldsymbol{x}$, where $\mathbf{P}_k(K) = [\mathrm{P}_k(K)]^2$, and $\boldsymbol{x}$ is the generic vector in R^2. In addition, we let b_K be the element bubble function defined as the unique polynomial in $\mathrm{P}_{k+1}(K)$ vanishing on ∂K with $\int_K b_K = 1$. Then, for each $K \in \mathcal{T}_h$ we consider the bubble space of order k, by

$$\mathbf{B}_k(K) := \mathrm{P}_k(K)\left(\frac{\partial b_K}{\partial x_2}, -\frac{\partial b_K}{\partial x_1}\right).$$

Appropriate finite element subspaces approximating the elasticity unknowns are as follows

$$\mathbb{H}_h^{\boldsymbol{\sigma}} := \{\boldsymbol{\tau}_h \in \mathbb{H}_0(\mathbf{div}, \Omega) : \quad \boldsymbol{\tau}_h|_K \in \mathbf{RT}_k(K) \oplus \mathbf{B}_k(K) \;\; \forall K \in \mathcal{T}_h\}, \tag{3.7}$$

$$\mathbf{H}_h^{\boldsymbol{u}} := \left\{\boldsymbol{v}_h \in \mathbf{L}^2(\Omega) : \quad \boldsymbol{v}_h|_K \in \mathbf{P}_k(K) \qquad \forall K \in \mathcal{T}_h\right\}, \tag{3.8}$$

$$\mathbb{H}_h^{\boldsymbol{\rho}} := \left\{\boldsymbol{\eta}_h \in \mathbb{L}^2_{\mathrm{skew}}(\Omega) : \quad \boldsymbol{\eta}_h \in \mathbf{C}(\Omega) \quad \text{and} \quad \boldsymbol{\eta}_h|_K \in \mathbb{P}_{k+1}(K) \qquad \forall K \in \mathcal{T}_h\right\}. \tag{3.9}$$

The discrete product space $\mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}}$ constitutes the classical PEERS elements introduced in [14] for the mixed finite element approximation of Dirichlet linear elasticity. In contrast, the approximation of the diffusion problem will be carried out using Lagrange finite elements of degree $\le k+1$, that is

$$\mathrm{H}_h^{\phi} := \left\{\psi_h \in \mathrm{C}(\Omega) \cap \mathrm{H}_0^1(\Omega) \quad \psi_h|_K \in \mathrm{P}_{k+1}(K) \qquad \forall K \in \mathcal{T}_h\right\}. \tag{3.10}$$

Useful approximation properties of these spaces are listed as follows (see *e.g.* [21,25]):

($\mathbf{AP}_h^{\boldsymbol{\sigma}}$) there exists $C > 0$, independent of h, such that for each $s \in (0, k+1]$, and for each $\boldsymbol{\sigma} \in \mathbb{H}^s(\Omega) \cap \mathbb{H}_0(\mathbf{div}, \Omega)$ with $\mathbf{div}\,(\boldsymbol{\sigma}) \in \mathbf{H}^s(\Omega)$, there holds

$$\mathrm{dist}(\boldsymbol{\sigma}, \mathbb{H}_h^{\boldsymbol{\sigma}}) := \inf_{\boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}} \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{\mathbf{div},\Omega} \le C h^s \left\{\|\boldsymbol{\sigma}\|_{s,\Omega} + \|\mathbf{div}\,(\boldsymbol{\sigma})\|_{s,\Omega}\right\}.$$

($\mathbf{AP}_h^{\boldsymbol{u}}$) there exists $C > 0$, independent of h, such that for each $s \in (0, k+1]$, and for each $\boldsymbol{u} \in \mathbf{H}^s(\Omega)$, there holds

$$\mathrm{dist}(\boldsymbol{u}, \mathbf{H}_h^{\boldsymbol{u}}) := \inf_{\boldsymbol{v}_h \in \mathbf{H}_h^{\boldsymbol{u}}} \|\boldsymbol{u} - \boldsymbol{v}_h\|_{0,\Omega} \le C h^s \,\|\boldsymbol{u}\|_{s,\Omega}\,.$$

($\mathbf{AP}_h^{\boldsymbol{\rho}}$) there exists $C > 0$, independent of h, such that for each $s \in (0, k+1]$, and for each $\boldsymbol{\rho} \in \mathbb{H}^s(\Omega)$, there holds

$$\operatorname{dist}(\boldsymbol{\rho}, \mathbb{H}_h^{\boldsymbol{\rho}}) := \inf_{\boldsymbol{\eta}_h \in \mathbb{H}_h^{\boldsymbol{\rho}}} \|\boldsymbol{\rho} - \boldsymbol{\eta}_h\|_{0,\Omega} \le C h^s \|\boldsymbol{\rho}\|_{s,\Omega} .$$

($\mathbf{AP}_h^{\phi}$) there exists $C > 0$, independent of h, such that for each $s \in (0, k+1]$, and for each $\phi \in \mathrm{H}^{s+1}(\Omega)$, there holds

$$\operatorname{dist}(\phi, \mathrm{H}_h^{\phi}) := \inf_{\psi_h \in \mathrm{H}_h^{\phi}} \|\phi - \psi_h\|_{1,\Omega} \le C h^s \|\phi\|_{s+1,\Omega} .$$

Next, we recall from [21, Sect. 4.5] that the discrete kernel of b is given by

$$V_h := \left\{ \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}} : \quad \mathbf{div}\, \boldsymbol{\tau}_h = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \int_\Omega \boldsymbol{\eta}_h : \boldsymbol{\tau}_h = 0 \quad \forall\, \boldsymbol{\eta}_h \in \mathbb{H}_h^{\boldsymbol{\rho}} \right\},$$

and according to (2.15) and Lemma 2.1, the bilinear form a is V_h-elliptic, implying that [**H.0**] is satisfied. Concerning assumption [**H.1**] we have the following result, proven in [21, Sect. 4.5].

Lemma 3.8. *There exists* $\widehat{\beta} > 0$ *such that*

$$\sup_{\boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}} \setminus \{\mathbf{0}\}} \frac{b(\boldsymbol{\tau}_h, (\boldsymbol{v}_h, \boldsymbol{\eta}_h))}{\|\boldsymbol{\tau}_h\|_{\mathbf{div},\Omega}} \ge \widehat{\beta} \left\|(\boldsymbol{v}_h, \boldsymbol{\eta}_h)\right\|_{\mathbf{L}^2(\Omega) \times \mathbb{L}^2_{\text{skew}}(\Omega)} \quad \forall\, (\boldsymbol{v}_h, \boldsymbol{\eta}_h) \in \mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}}.$$

3.5. *A priori* error analysis

Let $(\boldsymbol{\sigma}, (\boldsymbol{u}, \boldsymbol{\rho}), \phi) \in \mathbb{H}_0(\mathbf{div}, \Omega) \times (\mathbf{L}^2(\Omega) \times \mathbb{L}^2_{\text{skew}}(\Omega)) \times \mathrm{H}_0^1(\Omega)$ with $\phi \in W$, and $(\boldsymbol{\sigma}_h, (\boldsymbol{u}_h, \boldsymbol{\rho}_h), \phi_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times (\mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}}) \times \mathrm{H}_h^{\phi}$ with $\phi_h \in W_h$; be the solutions of (2.11) and (3.1), respectively. That is,

$$\begin{aligned} a(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, (\boldsymbol{u}, \boldsymbol{\rho})) &= G(\boldsymbol{\tau}) && \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}, \Omega), \\ b(\boldsymbol{\sigma}, (\boldsymbol{v}, \boldsymbol{\eta})) &= F_\phi(\boldsymbol{v}, \boldsymbol{\eta}) && \forall\, (\boldsymbol{v}, \boldsymbol{\eta}) \in \mathbf{L}^2(\Omega) \times \mathbb{L}^2_{\text{skew}}(\Omega), \\ a(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h, (\boldsymbol{u}_h, \boldsymbol{\rho}_h)) &= G(\boldsymbol{\tau}_h) && \forall\, \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}, \\ b(\boldsymbol{\sigma}_h, (\boldsymbol{v}_h, \boldsymbol{\eta}_h)) &= F_{\phi_h}(\boldsymbol{v}_h, \boldsymbol{\eta}_h) && \forall\, (\boldsymbol{v}_h, \boldsymbol{\eta}_h) \in \mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}} \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} A_{\boldsymbol{\sigma}}(\phi, \psi) &= G_{\boldsymbol{u}}(\psi) && \forall\, \psi \in \mathrm{H}_0^1(\Omega), \\ A_{\boldsymbol{\sigma}_h}(\phi_h, \psi_h) &= G_{\boldsymbol{u}_h}(\psi_h) && \forall\, \psi_h \in \mathrm{H}_h^{\phi}. \end{aligned} \tag{3.12}$$

Next, we recall a generalised Strang inequality (*cf.* [32, Thm. 11.2]), to be applied in (3.11).

Lemma 3.9. *For Hilbert spaces* X, Y, *let* $\boldsymbol{a} : X \times X \to \mathrm{R}, \boldsymbol{b} : X \times Y \to \mathrm{R}$ *be bounded bilinear forms and* $F \in X', G \in Y'$ *satisfying the hypotheses of the Babuška–Brezzi theory. Furthermore, let* $\{X_h\}_{h>0}$ *and* $\{Y_h\}_{h>0}$ *be sequences of finite-dimensional subspaces of* X *and* Y, *respectively, and suppose that* $\boldsymbol{a}, \boldsymbol{b}$ *and* $F_h \in X_h', G_h \in Y_h'$ *satisfy the hypotheses of the discrete Babuška–Brezzi theory uniformly on* X_h *and* Y_h, *that is, there exist positive constants* $\overline{\alpha}$ *and* $\overline{\beta}$ *independent of* h, *such that*

$$\sup_{\substack{\boldsymbol{\phi}_h \in V_h \\ \boldsymbol{\phi}_h \neq 0}} \frac{\boldsymbol{a}(\boldsymbol{\psi}_h, \boldsymbol{\phi}_h)}{\|\boldsymbol{\phi}_h\|_X} \ge \overline{\alpha} \|\boldsymbol{\psi}_h\|_X \quad \forall\, \boldsymbol{\psi}_h \in V_h \quad \text{and} \quad \sup_{\substack{\boldsymbol{\psi}_h \in X_h \\ \boldsymbol{\psi}_h \neq 0}} \frac{\boldsymbol{b}(\psi_h, \mu_h)}{\|\boldsymbol{\psi}_h\|_X} \ge \overline{\beta} \|\mu_h\|_Y \quad \forall\, \mu_h \in Y_h, \tag{3.13}$$

where V_h *is the discrete kernel of* $\boldsymbol{b}$. *Then, there exists a constant* C_{ST} *dependent only on* $\|\boldsymbol{a}\|$, $\|\boldsymbol{b}\|$, $\overline{\alpha}$ *and* $\overline{\beta}$ *such that if* $(\varphi, \lambda) \in X \times Y$ *and* $(\varphi_h, \lambda_h) \in X_h \times Y_h$ *are solutions to*

$$\begin{aligned} \boldsymbol{a}(\varphi, \psi) + \boldsymbol{b}(\psi, \lambda) &= F(\psi) && \forall\, \psi \in X, \\ \boldsymbol{b}(\varphi, \mu) &= G(\mu) && \forall\, \mu \in Y, \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{a}(\varphi_h, \psi_h) + \boldsymbol{b}(\psi_h, \lambda_h) &= F_h(\psi_h) \qquad && \forall\ \psi_h \in X_h, \\ \boldsymbol{b}(\varphi_h, \mu_h) &= G_h(\mu_h) \qquad && \forall\ \mu_h \in Y_h, \end{aligned}$$

respectively, then for each $h > 0$, *there holds*

$$\|\varphi - \varphi_h\|_X + \|\lambda - \lambda_h\|_Y \le C_{\mathrm{ST}} \left\{ \inf_{\substack{\psi_h \in X_h \\ \psi_h \neq 0}} \|\varphi - \psi_h\|_X + \inf_{\substack{\mu_h \in Y_h \\ \mu_h \neq 0}} \|\lambda - \mu_h\|_Y \right. \\ \left. + \sup_{\substack{\phi_h \in X_h \\ \phi_h \neq 0}} \frac{|F(\phi_h) - F_h(\phi_h)|}{\|\phi_h\|_X} + \sup_{\substack{\eta_h \in Y_h \\ \eta_h \neq 0}} \frac{|G(\eta_h) - G_h(\eta_h)|}{\|\eta_h\|_Y} \right\}.$$

In addition to the notations introduced in the approximation properties given in Section 3.4, we now define

$$\mathrm{dist}\left((\boldsymbol{\sigma}, (\boldsymbol{u}, \boldsymbol{\rho})), \mathbb{H}_h^{\boldsymbol{\sigma}} \times (\mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}})\right) := \inf_{(\boldsymbol{\tau}_h, (\boldsymbol{v}_h, \boldsymbol{\eta}_h)) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times (\mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}})} \left\| (\boldsymbol{\sigma}, (\boldsymbol{u}, \boldsymbol{\rho})) - (\boldsymbol{\tau}_h, (\boldsymbol{v}_h, \boldsymbol{\eta}_h)) \right\|_H,$$

or, equivalently,

$$\mathrm{dist}\left((\boldsymbol{\sigma}, (\boldsymbol{u}, \boldsymbol{\rho})), \mathbb{H}_h^{\boldsymbol{\sigma}} \times (\mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}})\right) := \mathrm{dist}(\boldsymbol{\sigma}, \mathbb{H}_h^{\boldsymbol{\sigma}}) + \mathrm{dist}(\boldsymbol{u}, \mathbf{H}_h^{\boldsymbol{u}}) + \mathrm{dist}\left(\boldsymbol{\rho}, \mathbb{H}_h^{\boldsymbol{\rho}}\right).$$

The following lemma provides an estimate for $\left\| (\boldsymbol{\sigma}, (\boldsymbol{u}, \boldsymbol{\rho})) - (\boldsymbol{\sigma}_h, (\boldsymbol{u}_h, \boldsymbol{\rho}_h)) \right\|_H$.

Lemma 3.10. *There exists* $C_{\mathrm{ST}} > 0$*, depending on* $\mu, \widehat{\alpha}$ *and* $\widehat{\beta}$ *(cf. (1.1), (3.4), (3.5)), such that*

$$\left\| (\boldsymbol{\sigma}, (\boldsymbol{u}, \boldsymbol{\rho})) - (\boldsymbol{\sigma}_h, (\boldsymbol{u}_h, \boldsymbol{\rho}_h)) \right\|_H \le C_{\mathrm{ST}} \left\{ \mathrm{dist}\left((\boldsymbol{\sigma}, (\boldsymbol{u}, \boldsymbol{\rho})), \mathbb{H}_h^{\boldsymbol{\sigma}} \times (\mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}})\right) + L_f \|\phi - \phi_h\|_{0,\Omega} \right\}. \tag{3.14}$$

Proof. We clearly observe that (3.4) and (3.5) imply that the hypothesis (3.13) in Lemma 3.9 is satisfied. Then, a straightforward application of Lemma 3.9 to (3.11), readily gives

$$\begin{aligned} &\left\| (\boldsymbol{\sigma}, (\boldsymbol{u}, \boldsymbol{\rho})) - (\boldsymbol{\sigma}_h, (\boldsymbol{u}_h, \boldsymbol{\rho}_h)) \right\|_H \\ &\quad \le C_{\mathrm{ST}} \left\{ \|(F_\phi - F_{\phi_h})|_{\mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}}}\| + \mathrm{dist}\left((\boldsymbol{\sigma}, (\boldsymbol{u}, \boldsymbol{\rho})), \mathbb{H}_h^{\boldsymbol{\sigma}} \times (\mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}})\right) \right\}. \end{aligned} \tag{3.15}$$

Next, and analogously to the proof of Lemma 2.6, we can assert that

$$\|(F_\phi - F_{\phi_h})|_{\mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}}}\| \le L_f \|\phi - \phi_h\|_{0,\Omega}, \tag{3.16}$$

and finally, by replacing (3.16) back into (3.15), we get the desired result. □

Lemma 3.11. *Let* α_2 *be the ellipticity constant of the bilinear form* $A_{\boldsymbol{\sigma}}$ *(cf. (2.18)). Then, there holds*

$$\|\phi - \phi_h\|_{1,\Omega} \le \frac{L_g}{\alpha_2} \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega} + \left(1 + \frac{\vartheta_2}{\alpha_2}\right) \mathrm{dist}(\phi, \mathrm{H}_h^{\phi}) + \frac{L_\vartheta}{\alpha_2} \|\phi\|_{1,\infty,\Omega} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega}. \tag{3.17}$$

Proof. We first observe by triangle inequality that

$$\|\phi - \phi_h\|_{1,\Omega} \le \|\phi - \psi_h\|_{1,\Omega} + \|\phi_h - \psi_h\|_{1,\Omega} \qquad \forall\, \psi_h \in \mathrm{H}_h^{\phi}. \tag{3.18}$$

Then, applying the ellipticity of $A_{\boldsymbol{\sigma}_h}$ and adding and subtracting the expression $G_{\boldsymbol{u}_h}(\phi_h - \psi_h) = A_{\boldsymbol{\sigma}_h}(\phi_h - \psi_h)$, (*cf.* (3.12)) we find that

$$\begin{aligned} \alpha_2 \|\phi_h - \psi_h\|_{1,\Omega}^2 &\le A_{\boldsymbol{\sigma}_h}(\phi_h - \psi_h, \phi_h - \psi_h) \\ &\le |G_{\boldsymbol{u}_h}(\phi_h - \psi_h) - G_{\boldsymbol{u}}(\phi_h - \psi_h)| + |A_{\boldsymbol{\sigma}}(\phi, \phi_h - \psi_h) - A_{\boldsymbol{\sigma}_h}(\psi_h, \phi_h - \psi_h)|. \end{aligned} \tag{3.19}$$

Next, analogously to (2.28), we get

$$|G_{\boldsymbol{u}_h}(\phi_h - \psi_h) - G_{\boldsymbol{u}}(\phi_h - \psi_h)| \le L_g \|\boldsymbol{u}_h - \boldsymbol{u}\|_{0,\Omega} \|\phi_h - \psi_h\|_{0,\Omega}. \tag{3.20}$$

In turn, adding and subtracting $\int_\Omega \vartheta(\boldsymbol{\sigma}_h)\nabla\phi\cdot\nabla(\phi_h-\psi_h)$, and applying the upper bound of ϑ (*cf.* (1.3)), we arrive at

$$\begin{aligned}&|A_{\boldsymbol{\sigma}}(\phi,\phi_h-\psi_h)-A_{\boldsymbol{\sigma}_h}(\psi_h,\phi_h-\psi_h)|\\&\qquad\le \vartheta_2|\phi-\psi_h|_{1,\Omega}|\phi_h-\psi_h|_{1,\Omega}+L_\vartheta\,\|\nabla\phi\|_{\infty,\Omega}\,\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,\Omega}\,|\phi_h-\psi_h|_{1,\Omega}.\end{aligned}\tag{3.21}$$

Thus, the inequalities (3.19), (3.20) and (3.21), imply that

$$\|\phi_h-\psi_h\|_{1,\Omega}\le\frac{L_g}{\alpha_2}\,\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega}+\frac{\vartheta_2}{\alpha_2}\,\|\phi-\psi_h\|_{1,\Omega}+\frac{L_\vartheta}{\alpha_2}\,\|\phi\|_{1,\infty,\Omega}\,\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,\Omega}\,.\tag{3.22}$$

Finally, replacing (3.22) back into (3.18) and taking the infimum on $\psi_h\in \mathrm{H}_h^\phi$, completes the proof. □

To derive the Céa estimation for the total error $\|\phi-\phi_h\|_{1,\Omega}+\left\|(\boldsymbol{\sigma},(\boldsymbol{u},\boldsymbol{\rho}))-(\boldsymbol{\sigma}_h,(\boldsymbol{u}_h,\boldsymbol{\rho}_h))\right\|_H$, we combine the inequalities provided by Lemmas 3.10 and 3.11. For sake of notational convenience we introduce the following constants

$$C_1:=\frac{L_g}{\alpha_2}C_{\mathrm{ST}},\qquad C_2:=\frac{L_\vartheta}{\alpha_2}C_\infty C_{\mathrm{ST}},\qquad C_3:=1+\frac{\vartheta_2}{\alpha_2}.\tag{3.23}$$

Hence, replacing the bound for $\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega}$ and $\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,\Omega}$ into (3.17), applying (2.23), and performing algebraic manipulations, we can deduce the bounds

$$\begin{aligned}\|\phi-\phi_h\|_{1,\Omega}&\le C_1\left\{\mathrm{dist}\left((\boldsymbol{\sigma},(\boldsymbol{u},\boldsymbol{\rho})),\mathbb{H}_h^{\boldsymbol{\sigma}}\times(\mathbf{H}_h^{\boldsymbol{u}}\times\mathbb{H}_h^{\boldsymbol{\rho}})\right)+L_f\,\|\phi-\phi_h\|_{0,\Omega}\right\}+C_3\,\mathrm{dist}(\phi,\mathrm{H}_h^\phi)\\&\quad+C_2\,c_{\mathbf{S}}\left\{\|\boldsymbol{u}_{\mathrm{D}}\|_{1/2,\Gamma}+f_2|\Omega|^{1/2}\right\}\left\{\mathrm{dist}\left((\boldsymbol{\sigma},(\boldsymbol{u},\boldsymbol{\rho})),\mathbb{H}_h^{\boldsymbol{\sigma}}\times(\mathbf{H}_h^{\boldsymbol{u}}\times\mathbb{H}_h^{\boldsymbol{\rho}})\right)+L_f\,\|\phi-\phi_h\|_{0,\Omega}\right\}\\&\le\left\{C_1+C_2\,c_{\mathbf{S}}\left(\|\boldsymbol{u}_{\mathrm{D}}\|_{1/2,\Gamma}+f_2|\Omega|^{1/2}\right)\right\}\left\{\mathrm{dist}\left((\boldsymbol{\sigma},(\boldsymbol{u},\boldsymbol{\rho})),\mathbb{H}_h^{\boldsymbol{\sigma}}\times(\mathbf{H}_h^{\boldsymbol{u}}\times\mathbb{H}_h^{\boldsymbol{\rho}})\right)\right\}\\&\quad+L_f\left\{C_1+C_2\,c_{\mathbf{S}}\left(\|\boldsymbol{u}_{\mathrm{D}}\|_{1/2,\Gamma}+f_2|\Omega|^{1/2}\right)\right\}\|\phi-\phi_h\|_{1,\Omega}+C_3\,\mathrm{dist}\,(\phi,\mathrm{H}_h^\phi).\end{aligned}\tag{3.24}$$

Consequently, we can establish the following result which provides the complete Céa estimate.

Theorem 3.12. *Assume that the data satisfy*

$$L_f\left\{C_1+C_2\,c_{\mathbf{S}}\left(\|\boldsymbol{u}_{\mathrm{D}}\|_{1/2,\Gamma}+f_2|\Omega|^{1/2}\right)\right\}<\frac{1}{2}.\tag{3.25}$$

Then, there exist positive constants C_4 and C_5 independent of h, such that

$$\begin{aligned}&\|\phi-\phi_h\|_{1,\Omega}+\left\|(\boldsymbol{\sigma},(\boldsymbol{u},\boldsymbol{\rho}))-(\boldsymbol{\sigma}_h,(\boldsymbol{u}_h,\boldsymbol{\rho}_h))\right\|_H\\&\qquad\qquad\le C_4\,\mathrm{dist}\,(\phi,\mathrm{H}_h^\phi)+C_5\,\mathrm{dist}\left((\boldsymbol{\sigma},(\boldsymbol{u},\boldsymbol{\rho})),\mathbb{H}_h^{\boldsymbol{\sigma}}\times(\mathbf{H}_h^{\boldsymbol{u}}\times\mathbb{H}_h^{\boldsymbol{\rho}})\right).\end{aligned}\tag{3.26}$$

Proof. The estimate for $\|\phi-\phi_h\|_{1,\Omega}$ follows from (3.24) and (3.25), and the proof is complete after inserting the bound back into (3.14). □

Theorem 3.13. *In addition to the hypotheses of Theorems* 2.9, 3.7 and 3.12*, assume that there exists $s>0$ such that $\boldsymbol{\sigma}\in\mathbb{H}^s(\Omega)$, $\mathbf{div}\,(\boldsymbol{\sigma})\in\mathbf{H}^s(\Omega)$, $\boldsymbol{u}\in\mathbf{H}^s(\Omega)$, $\boldsymbol{\rho}\in\mathbb{H}^s(\Omega)$ and $\phi\in\mathrm{H}^{1+s}(\Omega)$. Then, there exists $\widehat{C}>0$, independent of h, such that, with the finite element subspaces defined by* (3.7)*,* (3.8)*,* (3.9) *and* (3.10)*, there holds*

$$\begin{aligned}&\|\phi-\phi_h\|_{1,\Omega}+\left\|(\boldsymbol{\sigma},(\boldsymbol{u},\boldsymbol{\rho}))-(\boldsymbol{\sigma}_h,(\boldsymbol{u}_h,\boldsymbol{\rho}_h))\right\|_H\\&\qquad\qquad\le\widehat{C}h^{\min\{s,k+1\}}\left\{\|\boldsymbol{\sigma}\|_{s,\Omega}+\|\mathbf{div}\,\boldsymbol{\sigma}\|_{s,\Omega}+\|\boldsymbol{u}\|_{s,\Omega}+\|\boldsymbol{\rho}\|_{s,\Omega}+\|\phi\|_{1+s,\Omega}\right\}.\end{aligned}\tag{3.27}$$

Proof. It follows as a combination of the Céa estimate (3.26), and the approximation properties ($\mathbf{AP}_h^{\boldsymbol{\sigma}}$), ($\mathbf{AP}_h^{\boldsymbol{u}}$), ($\mathbf{AP}_h^{\boldsymbol{\rho}}$) and ($\mathbf{AP}_h^{\phi}$). □

4. An augmented mixed-primal formulation

In this section we follow the approach from previous works (see, e.g. [15,17,33] and the references therein) and put forward an augmented mixed-primal formulation for (1.7), which, as shown below, allows more freedom for choosing the finite element subspaces. We establish the augmented mixed-primal variational formulation of (1.1) and show that it is well-posed. Next, we define the corresponding Galerkin scheme, prove its solvability, introduce a specific mixed finite element method, and finally we establish the corresponding *a priori* error estimate.

4.1. The continuous setting

In order to increase flexibility in choosing discrete spaces for the approximation of the elasticity problem, we incorporate the following redundant terms in the variational formulation (2.6):

$$
\begin{aligned}
&\kappa_1 \int_\Omega \left(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \mathcal{C}^{-1}\boldsymbol{\sigma}\right) : \boldsymbol{\varepsilon}(\boldsymbol{v}) = 0 && \forall\, \boldsymbol{v} \in \mathbf{H}^1(\Omega),\\
&\kappa_2 \int_\Omega \mathbf{div}\,\boldsymbol{\sigma} \cdot \mathbf{div}\,\boldsymbol{\tau} = -\,\kappa_2 \int_\Omega \boldsymbol{f}(\phi) \cdot \mathbf{div}\,\boldsymbol{\tau} && \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}, \Omega),\\
&\kappa_3 \int_\Omega (\boldsymbol{\rho} - (\nabla \boldsymbol{u} - \boldsymbol{\varepsilon}(\boldsymbol{u}))) : \boldsymbol{\eta} = 0 && \forall\, \boldsymbol{\eta} \in \mathbb{L}^2_{\text{skew}}(\Omega),\\
&\kappa_4 \int_\Gamma \boldsymbol{u} \cdot \boldsymbol{v} = \kappa_4 \int_\Gamma \boldsymbol{u}_{\rm D} \cdot \boldsymbol{v} && \forall\, \boldsymbol{v} \in \mathbf{H}^1(\Omega),
\end{aligned}
\tag{4.1}
$$

where $(\kappa_1, \kappa_2, \kappa_3, \kappa_4)$ is a vector of positive parameters to be specified later on. It is important to observe here that the above terms now require that the displacement $\boldsymbol{u}$ live in $\mathbf{H}^1(\Omega)$.

Then, and alternatively to (2.6), we may consider the following augmented mixed formulation for the elasticity problem: find $(\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{\rho}) \in \mathbb{H}_0(\mathbf{div}, \Omega) \times \mathbf{H}^1(\Omega) \times \mathbb{L}^2_{\text{skew}}(\Omega)$ such that

$$
\widetilde{B}((\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{\rho}), (\boldsymbol{\tau}, \boldsymbol{v}, \boldsymbol{\eta})) = \widetilde{F}_\phi(\boldsymbol{\tau}, \boldsymbol{v}, \boldsymbol{\eta}) \quad \forall (\boldsymbol{\tau}, \boldsymbol{v}, \boldsymbol{\eta}) \in \mathbb{H}_0(\mathbf{div}, \Omega) \times \mathbf{H}^1(\Omega) \times \mathbb{L}^2_{\text{skew}}(\Omega), \tag{4.2}
$$

where the multilinear form and the associated right hand side functional are defined as

$$
\begin{aligned}
\widetilde{B}((\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{\rho}), (\boldsymbol{\tau}, \boldsymbol{v}, \boldsymbol{\eta})) &:= a(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, (\boldsymbol{u}, \boldsymbol{\rho})) - b(\boldsymbol{\sigma}, (\boldsymbol{v}, \boldsymbol{\eta})) + \kappa_1 \int_\Omega \left(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \mathcal{C}^{-1}\boldsymbol{\sigma}\right) : \boldsymbol{\varepsilon}(\boldsymbol{v})\\
&\quad + \kappa_2 \int_\Omega \mathbf{div}\,\boldsymbol{\sigma} \cdot \mathbf{div}\,\boldsymbol{\tau} + \kappa_3 \int_\Omega (\boldsymbol{\rho} - (\nabla \boldsymbol{u} - \boldsymbol{\varepsilon}(\boldsymbol{u}))) : \boldsymbol{\eta} + \kappa_4 \int_\Gamma \boldsymbol{u} \cdot \boldsymbol{v},
\end{aligned}
\tag{4.3}
$$

$$
\widetilde{F}_\phi(\boldsymbol{\tau}, \boldsymbol{v}, \boldsymbol{\eta}) := G(\boldsymbol{\tau}) - F_\phi(\boldsymbol{v}, \boldsymbol{\eta}) - \kappa_2 \int_\Omega \boldsymbol{f}(\phi) \cdot \mathbf{div}\,\boldsymbol{\tau} + \kappa_4 \int_\Gamma \boldsymbol{u}_{\rm D} \cdot \boldsymbol{v}. \tag{4.4}
$$

Hence, the augmented mixed-primal formulation for (1.7) reduces to (2.8) and (4.2), i.e.: find $(\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{\rho}, \phi) \in \mathbb{H}_0(\mathbf{div}, \Omega) \times \mathbf{H}^1(\Omega) \times \mathbb{L}^2_{\text{skew}}(\Omega) \times \mathrm{H}^1_0(\Omega)$ such that

$$
\begin{aligned}
\widetilde{B}((\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{\rho}), (\boldsymbol{\tau}, \boldsymbol{v}, \boldsymbol{\eta})) &= \widetilde{F}_\phi(\boldsymbol{\tau}, \boldsymbol{v}, \boldsymbol{\eta}) \quad \forall (\boldsymbol{\tau}, \boldsymbol{v}, \boldsymbol{\eta}) \in \mathbb{H}_0(\mathbf{div}, \Omega) \times \mathbf{H}^1(\Omega) \times \mathbb{L}^2_{\text{skew}}(\Omega),\\
A_{\boldsymbol{\sigma}}(\phi, \psi) &= G_{\boldsymbol{u}}(\psi) \qquad \forall\, \psi \in \mathrm{H}^1_0(\Omega).
\end{aligned}
\tag{4.5}
$$

We proceed to adapt the approach from Sections 2.2 and 2.3. Since now $\boldsymbol{u} \in \mathbf{H}^1(\Omega)$, we can define

$$
\mathbf{S} : \mathrm{H}^1_0(\Omega) \to \mathbb{H}_0(\mathbf{div}, \Omega) \times \mathbf{H}^1(\Omega) \times \mathbb{L}^2_{\text{skew}}(\Omega), \quad \mathbf{S}(\phi) := (\mathbf{S}_1(\phi), \mathbf{S}_2(\phi), \mathbf{S}_3(\phi)) := (\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{\rho}),
$$

where $(\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{\rho})$ is the unique solution of (4.2) with a given $\phi \in \mathrm{H}^1_0(\Omega)$. In turn, we define the operator

$$
\widetilde{\mathbf{S}} : \mathbb{H}_0(\mathbf{div}, \Omega) \times \mathbf{H}^1(\Omega) \to \mathrm{H}^1_0(\Omega), \quad \widetilde{\mathbf{S}}(\boldsymbol{\sigma}, \boldsymbol{u}) := \phi \quad \forall (\boldsymbol{\sigma}, \boldsymbol{u}) \in \mathbb{H}_0(\mathbf{div}, \Omega) \times \mathbf{H}^1(\Omega),
$$

where ϕ is the unique solution of (2.8) with the given $(\boldsymbol{\sigma}, \boldsymbol{u})$. Next, the definition of $\mathbf{T}$ and the fixed-point strategy follow exactly as in Section 2.2. The analysis of $\widetilde{\mathbf{S}}$ can be therefore omitted.

The following lemma will be instrumental in showing the well-posedness of (4.2) for a given ϕ.

Lemma 4.1. *There exists $c_2 > 0$ such that*

$$
\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|^2_{1,\Omega} + \|\boldsymbol{v}\|^2_{0,\Gamma} \geq c_2\, \|\boldsymbol{v}\|^2_{1,\Omega} \quad \forall \boldsymbol{v} \in \mathbf{H}^1(\Omega).
$$

Proof. See [15, Lemma 3.1 and (3.9)]. □

Lemma 4.2. *Assume that $\kappa_1 \in (0, 4\delta\mu)$ and $\kappa_3 \in \left(0, 2c_2\kappa_1\widetilde{\delta}\left(1 - \frac{\delta}{2}\right)\right)$ with $\delta, \widetilde{\delta} \in (0, 2)$, and that $\kappa_2, \kappa_4 > 0$. Then, for each $\phi \in \mathrm{H}^1_0(\Omega)$, problem (4.2) has a unique solution $\mathbf{S}(\phi) := (\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{\rho}) \in H := \mathbb{H}_0(\mathbf{div}, \Omega) \times \mathbf{H}^1(\Omega) \times \mathbb{L}^2_{\text{skew}}(\Omega)$. Moreover, there exists $k_{\mathbf{S}} > 0$, independent of ϕ, such that*

$$
\|\mathbf{S}(\phi)\|_H = \|(\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{\rho})\|_H \leq k_{\mathbf{S}} \left\{\|\boldsymbol{u}_{\rm D}\|_{1/2,\Gamma} + f_2|\Omega|^{1/2}\right\} \quad \forall \phi \in \mathrm{H}^1_0(\Omega).
$$

Proof. We first observe from (4.3) that B is a bilinear form. Next, applying Cauchy–Schwarz's inequality together with the trace theorem (with constant c_3), we can assert that

$$|\widetilde{B}((\boldsymbol{\sigma},\boldsymbol{u},\boldsymbol{\rho}),(\boldsymbol{\tau},\boldsymbol{v},\boldsymbol{\eta}))| \le \frac{1}{\mu}\|\boldsymbol{\sigma}\|_{0,\Omega}\|\boldsymbol{\tau}\|_{0,\Omega} + \|\boldsymbol{u}\|_{0,\Omega}\,\|\mathbf{div}\,\boldsymbol{\tau}\|_{0,\Omega} + \|\boldsymbol{\rho}\|_{0,\Omega}\,\|\boldsymbol{\tau}\|_{0,\Omega} + \|\boldsymbol{v}\|_{0,\Omega}\,\|\mathbf{div}\,\boldsymbol{\sigma}\|_{0,\Omega}$$
$$+ \|\boldsymbol{\eta}\|_{0,\Omega}\,\|\boldsymbol{\sigma}\|_{0,\Omega} + \kappa_1\,\|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_{0,\Omega}\,\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{0,\Omega} + \frac{\kappa_1}{\mu}\,\|\boldsymbol{\sigma}\|_{0,\Omega}\,\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{0,\Omega} + \kappa_2\,\|\mathbf{div}\,\boldsymbol{\sigma}\|_{0,\Omega}\,\|\mathbf{div}\,\boldsymbol{\tau}\|_{0,\Omega}$$
$$+ \kappa_3\,\|\boldsymbol{\rho}\|_{0,\Omega}\,\|\boldsymbol{\eta}\|_{0,\Omega} + \kappa_3|\boldsymbol{u}|_{1,\Omega}\,\|\boldsymbol{\eta}\|_{0,\Omega} + \kappa_3\,\|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_{0,\Omega}\,\|\boldsymbol{\eta}\|_{0,\Omega} + \kappa_4 c_3^2\,\|\boldsymbol{u}\|_{1,\Omega}\,\|\boldsymbol{v}\|_{1,\Omega}\,.$$

It follows that there exists $\|\widetilde{B}\| > 0$ depending on $\mu, \kappa_1, \kappa_2, \kappa_3, \kappa_4$ and c_3, such that

$$|\widetilde{B}((\boldsymbol{\sigma},\boldsymbol{u},\boldsymbol{\rho}),(\boldsymbol{\tau},\boldsymbol{v},\boldsymbol{\eta}))| \le \|\widetilde{B}\|\,\|(\boldsymbol{\sigma},\boldsymbol{u},\boldsymbol{\rho})\|_H\,\|(\boldsymbol{\tau},\boldsymbol{v},\boldsymbol{\eta})\|_H \qquad \forall\,(\boldsymbol{\sigma},\boldsymbol{u},\boldsymbol{\rho}),(\boldsymbol{\tau},\boldsymbol{v},\boldsymbol{\eta}) \in H,$$

implying that $\widetilde{B}$ is bounded independently of $\phi \in \mathrm{H}_0^1(\Omega)$. The H-ellipticity analysis of $\widetilde{B}$ will be conducted as in the proof of [16, Thm. 3.1]. For each $(\boldsymbol{\tau},\boldsymbol{v},\boldsymbol{\eta}) \in H$, Young's inequality yields

$$\widetilde{B}((\boldsymbol{\tau},\boldsymbol{v},\boldsymbol{\eta}),(\boldsymbol{\tau},\boldsymbol{v},\boldsymbol{\eta})) = \int_\Omega \mathcal{C}^{-1}\boldsymbol{\tau}:\boldsymbol{\tau} + \kappa_1\,\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{0,\Omega}^2 - \kappa_1\,\left\|\mathcal{C}^{-1}\boldsymbol{\tau}\right\|_{0,\Omega}\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{0,\Omega} + \kappa_2\,\|\mathbf{div}\,\boldsymbol{\tau}\|_{0,\Omega}^2$$
$$+ \kappa_3\,\|\boldsymbol{\eta}\|_{0,\Omega}^2 - \kappa_3\,\|\nabla\boldsymbol{v} - \boldsymbol{\varepsilon}(\boldsymbol{v})\|_{0,\Omega}\,\|\boldsymbol{\eta}\|_{0,\Omega} + \kappa_4\,\|\boldsymbol{v}\|_{0,\Gamma}^2$$
$$= \int_\Omega \mathcal{C}^{-1}\boldsymbol{\tau}:\boldsymbol{\tau} - \frac{\kappa_1}{2\delta}\left\|\mathcal{C}^{-1}\boldsymbol{\tau}\right\|_{0,\Omega}^2 + \kappa_1\,\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{0,\Omega}^2 - \frac{\kappa_1\delta}{2}\,\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{0,\Omega}^2 + \kappa_2\,\|\mathbf{div}\,\boldsymbol{\tau}\|_{0,\Omega}^2$$
$$+ \kappa_3\,\|\boldsymbol{\eta}\|_{0,\Omega}^2 - \frac{\kappa_3}{2\widetilde{\delta}}\,\|\nabla\boldsymbol{v} - \boldsymbol{\varepsilon}(\boldsymbol{v})\|_{0,\Omega}^2 - \frac{\kappa_3\widetilde{\delta}}{2}\,\|\boldsymbol{\eta}\|_{0,\Omega}^2 + \kappa_4\,\|\boldsymbol{v}\|_{0,\Gamma}^2\,,$$

from which, taking $\delta, \widetilde{\delta}, \kappa_1, \kappa_2, \kappa_3, \kappa_4$ as stated in the hypotheses, applying Lemmas 2.1 and 4.1, and using the relation $\|\nabla\boldsymbol{v} - \boldsymbol{\varepsilon}(\boldsymbol{v})\|_{0,\Omega}^2 = |\boldsymbol{v}|_{1,\Omega}^2 - \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{0,\Omega}^2$, we can deduce that

$$\widetilde{B}((\boldsymbol{\tau},\boldsymbol{v},\boldsymbol{\eta}),(\boldsymbol{\tau},\boldsymbol{v},\boldsymbol{\eta})) \ge \frac{1}{2\mu}\left(1 - \frac{\kappa_1}{4\delta\mu}\right)\|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega}^2 + \kappa_2\,\|\mathbf{div}\,\boldsymbol{\tau}\|_{0,\Omega}^2 + \kappa_1\left(1 - \frac{\delta}{2}\right)\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{0,\Omega}^2$$
$$+ \kappa_3\left(1 - \frac{\widetilde{\delta}}{2}\right)\|\boldsymbol{\eta}\|_{0,\Omega}^2 - \frac{\kappa_3}{2\widetilde{\delta}}|\boldsymbol{v}|_{1,\Omega}^2 + \kappa_4\,\|\boldsymbol{v}\|_{0,\Gamma}^2$$
$$= \widetilde{\alpha}_2\,\|\boldsymbol{\tau}\|_{\mathbf{div},\Omega}^2 + \left(c_2\widetilde{\alpha}_3 - \frac{\kappa_3}{2\widetilde{\delta}}\right)\|\boldsymbol{v}\|_{1,\Omega}^2 + \kappa_3\left(1 - \frac{\widetilde{\delta}}{2}\right)\|\boldsymbol{\eta}\|_{0,\Omega}^2\,,$$

where $\widetilde{\alpha}_1 := \min\{\frac{1}{2\mu}\left(1 - \frac{\kappa_1}{4\delta\mu}\right), \frac{\kappa_2}{2}\}$, $\widetilde{\alpha}_2 := \min\{c_1\widetilde{\alpha}_1, \frac{\kappa_2}{2}\}$, and $\widetilde{\alpha}_3 := \min\{\kappa_1\left(1 - \frac{\delta}{2}\right), \kappa_4\}$. In this way, defining $\widetilde{\alpha} := \min\{\widetilde{\alpha}_2, c_2\widetilde{\alpha}_3 - \frac{\kappa_3}{2\widetilde{\delta}}, \kappa_3\left(1 - \frac{\widetilde{\delta}}{2}\right)\}$, which depends on $\mu, \delta, \widetilde{\delta}, \kappa_1, \kappa_2, \kappa_3, \kappa_4, c_1$ and c_2, we conclude that

$$\widetilde{B}((\boldsymbol{\tau},\boldsymbol{v},\boldsymbol{\eta}),(\boldsymbol{\tau},\boldsymbol{v},\boldsymbol{\eta})) \ge \widetilde{\alpha}\,\|(\boldsymbol{\tau},\boldsymbol{v},\boldsymbol{\eta})\|_H^2 \qquad \forall\,(\boldsymbol{\tau},\boldsymbol{v},\boldsymbol{\eta}) \in H. \tag{4.6}$$

Next, given $\phi \in \mathrm{H}_0^1(\Omega)$, we look at the functional $\widetilde{F}_\phi$, which is certainly linear. Similarly to the proof of [17, Lemma 3.4], there exists a positive constant $\|\widetilde{F}\|$ depending on κ_2, κ_4 and c_3, such that

$$|\widetilde{F}_\phi(\boldsymbol{\tau},\boldsymbol{v},\boldsymbol{\eta})| \le \|\widetilde{F}\|\left\{\|\boldsymbol{u}_{\mathrm{D}}\|_{1/2,\Gamma} + f_2|\Omega|^{1/2}\right\}\|(\boldsymbol{\tau},\boldsymbol{v},\boldsymbol{\eta})\|_H\,. \tag{4.7}$$

The foregoing inequality shows the boundedness of $\widetilde{F}_\phi$ with

$$\|\widetilde{F}_\phi\| \le \|\widetilde{F}\|\left\{\|\boldsymbol{u}_{\mathrm{D}}\|_{1/2,\Gamma} + f_2|\Omega|^{1/2}\right\}. \tag{4.8}$$

Finally, a straightforward application of the Lax–Milgram Lemma proves that for each $\phi \in \mathrm{H}_0^1(\Omega)$, problem (4.2) has a unique solution $\mathbf{S}(\phi) := (\boldsymbol{\sigma},\boldsymbol{u},\boldsymbol{\rho}) \in H$. Moreover, the corresponding continuous dependence result together with the estimates (4.6) and (4.7) give

$$\|\mathbf{S}(\phi)\|_H = \|(\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{\rho})\|_H \le \frac{1}{\widetilde{\alpha}} \|\widetilde{F}_\phi\|_{H'} \le k_{\mathbf{S}} \left\{ \|\boldsymbol{u}_{\mathrm{D}}\|_{1/2,\Gamma} + f_2 |\Omega|^{1/2} \right\},$$

with $k_{\mathbf{S}} := \frac{\|\widetilde{F}\|}{\widetilde{\alpha}}$, thus completing the proof. □

Lemma 4.3. *Let $\widetilde{\alpha}$ be the ellipticity constant provided in Lemma 4.2. Then, there exists $K_{\mathbf{S}} > 0$ depending on L_f, κ_2 and $\widetilde{\alpha}$ (cf. (1.4), (4.1), (4.6)), such that*

$$\|\mathbf{S}(\phi) - \mathbf{S}(\varphi)\|_H \le K_{\mathbf{S}} \|\phi - \varphi\|_{0,\Omega} \qquad \forall\, \phi, \varphi \in \mathrm{H}^1_0(\Omega). \tag{4.9}$$

Proof. We follow [17, Lemma 3.9], and fix $\phi, \varphi \in \mathrm{H}^1_0(\Omega)$. We then take $(\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{\rho}) = \mathbf{S}(\phi)$ and $(\boldsymbol{\zeta}, \boldsymbol{w}, \boldsymbol{\chi}) = \mathbf{S}(\varphi)$, that is

$$\widetilde{B}((\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{\rho}), (\boldsymbol{\tau}, \boldsymbol{v}, \boldsymbol{\eta})) = \widetilde{F}_\phi(\boldsymbol{\tau}, \boldsymbol{v}, \boldsymbol{\eta}) \quad \text{and} \quad \widetilde{B}((\boldsymbol{\zeta}, \boldsymbol{w}, \boldsymbol{\chi}), (\boldsymbol{\tau}, \boldsymbol{v}, \boldsymbol{\eta})) = \widetilde{F}_\varphi(\boldsymbol{\tau}, \boldsymbol{v}, \boldsymbol{\eta}) \quad \forall (\boldsymbol{\tau}, \boldsymbol{v}, \boldsymbol{\eta}) \in H.$$

Exploiting the ellipticity of $\widetilde{B}$ we readily get

$$\begin{aligned} \widetilde{\alpha} \,\|(\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{\rho}) - (\boldsymbol{\zeta}, \boldsymbol{w}, \boldsymbol{\chi})\|_H^2 &\le \widetilde{B}((\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{\rho}), (\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{\rho}) - (\boldsymbol{\zeta}, \boldsymbol{w}, \boldsymbol{\chi})) - \widetilde{B}((\boldsymbol{\zeta}, \boldsymbol{w}, \boldsymbol{\chi}), (\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{\rho}) - (\boldsymbol{\zeta}, \boldsymbol{w}, \boldsymbol{\chi})) \\ &= (\widetilde{F}_\phi - \widetilde{F}_\varphi)((\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{\rho}) - (\boldsymbol{\zeta}, \boldsymbol{w}, \boldsymbol{\chi})), \end{aligned} \tag{4.10}$$

and the definition of $\widetilde{F}_\phi$ in combination with Cauchy–Schwarz's inequality and (1.4) implies that

$$\begin{aligned} &|(\widetilde{F}_\phi - \widetilde{F}_\varphi)((\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{\rho}) - (\boldsymbol{\zeta}, \boldsymbol{w}, \boldsymbol{\chi}))| \\ &\quad = \left| \int_\Omega (\boldsymbol{f}(\phi) - \boldsymbol{f}(\varphi)) \cdot (\boldsymbol{u} - \boldsymbol{w}) - \kappa_2 \int_\Omega (\boldsymbol{f}(\phi) - \boldsymbol{f}(\varphi)) \cdot \mathbf{div}(\boldsymbol{\sigma} - \boldsymbol{\zeta}) \right| \\ &\quad \le L_f (1 + \kappa_2^2)^{1/2} \|\phi - \varphi\|_{0,\Omega} \|(\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{\rho}) - (\boldsymbol{\zeta}, \boldsymbol{w}, \boldsymbol{\chi})\|_H \,. \end{aligned} \tag{4.11}$$

Back substitution of (4.11) into (4.10) then yields

$$\widetilde{\alpha} \,\|(\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{\rho}) - (\boldsymbol{\zeta}, \boldsymbol{w}, \boldsymbol{\chi})\|_H^2 \le L_f (1 + \kappa_2^2)^{1/2} \|\phi - \varphi\|_{0,\Omega} \|(\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{\rho}) - (\boldsymbol{\zeta}, \boldsymbol{w}, \boldsymbol{\chi})\|_H \,,$$

which finally gives (4.9). □

Lemma 4.4. *Let W be the closed ball defined in Lemma 2.5 and $K_{\mathbf{S}}$ be as in Lemma 4.3. Then, for each $\phi, \varphi \in \mathrm{H}^1_0(\Omega)$, there holds*

$$\|\mathbf{T}(\phi) - \mathbf{T}(\varphi)\|_{1,\Omega} \le \frac{1}{\alpha_2} K_{\mathbf{S}} \left(L_g + L_\vartheta \|\mathbf{T}(\varphi)\|_{1,\infty,\Omega} \right) \|\phi - \varphi\|_{0,\Omega} \,.$$

Proof. The definition of $\mathbf{T}$ together with Lemma 2.3 imply that $\mathbf{T}(W) \subseteq W$. The remainder of the proof proceeds exactly as the one of Lemma 2.7. □

Theorem 4.5. *The mixed-primal problem (2.11) has at least one solution $(\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{\rho}, \phi) \in \mathbb{H}_0(\mathbf{div}, \Omega) \times \mathbf{H}^1(\Omega) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega) \times \mathrm{H}^1_0(\Omega)$, satisfying*

$$\|\phi\|_{1,\Omega} \le r \quad \textit{and} \quad \|(\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{\rho})\|_H \le k_{\mathbf{S}} \left\{ \|\boldsymbol{u}_{\mathrm{D}}\|_{1/2,\Gamma} + f_2 |\Omega|^{1/2} \right\}.$$

Moreover, if the data satisfy

$$\frac{1}{\alpha_2} K_{\mathbf{S}} \left\{ L_g + L_\vartheta C_\infty k_{\mathbf{S}} \left(\|\boldsymbol{u}_{\mathrm{D}}\|_{1/2,\Gamma} + f_2 |\Omega|^{1/2} \right) \right\} < 1,$$

then the solution ϕ is unique in W.

Proof. It follows as in the proof of Theorem 2.9. □

4.2. The discrete scheme

We begin by observing that, thanks to the ellipticity of the bilinear forms $\widetilde{B}$ and $A_{\boldsymbol{\sigma}}$, we can consider arbitrary finite dimensional-subspaces

$$\mathbb{H}_h^{\boldsymbol{\sigma}} \subseteq \mathbb{H}_0(\mathbf{div}, \Omega), \quad \mathbf{H}_h^{\boldsymbol{u}} \subseteq \mathbf{H}^1(\Omega), \quad \mathbb{H}_h^{\boldsymbol{\rho}} \subseteq \mathbb{L}^2_{\mathrm{skew}}(\Omega) \quad \text{and} \quad \mathrm{H}_h^{\phi} \subseteq \mathrm{H}_0^1(\Omega),$$

for the augmented mixed-primal formulation. In particular, given an integer $k \geq 0$, we can define

$$\begin{aligned}
\mathbb{H}_h^{\boldsymbol{\sigma}} &:= \left\{ \boldsymbol{\tau}_h \in \mathbb{H}_0(\mathbf{div}, \Omega) : \quad \boldsymbol{c}^{\mathrm{t}} \boldsymbol{\tau}_h|_K \in \mathbf{RT}_k(K) \ \ \forall \boldsymbol{c} \in \mathrm{R}^n, \quad \forall K \in \mathcal{T}_h \right\}, \\
\mathbf{H}_h^{\boldsymbol{u}} &:= \{ \boldsymbol{v}_h \in \mathbf{C}(\Omega) \quad \boldsymbol{v}_h|_K \in \mathbf{P}_{k+1}(K) \qquad \forall K \in \mathcal{T}_h \}, \\
\mathbb{H}_h^{\boldsymbol{\rho}} &:= \left\{ \boldsymbol{\eta}_h \in \mathbb{L}^2_{\mathrm{skew}}(\Omega) \quad \boldsymbol{\eta}_h|_K \in \mathbb{P}_k(K) \qquad \forall K \in \mathcal{T}_h \right\}, \\
\mathrm{H}_h^{\phi} &:= \left\{ \psi_h \in \mathrm{C}(\Omega) \cap \mathrm{H}_0^1(\Omega) \quad \psi_h|_K \in \mathrm{P}_{k+1}(K) \qquad \forall K \in \mathcal{T}_h \right\}.
\end{aligned} \tag{4.12}$$

Then, a Galerkin scheme for (4.5) reads: find $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \boldsymbol{\rho}_h, \phi_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}} \times \mathrm{H}_h^{\phi}$ such that

$$\widetilde{B}((\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \boldsymbol{\rho}_h), (\boldsymbol{\tau}_h, \boldsymbol{v}_h, \boldsymbol{\eta}_h)) = \widetilde{F}_{\phi_h}(\boldsymbol{\tau}_h, \boldsymbol{v}_h, \boldsymbol{\eta}_h) \quad \forall (\boldsymbol{\tau}_h, \boldsymbol{v}_h, \boldsymbol{\eta}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}}, \tag{4.13}$$

$$A_{\boldsymbol{\sigma}_h}(\phi_h, \psi_h) = G_{\boldsymbol{u}_h}(\psi_h) \qquad \forall \psi_h \in \mathrm{H}_h^{\phi}. \tag{4.14}$$

We can now proceed analogously to Section 4.1 and define a fixed-point scheme for the analysis of the coupled problem (4.13)–(4.14). For this purpose, we define $\mathbf{S}_h : \mathrm{H}_h^{\phi} \to \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}}$ as

$$\mathbf{S}_h(\phi_h) := (\mathbf{S}_{1,h}(\phi_h), \mathbf{S}_{2,h}(\phi_h), \mathbf{S}_{3,h}(\phi_h)) := (\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \boldsymbol{\rho}_h) \quad \forall \phi_h \in \mathrm{H}_h^{\phi},$$

where the triple $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \boldsymbol{\rho}_h)$ is the unique solution of (4.13), with $\widetilde{B}$ and $\widetilde{F}_{\phi_h}$ defined by (4.3) and (4.4), respectively, with $\phi = \phi_h$. In turn, the operators $\widetilde{\mathbf{S}}_h$ and $\mathbf{T}_h$ are defined as in Section 3.2.

As the analysis of the operator $\widetilde{\mathbf{S}}_h$ follows verbatim from Section 3.2, we can omit the details here. Concerning $\mathbf{S}_h$, we start by investigating the well-posedness of (4.13).

Lemma 4.6. *Assume that $\kappa_1 \in (0, 4\delta\mu)$ and $\kappa_3 \in \left(0, 2c_2\kappa_1\widetilde{\delta}\left(1 - \frac{\delta}{2}\right)\right)$ with $\delta, \widetilde{\delta} \in (0, 2)$, and that $\kappa_2, \kappa_4 > 0$. Then, for each $\phi_h \in \mathrm{H}_h^{\phi}$ the problem (4.13) has a unique solution $\mathbf{S}(\phi_h) := (\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \boldsymbol{\rho}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}}$. Moreover, with the same constant $k_{\mathbf{S}} > 0$ provided by Lemma 4.2, there holds*

$$\|\mathbf{S}_h(\phi_h)\|_H = \left\|(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \boldsymbol{\rho}_h)\right\|_H \leq k_{\mathbf{S}} \left\{ \|\boldsymbol{u}_{\mathrm{D}}\|_{1/2,\Gamma} + f_2|\Omega|^{1/2} \right\} \quad \forall \phi_h \in \mathrm{H}_h^{\phi}.$$

Proof. It suffices to note that for each $\phi_h \in \mathrm{H}_h^{\phi}$, the multilinear form $\widetilde{B}$ is elliptic on $\mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}}$ with the same constant $\widetilde{\alpha}$ from Lemma 4.2 and that $\|\widetilde{F}_{\phi_h}\|_{\left(\mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}}\right)'}$ is bounded as in (4.8) with ϕ_h in place of ϕ. Hence, the result follows from a direct application of the Lax–Milgram Lemma. □

We now provide the discrete analogues of Lemmas 4.3, 4.4 and Theorem 4.5, whose proofs, which are almost verbatim of the corresponding continuous ones, are omitted.

Lemma 4.7. *Let $K_{\mathbf{S}}$ be the constant provided by Lemma 4.3. Then, there holds*

$$\|\mathbf{S}_h(\phi_h) - \mathbf{S}_h(\varphi_h)\|_H \leq K_{\mathbf{S}} \|\phi_h - \varphi_h\|_{0,\Omega} \qquad \forall \phi_h, \varphi_h \in \mathbb{H}_h^{\phi}.$$

Lemma 4.8. *Let W_h be as in Lemma 3.3. Then*

$$\|\mathbf{T}_h(\phi_h) - \mathbf{T}_h(\varphi_h)\|_{1,\Omega} \leq \frac{1}{\alpha_2} K_{\mathbf{S}} \left(L_g + L_\vartheta \|\nabla \mathbf{T}_h(\varphi_h)\|_{\infty,\Omega}\right) \|\phi_h - \varphi_h\|_{0,\Omega} \quad \forall \phi_h, \varphi_h \in \mathrm{H}_h^{\phi}.$$

Theorem 4.9. *Let W_h be as in Lemma 3.3. Then, the Galerkin scheme (4.13)–(4.14) has at least one solution $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \boldsymbol{\rho}_h, \phi_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}} \times \mathrm{H}_h^{\phi}$, and there holds*

$$\|\phi_h\|_{1,\Omega} \leq r \quad \text{and} \quad \left\|(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \boldsymbol{\rho}_h)\right\|_H \leq k_{\mathbf{S}} \left\{ \|\boldsymbol{u}_{\mathrm{D}}\|_{1/2,\Gamma} + f_2|\Omega|^{1/2} \right\}.$$

4.3. *A priori* error analysis

The goal of this section is to derive an estimate for $\left\|(\boldsymbol{\sigma},\boldsymbol{u},\boldsymbol{\rho})-(\boldsymbol{\sigma}_h,\boldsymbol{u}_h,\boldsymbol{\rho}_h)\right\|_H$, where $(\boldsymbol{\sigma},\boldsymbol{u},\boldsymbol{\rho})$ and $(\boldsymbol{\sigma}_h,\boldsymbol{u}_h,\boldsymbol{\rho}_h)$ are the solutions to the problems

$$\begin{aligned} \widetilde{B}((\boldsymbol{\sigma},\boldsymbol{u},\boldsymbol{\rho}),(\boldsymbol{\tau},\boldsymbol{v},\boldsymbol{\eta})) &= \widetilde{F}_\phi(\boldsymbol{\tau},\boldsymbol{v},\boldsymbol{\eta}) \qquad \forall\,(\boldsymbol{\tau},\boldsymbol{v},\boldsymbol{\eta})\in\mathbb{H}_0(\mathbf{div},\Omega)\times\mathbf{H}^1(\Omega)\times\mathbb{L}^2_{\mathrm{skew}}(\Omega),\\ \widetilde{B}((\boldsymbol{\sigma}_h,\boldsymbol{u}_h,\boldsymbol{\rho}_h),(\boldsymbol{\tau}_h,\boldsymbol{v}_h,\boldsymbol{\eta}_h)) &= \widetilde{F}_{\phi_h}(\boldsymbol{\tau}_h,\boldsymbol{v}_h,\boldsymbol{\eta}_h) \quad \forall\,(\boldsymbol{\tau}_h,\boldsymbol{v}_h,\boldsymbol{\eta}_h)\in\mathbb{H}_h^{\boldsymbol{\sigma}}\times\mathbf{H}_h^{\boldsymbol{u}}\times\mathbb{H}_h^{\boldsymbol{\rho}}, \end{aligned} \tag{4.15}$$

respectively. For this purpose, we recall (again from [32]) a Strang-type lemma, which will be applied to (4.15).

Lemma 4.10. *Let H be a Hilbert space, $F\in H'$ and $\boldsymbol{a}: H\times H\to \mathrm{R}$ be a bounded and elliptic bilinear form. In addition, let $\{H_h\}_{h>0}$ be a sequence of finite dimensional subspaces of H and for each $h>0$ consider a bounded bilinear form $\boldsymbol{a}_h: H_h\times H_h\to\mathrm{R}$ and a functional $F_h\in H_h'$. Assume that the family $\{\boldsymbol{a}_h\}_{h>0}$ is uniformly elliptic, that is, there exists a constant $\alpha>0$, independent of h, such that*

$$\boldsymbol{a}_h(v_h,v_h)\ge\alpha\,\|v_h\|_H^2 \quad \forall\,v_h\in H_h, \quad \forall\,h>0.$$

In turn, let $u\in H$ and $u_h\in H_h$ such that

$$\boldsymbol{a}(u,v)=F(v)\quad\forall\,v\in H \quad\text{and}\quad \boldsymbol{a}_h(u_h,v_h)=F_h(v_h)\quad\forall\,v_h\in H_h.$$

Then, for each $h>0$, there holds

$$\|u-u_h\|_H \le \widetilde{C}_{\mathrm{ST}}\left\{\sup_{\substack{w_h\in H_h\\ w_h\neq 0}}\frac{|F(w_h)-F_h(w_h)|}{\|w_h\|_H}+\inf_{\substack{v_h\in H_h\\ v_h\neq 0}}\left(\|u-v_h\|_V+\sup_{\substack{w_h\in H_h\\ w_h\neq 0}}\frac{|\boldsymbol{a}(v_h,w_h)-\boldsymbol{a}_h(v_h,w_h)|}{\|w_h\|_H}\right)\right\}.$$

where $\widetilde{C}_{\mathrm{ST}}:=\alpha^{-1}\max\{1,\|\boldsymbol{a}\|\}$.

Proof. See [32, Thm. 11.1]. □

As in Sections 3.4 and 3.5, we now set

$$\operatorname{dist}\left((\boldsymbol{\sigma},\boldsymbol{u},\boldsymbol{\rho}),\mathbb{H}_h^{\boldsymbol{\sigma}}\times\mathbf{H}_h^{\boldsymbol{u}}\times\mathbb{H}_h^{\boldsymbol{\rho}}\right) := \inf_{(\boldsymbol{\tau}_h,\boldsymbol{v}_h,\boldsymbol{\eta}_h)\in\mathbb{H}_h^{\boldsymbol{\sigma}}\times\mathbf{H}_h^{\boldsymbol{u}}\times\mathbb{H}_h^{\boldsymbol{\rho}}}\left\|(\boldsymbol{\sigma},\boldsymbol{u},\boldsymbol{\rho})-(\boldsymbol{\tau}_h,\boldsymbol{v}_h,\boldsymbol{\eta}_h)\right\|_H,$$

or, equivalently

$$\operatorname{dist}\left((\boldsymbol{\sigma},\boldsymbol{u},\boldsymbol{\rho}),\mathbb{H}_h^{\boldsymbol{\sigma}}\times\mathbf{H}_h^{\boldsymbol{u}}\times\mathbb{H}_h^{\boldsymbol{\rho}}\right) := \operatorname{dist}(\boldsymbol{\sigma},\mathbb{H}_h^{\boldsymbol{\sigma}})+\operatorname{dist}(\boldsymbol{u},\mathbf{H}_h^{\boldsymbol{u}})+\operatorname{dist}\left(\boldsymbol{\rho},\mathbb{H}_h^{\boldsymbol{\rho}}\right),$$

where, having in mind that now $\mathbf{H}_h^{\boldsymbol{u}}\subseteq\mathbf{H}^1(\Omega)$, we set $\operatorname{dist}(\boldsymbol{u},\mathbf{H}_h^{\boldsymbol{u}}):=\inf_{\boldsymbol{v}_h\in\mathbf{H}_h^{\boldsymbol{u}}}\|\boldsymbol{u}-\boldsymbol{v}_h\|_{1,\Omega}$. The other two distances are exactly as defined in Section 3.4.

Lemma 4.11. *Let $\widetilde{C}_{\mathrm{ST}}:=\widetilde{\alpha}^{-1}\max\{1,\|\widetilde{B}\|\}$, where $\widetilde{\alpha}$ is the constant yielding the ellipticity of $\widetilde{B}$ (cf. (4.6)). Then, there holds*

$$\begin{aligned}&\left\|(\boldsymbol{\sigma},\boldsymbol{u},\boldsymbol{\rho})-(\boldsymbol{\sigma}_h,\boldsymbol{u}_h,\boldsymbol{\rho}_h)\right\|_H\\ &\qquad\le\widetilde{C}_{\mathrm{ST}}\left\{\operatorname{dist}\left((\boldsymbol{\sigma},\boldsymbol{u},\boldsymbol{\rho}),\mathbb{H}_h^{\boldsymbol{\sigma}}\times\mathbf{H}_h^{\boldsymbol{u}}\times\mathbb{H}_h^{\boldsymbol{\rho}}\right)+L_f(1+\kappa_2^2)^{1/2}\,\|\phi-\phi_h\|_{0,\Omega}\right\}.\end{aligned} \tag{4.16}$$

Proof. We note that the bilinear form $\widetilde{B}$ and the functionals $\widetilde{F}_\phi$ and $\widetilde{F}_{\phi_h}$ satisfy the hypotheses of Lemma 4.10. Then, a straightforward application of Lemma 4.10 to the context (4.15) gives

$$\begin{aligned}&\left\|(\boldsymbol{\sigma},\boldsymbol{u},\boldsymbol{\rho})-(\boldsymbol{\sigma}_h,\boldsymbol{u}_h,\boldsymbol{\rho}_h)\right\|_H\\ &\qquad\le\widetilde{C}_{\mathrm{ST}}\left\{\|(\widetilde{F}_\phi-\widetilde{F}_{\phi_h})|_{\mathbb{H}_h^{\boldsymbol{\sigma}}\times\mathbf{H}_h^{\boldsymbol{u}}\times\mathbb{H}_h^{\boldsymbol{\rho}}}\|+\operatorname{dist}\left((\boldsymbol{\sigma},\boldsymbol{u},\boldsymbol{\rho}),\mathbb{H}_h^{\boldsymbol{\sigma}}\times\mathbf{H}_h^{\boldsymbol{u}}\times\mathbb{H}_h^{\boldsymbol{\rho}}\right)\right\}.\end{aligned} \tag{4.17}$$

Next, similarly as in the proof of Lemma 4.3, we deduce that

$$\|(\widetilde{F}_\phi-\widetilde{F}_{\phi_h})|_{\mathbb{H}_h^{\boldsymbol{\sigma}}\times\mathbf{H}_h^{\boldsymbol{u}}\times\mathbb{H}_h^{\boldsymbol{\rho}}}\|\le L_f(1+\kappa_2^2)^{1/2}\,\|\phi-\phi_h\|_{0,\Omega}. \tag{4.18}$$

Finally, by replacing (4.18) back into (4.17), we get (4.16) and the lemma follows. □

At this point, we realise that in the present context the estimate for $\|\phi - \phi_h\|_{1,\Omega}$ stays exactly as in (3.17). Consequently, the corresponding Céa estimate for the total error

$$\|\phi - \phi_h\|_{1,\Omega} + \left\|(\boldsymbol{\sigma}, (\boldsymbol{u}, \boldsymbol{\rho})) - (\boldsymbol{\sigma}_h, (\boldsymbol{u}_h, \boldsymbol{\rho}_h))\right\|_H$$

is derived by combining (3.17) and (4.16). By virtue of the aforementioned, we can establish the analogues of Theorems 3.12 and 3.13, whose proofs are omitted.

Theorem 4.12. *Let C_1 and C_2 be the constants defined in (3.23), and assume that the data satisfy*

$$L_f(1+\kappa_2^2)^{1/2}\left\{C_1 + C_2\, k_{\mathbf{S}}\left(\|\boldsymbol{u}_{\rm D}\|_{1/2,\Gamma} + f_2|\Omega|^{1/2}\right)\right\} < \frac{1}{2}.$$

Then, there exist positive constants C_6 and C_7, independent of h, such that

$$\|\phi - \phi_h\|_{1,\Omega} + \left\|(\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{\rho}) - (\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \boldsymbol{\rho}_h)\right\|_H \le C_6 \,\mathrm{dist}\,(\phi, \mathrm{H}_h^{\phi}) + C_7\,\mathrm{dist}\left((\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{\rho}), \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}}\right).$$

Theorem 4.13. *In addition to the hypotheses of Theorems 4.5, 4.9 and 4.12, assume that there exists $s > 0$ such that $\boldsymbol{\sigma} \in \mathbb{H}^s(\Omega)$, $\mathbf{div}\,(\boldsymbol{\sigma}) \in \mathbf{H}^s(\Omega)$, $\boldsymbol{u} \in \mathbf{H}^{1+s}(\Omega)$, $\boldsymbol{\rho} \in \mathbb{H}^s(\Omega)$ and $\phi \in \mathrm{H}^{1+s}(\Omega)$. Then, there exists $\widehat{C} > 0$, independent of h, such that, with the finite element subspaces defined by (4.12), there holds*

$$\|\phi - \phi_h\|_{1,\Omega} + \left\|(\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{\rho}) - (\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \boldsymbol{\rho}_h)\right\|_H \le \widehat{C} h^{\min\{s,k+1\}}\left\{\|\boldsymbol{\sigma}\|_{s,\Omega} + \|\mathbf{div}\,\boldsymbol{\sigma}\|_{s,\Omega} + \|\boldsymbol{u}\|_{1+s,\Omega} + \|\boldsymbol{\rho}\|_{s,\Omega} + \|\phi\|_{1+s,\Omega}\right\}. \quad (4.19)$$

5. Numerical results

In this section we provide a set of computational tests. The first one serves to illustrate the convergence rates anticipated by our previous analysis for the mixed-primal and the augmented Galerkin schemes, whereas the remaining examples address a few cases not covered by our analysis (mixed boundary conditions, non-convex domains, and the 3D case).

Example 1: Error history for a constructed solution in 2D. We consider (1.7) in the unit square $\Omega = (0,1)^2$ and propose exact solutions and coupling terms (tensorial diffusivity, body load, and diffusive source) as follows

$$\boldsymbol{u} = \begin{pmatrix} d_1 \sin(\pi x_1)\cos(\pi x_2) + \dfrac{x_1^2}{2\lambda} \\ -d_1\cos(\pi x_1)\sin(\pi x_2) + \dfrac{x_2^2}{2\lambda}\end{pmatrix}, \quad \boldsymbol{\sigma} = \lambda\,\mathrm{tr}\boldsymbol{\varepsilon}(\boldsymbol{u})\mathbb{I} + 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}), \quad \boldsymbol{\rho} = \nabla u - \boldsymbol{\varepsilon}(\boldsymbol{u}),$$
$$\phi = x_1(1-x_2)x_2(1-x_2),\ \vartheta(\boldsymbol{\sigma}) = D_0\mathbb{I} + D_2\boldsymbol{\sigma}^2,\ \boldsymbol{f}(\phi) = d_2\begin{pmatrix}\cos^2(\phi)\\ -\sin(\phi)\end{pmatrix},\ g(\boldsymbol{u}) = d_2\left(1 + \frac{1}{1+|\boldsymbol{u}|}\right). \quad (5.1)$$

These closed-form solutions satisfy the boundary conditions $\boldsymbol{u}_{\rm D} = \boldsymbol{u}$ on Γ and $\phi = 0$ on Γ. Moreover, the elasticity and diffusion equations are considered non-homogeneous and the extra source terms are chosen according to (5.1). This treatment does not compromise the continuous and discrete analyses, as the smoothness of the exact solution provides right-hand sides with terms in $L^2(\Omega)$, thus only requiring a slight modification of the functionals in the variational formulation. We note that the forcing and source terms satisfy (1.4)–(1.5). Additionally, we pick out the following value to the model parameters: displacement and forcing term scalings $d_1 = 0.05$, $d_2 = 0.1$; Young's modulus $E = 1e3$; Poisson's ratio $\nu = 0.4$; the constants specifying ϑ given by $D_0 = 1.0$ and $D_2 = 0.1$, and the Lamé constants $\lambda = E\nu(1+\nu)^{-1}(1-2\nu)^{-1}$ and $\mu = E/(2+2\nu)$. We consider a heuristic value for Korn's constant (*cf.* Lemma 4.1) as $c_2 = 0.1$; and using the proof of Lemma 4.2, the stabilisation parameters assume the values $\delta = \widetilde{\delta} = 1$, $\kappa_1 = 2\mu$, $\kappa_2 = 0.5\mu$, $\kappa_3 = 0.1\mu$, and $\kappa_4 = \mu$. We generate a sequence of uniformly refined meshes and proceed to define errors and convergence rates as usual:

$$\mathrm{e}(\boldsymbol{\sigma}) = \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div},\Omega},\ \mathrm{e}(\boldsymbol{u}) = \|\boldsymbol{u} - \boldsymbol{u}_h\|_{j,\Omega},\ \mathrm{e}(\boldsymbol{\rho}) = \left\|\boldsymbol{\rho} - \boldsymbol{\rho}_h\right\|_{0,\Omega},\ \mathrm{e}(\phi) = \|\phi - \phi_h\|_{1,\Omega},$$
$$r(\cdot) = \frac{\log(\mathrm{e}(\cdot)/\widehat{\mathrm{e}}(\cdot))}{\log(h/\widehat{h})},$$

Table 1
Example 1: Degrees of freedom, meshsizes, errors, rates of convergence, and number of Picard iterations for the mixed-primal PEERS-P_1 and augmented $\mathbf{RT}_k - \mathbf{P}_{k+1} - \mathbb{P}_k - \mathrm{P}_{k+1}$ approximations of the coupled problem with $k = 0, 1$, and using $\nu = 0.4$ and $\kappa_2 = 0.5\mu, \kappa_4 = \mu$. In the first block of the table, the displacement error is measured in the $\mathbf{L}^2$-norm.

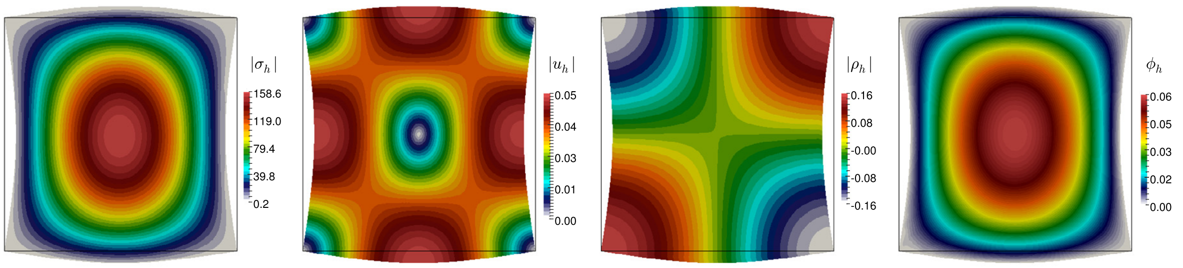
N	h	$\mathrm{e}(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$\mathrm{e}(\boldsymbol{u})$	$r(\boldsymbol{u})$	$\mathrm{e}(\boldsymbol{\rho})$	$r(\boldsymbol{\rho})$	$\mathrm{e}(\phi)$	$r(\phi)$	iter
Mixed-primal PEERS-Lagrange scheme with $k = 0$										
129	0.7071	124.43	–	1.72e−2	–	5.49e−2	–	0.1125	–	4
457	0.3536	65.778	0.91	9.11e−3	0.92	2.87e−2	0.94	6.72e−2	0.74	5
1713	0.1768	33.305	0.98	4.61e−3	0.98	1.45e−2	0.98	3.81e−2	0.82	6
6625	0.0883	16.703	0.99	2.32e−3	0.99	7.26e−3	0.99	1.87e−2	1.02	6
26049	0.0441	8.3584	0.99	1.15e−3	0.99	3.63e−3	0.99	8.35e−3	1.16	6
103297	0.0221	4.1802	0.99	5.78e−4	0.99	1.81e−3	0.99	3.91e−3	1.09	6
Augmented scheme with $k = 0$										
67	0.7071	132.53	–	0.1043	–	0.1120	–	0.1105	–	5
219	0.3536	70.733	0.91	0.0643	0.69	0.1036	0.61	0.0708	0.64	5
787	0.1768	35.492	0.99	0.0323	0.99	0.0789	0.93	0.0427	0.82	6
2979	0.0883	17.604	1.01	0.0157	1.04	0.0463	0.97	0.0230	0.99	6
11587	0.0441	8.7683	1.00	0.0077	1.01	0.0242	0.93	0.0108	1.08	6
45699	0.0221	4.3792	1.00	3.86e−3	1.00	0.0129	0.98	4.62e−3	1.23	6
Augmented scheme with $k = 1$										
195	0.7071	38.856	–	0.0309	–	0.0169	–	0.0358	–	6
691	0.3536	10.373	1.90	0.0088	1.81	0.0074	1.49	0.0100	1.83	6
2595	0.1768	2.6473	1.97	0.0023	1.93	0.0029	1.53	0.0024	2.01	6
10051	0.0883	0.6637	1.99	0.0005	1.97	0.0009	1.67	0.0006	2.03	6
39555	0.0441	0.1658	2.00	0.0001	1.99	0.0002	1.86	0.0001	2.02	8
156931	0.0221	0.0414	2.00	3.72e−5	1.99	6.65e−5	1.94	3.68e−5	2.03	6

where e and $\widehat{\mathrm{e}}$ denote errors computed on two consecutive meshes of sizes h and $\widehat{h}$; and where $j = 0, 1$ will be used to measure the displacement error for the mixed-primal and augmented mixed-primal schemes, respectively.

On each refinement level we generate approximate solutions with the lowest-order PEERS-Lagrange elements indicated in Section 3.4, and also with the $\mathbf{RT}_k - \mathbf{P}_{k+1} - \mathbb{P}_k - \mathrm{P}_{k+1}$ scheme specified in Section 4.2, for $k = 0, 1$. The output of this error study is collected in Table 1 (where we tabulate errors, experimental convergence rates, and iteration count). We observe an asymptotic $O(h^{k+1})$ convergence for all individual errors (stress, displacement, rotation, and concentration), which agrees with the theoretical error bounds derived in Section 3.5 (*cf.* (3.27)) and Section 4.3 (*cf.* (4.19)). Around six Picard iterations are necessary to reach the prescribed tolerance Tol=1e−6 imposed on the ℓ^∞-norm of the total residual. At each fixed-point step the resulting linear systems were solved with the direct method SuperLU. For completeness, we also depict in Fig. 5.1 the obtained numerical solutions computed with the lowest-order augmented method. We also mention that the proposed methods maintain their accuracy in the incompressibility limit. This is confirmed by replicating the same experimental analysis, now considering $\nu = 0.49999$. The error history for this case is displayed in Table 2, where we observe that the magnitude of errors and convergence rates are comparable to those in Table 1. However, if the stabilisation parameters are kept as in the first case, then the number of Picard iterations needed to achieve the prescribed tolerance for the augmented schemes is considerably higher. Similar iteration counts as those in the non-augmented case can be obtained with much smaller values of κ_2 and κ_4: here we choose $\kappa_2 = \kappa_4 = 0.001\mu$.

Example 2: Convergence in a non-convex domain. The goal of this example is to observe the behaviour of the numerical method producing solutions on a non-convex domain (we recall that convexity was required in the analysis of the fixed-point operators defining the coupled continuous problem). To this end we consider a ring-shaped membrane bounded by an outer circle of radius 1 and an inner circle of radius 0.5. Initial guesses for stress, displacement, and concentration are zero. Differently from Example 1, we now apply the following tensorial diffusivity, body load, source of species, and prescribed boundary displacement on the outer ring

$$\vartheta(\boldsymbol{\sigma}) = D_0\,\mathbb{I} + D_1\boldsymbol{\sigma} + D_2\boldsymbol{\sigma}^2,\ \boldsymbol{f}(\phi) = d_2\begin{pmatrix}\phi\\ \phi(1-\phi)\end{pmatrix},\ g(\boldsymbol{u}) = d_3|\boldsymbol{u}|,\ \boldsymbol{u}_{\mathrm{D}} = \begin{pmatrix}d_1\sin(\pi x_1)\cos(\pi x_2)\\ -d_1\cos(\pi x_1)\sin(\pi x_2)\end{pmatrix},$$



(a) (b) (c) (d)

Fig. 5.1. Example 1: $\mathbf{RT}_0 - \mathbf{P}_1 - \mathbb{P}_0 - \mathrm{P}_1$ approximation of stress magnitude $|\boldsymbol{\sigma}_h|$ (a), displacement magnitude $|\boldsymbol{u}_h|$ (b), relevant component of the rotation tensor $\boldsymbol{\rho}_h$ (c), and concentration of the diffusive substance ϕ_h (d); using $\nu = 0.4$. All fields are plotted on the deformed domain.

Table 2
Example 1: Error history produced using a higher Poisson ratio $\nu = 0.49999$ and setting $\kappa_2 = \kappa_4 = 0.001\mu$. In the first block of the table, the displacement error is measured in the $\mathbf{L}^2$-norm.

N	h	$\mathrm{e}(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$\mathrm{e}(\boldsymbol{u})$	$r(\boldsymbol{u})$	$\mathrm{e}(\boldsymbol{\rho})$	$r(\boldsymbol{\rho})$	$\mathrm{e}(\phi)$	$r(\phi)$	iter
Mixed-primal PEERS-Lagrange scheme with $k = 0$										
129	0.7071	9189.7	–	6.05e−2	–	0.1477	–	1.9940	–	6
457	0.3536	605.93	2.98	9.14e−3	2.72	2.88e−2	2.35	9.59e−2	4.37	6
1713	0.1768	30.604	4.30	4.61e−3	0.98	1.45e−2	0.99	3.67e−2	1.30	6
6625	0.0883	15.390	0.99	2.31e−3	0.99	7.26e−3	0.99	1.89e−2	0.96	6
26049	0.0441	7.7948	0.98	1.15e−3	0.99	3.63e−3	0.99	8.41e−3	1.17	6
103297	0.0221	3.9011	0.99	5.78e−4	0.99	1.81e−3	0.99	3.91e−3	1.10	6
Augmented scheme with $k = 0$										
67	0.7071	5525.5	–	1.6922	–	7.7691	–	0.1523	–	4
219	0.3536	853.17	5.02	0.1672	3.41	0.9461	4.14	8.05e−2	0.72	5
787	0.1768	33.563	4.62	7.50e−2	1.29	0.3937	1.25	3.75e−2	1.04	6
2979	0.0883	16.784	0.99	3.39e−2	1.04	0.1467	1.16	1.97e−2	0.92	6
11587	0.0441	8.2505	1.02	1.95e−2	0.93	7.43e−2	0.94	1.03e−2	0.94	6
45699	0.0221	4.0961	1.01	9.73e−3	0.99	3.73e−2	0.98	4.54e−3	1.18	6
Augmented scheme with $k = 1$										
195	0.7071	172.52	–	1.2010	–	1.4012	–	7.34e−2	–	10
691	0.3536	9.4288	3.94	2.33e−2	5.68	2.28e−2	5.93	1.84e−2	1.44	6
2595	0.1768	1.8711	2.59	2.36e−3	3.30	2.86e−3	2.99	4.19e−3	2.04	6
10051	0.0883	0.8375	2.14	5.90e−4	1.99	9.05e−4	1.69	7.26e−4	1.90	6
39555	0.0441	0.1559	2.24	1.48e−4	1.99	2.52e−4	1.84	1.49e−4	2.12	6
156931	0.0221	3.91e−2	1.99	3.72e−5	1.99	6.65e−5	1.92	3.79e−5	1.97	6

whereas on the inner ring the structure is clamped. We impose a concentration of 1 on the outer ring and zero on the inner boundary. The coefficients defining the problem assume the values $D_0 = d_1 = 0.1$, $D_1 = D_2 = 0.05$, $d_2 = 0.025$, $d_3 = -1$, $E = 100$ and $\nu = 0.33$, and the numerical solutions generated with the lowest-order PEERS-Lagrange scheme are presented in Fig. 5.2(a).

In view of assessing the convergence of the lowest-order primal-mixed method, and in the absence of a closed-form expression for the solution of this problem, we consider a reference solution computed in a highly refined mesh (of around 50 K elements) and proceed to compute approximate solutions on coarser meshes. The obtained errors (with respect to the reference solutions projected to each coarse mesh) and convergence rates are shown in Fig. 5.2(b), where one sees that all fields exhibit an $O(h)$ accuracy, and note that the stress error is dominant. For all refinement levels the fixed-point algorithm took less than five iterations to converge.

We exploit the same setting to study the influence of different values for the additional diffusion parameters $D_1 = D_2$ (representing scenarios where the stress-assisted diffusion decreases in intensity). Fig. 5.3 compares three different cases, where a substantial difference is observed in the generated diffusion patterns. A similar effect as the

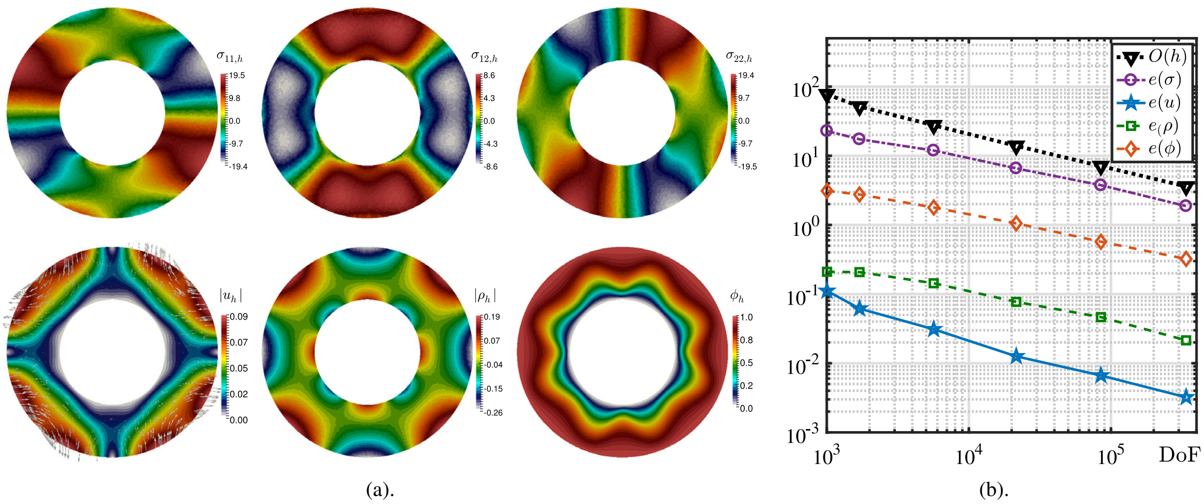
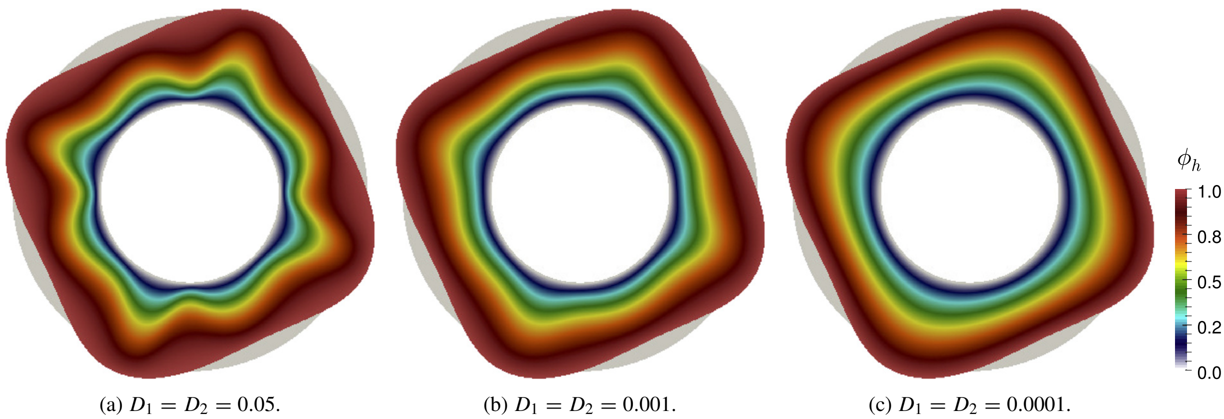


Fig. 5.2. Example 2: Approximate solutions (stress components, displacement magnitude with directions, rotation, and concentration) using a lowest order PEERS-Lagrange scheme displayed on the undeformed domain (a); and individual errors computed with respect to a reference solution (b).



(a) $D_1 = D_2 = 0.05$. (b) $D_1 = D_2 = 0.001$. (c) $D_1 = D_2 = 0.0001$.

Fig. 5.3. Example 2: Concentration profiles of the diffusive substance ϕ_h plotted on the deformed domain, for different values of the additional diffusivity constants.

one produced with very low values of D_1 and D_2 (the profiles in Fig. 5.3(c) show a very smooth diffusion going uniformly from $\phi = 1$ on the outer circle, to $\phi = 0$ on the inner boundary) can be achieved by softening the material, prescribing a Young modulus of $E = 1$.

Example 3: Stress-assisted diffusion and experimental convergence on a 3D slab. In much the same way as in Example 2, here we will confirm that the other assumption in Theorem 2.4 (the restriction to two spatial dimensions) can be obviated at the implementation stage, and that it does not compromise the behaviour of the proposed methods. Focusing on an applicative test, let us regard a porous block occupying the domain $\Omega = (0, 250) \times (0, 250) \times (0, 50)$ and construct an unstructured tetrahedral mesh of 55K elements. The stress-dependent diffusivity is considered as in Example 2: $\vartheta(\boldsymbol{\sigma}) = D_0\,\mathbb{I} + D_1\boldsymbol{\sigma} + D_2\boldsymbol{\sigma}^2$, the concentration-dependent body load is $\boldsymbol{f}(\phi) = d_2(\phi, \phi, \phi(1-\phi))^{\rm t}$, and the displacement-dependent source is now $g(\boldsymbol{u}) = d_3 \operatorname{div} \boldsymbol{u}$. We will take the parameter values $D_0 = 0.5$, $D_1 = 0.025$, $D_2 = -0.015$, $d_2 = 0.1$, $d_3 = 0.25$, $E = 1e4$, and $\nu = 0.49$. Boundary conditions for the elasticity problem differ from the ones analysed in the paper: The block is clamped on the surface $x_1 = 0$, a normal traction force is imposed on the surface $x_1 = 250$, $\boldsymbol{\sigma}\boldsymbol{\nu} = 3/4\mu\boldsymbol{\nu}$, and zero normal stresses are considered elsewhere on the

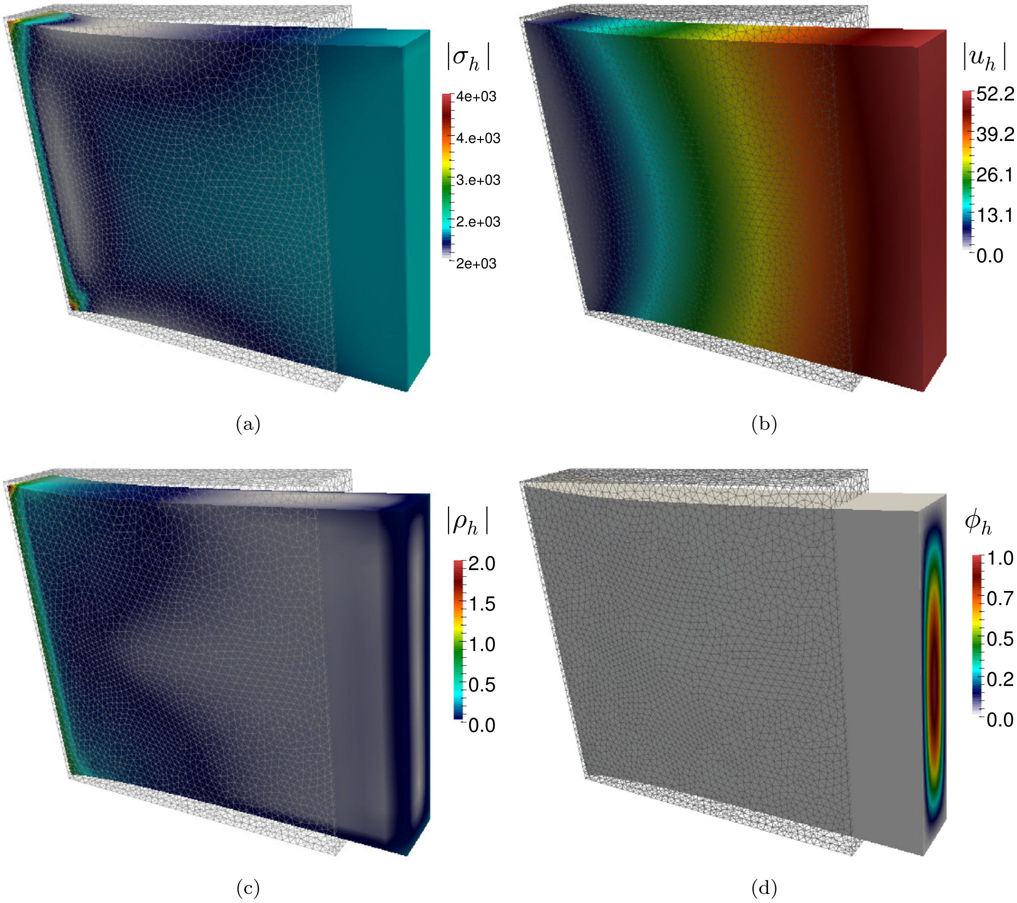


Fig. 5.4. Example 3: Augmented mixed-primal approximation of stress magnitude $|\boldsymbol{\sigma}_h|$ (a), displacement magnitude $|\boldsymbol{u}_h|$ (b), rotation tensor magnitude $|\boldsymbol{\rho}_h|$ (c), and concentration of the diffusive substance ϕ_h (d); all plotted on the deformed domain and showing the undeformed, skeleton mesh.

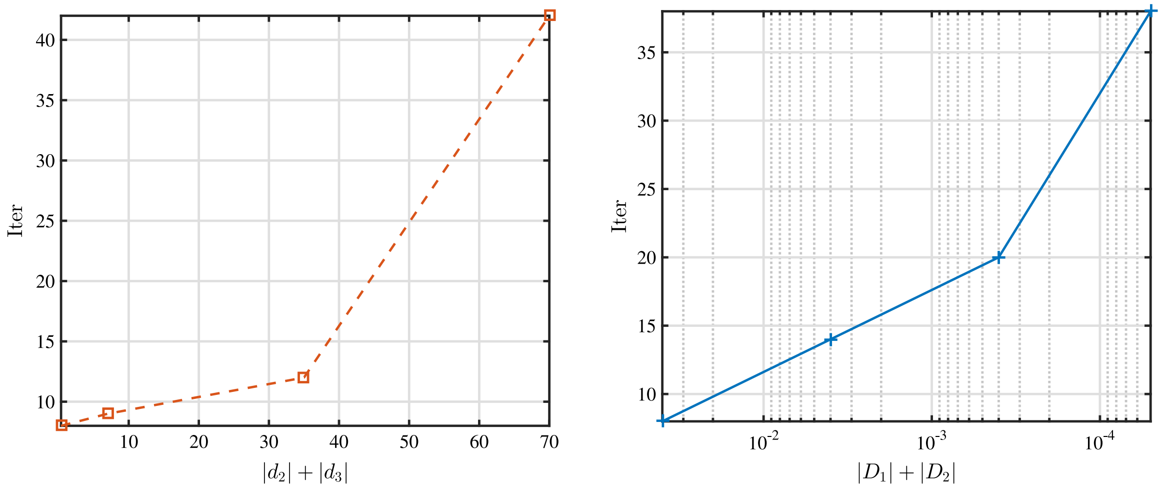
boundary, $\boldsymbol{\sigma}\boldsymbol{\nu} = \mathbf{0}$. On the surface $x_1 = 0$ we fix the concentration $\phi = x_2(250 - x_2)x_3(50 - x_3)/(25 \cdot 125)^2$, we impose zero-flux boundary conditions on the face $x_1 = 250$, $\tilde{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu} = 0$; and consider an homogeneous Dirichlet boundary condition for concentration on the remainder of $\partial\Omega$. Once again we consider the augmented mixed-primal method of lowest order, for which the penalisation constants adopt the values $\kappa_1 = 2\mu$, $\kappa_2 = 0.5\mu$, $\kappa_3 = 0.01\mu$, and $\kappa_4 = 1$. The linear systems encountered at each Picard step are solved with the GMRES method preconditioned with an incomplete LU factorisation. The computational results are summarised in Fig. 5.4, indicating that stresses are concentrated on the corners of the boundaries where Dirichlet conditions are set for displacements, and rotations are higher in the vicinities of the rectangles at $x_1 = 0$ and $x_1 = 250$. For this case the Picard method takes eight iterations to converge.

We also assess the accuracy of the method through an experimental error analysis. Since, for this particular problem configuration, a closed form solution to (1.1) is not available, we produce an approximate solution using a highly refined mesh (of 290K elements) and consider it as a reference solution for error computation. We also generate a sequence of much coarser quasi-uniformly refined meshes (but not necessarily nested) on which we compute approximate solutions. The result of this error analysis is collected in Table 3. The observed convergence rates, here only presented for the lowest-order augmented scheme, approach the optimal values as the number of degrees of freedom increases. In addition, the fixed-point iteration count remains near the base case (of eight steps) for all levels of mesh refinement.

Next we investigate the effect of the stress–diffusion coupling (which is actually encoded in the magnitude of the parameters D_1, D_2 and d_2, d_3) on the performance of the fixed-point iteration count. We conduct six rounds of

Table 3
Example 3: Experimental error history against a reference (fine mesh) solution, and number of Picard iterations per refinement level. Lowest-order augmented method.

N	e($\boldsymbol{\sigma}$)	$r(\boldsymbol{\sigma})$	e($\boldsymbol{u}$)	$r(\boldsymbol{u})$	e($\boldsymbol{\rho}$)	$r(\boldsymbol{\rho})$	e(ϕ)	$r(\phi)$	iter
487	1968.2	–	20.384	–	1.0379	–	0.2571	–	7
2837	639.81	0.64	6.9320	0.76	0.2825	0.93	0.0842	0.84	8
22156	204.12	0.79	2.1943	0.80	0.0873	0.92	0.0293	0.83	7
150109	70.451	0.95	0.7904	0.92	0.0315	0.79	0.0096	0.90	8
907803	25.298	0.94	0.2246	0.93	0.0102	0.89	0.0034	0.92	8



(a) With $D_1 = 0.025$, $D_2 = -0.015$.

(b) With $d_2 = 0.1$, $d_3 = 0.25$.

Fig. 5.5. Example 3: Iteration count produced when varying the coupling parameters defining the concentration-dependent body load and displacement-dependent source (a), and the stress-assisted diffusivity parameters (b).

simulations, first fixing the tensorial diffusivity constants D_1, D_2 and increasing d_2, d_3; and then fixing d_2, d_3 and decreasing D_1, D_2 (large contributions from stresses will only increase diffusion, therefore making the generalised Poisson problem more stable). Fig. 5.5 presents the response of the method in terms of number of fixed-point iterations needed to reach the tolerance Tol=1e−6. We observe that as the coupling terms depart from the base case, the solver performs a larger number of steps.

6. Concluding remarks

Stress-enhanced or stress-assisted diffusion effects constitute the main mechanism in many applicative problems. Here we have focused on a coupled system consisting of the three-field equations of linear elastostatics imposing weakly the symmetry of the Cauchy stress, and a generalised diffusion problem where the diffusion tensor depends nonlinearly on the stress. We have analysed the mathematical properties of this system (existence, uniqueness, and regularity of weak solutions) by means of fixed-point theory and the classical theory for elliptic and saddle-point PDEs. We have also introduced two main families of finite element schemes for the discretisation of the model problem: one that adopts the mixed-primal character of the set of governing equations, and another one based on augmentation and penalisation. The properties of the resulting discrete problems were also established, and we have rigorously proved convergence estimates under suitable assumptions. Finally, we have presented some 2D and 3D tests that exemplify the accuracy of the methods under different regimes.

A number of generalisations are envisaged at this point. First, the physical context of Example 3 was motivated by the study of stress-assisted diffusion in actively deforming hyperelastic media (see e.g. [10,34]). The analysis of this class of problems constitutes one of the forthcoming extensions of the present work, where the regime of nonlinear elasticity and the difficulties associated to nearly incompressible and incompressible material poses a great challenge. Secondly, it is left to investigate other constitutive relations for the tensor diffusivity ϑ, possibly depending also on the concentration and entailing the study of non-monotone operators and embedded fixed-point schemes [35]. The regularity assumptions and the structure of the mixed-primal formulations will also need to be rewritten once we incorporate concentration gradient modulations of the body loads $\boldsymbol{f}(\phi) = O(\nabla\phi)$, as in [24]. We also plan to derive suitable *a posteriori* error estimates, and state a multiscale counterpart of (1.1) together with suitable finite element schemes of special interest in the modelling of lithium batteries. Finally we mention that extensions dealing with different treatment of boundary conditions, mixed–mixed formulations, and time-dependent generalisations are currently under development.

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References

[1] C. Truesdell, Mechanical basis of diffusion, J. Chem. Phys. 37 (1962) 2336–2344.
[2] Y. Podstrigach, V. Pavlina, Differential equations of thermodynamic processes in $n-$component solid solutions, Sov. Mater. Sci. 1 (4) (1966) 259–264.
[3] E.C. Aifantis, On the problem of diffusion in solids, Acta Mech. 37 (1980) 265–296.
[4] J. Toribio, V. Kharin, A hydrogen diffusion model for applications in fusion nuclear technology, Fusion Eng. Des. 51–52 (2000) 213–218.
[5] Y. An, H. Jiang, A finite element simulation on transient large deformation and mass diffusion in electrodes for lithium ion batteries, Modelling Simul. Mater. Sci. Eng. 21 (7) (2013) 074007.
[6] S. Roy, K. Vengadassalam, Y. Wang, S. Park, K.M. Liechti, Characterization and modeling of strain assisted diffusion in an epoxy adhesive layer, Int. J. Solids Struct. 43 (2006) 27–52.
[7] B.W. Van de Fliert, R. Van der Hout, Stress-driven diffusion in a drying liquid paint layer, Eur. J. Appl. Math. 9 (1998) 447–461.
[8] W. Hong, X. Zhao, J. Zhou, Z. Suo, A theory of coupled diffusion and large deformation in polymeric gels, J. Mech. Phys. Solids 56 (5) (2008) 1779–1793.
[9] F. Xuan, S.S. Shao, Z. Wang, S.T. Tu, Coupling effects of chemical stresses and external mechanical stresses on diffusion, J. Phys. D: Appl. Phys. 42 (1) (2009) 015401.
[10] C. Cherubini, S. Filippi, A. Gizzi, R. Ruiz-Baier, A note on stress-driven anisotropic diffusion and its role in active deformable media, J. Theoret. Biol. 430 (7) (2017) 221–228.
[11] J. Hill, Plane steady solutions for stress-assisted diffusion, Mech. Res. Commun. 6 (3) (1979) 147–150.
[12] R.W. Cox, Stress-assisted diffusion: A free boundary problem, SIAM J. Appl. Math. 51 (6) (1991) 1522–1537.
[13] M. Lewicka, P.B. Mucha, A local and global well-posedness results for the general stress-assisted diffusion systems, J. Elasticity 123 (1) (2016) 19–41.
[14] D.N. Arnold, F. Brezzi, J. Douglas, PEERS: A new mixed finite element method for plane elasticity, Jpn. J. Appl. Math. 1 (1984) 347–367.
[15] G.N. Gatica, An augmented mixed finite element method for linear elasticity with non-homogeneous Dirichlet conditions, Electron. Trans. Numer. Anal. 26 (2007) 421–438.
[16] G.N. Gatica, A. Marquez, S. Meddahi, An augmented mixed finite element method for 3D linear elasticity problems, J. Comput. Appl. Math. 231 (2) (2009) 526–540.
[17] M. Álvarez, G.N. Gatica, R. Ruiz-Baier, An augmented mixed-primal finite element method for a coupled flow-transport problem, ESAIM Math. Model. Numer. Anal. 49 (5) (2015) 1399–1427.
[18] M. Álvarez, G.N. Gatica, R. Ruiz-Baier, A mixed-primal finite element approximation of a sedimentation-consolidation system, Math. Models Methods Appl. Sci. 26 (5) (2016) 867–900.
[19] C. Bacuta, J.H. Bramble, Regularity estimates for solutions of the equations of linear elasticity in convex plane polygonal domains, Z. Angew. Math. Phys. 54 (5) (2003) 874–878.
[20] A. Ern, J.-L. Guermond, Theory and Practice of Finite Elements. Applied Mathematical Sciences, Springer-Verlag, 2004.
[21] G.N. Gatica, A Simple Introduction to the Mixed Finite Element Method: Theory and Applications. Springer Briefs in Mathematics, Springer, Cham, 2014.
[22] C. Gritton, J. Guilkey, J. Hopper, D. Bedrov, R.M. Kirby, M. Berzins, Using the material point method to model chemical/mechanical coupling in the deformation of a silicon anode, Modelling Simul. Mater. Sci. Eng. 25 (4) (2017) 1–22.
[23] D. Klepach, T.I. Zohdi, Strain assisted diffusion: Modeling and simulation of deformation-dependent diffusion in composite media, Composites B 56 (2014) 413–423.
[24] R. Ruiz Baier, Primal-mixed formulations for reaction–diffusion systems on deforming domains, J. Comput. Phys. 299 (2015) 320–338.
[25] F. Brezzi, M. Fortin, Mixed and Hybrid Finite Element Methods, Springer-Verlag, 1991.
[26] P. Grisvard, Smoothness of the solution of a monotonic boundary value problem for a second order elliptic equation in a general convex domain, in: Lecture Notes in Math, vol. 564, Springer-Verlag, 1976, pp. 135–151.

[27] P. Grisvard, Elliptic Problems in Nonsmooth Domains. Monographs and Studies in Mathematics, Vol. 24, Pitman (Advanced Publishing Program, Boston, MA, 1985.

[28] J. Nečas, Les Méthodes Directes en Théorie des Équations Elliptiques. Masson et Cie, Éditeurs, Academia, Éditeurs, Prague, Paris, 1967.

[29] R.A. Adams, J.J.F. Fournier, Sobolev Spaces, Academic Press, Elsevier Ltd, 2003.

[30] S. Larsson, V. Thomée, Partial Differential Equations with Numerical Methods, Springer-Verlag, Berlin, 2003.

[31] P. Ciarlet, Linear and Nonlinear Functional Analysis with Applications, SIAM Philadelphia, 2013.

[32] J.E. Roberts, J.M. Thomas, Mixed and hybrid methods, in: P.G. Ciarlet, J.L. Lions (Eds.), Hand-Book of Numerical Analysis, Vol. II, in: Finite Element Methods (Part 1), North-Holland, Amsterdam, 1991.

[33] G.N. Gatica, Analysis of a new augmented mixed finite element method for linear elasticity allowing $RT_0 - P_1 - P_0$ approximations, M2AN Math. Model. Numer. Anal. 40 (1) (2006) 1–28.

[34] A. Loppini, A. Gizzi, R. Ruiz-Baier, C. Cherubini, F.H. Fenton, S. Filippi, Modeling cardiac electromechanics with stress-assisted diffusivity and stretch-activated channels, Frontiers Physiol. (2018) in press.

[35] T. Roubiček, Nonlinear Partial Differential Equations with Applications, in: Int. Ser. Numer. Math., vol. 153, Birkhäuser, Basel, 2005.