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Primal-mixed finite element methods for the coupled Biot and Poisson–Nernst–Planck equations ☆

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ABSTRACT

We propose mixed finite element methods for the coupled Biot poroelasticity and Poisson–Nernst–Planck equations (modeling ion transport in deformable porous media). For the poroelasticity, we consider a primal-mixed, four-field formulation in terms of the solid displacement, the fluid pressure, the Darcy flux, and the total pressure. In turn, the Poisson–Nernst–Planck equations are formulated in terms of the electrostatic potential, the electric field, the ionized particle concentrations, their gradients, and the total ionic fluxes. The weak formulation, posed in Banach spaces, exhibits the structure of a perturbed block-diagonal operator consisting of perturbed and generalized saddle-point problems for the Biot equations, a generalized saddle-point system for the Poisson equations, and a perturbed twofold saddle-point problem for the Nernst–Planck equations. One of the main novelties here is the well-posedness analysis, hinging on the Banach fixed-point theorem along with small data assumptions, the Babuška–Brezzi theory in Banach spaces, and a slight variant of recent abstract results for perturbed saddle-point problems, again in Banach spaces. The associated Galerkin scheme is addressed similarly, employing the Banach fixed-point theorem to yield discrete well-posedness. A priori error estimates are derived, and simple numerical examples validate the theoretical error bounds, and illustrate the performance of the proposed schemes.

1. Introduction

Scope. We study a mathematical model for the transport of electrolytes through an electrically charged fully saturated and deformable porous medium. The electro-hydrostatics are described by the Nernst–Planck relations (mass balance for the counterions) and a mixed Poisson problem (Gauss law) while the fluid movement of the electrolyte solution within the pores of the poroelastic structure are modeled with the Biot equations – one of the most common models for coupled fluid flow and mechanical deformations of porous structures – written in mixed form. Homogenized models of ion transport in poroelastic media can be found in [41,47] (see also [1] for theory and application in nuclear waste disposal in argillaceous rocks). Other applications of macroscopic models where fixed charges yield Debye layers include polymer gels, mechanical actuators for soft robotics, and charged proteoglycans in solid scaffolds of hydrated biological tissues such as articular cartilage [38,39,49,50,52]. As far as we know, no mixed finite element methods (that

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is, formulations that include other variables of interest in addition to solid displacement, fluid pressure, electrostatic potential, and ionic concentrations) have been developed – including formulation and theoretical analysis – for this particular problem.

Mixed methods for poromechanical equations (and solving not just for the displacement-pressure pair) are abundant in the recent literature (see, for example, the very different formulations in [2,6,10,12,14,42,34,37,43,51,53] and the references therein). In this respect we focus on formulations that maintain robustness with respect to the Lamé parameters and permeability of the porous matrix. From those works we refer to [36,11] that use displacement, fluid pressure, total pressure and the relative fluid velocity (Darcy flux) as unknowns.

We also stress that the coupling of Biot equations in mixed form to other physical effects (interface contact, thermal properties, second- and fourth-order transport, etc.) can be substantially more difficult to analyze. Again, focusing on mixed methods, we refer, for example, to [13,33,45,44,48]. In some cases, augmented methods allow the recovery of a Hilbertian framework. For our problem, however, it is difficult to readily construct a Hilbert norm for the Darcy filtration velocity due to the advective coupling in the Nernst–Planck equations.

The problem we tackle here has similar components as in the aforementioned works, also including the Biot-heat equations analyzed in [17] based on a Banach spaces approach, which we follow herein. In this regard, we refer as well to similar multiphysics coupled problems addressed with generalizations of the fixed-point and saddle-point abstract framework to Banach spaces [16,18,20]. On the other hand, the analysis of fully mixed methods for the Poisson–Nernst–Planck equations coupled with Stokes and Navier–Stokes equations has been recently advanced in [26,25], respectively, also using a Banach spaces framework. In contrast with these formulations, in the present model the linear momentum balance of the poroelasticity problem involves the gradients of the ionic concentrations, which suggests to use the gradient of the ionic concentrations as additional variable; yielding again a first-order structure of the coupled equations, but now exhibiting a twofold saddle point form. In general, the type of methods we propose here inherits appealing features such as more flexibility in data assumptions and solution regularity, obtaining all variables of interest without postprocessing, and preserving balance equations exactly. The analysis uses the Babuška–Brezzi and related theories including the extension to perturbed saddle-point problems, all in Banach spaces. These arguments are combined with the classical Banach theorem to establish the existence of a unique solution.

Most PNP–Stokes or Navier–Stokes formulations restrict the analysis to only space-dependent parameters and we have therefore followed that assumption. Some recent works [40] suggest a generalization to concentration-dependent density, viscosity, and permittivity. Extending that framework to the Biot–PNP system is still missing from the literature and this might be of the possible extensions of the current work. Finally, we note that the steady state regime presents a number of challenges that turn the analysis sufficiently long. We anticipate that extending this analysis to the transient case is possible but for it we would first require properties of invertibility of resolvents and similar arguments that we still need to address as done here in the steady case. The transient regime is briefly addressed in our numerical examples.

Plan of the paper. We have organized the contents of this paper as follows. In Section 2, we present the Biot–Poisson–Nernst–Planck equations. In particular, the auxiliary unknowns are introduced here. In Section 3, we establish the primal-mixed variational formulation of the problem by breaking down the analysis according to the three sets of equations comprising the coupled model. In Section 4, we employ a fixed-point strategy to examine the solvability of the continuous formulation. The Galerkin scheme is introduced in Section 5 and a fixed-point approach analogous to that of Section 4 is employed to investigate its well-posedness. Under appropriate stability conditions on the finite element subspaces used, the existence and uniqueness of the solution are proven by applying Banach's fixed-point theorem, along with the discrete versions of the theories employed in the continuous analysis. The error analysis is also conducted there and a corresponding Céa estimate is derived. Next, in Section 6, we introduce specific finite element subspaces that meet the used assumptions. Rates of convergence of the resulting discrete scheme are also established. Finally, several numerical examples confirming these theoretical findings and illustrating the good performance of the method are presented in Section 7.

Notation conventions and preliminaries. Throughout the paper Ω is an open and bounded Lipschitz-continuous domain of R^d, $d \in \{2,3\}$, which satisfies a uniform exterior ball condition, and whose outward unit normal on its boundary Γ is denoted $\boldsymbol{n}$. We remark in advance that the above geometric assumption on Ω is a technical tool to be employed only to prove the continuous and discrete versions of a particular inf-sup condition arising from the analysis (cf. Lemmas 4.2 and 6.1). Standard notation will be adopted for Lebesgue spaces $\mathrm{L}^t(\Omega)$, with $t \in [1,+\infty)$, and Sobolev spaces $\mathrm{W}^{\ell,t}(\Omega)$, with $\ell \geq 0$, whose corresponding norms and seminorms, either for the scalar, vector, or tensorial version, are denoted by $\|\cdot\|_{0,t;\Omega}$, $\|\cdot\|_{\ell,t;\Omega}$, and $|\cdot|_{\ell,t;\Omega}$, respectively. Note that $\mathrm{W}^{0,t}(\Omega) = \mathrm{L}^t(\Omega)$, and that when $t=2$, we simply write $\mathrm{H}^\ell(\Omega)$ instead of $\mathrm{W}^{\ell,2}(\Omega)$, with its norm and seminorm denoted by $\|\cdot\|_{\ell,\Omega}$ and $|\cdot|_{\ell,\Omega}$, respectively. Now, letting $t, t' \in (1,+\infty)$ conjugate to each other, that is such that $1/t + 1/t' = 1$, we let $\mathrm{W}^{1/t',t}(\Gamma)$ and $\mathrm{W}^{-1/t',t'}(\Gamma)$ be the trace space of $\mathrm{W}^{1,t}(\Omega)$ and its dual, respectively, and denote the duality pairing between them by $\langle\cdot,\cdot\rangle$. In particular, when $t = t' = 2$, we simply write $\mathrm{H}^{1/2}(\Gamma)$ and $\mathrm{H}^{-1/2}(\Gamma)$ instead of $\mathrm{W}^{1/2,2}(\Gamma)$ and $\mathrm{W}^{-1/2,2}(\Gamma)$, respectively. Also, given any generic scalar functional space M, we let $\mathbf{M}$ be its vector counterpart.

2. The model problem

Consider a homogeneous porous medium (a mixture of incompressible grains and charged interstitial fluid) occupying the domain Ω. There, we assume the presence of positively and negatively charged ions (e.g., binary monovalent completely dissociated electrolytes Na^+ and Cl^-). For a given body force $\boldsymbol{f}$ and mass source g, neglecting convective, gravitational, and inertial terms, the steady-state linear momentum balance for the mixture and mass balance for the fluid (using the modified Darcy law) are expressed as

$$-\mathbf{div}(\boldsymbol{\sigma}) = \boldsymbol{f} \quad \text{in} \quad \Omega \quad \text{and} \quad c_0\, p + \alpha\, \mathrm{div}(\boldsymbol{u}) - \mathrm{div}\left(\frac{\kappa}{\nu}\nabla p\right) = g \quad \text{in} \quad \Omega, \tag{2.1}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor of the mixture, $\boldsymbol{u}$ is the unknown vector of displacement of the solid, and p is the reference bulk pressure of the fluid. The remaining parameters are the permeability κ, the constrained specific storage coefficient c_0, the Biot–Willis parameter α, and the viscosity of the pore fluid ν. Following the modified Terzaghi decomposition, the constitutive equation for $\boldsymbol{\sigma}$ is conformed by the effective poroelastic stress through Hooke's law for infinitesimal deformation and Biot's consolidation, plus an active macroscopic stress tensor governing the electrochemical interaction between the electrolyte solution and charged molecules as follows (the dependence on the electric field – known as Maxwell's stress – can be found in, e.g., [1,41,47], and that on the ionic concentrations in [50])

$$\boldsymbol{\sigma} = 2\mu\, \boldsymbol{\varepsilon}(\boldsymbol{u}) \;+\; \lambda\, \mathrm{div}(\boldsymbol{u})\mathbb{I} - \alpha\, p\mathbb{I} \;+\; \varepsilon\, \nabla\chi \otimes \nabla\chi - \frac{\varepsilon}{2}|\nabla\chi|^2\, \mathbb{I} - \delta(\xi_1 - \xi_2)\mathbb{I} \quad \text{in} \quad \Omega, \tag{2.2}$$

where ε is the electric conductivity, δ is an osmotic parameter, $\boldsymbol{\varepsilon}(\boldsymbol{u}) := \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^{\mathrm{t}})$ is the tensor of infinitesimal strains, and λ, μ are the Lamé constants of the solid. The fields ξ_1 and ξ_2 are the solute concentrations of positive and negatively charged ions, respectively, and χ is the macroscopic dimensionless electrostatic potential. They satisfy current conservation and mass balance of the charged species as follows

$$\begin{aligned} -\mathrm{div}(\varepsilon\,\nabla\chi) &= \xi_1 - \xi_2 && \text{in} \quad \Omega, \\ \xi_1 - \frac{\kappa}{\nu}\nabla p\cdot\nabla\xi_1 - \mathrm{div}\big(\kappa_1(\nabla\xi_1 + q_1\,\xi_1\nabla\chi)\big) &= f_1 && \text{in} \quad \Omega, \\ \xi_2 - \frac{\kappa}{\nu}\nabla p\cdot\nabla\xi_2 - \mathrm{div}\big(\kappa_2(\nabla\xi_2 + q_2\,\xi_2\nabla\chi)\big) &= f_2 && \text{in} \quad \Omega, \end{aligned} \tag{2.3}$$

where $q_1 = 1$, $q_2 = -1$, f_1, f_2 are external charge sources, and κ_1, κ_2 are the diffusivities of the cations and anions, respectively. Here we have assumed that the balance equations are scaled with the porosity (assumed constant) and the scaling is absorbed in the external sources. Note that the second term on the left-hand sides of the second and third rows of (2.3) is the advection using the filtration (Darcy's seepage) flux, which indicates that the ionic particles diffuse in the mixture and are advected in the interstitial fluid.

We emphasize here that a recent study [17] addresses the fully-mixed coupling of Biot and convection-diffusion equations using the Darcy seepage velocity and the total stress. There, we employed a fully-mixed formulation (the Biot equation utilizes a mixed approach solving for total stress and displacement). Therefore, a natural progression from the findings of that study is to also incorporate a fully-mixed approach for the present Biot–Poisson–Nernst–Planck equations presented in this work.

Now, we follow [36,11] and, in order to maintain robustness in the regime of nearly incompressibility and to achieve mass conservativity, we adopt a four-field formulation for the poroelasticity system (2.1) introducing the total pressure θ, and the Darcy seepage velocity $\boldsymbol{z}$ as the following additional unknowns

$$\theta := -\lambda\, \mathrm{div}(\boldsymbol{u}) + \alpha\, p \quad \text{in} \quad \Omega \quad \text{and} \quad \boldsymbol{z} := -\frac{\kappa}{\nu}\nabla p \quad \text{in} \quad \Omega. \tag{2.4}$$

We stress that the approach described above has been motivated by the fact that some of the targeted applications include mesoscale soft tissue, which are precisely nearly incompressible. In turn, we notice that for a sufficiently smooth vector function $\boldsymbol{w}$ we have

$$\mathbf{div}(\boldsymbol{w}\otimes\boldsymbol{w}) = (\mathrm{div}\,\boldsymbol{w})\,\boldsymbol{w} \;+\; (\nabla\boldsymbol{w})\,\boldsymbol{w} \quad \text{and} \quad \nabla(|\boldsymbol{w}|^2) = 2\,(\nabla\boldsymbol{w})^{\mathrm{t}}\,\boldsymbol{w}.$$

Thus, since $\nabla\boldsymbol{w}$ is symmetric for $\boldsymbol{w} = \nabla\chi$, a combination of the first equation of (2.1) with (2.2) and the definition of the total pressure θ allows obtaining

$$-\mathbf{div}(2\mu\,\boldsymbol{\varepsilon}(\boldsymbol{u}) - \theta\,\mathbb{I}) - \mathrm{div}(\varepsilon\,\nabla\chi)\nabla\chi + \delta(\nabla\xi_1 - \nabla\xi_2) = \boldsymbol{f} \quad \text{in} \quad \Omega. \tag{2.5}$$

Next, for the mass balance (cf. second equation of (2.1)) we use the definition of the total pressure θ and of the Darcy flux $\boldsymbol{z}$ (cf. (2.4)) to have

$$\left(c_0 + \frac{\alpha^2}{\lambda}\right)p - \frac{\alpha}{\lambda}\theta + \mathrm{div}(\boldsymbol{z}) = g \quad \text{in} \quad \Omega.$$

In addition, for sake of pressure uniqueness in the limiting cases when $c_0 = 0$ and $\lambda \to \infty$, we impose:

$$\int_\Omega p = 0. \tag{2.6}$$

Likewise, we make use of the electric current $\boldsymbol{\varphi} := \varepsilon\,\nabla\chi$, which, jointly with the first row of (2.3), gives

$$-\mathrm{div}(\boldsymbol{\varphi}) = \xi_1 - \xi_2 \quad \text{in} \quad \Omega.$$

In turn, for each $i \in \{1,2\}$, we define the ionic concentration gradients $\boldsymbol{t}_i$, and total (diffusive plus advective) flux of ionic species $\boldsymbol{\sigma}_i$, which are defined as follows

$$\boldsymbol{t}_i := \nabla\xi_i \quad \text{in} \quad \Omega \quad \text{and} \quad \boldsymbol{\sigma}_i := \kappa_i(\boldsymbol{t}_i + q_i\varepsilon^{-1}\xi_i\,\boldsymbol{\varphi}) - \xi_i\,\boldsymbol{z} \quad \text{in} \quad \Omega.$$

This is similar to [26], where the $\boldsymbol{t}_i$ are not used. Here we need these chemical fluxes to manage the last term on the right-hand side of the momentum balance (2.5). Finally, for each $i \in \{1,2\}$ we use the identity

$$\mathrm{div}(\xi_i\,\boldsymbol{z}) = \boldsymbol{z}\cdot\nabla\xi_i + \xi_i\,\mathrm{div}(\boldsymbol{z}),$$

which, in combination with the second and third rows of (2.3), yields

$$\xi_i - \mathrm{div}(\boldsymbol{\sigma}_i) - \xi_i\,\mathrm{div}(\boldsymbol{z}) = f_i \quad \text{in} \quad \Omega.$$

In summary, the steps above lead to the following Biot–Poisson–Nernst–Planck equations in terms of the unknowns $\boldsymbol{u}$, θ, $\boldsymbol{z}$, p, $\boldsymbol{\varphi}$, χ, $\boldsymbol{t}_i$, $\boldsymbol{\sigma}_i$ and ξ_i, $i \in \{1,2\}$, as

$$-\mathbf{div}(2\mu\,\boldsymbol{\varepsilon}(\boldsymbol{u}) - \theta\,\mathbb{I}) + \varepsilon^{-1}(\xi_1-\xi_2)\,\boldsymbol{\varphi} + \delta(\boldsymbol{t}_1-\boldsymbol{t}_2) = \boldsymbol{f} \quad \text{in} \quad \Omega, \tag{2.7a}$$

$$\theta - \alpha\,p + \lambda\,\mathrm{div}(\boldsymbol{u}) = 0 \quad \text{in} \quad \Omega, \tag{2.7b}$$

$$\frac{\nu}{\kappa}\boldsymbol{z} + \nabla p = \mathbf{0} \quad \text{in} \quad \Omega, \tag{2.7c}$$

$$\left(c_0 + \frac{\alpha^2}{\lambda}\right)p - \frac{\alpha}{\lambda}\theta + \mathrm{div}(\boldsymbol{z}) = g \quad \text{in} \quad \Omega, \tag{2.7d}$$

$$\boldsymbol{\varphi} - \varepsilon\,\nabla\chi = \mathbf{0} \quad \text{in} \quad \Omega, \tag{2.7e}$$

$$-\mathrm{div}(\boldsymbol{\varphi}) = \xi_1 - \xi_2 \quad \text{in} \quad \Omega, \tag{2.7f}$$

$$\boldsymbol{t}_i - \nabla\xi_i = \mathbf{0} \quad \text{in} \quad \Omega, \tag{2.7g}$$

$$-\boldsymbol{\sigma}_i + \kappa_i\boldsymbol{t}_i + q_i\,\kappa_i\,\varepsilon^{-1}\,\xi_i\,\boldsymbol{\varphi} - \xi_i\,\boldsymbol{z} = \mathbf{0} \quad \text{in} \quad \Omega, \tag{2.7h}$$

$$\xi_i - \mathrm{div}(\boldsymbol{\sigma}_i) - \xi_i\,\mathrm{div}(\boldsymbol{z}) = f_i \quad \text{in} \quad \Omega. \tag{2.7i}$$

We endow (2.7a)-(2.7d) with the following boundary conditions

$$\boldsymbol{u} = \mathbf{0} \quad \text{and} \quad \boldsymbol{z}\cdot\boldsymbol{n} = 0 \quad \text{on} \quad \Gamma, \tag{2.8}$$

and Dirichlet boundary conditions with given data χ_{D}, $\xi_{i,\mathrm{D}}$, $i \in \{1,2\}$, are considered for (2.7e)-(2.7i):

$$\chi = \chi_{\mathrm{D}} \quad \text{and} \quad \xi_i = \xi_{i,\mathrm{D}} \quad \text{on} \quad \Gamma. \tag{2.9}$$

3. The weak formulation

In this section, we derive a primal-mixed formulation of the system (2.7) - (2.9). To this end, we first provide some preliminaries, and then split the analysis according to the respective decoupled problems.

3.1. Preliminaries

We start by considering, for each $t \in [1,+\infty)$, the Banach spaces

$$\begin{aligned}
\mathbf{H}(\mathrm{div}_t;\Omega) &:= \left\{\boldsymbol{\tau} \in \mathbf{L}^2(\Omega): \quad \mathrm{div}(\boldsymbol{\tau}) \in \mathrm{L}^t(\Omega)\right\},\\
\mathbf{H}^t(\mathrm{div};\Omega) &:= \left\{\boldsymbol{\tau} \in \mathbf{L}^t(\Omega): \quad \mathrm{div}(\boldsymbol{\tau}) \in \mathrm{L}^2(\Omega)\right\},\\
\mathbf{H}^t(\mathrm{div}_t;\Omega) &:= \left\{\boldsymbol{\tau} \in \mathbf{L}^t(\Omega): \quad \mathrm{div}(\boldsymbol{\tau}) \in \mathrm{L}^t(\Omega)\right\},
\end{aligned}$$

which are endowed with the natural norms:

$$\begin{aligned}
\|\boldsymbol{\tau}\|_{\mathrm{div}_t;\Omega} &:= \|\boldsymbol{\tau}\|_{0;\Omega} + \|\mathrm{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \qquad \forall\,\boldsymbol{\tau} \in \mathbf{H}(\mathrm{div}_t;\Omega),\\
\|\boldsymbol{\tau}\|_{t,\mathrm{div};\Omega} &:= \|\boldsymbol{\tau}\|_{0,t;\Omega} + \|\mathrm{div}(\boldsymbol{\tau})\|_{0;\Omega} \qquad \forall\,\boldsymbol{\tau} \in \mathbf{H}^t(\mathrm{div};\Omega),\\
\|\boldsymbol{\tau}\|_{t,\mathrm{div}_t;\Omega} &:= \|\boldsymbol{\tau}\|_{0,t;\Omega} + \|\mathrm{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \qquad \forall\,\boldsymbol{\tau} \in \mathbf{H}^t(\mathrm{div}_t;\Omega).
\end{aligned}$$

We recall that, proceeding as in [28, eqn. (1.43), Section 1.3.4] (see also [23, Section 3.1]), one can prove that for each $t \in \begin{cases} (1,+\infty) & \text{in } \mathrm{R}^2,\\ [6/5,+\infty) & \text{in } \mathrm{R}^3,\end{cases}$ there holds

$$\langle \boldsymbol{\tau}\cdot\boldsymbol{n}, v\rangle = \int_\Omega \left\{\boldsymbol{\tau}\cdot\nabla v + v\,\mathrm{div}(\boldsymbol{\tau})\right\} \qquad \forall\,(\boldsymbol{\tau},v) \in \mathbf{H}(\mathrm{div}_t;\Omega)\times\mathrm{H}^1(\Omega), \tag{3.2}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $\mathrm{H}^{1/2}(\Gamma)$ and $\mathrm{H}^{-1/2}(\Gamma)$. In turn, given $t, t' \in (1,+\infty)$ conjugate to each other, there also holds (cf. [27, Corollary B.57])

$$\langle \boldsymbol{\tau}\cdot\boldsymbol{n}, v\rangle = \int_\Omega \Big\{ \boldsymbol{\tau}\cdot\nabla v \,+\, v\,\mathrm{div}(\boldsymbol{\tau}) \Big\} \qquad \forall\,(\boldsymbol{\tau},v)\in \mathbf{H}^t(\mathrm{div}_t;\Omega)\times \mathrm{W}^{1,t'}(\Omega)\,, \tag{3.3}$$

where $\langle\cdot,\cdot\rangle$ denotes in (3.3) the duality pairing between $\mathrm{W}^{-1/t,t}(\Gamma)$ and $\mathrm{W}^{1/t,t'}(\Gamma)$.

Note that handling the terms involving $(\xi_1-\xi_2)\,\boldsymbol{\varphi}$, $\xi_i\,\boldsymbol{\varphi}$, $\xi_i\,\boldsymbol{z}$, and $\xi_i\,\mathrm{div}(\boldsymbol{z})$, will determine adequate Sobolev and Lebesgue exponents specifying trial and test spaces. Given test functions $\boldsymbol{v}$, $\boldsymbol{s}_i$ and η_i associated with $\boldsymbol{u}$, $\boldsymbol{t}_i$ and ξ_i, respectively, an application of the Cauchy–Schwarz and Hölder inequalities yield

$$\left|\int_\Omega (\xi_1-\xi_2)\,\boldsymbol{\varphi}\cdot\boldsymbol{v}\right| \le \|\xi_1-\xi_2\|_{0,2l;\Omega}\,\|\boldsymbol{\varphi}\|_{0,2j;\Omega}\,\|\boldsymbol{v}\|_{0,\Omega}\,, \tag{3.4a}$$

$$\left|\int_\Omega \xi_i\,\boldsymbol{\varphi}\cdot\boldsymbol{s}_i\right| \le \|\xi_i\|_{0,2l;\Omega}\,\|\boldsymbol{\varphi}\|_{0,2j;\Omega}\,\|\boldsymbol{s}_i\|_{0,\Omega}\,, \tag{3.4b}$$

$$\left|\int_\Omega \xi_i\,\boldsymbol{z}\cdot\boldsymbol{s}_i\right| \le \|\xi_i\|_{0,2l;\Omega}\,\|\boldsymbol{z}\|_{0,2j;\Omega}\,\|\boldsymbol{s}_i\|_{0,\Omega}\,, \tag{3.4c}$$

$$\left|\int_\Omega \xi_i\,\mathrm{div}(\boldsymbol{z})\,\eta_i\right| \le \|\xi_i\|_{0,2l;\Omega}\,\|\mathrm{div}(\boldsymbol{z})\|_{0,\Omega}\,\|\eta_i\|_{0,2j;\Omega}\,, \tag{3.4d}$$

where $l, j \in (1,+\infty)$ are conjugate to each other. In this way, denoting

$$r := 2j\,,\quad s := \frac{2j}{2j-1}\ \text{(conjugate of } r)\,,\quad \rho := 2l\,,\quad \varrho := \frac{2l}{2l-1}\ \text{(conjugate of } \rho)\,, \tag{3.5}$$

it follows that the above expressions are integrable for $\xi_i \in \mathrm{L}^\rho(\Omega)$, $\boldsymbol{\varphi}\in\mathbf{L}^r(\Omega)$, $\boldsymbol{z}\in\mathbf{H}^r(\mathrm{div};\Omega)$, $\boldsymbol{v}\in\mathbf{L}^2(\Omega)$, $\boldsymbol{s}_i\in\mathbf{L}^2(\Omega)$ and, assuming that $\rho > r$ (a condition to be satisfied below in (3.6)), we can consider $\eta_i\in\mathrm{L}^\rho(\Omega)$. Moreover, since we aim to apply (3.2) to $\boldsymbol{\tau}_i\in\mathbf{H}(\mathrm{div}_\varrho;\Omega)$ and $\xi_i\in\mathrm{L}^\rho(\Omega)$, we need that $\mathrm{H}^1(\Omega)$ is continuously embedded in $\mathrm{L}^\rho(\Omega)$. The latter is guaranteed for $\rho\in[1,+\infty)$ when $n=2$, and $\rho\in[1,6]$ when $n=3$.

On the other hand, in the forthcoming analysis we require a result on the $\mathrm{W}^{1,r}(\Omega)$-solvability of an auxiliary Poisson equation (in showing a continuous inf-sup condition). For this we need that $4/3\le r\le 4$ when $n=2$, and $3/2\le r\le 3$ when $n=3$. Thus, since $r=\frac{\rho}{l-1}$, intersecting this with the previous restrictions on ρ, we find the following feasible ranges for r, s, ρ and ϱ:

$$\left\{\begin{array}{lll} r\in(2,4] \ \text{and}\ s\in[4/3,2) & \text{if } n=2\,,\\ r=3 \ \text{and}\ s=3/2 & \text{if } n=3\,,\end{array}\right.\qquad \left\{\begin{array}{lll} \rho\in[4,+\infty) \ \text{and}\ \varrho\in(1,4/3] & \text{if } n=2\,,\\ \rho=6 \ \text{and}\ \varrho=6/5 & \text{if } n=3\,.\end{array}\right. \tag{3.6}$$

In turn, in view of the essential boundary conditions for displacement and Darcy flux in (2.8), we consider the following closed subspaces of Hilbert and Banach spaces

$$\mathbf{H}^1_0(\Omega) := \Big\{ \boldsymbol{v}\in\mathbf{H}^1(\Omega) : \quad \boldsymbol{v}|_\Gamma = \mathbf{0} \Big\}, \tag{3.7a}$$

$$\mathbf{H}^s_0(\mathrm{div}_s;\Omega) := \Big\{ \boldsymbol{w}\in\mathbf{H}^s(\mathrm{div}_s;\Omega) : \quad (\boldsymbol{w}\cdot\boldsymbol{n})|_\Gamma = 0 \Big\}, \tag{3.7b}$$

$$\mathbf{H}^r_0(\mathrm{div};\Omega) := \Big\{ \boldsymbol{w}\in\mathbf{H}^r(\mathrm{div};\Omega) : \quad (\boldsymbol{w}\cdot\boldsymbol{n})|_\Gamma = 0 \Big\}. \tag{3.7c}$$

The boundary specification is to be understood in the sense of traces. In addition, for $t\in[1,+\infty)$ we define

$$\mathrm{L}^t_0(\Omega) := \Big\{ q\in\mathrm{L}^t(\Omega) : \quad \int_\Omega q = 0 \Big\}. \tag{3.8}$$

As announced earlier, in what follows we rewrite each variational formulation of Biot, Poisson and Nernst–Planck equations independently, ending up with three systems whose coupling is carried out via a fixed-point iteration. We also provide preliminary properties of the corresponding bilinear forms.

3.2. Primal-mixed formulation of the poroelasticity equations

In this section, we follow very closely [36, Section 2] to derive the variational formulation of the poroelasticity equations (2.7a)-(2.7d) and (2.8), which, given $\boldsymbol{\varphi}$, ξ_1, ξ_2, $\boldsymbol{t}_1$, and $\boldsymbol{t}_2$, consist of finding $\boldsymbol{u}$, θ, $\boldsymbol{z}$, and p, all the above in suitable spaces, such that

$$-\mathbf{div}(2\mu\,\boldsymbol{\varepsilon}(\boldsymbol{u}) - \theta\,\mathbb{I}) + \varepsilon^{-1}(\xi_1 - \xi_2)\,\boldsymbol{\varphi} + \delta(\boldsymbol{t}_1 - \boldsymbol{t}_2) = \boldsymbol{f} \quad \text{in} \quad \Omega, \tag{3.9a}$$

$$\theta - \alpha\, p + \lambda\,\mathrm{div}(\boldsymbol{u}) = 0 \quad \text{in} \quad \Omega, \tag{3.9b}$$

$$\frac{\nu}{\kappa}\boldsymbol{z} + \nabla p = \mathbf{0} \quad \text{in} \quad \Omega, \tag{3.9c}$$

$$\left(c_0 + \frac{\alpha^2}{\lambda}\right) p - \frac{\alpha}{\lambda}\theta + \mathrm{div}(\boldsymbol{z}) = g \quad \text{in} \quad \Omega, \tag{3.9d}$$

$$\boldsymbol{u} = \mathbf{0} \quad \text{and} \quad \boldsymbol{z}\cdot\boldsymbol{n} = 0 \quad \text{on} \quad \Gamma. \tag{3.9e}$$

We begin by testing (3.9a) against $\boldsymbol{v} \in \mathbf{H}^1_0(\Omega)$ (cf. (3.7a)), which satisfies the bound given by (3.4a). In this way, applying (3.2) with $t = 2$, and employing the first boundary condition in (3.9e), we obtain

$$2\mu \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v}) - \int_\Omega \theta\,\mathrm{div}(\boldsymbol{v}) = \int_\Omega \Big(\boldsymbol{f} - \varepsilon^{-1}(\xi_1 - \xi_2)\,\boldsymbol{\varphi} - \delta(\boldsymbol{t}_1 - \boldsymbol{t}_2)\Big)\cdot\boldsymbol{v} \qquad \forall\, \boldsymbol{v} \in \mathbf{H}^1_0(\Omega). \tag{3.10}$$

Thanks to the Cauchy–Schwarz's inequality and (3.4a), each term in (3.10) makes sense for $\boldsymbol{u} \in \mathbf{H}^1_0(\Omega)$, $\theta \in \mathrm{L}^2(\Omega)$, $\boldsymbol{f} \in \mathbf{L}^2(\Omega)$, $\boldsymbol{\varphi} \in \mathbf{L}^r(\Omega)$, $\xi_i \in \mathrm{L}^\rho(\Omega)$, and $\boldsymbol{t}_i \in \mathbf{L}^2(\Omega)$, $i \in \{1,2\}$. Next, we test (3.9b) against $\vartheta \in \mathrm{L}^2(\Omega)$, which gives

$$-\int_\Omega \vartheta\,\mathrm{div}(\boldsymbol{u}) - \frac{1}{\lambda}\int_\Omega \theta\,\vartheta + \frac{\alpha}{\lambda}\int_\Omega p\,\vartheta = 0 \qquad \forall\, \vartheta \in \mathrm{L}^2(\Omega). \tag{3.11}$$

On the other hand, recalling from (3.4c) and (3.4d) that $\boldsymbol{z} \in \mathbf{H}^r(\mathrm{div};\Omega)$, and bearing in mind the second boundary condition in (3.9e), we deduce that $\boldsymbol{z}$ must be sought in $\mathbf{H}^r_0(\mathrm{div};\Omega)$ (cf. (3.7b)), whence (3.9c) suggests to look originally for $p \in \mathrm{W}^{1,r}(\Omega)$. In this way, testing (3.9c) against $\boldsymbol{w} \in \mathbf{H}^s_0(\mathrm{div}_s;\Omega)$ (cf. (3.7c)), and employing (3.3), we formally get

$$\frac{\nu}{\kappa}\int_\Omega \boldsymbol{z}\cdot\boldsymbol{w} - \int_\Omega p\,\mathrm{div}(\boldsymbol{w}) = 0 \qquad \forall\, \boldsymbol{w} \in \mathbf{H}^s_0(\mathrm{div}_s;\Omega), \tag{3.12}$$

from whose second term and (2.6), we notice that it suffices to look for the pressure p in the space $\mathrm{L}^r_0(\Omega)$ (cf. (3.8)). In turn, since $\mathrm{div}(\boldsymbol{z})$ belongs to $\mathrm{L}^2(\Omega)$, we test (3.9d) against $q \in \mathrm{L}^2_0(\Omega)$ obtaining

$$\frac{\alpha}{\lambda}\int_\Omega \theta\,q - \int_\Omega q\,\mathrm{div}(\boldsymbol{z}) - \left(c_0 + \frac{\alpha^2}{\lambda}\right)\int_\Omega p\,q = -\int_\Omega g\,q \qquad \forall\, q \in \mathrm{L}^2_0(\Omega), \tag{3.13}$$

which requires assuming that $g \in \mathrm{L}^2(\Omega)$. In addition, knowing that $\vartheta \in \mathrm{L}^2(\Omega)$, $p \in \mathrm{L}^r_0(\Omega)$, and $q \in \mathrm{L}^2_0(\Omega)$, and recalling from (3.6) that $r > 2$, which certainly yields $\mathrm{L}^r(\Omega) \subset \mathrm{L}^2(\Omega)$, we realize that the third terms of (3.11) and (3.13) make sense as well. According to the foregoing discussion, and aiming to conveniently rewrite the system of equations (3.10) - (3.13), we now introduce the spaces

$$\begin{aligned} &\mathbf{X} := \mathbf{H}^1_0(\Omega), \quad \mathbf{X}_2 := \mathbf{H}^r_0(\mathrm{div};\Omega), \quad \mathbf{X}_1 := \mathbf{H}^s_0(\mathrm{div}_s;\Omega), \\ &\mathbf{Q} = \mathrm{L}^2(\Omega), \quad \mathbf{Q}_1 := \mathrm{L}^r_0(\Omega), \quad \text{and} \quad \mathbf{Q}_2 := \mathrm{L}^2_0(\Omega), \end{aligned}$$

which are endowed, respectively, with the norms

$$\begin{aligned} &\|\boldsymbol{v}\|_{\mathbf{X}} := \|\boldsymbol{v}\|_{1,\Omega}, \quad \|\boldsymbol{z}\|_{\mathbf{X}_2} := \|\boldsymbol{z}\|_{r,\mathrm{div};\Omega}, \quad \|\boldsymbol{w}\|_{\mathbf{X}_1} := \|\boldsymbol{w}\|_{s,\mathrm{div}_s;\Omega}, \\ &\|\vartheta\|_{\mathbf{Q}} := \|\vartheta\|_{0,\Omega}, \quad \|p\|_{\mathbf{Q}_1} := \|p\|_{0,r;\Omega}, \quad \text{and} \quad \|q\|_{\mathbf{Q}_2} := \|q\|_{0,\Omega}. \end{aligned}$$

In this way, given $\boldsymbol{\varphi} \in \mathbf{L}^r(\Omega)$, $\boldsymbol{\xi} = (\xi_1, \xi_2) \in \mathrm{L}^\rho(\Omega) \times \mathrm{L}^\rho(\Omega)$, $\boldsymbol{t} = (\boldsymbol{t}_1, \boldsymbol{t}_2) \in \mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Omega)$, and $p \in \mathrm{L}^r_0(\Omega)$, (3.10) and (3.11) can be reformulated as: Find $(\boldsymbol{u},\theta) \in \mathbf{X} \times \mathbf{Q}$ such that

$$\begin{array}{lllll} \mathbf{a}_s(\boldsymbol{u},\boldsymbol{v}) & + \ \mathbf{b}_s(\boldsymbol{v},\theta) & & = \ \mathbf{F}_{\boldsymbol{\varphi},\xi,\boldsymbol{t}}(\boldsymbol{v}) & \forall\, \boldsymbol{v} \in \mathbf{X}, \\ \mathbf{b}_s(\boldsymbol{u},\vartheta) & - \ \mathbf{c}_s(\theta,\vartheta) & + \mathbf{e}_s(p,\vartheta) & = \ 0 & \forall\, \vartheta \in \mathbf{Q}, \end{array} \tag{3.14}$$

where the bilinear forms $\mathbf{a}_s : \mathbf{X} \times \mathbf{X} \to \mathrm{R}$, $\mathbf{b}_s : \mathbf{X} \times \mathbf{Q} \to \mathrm{R}$, $\mathbf{c}_s : \mathbf{Q} \times \mathbf{Q} \to \mathrm{R}$, and $\mathbf{e}_s : \mathbf{Q}_1 \times \mathbf{Q} \to \mathrm{R}$, and the functional $\mathbf{F}_{\boldsymbol{\varphi},\xi,\boldsymbol{t}} : \mathbf{X} \to \mathrm{R}$, are defined, respectively, as

$$\begin{aligned} \mathbf{a}_s(\boldsymbol{u},\boldsymbol{v}) &:= 2\mu \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v}) && \forall\, (\boldsymbol{u},\boldsymbol{v}) \in \mathbf{X} \times \mathbf{X}, \\ \mathbf{b}_s(\boldsymbol{v},\vartheta) &:= -\int_\Omega \vartheta\,\mathrm{div}(\boldsymbol{v}) && \forall\, (\boldsymbol{v},\vartheta) \in \mathbf{X} \times \mathbf{Q}, \\ \mathbf{c}_s(\theta,\vartheta) &:= \frac{1}{\lambda}\int_\Omega \theta\,\vartheta && \forall\, (\theta,\vartheta) \in \mathbf{Q} \times \mathbf{Q}, \\ \mathbf{e}_s(p,\vartheta) &:= \frac{\alpha}{\lambda}\int_\Omega p\,\vartheta && \forall\, (p,\vartheta) \in \mathbf{Q}_1 \times \mathbf{Q}, \quad \text{and} \end{aligned} \tag{3.15}$$

$$\mathbf{F}_{\varphi,\xi,t}(\boldsymbol{v}) := \int_\Omega \Big(\boldsymbol{f} - \varepsilon^{-1}(\xi_1 - \xi_2)\boldsymbol{\varphi} - \delta(\boldsymbol{t}_1 - \boldsymbol{t}_2)\Big)\cdot \boldsymbol{v} \qquad \forall\, \boldsymbol{v} \in \mathbf{X}. \tag{3.16}$$

Similarly, given $\theta \in \mathrm{L}^2(\Omega)$, (3.12) and (3.13) can be reformulated as: Find $(\boldsymbol{z}, p) \in \mathbf{X}_2 \times \mathbf{Q}_1$ such that

$$\begin{aligned} \mathbf{a}_f(\boldsymbol{z},\boldsymbol{w}) \;+\; \mathbf{d}_1(\boldsymbol{w},p) \qquad\qquad &= \; 0 \qquad \forall\, \boldsymbol{w} \in \mathbf{X}_1, \\ \mathbf{d}_2(\boldsymbol{z},q) \qquad\qquad + \mathbf{e}_f((\theta,p),q) &= \; \mathbf{G}(q) \qquad \forall\, q \in \mathbf{Q}_2, \end{aligned} \tag{3.17}$$

where the bilinear forms $\mathbf{a}_f : \mathbf{X}_2 \times \mathbf{X}_1 \to \mathrm{R}$, $\mathbf{d}_i : \mathbf{X}_i \times \mathbf{Q}_i \to \mathrm{R}$, $i \in \{1,2\}$, and $\mathbf{e}_f : (\mathbf{Q}\times\mathbf{Q}_1)\times\mathbf{Q}_2 \to \mathrm{R}$, and the functional $\mathbf{G} : \mathbf{Q}_2 \to \mathrm{R}$, are given, respectively, by

$$\begin{aligned} \mathbf{a}_f(\boldsymbol{z},\boldsymbol{w}) &:= \frac{\nu}{\kappa}\int_\Omega \boldsymbol{z}\cdot\boldsymbol{w} && \forall\,(\boldsymbol{z},\boldsymbol{w}) \in \mathbf{X}_2\times\mathbf{X}_1, \\ \mathbf{d}_i(\boldsymbol{w},q) &:= -\int_\Omega q\,\mathrm{div}(\boldsymbol{w}) && \forall\,(\boldsymbol{w},q)\in \mathbf{X}_i\times\mathbf{Q}_i, \\ \mathbf{e}_f((\theta,p),q) &:= \frac{\alpha}{\lambda}\int_\Omega \theta\, q - \left(c_0 + \frac{\alpha^2}{\lambda}\right)\int_\Omega p\, q && \forall\,((\theta,p),q) \in (\mathbf{Q}\times\mathbf{Q}_1)\times\mathbf{Q}_2, \quad \text{and} \\ \mathbf{G}(q) &:= \int_\Omega g\, q && \forall\, q \in \mathbf{Q}_2. \end{aligned} \tag{3.18}$$

Summarizing, given $\boldsymbol{\varphi} \in \mathbf{L}^r(\Omega)$, $\boldsymbol{\xi} = (\xi_1,\xi_2) \in \mathrm{L}^\rho(\Omega)\times\mathrm{L}^\rho(\Omega)$, and $\boldsymbol{t} = (\boldsymbol{t}_1,\boldsymbol{t}_2) \in \mathbf{L}^2(\Omega)\times\mathbf{L}^2(\Omega)$, the primal-mixed formulation for the poroelasticity equations (cf. (3.9)) reduces to gathering (3.14) and (3.17), that is: Find $((\boldsymbol{u},\theta),(\boldsymbol{z},p)) \in (\mathbf{X}\times\mathbf{Q})\times(\mathbf{X}_2\times\mathbf{Q}_1)$ such that

$$\begin{aligned} \mathbf{a}_s(\boldsymbol{u},\boldsymbol{v}) \;+\; \mathbf{b}_s(\boldsymbol{v},\theta) \qquad\qquad\qquad\qquad &= \mathbf{F}_{\varphi,\xi,t}(\boldsymbol{v}) && \forall\,\boldsymbol{v}\in\mathbf{X}, \\ \mathbf{b}_s(\boldsymbol{u},\vartheta) \;-\; \mathbf{c}_s(\theta,\vartheta) \qquad\qquad + \mathbf{e}_s(p,\vartheta) &= 0 && \forall\,\vartheta \in \mathbf{Q}, \\ \mathbf{a}_f(\boldsymbol{z},\boldsymbol{w}) + \mathbf{d}_1(\boldsymbol{w},p) \qquad\qquad &= 0 && \forall\,\boldsymbol{w}\in\mathbf{X}_1, \\ \mathbf{d}_2(\boldsymbol{z},q) \qquad\qquad + \mathbf{e}_f((\theta,p),q) &= \mathbf{G}(q) && \forall\, q\in\mathbf{Q}_2. \end{aligned} \tag{3.19}$$

It is important to stress here that, ignoring the bilinear forms $\mathbf{e}_s$ and $\mathbf{e}_f$, the left-hand side of (3.19) shows a block-diagonal structure with perturbed and generalized saddle-point problems, respectively, as the first and second block. We take advantage of this fact later on in Section 4.2.

Direct applications of the Hölder and Cauchy–Schwarz inequalities allow us to conclude that the above bilinear forms and the functional $\mathbf{G}$ are bounded with positive constants given by

$$\begin{aligned} &\|\mathbf{a}_s\| := 2\mu, \quad \|\mathbf{b}_s\| := 1, \quad \|\mathbf{c}_s\| := \tfrac{1}{\lambda}, \quad \|\mathbf{e}_s\| := C_r(\Omega)\,\frac{\alpha}{\lambda}, \quad \|\mathbf{a}_f\| := \frac{\nu}{\kappa}, \\ &\|\mathbf{d}_1\|, \|\mathbf{d}_2\| := 1, \quad \|\mathbf{e}_f\| := \max\left\{\frac{\alpha}{\lambda}, C_r(\Omega)\left(c_0 + \frac{\alpha^2}{\lambda}\right)\right\}, \quad \text{and} \quad \|\mathbf{G}\| = \|g\|_{0,\Omega}, \end{aligned} \tag{3.20}$$

where $C_r(\Omega) := |\Omega|^{\frac{r-2}{2r}}$. In addition, for each $\boldsymbol{v} \in \mathbf{X}_1$ there holds

$$\begin{aligned} |\mathbf{F}_{\varphi,\xi,t}(\boldsymbol{v})| \le \|\mathbf{F}\|\Big\{\|\boldsymbol{f}\|_{0,\Omega} + \|\boldsymbol{\varphi}\|_{0,r;\Omega}\,\|\xi_1-\xi_2\|_{0,\rho;\Omega} + \|\boldsymbol{t}_1-\boldsymbol{t}_2\|_{0,\Omega}\Big\}\|\boldsymbol{v}\|_{\mathbb{X}_1}, \quad \text{with} \\ \|\mathbf{F}\| := \max\left\{1,\varepsilon^{-1},\delta\right\}. \end{aligned} \tag{3.21}$$

3.3. *Mixed formulation of the electrostatic potential equations*

From (2.7e) - (2.7f) and the boundary condition for χ in (2.9) we recall

$$\boldsymbol{\varphi} - \varepsilon\,\nabla\chi = \mathbf{0} \quad \text{in} \quad \Omega, \quad -\mathrm{div}(\boldsymbol{\varphi}) = \xi_1 - \xi_2 \quad \text{in} \quad \Omega, \quad \chi = \chi_{\mathrm{D}} \quad \text{on} \quad \Gamma. \tag{3.22}$$

Then, following [26, Section 3.3], we set the trial and test spaces

$$\mathrm{X}_1 := \mathbf{H}^s(\mathrm{div}_s;\Omega), \quad \mathrm{X}_2 := \mathbf{H}^r(\mathrm{div}_r;\Omega), \quad \mathrm{M}_1 := \mathrm{L}^r(\Omega) \quad \text{and} \quad \mathrm{M}_2 := \mathrm{L}^s(\Omega),$$

which are provided with the norms

$$\|\boldsymbol{\psi}\|_{\mathrm{X}_1} := \|\boldsymbol{\psi}\|_{s,\mathrm{div}_s;\Omega}, \quad \|\boldsymbol{\varphi}\|_{\mathrm{X}_2} := \|\boldsymbol{\varphi}\|_{r,\mathrm{div}_r;\Omega}, \quad \|\chi\|_{\mathrm{M}_1} := \|\chi\|_{0,r;\Omega} \quad \text{and} \quad \|\gamma\|_{\mathrm{M}_2} := \|\gamma\|_{0,s;\Omega},$$

and deduce that, given $\boldsymbol{\xi} = (\xi_1,\xi_2) \in \mathrm{L}^\rho(\Omega)\times\mathrm{L}^\rho(\Omega)$, the weak formulation of (3.22) reduces to the generalized saddle-point problem: Find $(\boldsymbol{\varphi},\chi) \in \mathrm{X}_2\times\mathrm{M}_1$ such that

$$\begin{aligned} a(\boldsymbol{\varphi},\boldsymbol{\psi}) + b_1(\boldsymbol{\psi},\chi) &= G(\boldsymbol{\psi}) && \forall\,\boldsymbol{\psi}\in\mathrm{X}_1, \\ b_2(\boldsymbol{\varphi},\gamma) &= F_\xi(\gamma) && \forall\,\gamma\in\mathrm{M}_2, \end{aligned} \tag{3.23}$$

where the bilinear forms $a : \mathrm{X}_2 \times \mathrm{X}_1 \to \mathrm{R}$, and $b_i : \mathrm{X}_i \times \mathrm{M}_i \to \mathrm{R}$, with $i \in \{1,2\}$, and the linear functionals $G : \mathrm{X}_1 \to \mathrm{R}$ and $F_\xi : \mathrm{M}_2 \to \mathrm{R}$, are given, respectively, by

$$\begin{aligned} a(\boldsymbol{\varphi},\boldsymbol{\psi}) &:= \int_\Omega \varepsilon^{-1}\,\boldsymbol{\varphi}\cdot\boldsymbol{\psi} \qquad \forall\,(\boldsymbol{\varphi},\boldsymbol{\psi}) \in \mathrm{X}_2\times\mathrm{X}_1\,,\\ b_i(\boldsymbol{\psi},\gamma) &:= \int_\Omega \gamma\,\mathrm{div}(\boldsymbol{\psi}) \qquad \forall\,(\boldsymbol{\psi},\gamma)\in \mathrm{X}_i\times\mathrm{M}_i\,, i\in\{1,2\}\,,\\ G(\boldsymbol{\psi}) &:= \langle \boldsymbol{\psi}\cdot\boldsymbol{n},\chi_{\mathrm{D}}\rangle \qquad \forall\,\boldsymbol{\psi}\in\mathrm{X}_1\,,\\ F_\xi(\gamma) &:= -\int_\Omega (\xi_1-\xi_2)\,\gamma \qquad \forall\,\gamma\in\mathrm{M}_2\,. \end{aligned}$$

Straightforward applications of Hölder's inequality allow us to conclude that a and b_i, with $i \in \{1,2\}$, are bounded with constants given by

$$\|a\| := \varepsilon^{-1} \quad \text{and} \quad \|b_1\|,\, \|b_2\| := 1\,. \tag{3.24}$$

By similar arguments there holds

$$|F_\xi(\gamma)| \le \|F\|\,\|\xi_1-\xi_2\|_{0,\rho;\Omega}\,\|\gamma\|_{\mathrm{M}_2} \quad \forall\,\gamma\in\mathrm{M}_2\,, \quad \text{with} \quad \|F\| := |\Omega|^{\frac{\rho-r}{\rho r}}\,. \tag{3.25}$$

In turn, regarding the boundedness of G, we invoke [27, Lemma A.36] and the surjectivity of the trace operator mapping $\mathrm{W}^{1,r}(\Omega)$ onto $\mathrm{W}^{1/s,r}(\Gamma)$, which imply the existence of a constant c_r, such that for the given $\chi_{\mathrm{D}} \in \mathrm{W}^{1/s,r}(\Gamma)$, there exists $v_{\mathrm{D}} \in \mathrm{W}^{1,r}(\Omega)$ satisfying $v_{\mathrm{D}}|_\Gamma = \chi_{\mathrm{D}}$ and the estimate $\|v_{\mathrm{D}}\|_{1,r;\Omega} \le c_r\,\|\chi_{\mathrm{D}}\|_{1/s,r;\Gamma}$, which, thanks to (3.3), yields

$$|G(\boldsymbol{\psi})| \le \|G\|\,\|\boldsymbol{\psi}\|_{\mathrm{X}_1} \quad \forall\,\boldsymbol{\psi}\in\mathrm{X}_1\,, \quad \text{with} \quad \|G\| := c_r\,\|\chi_{\mathrm{D}}\|_{1/s,r;\Gamma}\,. \tag{3.26}$$

3.4. Mixed formulation of the ionized particles concentration equations

In what follows we deduce the weak formulation of the Nernst–Planck equations (2.7g) - (2.7i), and the Dirichlet boundary condition for ξ_i in (2.9), for $i \in \{1,2\}$, which, given $\boldsymbol{\varphi} \in \mathbf{H}^r(\mathrm{div}_r;\Omega)$ and $\boldsymbol{z} \in \mathbf{H}^r_0(\mathrm{div};\Omega)$, consist in finding $\boldsymbol{t}_i \in \mathbf{L}^2(\Omega)$, $\xi_i \in \mathrm{L}^\rho(\Omega)$, and $\boldsymbol{\sigma}_i$ in a suitable space to be made precise, such that

$$\boldsymbol{t}_i - \nabla\xi_i = \mathbf{0} \quad \text{in} \quad \Omega\,, \tag{3.27a}$$

$$-\boldsymbol{\sigma}_i + \kappa_i\,\boldsymbol{t}_i + q_i\,\kappa_i\,\varepsilon^{-1}\,\xi_i\,\boldsymbol{\varphi} - \xi_i\,\boldsymbol{z} = \mathbf{0} \quad \text{in} \quad \Omega\,, \tag{3.27b}$$

$$\xi_i - \mathrm{div}(\boldsymbol{\sigma}_i) - \xi_i\,\mathrm{div}(\boldsymbol{z}) = f_i \quad \text{in} \quad \Omega\,. \tag{3.27c}$$

$$\xi_i = \xi_{i,\mathrm{D}} \quad \text{on} \quad \Gamma\,. \tag{3.27d}$$

Note that the spaces to which $\boldsymbol{t}_i$ and ξ_i are indicated to belong, for $i \in \{1,2\}$, were derived in Section 3.2 after analyzing the validity of (3.10). These belongings are confirmed next, but we need to suppose momentarily that $\xi_i \in \mathrm{H}^1(\Omega)$, which implies assuming as well that $\xi_{i,\mathrm{D}} \in \mathrm{H}^{1/2}(\Gamma)$. Indeed, we begin by testing (3.27a) against $\boldsymbol{\tau}_i \in \mathbf{H}(\mathrm{div}_\varrho;\Omega)$, so that applying (3.2) with $t = \varrho$ to the aforementioned $\boldsymbol{\tau}_i$ and $\xi_i \in \mathrm{H}^1(\Omega)$, and using the Dirichlet boundary condition for ξ_i (cf. (3.27d)), we get

$$\int_\Omega \boldsymbol{t}_i\cdot\boldsymbol{\tau}_i + \int_\Omega \xi_i\,\mathrm{div}(\boldsymbol{\tau}_i) = \langle \boldsymbol{\tau}_i\cdot\boldsymbol{n},\xi_{i,\mathrm{D}}\rangle \qquad \forall\,\boldsymbol{\tau}_i\in\mathbf{H}(\mathrm{div}_\varrho;\Omega)\,, \tag{3.28}$$

from which it suffices to look for ξ_i in $\mathrm{L}^\rho(\Omega)$, as previously announced. In turn, bearing in mind (3.4b) and (3.4c), we test (3.27b) against $\boldsymbol{s}_i \in \mathbf{L}^2(\Omega)$, thus arriving at

$$\kappa_i\int_\Omega \boldsymbol{t}_i\cdot\boldsymbol{s}_i - \int_\Omega \boldsymbol{\sigma}_i\cdot\boldsymbol{s}_i + q_i\,\varepsilon^{-1}\,\kappa_i\int_\Omega \xi_i\,\boldsymbol{\varphi}\cdot\boldsymbol{s}_i - \int_\Omega \xi_i\,\boldsymbol{z}\cdot\boldsymbol{s}_i = 0 \qquad \forall\,\boldsymbol{s}_i\in\mathbf{L}^2(\Omega)\,, \tag{3.29}$$

from where it only remains to observe that the second term on the left-hand side makes sense for $\boldsymbol{\sigma}_i \in \mathbf{L}^2(\Omega)$. Furthermore, assuming that f_i belongs to $\mathrm{L}^\varrho(\Omega)$, we test (3.27c) against $\eta_i \in \mathrm{L}^\rho(\Omega)$ and obtain

$$\int_\Omega \eta_i\,\mathrm{div}(\boldsymbol{\sigma}_i) - \int_\Omega \xi_i\,\eta_i + \int_\Omega \xi_i\,\mathrm{div}(\boldsymbol{z})\,\eta_i = -\int_\Omega f_i\,\eta_i \qquad \forall\,\eta_i\in\mathrm{L}^\rho(\Omega)\,, \tag{3.30}$$

whose first term on the left-hand side is well-defined if $\mathrm{div}(\boldsymbol{\sigma}_i)$ belongs to $\mathrm{L}^\varrho(\Omega)$, whence we look for $\boldsymbol{\sigma}$ in $\mathbf{H}(\mathrm{div}_\varrho;\Omega)$. In addition, being $\rho \ge r > 2$ (cf. (3.6)), Cauchy–Schwarz and Hölder inequalities indicate that the second and third terms make sense as well. Consequently, we now introduce the spaces

$$\mathcal{H}_1 := \mathbf{L}^2(\Omega)\,, \quad \mathcal{H}_2 := \mathbf{H}(\mathrm{div}_\varrho;\Omega)\,, \quad \mathcal{M} := \mathrm{L}^\rho(\Omega)\,,$$

which are endowed, respectively, with the norms

$$\|s\|_{\mathcal{H}_1} := \|s\|_{0,\Omega} \quad \forall\, s \in \mathcal{H}_1\,, \quad \|\tau\|_{\mathcal{H}_2} := \|\tau\|_{\mathrm{div}_\rho;\Omega} \quad \forall\, \tau \in \mathcal{H}_2\,, \quad \|\eta\|_{\mathcal{M}} := \|\eta\|_{0,\rho;\Omega} \quad \forall\, \eta \in \mathcal{M}\,.$$

We also define

$$\boldsymbol{\mathcal{H}} := \mathcal{H}_1 \times \mathcal{H}_2 \quad \text{with product norm} \quad \|\vec{s}\|_{\boldsymbol{\mathcal{H}}} := \|s\|_{\mathcal{H}_1} + \|\tau\|_{\mathcal{H}_2} \quad \forall\, \vec{s} := (s, \tau) \in \boldsymbol{\mathcal{H}}\,,$$

and set the notations

$$\vec{t}_i := (t_i, \sigma_i)\,, \quad \vec{r}_i := (r_i, \zeta_i)\,, \quad \vec{s}_i := (s_i, \tau_i) \in \boldsymbol{\mathcal{H}}\,.$$

Then, adding (3.28) and (3.29), and gathering the result with (3.30), we conclude that, given $(\boldsymbol{z}, \boldsymbol{\varphi}) \in \mathbf{X}_2 \times \mathrm{X}_2$, the mixed formulation of (3.27a) - (3.27d) reduces to: Find $(\vec{t}_i, \xi_i) \in \boldsymbol{\mathcal{H}} \times \mathcal{M}$ such that

$$\begin{aligned} \mathcal{A}(\vec{t}_i, \vec{s}_i) \;&+\; \mathcal{B}(\vec{s}_i, \xi_i) \;+ \mathcal{E}_{z,\varphi}(\vec{s}_i, \xi_i) \;=\; \mathcal{G}(\vec{s}_i) \qquad \forall\, \vec{s}_i \in \boldsymbol{\mathcal{H}}\,, \\ \mathcal{B}(\vec{t}_i, \eta_i) \;&-\; \mathcal{C}(\xi_i, \eta_i) \;+ \mathcal{D}_z(\xi_i, \eta_i) \;=\; \mathcal{F}(\eta_i) \qquad \forall\, \eta_i \in \mathcal{M}\,, \end{aligned} \tag{3.31}$$

where the bilinear forms $\mathcal{A} : \boldsymbol{\mathcal{H}} \times \boldsymbol{\mathcal{H}} \to \mathrm{R}$, $\mathcal{B} : \boldsymbol{\mathcal{H}} \times \mathcal{M} \to \mathrm{R}$, $\mathcal{C} : \mathcal{M} \times \mathcal{M}$, $\mathcal{D}_z : \mathcal{M} \times \mathcal{M} \to \mathrm{R}$, and $\mathcal{E}_{z,\varphi} : \boldsymbol{\mathcal{H}} \times \mathcal{M} \to \mathrm{R}$, are defined, respectively, as

$$\begin{aligned} \mathcal{A}(\vec{t}_i, \vec{s}_i) \;&:=\; \kappa_i \int_\Omega t_i \cdot s_i - \int_\Omega \sigma_i \cdot s_i + \int_\Omega \tau_i \cdot t_i \qquad \forall\, \vec{t}_i, \vec{s}_i \in \boldsymbol{\mathcal{H}}\,, \\ \mathcal{B}(\vec{s}_i, \eta_i) \;&:=\; \int_\Omega \eta_i \,\mathrm{div}(\tau_i) \qquad \forall\, (\vec{s}_i, \eta_i) \in \boldsymbol{\mathcal{H}} \times \mathcal{M}\,, \\ \mathcal{C}(\xi_i, \eta_i) \;&:=\; \int_\Omega \xi_i\, \eta_i \qquad \forall\, \xi_i, \eta_i \in \mathcal{M}\,, \\ \mathcal{D}_z(\xi_i, \eta_i) \;&:=\; \int_\Omega \xi_i \,\mathrm{div}(\boldsymbol{z})\, \eta_i \qquad \forall\, \xi_i,\, \eta_i \in \mathcal{M}\,, \quad \text{and} \\ \mathcal{E}_{z,\varphi}(\vec{s}_i, \eta_i) \;&:=\; -\int_\Omega \eta_i\, \boldsymbol{z} \cdot s_i + q_i\, \varepsilon^{-1} \kappa_i \int_\Omega \eta_i\, \boldsymbol{\varphi} \cdot s_i \qquad \forall\, (\vec{s}_i, \eta_i) \in \boldsymbol{\mathcal{H}} \times \mathcal{M}\,, \end{aligned} \tag{3.32}$$

whereas the functionals $\mathcal{G} : \boldsymbol{\mathcal{H}} \to \mathrm{R}$ and $\mathcal{F} : \mathcal{M} \to \mathrm{R}$ are given, respectively, by

$$\mathcal{G}(\vec{s}_i) := \langle \tau_i \cdot \boldsymbol{n}, \xi_{i,\mathrm{D}} \rangle \quad \text{and} \quad \mathcal{F}(\eta_i) := -\int_\Omega f_i\, \eta_i\,.$$

We remark here that, ignoring the bilinear forms $\mathcal{E}_{z,\varphi}$ and $\mathcal{D}_z$, the structure of the left-hand side of (3.31) corresponds to a perturbed saddle-point problem.

Applying once again the Cauchy–Schwarz and Hölder inequalities, and using the continuous injection $i_\rho : \mathrm{H}^1(\Omega) \to \mathrm{L}^\rho(\Omega)$, we readily show that the bilinear forms $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, and the functionals $\mathcal{G}$ and $\mathcal{F}$, are all bounded with respective constants given by

$$\begin{aligned} &\|\mathcal{A}\| := \max\left\{\kappa_i, 1\right\}, \quad \|\mathcal{B}\| := 1\,, \quad \|\mathcal{C}\| := |\Omega|^{\frac{\rho-2}{\rho}}\,, \\ &\|\mathcal{G}\| := \left(1 + \|i_\rho\|\right) \|\xi_{i,\mathrm{D}}\|_{1/2,\Gamma}\,, \quad \text{and} \quad \|\mathcal{F}\| := \|f_i\|_{0,\varrho;\Omega}\,. \end{aligned} \tag{3.33}$$

Likewise, there hold

$$\begin{aligned} |\mathcal{D}_z(\xi_i, \eta_i)| \;&\le\; \|\mathcal{D}\|\, \|\boldsymbol{z}\|_{\mathbf{X}_2}\, \|\xi_i\|_{\mathcal{M}}\, \|\eta_i\|_{\mathcal{M}} \qquad \forall\, \xi_i, \eta_i \in \mathcal{M}\,, \\ |\mathcal{E}_{z,\varphi}(\vec{s}_i, \eta_i)| \;&\le\; \|\mathcal{E}\|\, \|(\boldsymbol{z}, \boldsymbol{\varphi})\|_{\mathbf{X}_2 \times \mathrm{X}_2}\, \|\vec{s}_i\|_{\boldsymbol{\mathcal{H}}}\, \|\eta_i\|_{\mathcal{M}} \qquad \forall\, (\vec{s}_i, \eta_i) \in \boldsymbol{\mathcal{H}} \times \mathcal{M}\,, \end{aligned} \tag{3.34}$$

with

$$\|\mathcal{D}\| := 1\,, \quad \text{and} \quad \|\mathcal{E}\| := \max\left\{\varepsilon^{-1} \kappa_i, 1\right\}. \tag{3.35}$$

3.5. Weak formulation of the full coupled problem

According to Sections 3.2, 3.3, and 3.4, we conclude that, under the assumption that $\boldsymbol{f} \in \mathbf{L}^2(\Omega)$, $g \in \mathrm{L}^2(\Omega)$, $\chi_\mathrm{D} \in \mathrm{W}^{1/s,r}(\Gamma)$, $\xi_{i,\mathrm{D}} \in \mathrm{H}^{1/2}(\Gamma)$, and $f_i \in \mathrm{L}^\varrho(\Omega)$, $i \in \{1,2\}$, the primal-mixed formulation of the Biot–Poisson–Nernst–Planck problem (2.7a) - (2.9) is obtained by gathering (3.19), (3.23), and (3.31): Find $(\boldsymbol{u}, \theta) \in \mathbf{X} \times \mathbf{Q}$, $(\boldsymbol{z}, p) \in \mathbf{X}_2 \times \mathbf{Q}_1$, $(\boldsymbol{\varphi}, \chi) \in \mathrm{X}_2 \times \mathrm{M}_1$ and $(\vec{t}_i, \xi_i) \in \boldsymbol{\mathcal{H}} \times \mathcal{M}$ such that

$$
\begin{array}{llllll}
\mathbf{a}_s(\boldsymbol{u},\boldsymbol{v}) + \mathbf{b}_s(\boldsymbol{v},\theta) & & & = & \mathbf{F}_{\boldsymbol{\varphi},\boldsymbol{\xi},\boldsymbol{t}}(\boldsymbol{v}) & \forall\,\boldsymbol{v}\in\mathbf{X}\,,\\
\mathbf{b}_s(\boldsymbol{u},\vartheta) - \mathbf{c}_s(\theta,\vartheta) & & +\mathbf{e}_s(p,\vartheta) & = & 0 & \forall\,\vartheta\in\mathbf{Q}\,,\\
 & \mathbf{a}_f(\boldsymbol{z},\boldsymbol{w}) + \mathbf{d}_1(\boldsymbol{w},p) & & = & 0 & \forall\,\boldsymbol{w}\in\mathbf{X}_1\,,\\
 & \mathbf{d}_2(\boldsymbol{z},q) & +\mathbf{e}_f((\theta,p),q) & = & \mathbf{G}(q) & \forall\,q\in\mathbf{Q}_2\,,\\
a(\boldsymbol{\varphi},\boldsymbol{\psi}) + b_1(\boldsymbol{\psi},\chi) & & & = & G(\boldsymbol{\psi}) & \forall\,\boldsymbol{\psi}\in \mathrm{X}_1\,,\\
b_2(\boldsymbol{\varphi},\gamma) & & & = & F_{\boldsymbol{\xi}}(\gamma) & \forall\,\gamma\in \mathrm{M}_2\,,\\
\mathcal{A}(\vec{\boldsymbol{t}}_i,\vec{\boldsymbol{s}}_i) + \mathcal{B}(\vec{\boldsymbol{s}}_i,\xi_i) & & +\,\mathcal{E}_{\boldsymbol{z},\boldsymbol{\varphi}}(\vec{\boldsymbol{s}}_i,\xi_i) & = & \mathcal{G}(\vec{\boldsymbol{s}}_i) & \forall\,\vec{\boldsymbol{s}}_i\in\boldsymbol{\mathcal{H}}\,,\\
\mathcal{B}(\vec{\boldsymbol{t}}_i,\eta_i) - \mathcal{C}(\xi_i,\eta_i) & & +\,\mathcal{D}_{\boldsymbol{z}}(\xi_i,\eta_i) & = & \mathcal{F}(\eta_i) & \forall\,\eta_i\in\mathcal{M}\,.
\end{array}
\tag{3.36}
$$

4. Continuous solvability analysis

We proceed similarly as in [26] (see also [20,31]), and adopt a fixed-point strategy to study the solvability of (3.36). To this end, we define operators solving the decoupled problems leading to a fixed-point equation equivalent to (3.36).

4.1. Fixed-point approach

Let us define the spaces $\boldsymbol{\mathcal{H}}_1 := \mathcal{H}_1\times\mathcal{H}_1$ and $\boldsymbol{\mathcal{M}} := \mathcal{M}\times\mathcal{M}$, endowed with the product norms

$$\|\boldsymbol{r}\|_{\boldsymbol{\mathcal{H}}_1} := \|\boldsymbol{r}_1\|_{\mathcal{H}_1} + \|\boldsymbol{r}_2\|_{\mathcal{H}_1}, \qquad \|\boldsymbol{\eta}\|_{\boldsymbol{\mathcal{M}}} := \|\eta_1\|_{\mathcal{M}} + \|\eta_2\|_{\mathcal{M}},$$

for all $\boldsymbol{r} := (\boldsymbol{r}_1,\boldsymbol{r}_2)\in\boldsymbol{\mathcal{H}}_1$ and $\boldsymbol{\eta} := (\eta_1,\eta_2)\in\boldsymbol{\mathcal{M}}$, and additionally set the notations

$$\boldsymbol{t} := (\boldsymbol{t}_1,\boldsymbol{t}_2)\in\boldsymbol{\mathcal{H}}_1 \qquad\text{and}\qquad \boldsymbol{\xi} := (\xi_1,\xi_2)\in\boldsymbol{\mathcal{M}}.$$

Now, let $\mathbf{S}:\mathrm{X}_2\times\boldsymbol{\mathcal{M}}\times\boldsymbol{\mathcal{H}}_1\to\mathbf{X}_2$ be the operator defined for each $(\boldsymbol{\phi},\boldsymbol{\eta},\boldsymbol{r})\in\mathrm{X}_2\times\boldsymbol{\mathcal{M}}\times\boldsymbol{\mathcal{H}}_1$ by

$$\mathbf{S}(\boldsymbol{\phi},\boldsymbol{\eta},\boldsymbol{r}) := \boldsymbol{z}, \tag{4.1}$$

where $((\boldsymbol{u},\theta),(\boldsymbol{z},p))\in(\mathbf{X}\times\mathbf{Q})\times(\mathbf{X}_2\times\mathbf{Q}_1)$ is the unique solution (confirmed below) of (3.19) when $\mathbf{F}_{\boldsymbol{\varphi},\boldsymbol{\xi},\boldsymbol{t}}$ is replaced by $\mathbf{F}_{\boldsymbol{\phi},\boldsymbol{\eta},\boldsymbol{r}}$, that is

$$
\begin{array}{llllll}
\mathbf{a}_s(\boldsymbol{u},\boldsymbol{v}) + \mathbf{b}_s(\boldsymbol{v},\theta) & & & = & \mathbf{F}_{\boldsymbol{\phi},\boldsymbol{\eta},\boldsymbol{r}}(\boldsymbol{v}) & \forall\,\boldsymbol{v}\in\mathbf{X}\,,\\
\mathbf{b}_s(\boldsymbol{u},\vartheta) - \mathbf{c}_s(\theta,\vartheta) & & +\mathbf{e}_s(p,\vartheta) & = & 0 & \forall\,\vartheta\in\mathbf{Q}\,,\\
 & \mathbf{a}_f(\boldsymbol{z},\boldsymbol{w}) + \mathbf{d}_1(\boldsymbol{w},p) & & = & 0 & \forall\,\boldsymbol{w}\in\mathbf{X}_1\,,\\
 & \mathbf{d}_2(\boldsymbol{z},q) & +\mathbf{e}_f((\theta,p),q) & = & \mathbf{G}(q) & \forall\,q\in\mathbf{Q}_2\,.
\end{array}
\tag{4.2}
$$

In turn, let $\widetilde{\mathbf{S}}:\boldsymbol{\mathcal{M}}\to\mathrm{X}_2$ be the operator defined for each $\boldsymbol{\eta}\in\boldsymbol{\mathcal{M}}$ by

$$\widetilde{\mathbf{S}}(\boldsymbol{\eta}) := \boldsymbol{\varphi}\,,$$

where $(\boldsymbol{\varphi},\chi)\in\mathrm{X}_2\times\mathrm{M}_1$ is the unique solution (confirmed below) of (3.23) with $F_{\boldsymbol{\eta}}$ instead of $F_{\boldsymbol{\xi}}$, i.e.,

$$
\begin{array}{llll}
a(\boldsymbol{\varphi},\boldsymbol{\psi}) + b_1(\boldsymbol{\psi},\chi) & = & G(\boldsymbol{\psi}) & \forall\,\boldsymbol{\psi}\in\mathrm{X}_1\,,\\
b_2(\boldsymbol{\varphi},\gamma) & = & F_{\boldsymbol{\eta}}(\gamma) & \forall\,\gamma\in\mathrm{M}_2\,.
\end{array}
\tag{4.3}
$$

Furthermore, we let $\mathrm{T}_i:\mathbf{X}_2\times\mathrm{X}_2\to\mathcal{H}_1$ and $\Xi_i:\mathbf{X}_2\times\mathrm{X}_2\to\mathcal{M}$, $i\in\{1,2\}$, be the operators defined for each $(\boldsymbol{w},\boldsymbol{\phi})\in\mathbf{X}_2\times\mathrm{X}_2$ by

$$\mathrm{T}_i(\boldsymbol{w},\boldsymbol{\phi}) := \boldsymbol{t}_i \quad\text{and}\quad \Xi_i(\boldsymbol{w},\boldsymbol{\phi}) := \xi_i\,,$$

where $(\vec{\boldsymbol{t}}_i,\xi_i) = ((\boldsymbol{t}_i,\boldsymbol{\sigma}_i),\xi_i)\in\boldsymbol{\mathcal{H}}\times\mathcal{M}$ is the unique solution (to be confirmed below) of problem (3.31) when $\mathcal{E}_{\boldsymbol{z},\boldsymbol{\varphi}}$ and $\mathcal{D}_{\boldsymbol{z}}$ are replaced by $\mathcal{E}_{\boldsymbol{w},\boldsymbol{\phi}}$ and $\mathcal{D}_{\boldsymbol{w}}$, respectively, that is

$$
\begin{array}{llll}
\mathcal{A}(\vec{\boldsymbol{t}}_i,\vec{\boldsymbol{s}}_i) + \mathcal{B}(\vec{\boldsymbol{s}}_i,\xi_i) + \mathcal{E}_{\boldsymbol{w},\boldsymbol{\phi}}(\vec{\boldsymbol{s}}_i,\xi_i) & = & \mathcal{G}(\vec{\boldsymbol{s}}_i) & \forall\,\vec{\boldsymbol{s}}_i\in\boldsymbol{\mathcal{H}}\,,\\
\mathcal{B}(\vec{\boldsymbol{t}}_i,\eta_i) - \mathcal{C}(\xi_i,\eta_i) + \mathcal{D}_{\boldsymbol{w}}(\xi_i,\eta_i) & = & \mathcal{F}(\eta_i) & \forall\,\eta_i\in\mathcal{M}\,.
\end{array}
\tag{4.4}
$$

As a consequence, we can set the operators $\boldsymbol{\Xi}:\mathbf{X}_2\times\mathrm{X}_2\to\boldsymbol{\mathcal{M}}$ and $\mathbf{T}:\mathbf{X}_2\times\mathrm{X}_2\to\boldsymbol{\mathcal{H}}_1$ as

$$\boldsymbol{\Xi}(\boldsymbol{w},\boldsymbol{\phi}) := (\Xi_1(\boldsymbol{w},\boldsymbol{\phi}),\Xi_2(\boldsymbol{w},\boldsymbol{\phi})) = \boldsymbol{\xi} \quad\text{and}\quad \mathbf{T}(\boldsymbol{w},\boldsymbol{\phi}) := (\mathrm{T}_1(\boldsymbol{w},\boldsymbol{\phi}),\mathrm{T}_2(\boldsymbol{w},\boldsymbol{\phi})) = \boldsymbol{t},$$

for all $(\boldsymbol{w},\boldsymbol{\phi})\in\mathbf{X}_2\times\mathrm{X}_2$. Finally, we introduce the operator $\boldsymbol{\Pi}:\mathbf{X}_2\times\mathrm{X}_2\to\mathbf{X}_2\times\mathrm{X}_2$ defined for each $(\boldsymbol{w},\boldsymbol{\phi})\in\mathbf{X}_2\times\mathrm{X}_2$ by

$$\boldsymbol{\Pi}(\boldsymbol{w},\boldsymbol{\phi}) := \big(\mathbf{S}(\boldsymbol{\phi},\boldsymbol{\Xi}(\boldsymbol{w},\boldsymbol{\phi}),\mathbf{T}(\boldsymbol{w},\boldsymbol{\phi})),\widetilde{\mathbf{S}}(\boldsymbol{\Xi}(\boldsymbol{w},\boldsymbol{\phi}))\big)\,, \tag{4.5}$$

and realize that solving (3.36) is equivalent to finding a fixed point of $\boldsymbol{\Pi}$, that is, $(\boldsymbol{z},\boldsymbol{\varphi})\in\mathbf{X}_2\times\mathrm{X}_2$ such that

$$\mathbf{\Pi}(\boldsymbol{z},\boldsymbol{\varphi}) = (\boldsymbol{z},\boldsymbol{\varphi}). \tag{4.6}$$

4.2. Well-definedness of the operator S

We first apply an abstract result on perturbed saddle-point problems (cf. [9, Theorem 4.3.1]) and the generalized Babuška–Brezzi theory (cf. [7, Theorem 2.1, Corollary 2.1]) to the bilinear form arising from (4.2) when $\mathbf{e}_s$ and $\mathbf{e}_f$ are dropped, and then employ the Banach–Nečas–Babuška theorem (cf. [27, Theorem 2.6]) to conclude that (4.2) is well-posed, which is equivalent to stating that $\mathbf{S}$ (cf. (4.1)) is well-defined. For this purpose, we now introduce the spaces

$$\mathbb{X} := \mathbf{X}\times\mathbf{Q}\times\mathbf{X}_2\times\mathbf{Q}_1 \quad \text{and} \quad \mathbb{Q} := \mathbf{X}\times\mathbf{Q}\times\mathbf{X}_1\times\mathbf{Q}_2,$$

which are endowed with the norms

$$\|\vec{\boldsymbol{u}}\|_{\mathbb{X}} := \|\boldsymbol{u}\|_{\mathbf{X}} + \|\theta\|_{\mathbf{Q}} + \|z\|_{\mathbf{X}_2} + \|p\|_{\mathbf{Q}_1} \quad \forall\, \vec{\boldsymbol{u}} := (\boldsymbol{u},\theta,\boldsymbol{z},p)\in\mathbb{X}, \quad \text{and}$$
$$\|\vec{\boldsymbol{v}}\|_{\mathbb{Q}} := \|\boldsymbol{v}\|_{\mathbf{X}} + \|\vartheta\|_{\mathbf{Q}} + \|\boldsymbol{w}\|_{\mathbf{X}_1} + \|q\|_{\mathbf{Q}_2} \quad \forall\, \vec{\boldsymbol{v}} := (\boldsymbol{v},\vartheta,\boldsymbol{w},q)\in\mathbb{Q}.$$

Then, as announced, we let $\mathbf{A}:\mathbb{X}\times\mathbb{Q}\to \mathrm{R}$ be the bounded bilinear form arising from (4.2) after adding the left-hand sides of its equations, but without including $\mathbf{e}_s$ and $\mathbf{e}_f$, that is

$$\mathbf{A}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}) := \mathbf{a}_s(\boldsymbol{u},\boldsymbol{v}) + \mathbf{b}_s(\boldsymbol{v},\theta) + \mathbf{b}_s(\boldsymbol{u},\vartheta) - \mathbf{c}_s(\theta,\vartheta) + \mathbf{a}_f(\boldsymbol{z},\boldsymbol{w}) + \mathbf{d}_1(\boldsymbol{w},p) + \mathbf{d}_2(\boldsymbol{z},q), \tag{4.7}$$

for all $(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})\in\mathbb{X}\times\mathbb{Q}$. Note that the boundedness of $\mathbf{A}$ follows from that of $\mathbf{a}_s$, $\mathbf{b}_s$, $\mathbf{c}_s$, $\mathbf{a}_f$, $\mathbf{d}_1$, and $\mathbf{d}_2$ (cf. (3.20)). In addition, as remarked in Section 3.2, we note that $\mathbf{A}$ shows the matrix representation

$$\left(\begin{array}{cc|cc} \mathbf{a}_s & \mathbf{b}_s' & & \\ \mathbf{b}_s & -\mathbf{c}_s & & \\ \hline & & \mathbf{a}_f & \mathbf{d}_1' \\ & & \mathbf{d}_2 & \end{array}\right), \tag{4.8}$$

whose block-diagonal structure, composed by the perturbed and generalized saddle-point matrix operators given, respectively, by $\begin{pmatrix} \mathbf{a}_s & \mathbf{b}_s' \\ \mathbf{b}_s & -\mathbf{c}_s \end{pmatrix}$ and $\begin{pmatrix} \mathbf{a}_f & \mathbf{d}_1' \\ \mathbf{d}_2 & \end{pmatrix}$, is evident. This is an advantageous feature in proving global inf-sup conditions. More precisely, introducing

$$\mathcal{S}_1(\vec{\boldsymbol{u}}) := \sup_{\substack{\vec{\boldsymbol{v}}\in\mathbb{Q}\\ \vec{\boldsymbol{v}}\neq\mathbf{0}}} \frac{\mathbf{A}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})}{\|\vec{\boldsymbol{v}}\|_{\mathbb{Q}}} \quad \forall\,\vec{\boldsymbol{u}}\in\mathbb{X} \qquad \text{and} \qquad \mathcal{S}_2(\vec{\boldsymbol{v}}) := \sup_{\substack{\vec{\boldsymbol{u}}\in\mathbb{X}\\ \vec{\boldsymbol{u}}\neq\mathbf{0}}} \frac{\mathbf{A}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})}{\|\vec{\boldsymbol{u}}\|_{\mathbb{X}}} \quad \forall\,\vec{\boldsymbol{v}}\in\mathbb{Q},$$

we aim to prove next the existence of a positive constant $\boldsymbol{\alpha}_{\mathbf{A}}$ such that

$$\mathcal{S}_1(\vec{\boldsymbol{u}}) \geq \boldsymbol{\alpha}_{\mathbf{A}}\|\vec{\boldsymbol{u}}\|_{\mathbb{X}} \qquad \forall\,\vec{\boldsymbol{u}}\in\mathbb{X}, \quad \text{and,} \tag{4.9a}$$

$$\mathcal{S}_2(\vec{\boldsymbol{v}}) \geq \boldsymbol{\alpha}_{\mathbf{A}}\|\vec{\boldsymbol{v}}\|_{\mathbb{Q}} \qquad \forall\,\vec{\boldsymbol{v}}\in\mathbb{Q}. \tag{4.9b}$$

To this end, and according to (4.8), we decompose $\mathbf{A}$ as

$$\mathbf{A}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}) := \mathbf{A}_s((\boldsymbol{u},\theta),(\boldsymbol{v},\vartheta)) + \mathbf{A}_f((\boldsymbol{z},p),(\boldsymbol{w},q)) \qquad \forall\,(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})\in\mathbb{X}\times\mathbb{Q}, \tag{4.10}$$

where $\mathbf{A}_s:(\mathbf{X}\times\mathbf{Q})\times(\mathbf{X}\times\mathbf{Q})\to\mathrm{R}$ and $\mathbf{A}_f:(\mathbf{X}_2\times\mathbf{Q}_1)\times(\mathbf{X}_1\times\mathbf{Q}_2)\to\mathrm{R}$ are defined by

$$\mathbf{A}_s\big((\boldsymbol{u},\theta),(\boldsymbol{v},\vartheta)\big) := \mathbf{a}_s(\boldsymbol{u},\boldsymbol{v}) + \mathbf{b}_s(\boldsymbol{v},\theta) + \mathbf{b}_s(\boldsymbol{u},\vartheta) - \mathbf{c}_s(\theta,\vartheta),$$

for all $(\boldsymbol{u},\theta),\,(\boldsymbol{v},\vartheta)\in\mathbf{X}\times\mathbf{Q}$, and

$$\mathbf{A}_f\big((\boldsymbol{z},p),(\boldsymbol{w},q)\big) := \mathbf{a}_f(\boldsymbol{z},\boldsymbol{w}) + \mathbf{d}_1(\boldsymbol{w},p) + \mathbf{d}_2(\boldsymbol{z},q),$$

for all $\big((\boldsymbol{z},p),\,(\boldsymbol{w},q)\big)\in(\mathbf{X}_2\times\mathbf{Q}_1)\times(\mathbf{X}_1\times\mathbf{Q}_2)$. Thus, thanks to (4.10), it is straightforward to see that

$$\mathcal{S}_1(\vec{\boldsymbol{u}}) \geq \frac{1}{2}\left\{ \sup_{\substack{(\boldsymbol{v},\vartheta)\in\mathbf{X}\times\mathbf{Q}\\ (\boldsymbol{v},\vartheta)\neq\mathbf{0}}} \frac{\mathbf{A}_s((\boldsymbol{u},\theta),(\boldsymbol{v},\vartheta))}{\|(\boldsymbol{v},\vartheta)\|_{\mathbf{X}\times\mathbf{Q}}} + \sup_{\substack{(\boldsymbol{w},q)\in\mathbf{X}_1\times\mathbf{Q}_2\\ (\boldsymbol{w},q)\neq\mathbf{0}}} \frac{\mathbf{A}_f((\boldsymbol{z},p),(\boldsymbol{w},q))}{\|(\boldsymbol{w},q)\|_{\mathbf{X}_1\times\mathbf{Q}_2}} \right\} \quad \forall\,\vec{\boldsymbol{u}}\in\mathbb{X}, \tag{4.11}$$

whence, in order to prove (4.9a), it suffices to show that there exist positive constants $\boldsymbol{\alpha}_s$ and $\boldsymbol{\alpha}_f$ such that

$$\sup_{\substack{(\boldsymbol{v},\vartheta)\in\mathbf{X}\times\mathbf{Q}\\ (\boldsymbol{v},\vartheta)\neq\mathbf{0}}} \frac{\mathbf{A}_s((\boldsymbol{u},\theta),(\boldsymbol{v},\vartheta))}{\|(\boldsymbol{v},\vartheta)\|_{\mathbf{X}\times\mathbf{Q}}} \geq \boldsymbol{\alpha}_s\,\|(\boldsymbol{u},\theta)\|_{\mathbf{X}\times\mathbf{Q}} \qquad \forall\,(\boldsymbol{u},\theta)\in\mathbf{X}\times\mathbf{Q} \quad \text{and} \tag{4.12a}$$

$$\sup_{\substack{(\boldsymbol{w},q)\in\mathbf{X}_1\times\mathbf{Q}_2\\(\boldsymbol{w},q)\neq\mathbf{0}}}\frac{\mathbf{A}_f((\boldsymbol{z},p),(\boldsymbol{w},q))}{\|(\boldsymbol{w},q)\|_{\mathbf{X}_1\times\mathbf{Q}_2}}\geq\boldsymbol{\alpha}_f\,\|(\boldsymbol{z},p)\|_{\mathbf{X}_2\times\mathbf{Q}_1}\qquad\forall\,(\boldsymbol{z},p)\in\mathbf{X}_2\times\mathbf{Q}_1\,. \tag{4.12b}$$

Because of the matrix representation of $\mathbf{A}_s$ (cf. upper block in (4.8)), establishing (4.12a) is equivalent to proving that $\mathbf{a}_s$, $\mathbf{b}_s$ and $\mathbf{c}_s$ satisfy the hypotheses of the abstract result in Hilbert spaces [9, Theorem 4.3.1]. Indeed, we first notice from (3.15) that $\mathbf{a}_s$ and $\mathbf{c}_s$ are clearly symmetric and positive semi-definite. In addition, applying the Körn and Poincaré inequalities, which say, respectively, that $\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|^2_{0,\Omega}\geq\frac{1}{2}|\boldsymbol{v}|^2_{1,\Omega}$ and $|\boldsymbol{v}|^2_{1,\Omega}\geq C_P\,\|\boldsymbol{v}\|^2_{1,\Omega}$ for all $\boldsymbol{v}\in\mathbf{H}^1_0(\Omega)$, where C_P is a fixed positive constant, we readily deduce that

$$\mathbf{a}_s(\boldsymbol{v},\boldsymbol{v})\,=\,2\mu\,\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|^2_{0,\Omega}\geq\alpha_s\|\boldsymbol{v}\|^2_{\mathbf{X}}\quad\forall\,\boldsymbol{v}\in\mathbf{X}, \tag{4.13}$$

with the constant $\alpha_s\;=\mu\,C_P$, thus proving that $\mathbf{a}_s$ is $\mathbf{X}$-elliptic. Furthermore, we know from [32, Chapter I, eqn. (5.14)] that there exists a positive constant β_s such that

$$\sup_{\substack{\boldsymbol{v}\in\mathbf{X}\\\boldsymbol{v}\neq\mathbf{0}}}\frac{\mathbf{b}_s(\boldsymbol{v},\vartheta)}{\|\boldsymbol{v}\|_{\mathbf{X}}}\geq\beta_s\,\|\vartheta\|_{\mathbf{Q}}\qquad\forall\,\vartheta\in\mathbf{Q}. \tag{4.14}$$

Therefore, under the hypotheses of [9, Theorem 4.3.1], the estimates from [9, Proposition 2.11, eqn. (4.3.21)] imply that there exists a positive constant $\boldsymbol{\alpha}_s$, depending on $\|\mathbf{a}_s\|$, $\|\mathbf{c}_s\|$, α_s, and β_s, such that (4.12a) holds.

In turn, due to the matrix representation of $\mathbf{A}_f$ (cf. lower block in (4.8)), we realize that proving (4.12b) is equivalent to verifying that $\mathbf{a}_f$, $\mathbf{d}_1$, and $\mathbf{d}_2$ satisfy the hypotheses of the generalized Babuška–Brezzi theory (cf. [7, Theorem 2.1]). In fact, we first observe that the kernels of the bilinear forms $\mathbf{d}_i$ (cf. (3.18)), $i\in\{1,2\}$, are given, respectively, by

$$\mathbf{K}_1\;:=\Big\{\boldsymbol{w}\in\mathbf{H}^s_0(\mathrm{div}_s;\Omega):\;\mathrm{div}(\boldsymbol{w})=0\quad\text{in}\quad\Omega\Big\},\quad\mathbf{K}_2\;:=\Big\{\boldsymbol{w}\in\mathbf{H}^r_0(\mathrm{div};\Omega):\;\mathrm{div}(\boldsymbol{w})=0\quad\text{in}\quad\Omega\Big\}.$$

Lemma 4.1. *There exists a positive constant α_f such that*

$$\sup_{\boldsymbol{w}\in\mathbf{K}_1\backslash\{\mathbf{0}\}}\frac{\mathbf{a}_f(\boldsymbol{z},\boldsymbol{w})}{\|\boldsymbol{w}\|_{\mathbf{X}_1}}\geq\alpha_f\,\|\boldsymbol{z}\|_{\mathbf{X}_2}\qquad\forall\,\boldsymbol{z}\in\mathbf{K}_2\,,\quad\textit{and} \tag{4.15a}$$

$$\sup_{\boldsymbol{z}\in\mathbf{K}_2}\mathbf{a}_f(\boldsymbol{z},\boldsymbol{w})>0\qquad\forall\,\boldsymbol{w}\in\mathbf{K}_1\,,\boldsymbol{w}\neq\mathbf{0}. \tag{4.15b}$$

Proof. A minor modification of the proof of [30, Lemma 2.6], yields (4.15a) with $\alpha_f\;:=\frac{\nu}{\kappa\|D_s\|}$, where D_s is the bounded linear operator defined in [30, Lemma 2.3]. In turn, proceeding similarly, showing that

$$\sup_{\boldsymbol{z}\in\mathbf{K}_2}\mathbf{a}_f(\boldsymbol{z},\boldsymbol{w})\geq\frac{\nu}{\kappa}\,\|\boldsymbol{w}\|^s_{0,s;\Omega}\quad\forall\,\boldsymbol{w}\in\mathbf{K}_1,$$

gives (4.15b). □

Furthermore, the continuous inf-sup condition for $\mathbf{d}_1$ can be found in [30, Lemma 2.7], whereas the one for $\mathbf{d}_2$, to be commented next, uses a uniform exterior ball condition on Ω (cf. last paragraphs in Section 1). Lemmas 4.2 and 6.1 are the only places where this hypothesis is employed.

Lemma 4.2. *There exists a constant $\beta_2>0$ such that*

$$\sup_{\substack{\boldsymbol{w}\in\mathbf{X}_2\\\boldsymbol{w}\neq\mathbf{0}}}\frac{\mathbf{d}_2(\boldsymbol{w},q)}{\|\boldsymbol{w}\|_{\mathbf{X}_2}}\geq\beta_2\,\|q\|_{\mathbf{Q}_2}\qquad\forall\,q\in\mathbf{Q}_2\,. \tag{4.16}$$

Proof. Thanks to the aforementioned geometric assumption on Ω, we can use [35, Theorem 1.1], and then proceed similarly to the proof of [30, Lemma 2.7]. We omit further details and refer to [29, Lemma4.2]. □

Consequently, the required hypotheses of [7, Theorem 2.1] are satisfied, and hence the a priori estimates provided by [7, Corollary 2.1] imply that there exists a positive constant $\boldsymbol{\alpha}_f$, depending on $\|\mathbf{a}_f\|$, α_f, β_1 (the constant of the continuous inf-sup condition for $\mathbf{d}_1$ in [30, Lemma 2.7]), and β_2, such that (4.12b) holds.

Thus, having proved (4.12a) and (4.12b), the required inf-sup condition (4.9a) follows straightforwardly from (4.11), which gives the constant $\boldsymbol{\alpha}_{\mathbf{A}}\;:=\;\frac{1}{2}\min\big\{\boldsymbol{\alpha}_s,\boldsymbol{\alpha}_f\big\}$. Similarly, using that $\mathbf{A}_s$ is symmetric, and that the transpose of $\mathbf{A}_f$, defined as $\mathbf{A}^{\mathtt{t}}_f\big((\boldsymbol{w},q),(\boldsymbol{z},p)\big)\;:=\mathbf{A}_f\big((\boldsymbol{z},p),(\boldsymbol{w},q)\big)$, also satisfies the hypotheses of the generalized Babuška–Brezzi theory, we can prove (4.9b) by using analogue arguments to those yielding (4.9a). In particular, note that the matrix representation of $\mathbf{A}^{\mathtt{t}}_f$ arises from the one of $\mathbf{A}_f$ after exchanging $\mathbf{d}_1$ and $\mathbf{d}_2$, that is $\begin{pmatrix}\mathbf{a}_f & \mathbf{d}'_2\\ \mathbf{d}_1 & \end{pmatrix}$, and hence the hypotheses of [7, Theorem 2.1, Section 2.1] are clearly attained.

Now, we set the product spaces $\mathbb{X} := (\mathbf{X}\times\mathbf{Q})\times(\mathbf{X}_2\times\mathbf{Q}_1)$ and $\mathbb{Q} := (\mathbf{X}\times\mathbf{Q})\times(\mathbf{X}_1\times\mathbf{Q}_2)$, so that, given $(\boldsymbol{\phi},\boldsymbol{\eta},\boldsymbol{r})\in \mathrm{X}_2\times\boldsymbol{\mathcal{M}}\times\boldsymbol{\mathcal{H}}_1$, (4.2) is equivalent to finding $\vec{\boldsymbol{u}} = ((\boldsymbol{u},\theta),(\boldsymbol{z},p))\in\mathbb{X}$ such that

$$\mathbf{A}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}) + \mathbf{e}_s(p,\vartheta) + \mathbf{e}_f((\theta,p),q) = \mathbf{F}_{\boldsymbol{\phi},\boldsymbol{\eta},\boldsymbol{r}}(\boldsymbol{v}) + \mathbf{G}(q) \qquad \forall\,\vec{\boldsymbol{v}} = ((\boldsymbol{v},\vartheta),(\boldsymbol{w},q))\in\mathbb{Q}\,. \tag{4.17}$$

Hence, employing (4.9a), (4.9b), and the boundedness of $\|\mathbf{e}_s\|$ and $\|\mathbf{e}_f\|$ (cf. (3.20)), and assuming that

$$\max\left\{\|\mathbf{e}_s\|,\|\mathbf{e}_f\|\right\} := C_r(\Omega)\,\max\left\{c_0 + \frac{\alpha^2}{\lambda},\frac{\alpha}{\lambda}\right\} \le \frac{\alpha_{\mathbf{A}}}{2}\,, \tag{4.18}$$

we deduce that

$$\sup_{\vec{\boldsymbol{v}}\in\mathbb{Q}\backslash\{\mathbf{0}\}} \frac{\mathbf{A}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}) + \mathbf{e}_s(p,\vartheta) + \mathbf{e}_f((\theta,p),q)}{\|\vec{\boldsymbol{v}}\|_{\mathbb{Q}}} \ge \frac{\alpha_{\mathbf{A}}}{2}\|\vec{\boldsymbol{u}}\|_{\mathbb{X}} \qquad \forall\,\vec{\boldsymbol{u}}\in\mathbb{X}, \quad \text{and} \tag{4.19a}$$

$$\sup_{\vec{\boldsymbol{u}}\in\mathbb{X}\backslash\{\mathbf{0}\}} \frac{\mathbf{A}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}) + \mathbf{e}_s(p,\vartheta) + \mathbf{e}_f((\theta,p),q)}{\|\vec{\boldsymbol{u}}\|_{\mathbb{Q}}} \ge \frac{\alpha_{\mathbf{A}}}{2}\|\vec{\boldsymbol{v}}\|_{\mathbb{Q}} \qquad \forall\,\vec{\boldsymbol{v}}\in\mathbb{Q}\,. \tag{4.19b}$$

Note that (4.18) becomes feasible for sufficiently small c_0 and for the quasi-incompressible case ($\lambda\to+\infty$).

We are now in a position to establish the well-definedness of $\mathbf{S}$.

Lemma 4.3. *Assume that the data satisfy (4.18). Then, for each $(\boldsymbol{\phi},\boldsymbol{\eta},\boldsymbol{r})\in \mathrm{X}_2\times\boldsymbol{\mathcal{M}}\times\boldsymbol{\mathcal{H}}_1$, there exists a unique $((\boldsymbol{u},\theta),(\boldsymbol{z},p))\in(\mathbf{X}\times\mathbf{Q})\times(\mathbf{X}_2\times\mathbf{Q}_1)$ solution to (4.2), and hence we can define $\mathbf{S}(\boldsymbol{\phi},\boldsymbol{\eta},\boldsymbol{r}) := \boldsymbol{z}\in\mathbf{X}_2$. Moreover, there exists a positive constant $C_{\mathbf{S}}$, depending on $\alpha_{\mathbf{A}}$, ε, and δ, such that*

$$\begin{aligned}\|\mathbf{S}(\boldsymbol{\phi},\boldsymbol{\eta},\boldsymbol{r})\|_{\mathbf{X}_2} &= \|\boldsymbol{z}\|_{\mathbf{X}_2} \le \|(\boldsymbol{u},\theta)\|_{\mathbf{X}\times\mathbf{Q}} + \|(\boldsymbol{z},p)\|_{\mathbf{X}_2\times\mathbf{Q}_1}\\ &\le C_{\mathbf{S}}\left\{\|\boldsymbol{f}\|_{0,\Omega} + \|g\|_{0,\Omega} + \|\boldsymbol{\eta}\|_{\boldsymbol{\mathcal{M}}}\,\|\boldsymbol{\phi}\|_{0,r;\Omega} + \|\boldsymbol{r}\|_{\boldsymbol{\mathcal{H}}_1}\right\}.\end{aligned} \tag{4.20}$$

Proof. Thanks to the boundedness of $\mathbf{A}$, $\mathbf{e}_s$, and $\mathbf{e}_f$, and the global inf-sup conditions (4.19a) and (4.19b), a direct application of the Banach–Nečas–Babuška theorem (cf. [27, Theorem 2.6]) provides the existence of a unique solution to (4.2). The a priori estimate (4.20) follows from [27, eqn. (2.5)] along with the boundedness of $\mathbf{F}_{\boldsymbol{\phi},\boldsymbol{\eta},\boldsymbol{r}}$ (cf. (3.21)) and $\mathbf{G}$ (cf. (3.20)). □

4.3. Well-definedness of the operator $\widetilde{\mathbf{S}}$

We now prove that (4.3) is well-posed (equivalently, that $\widetilde{\mathbf{S}}$ is well-defined) resorting to the analysis in [26, Section 4.2.2]. The inf-sup conditions for a, b_1, and b_2 that are required by the Babuška–Brezzi theory (cf. [7, Theorem 2.1, Corollary 2.1, Section 2.1]) for the unique solvability of (4.3), were established in [26, Lemmas 4.3 and 4.4] with constants that here we denote $\widetilde{\alpha}$, $\widetilde{\beta}_1$, and $\widetilde{\beta}_2$, respectively. In particular, recall that the analysis for a involves the kernels K_i of the forms b_i, $i\in\{1,2\}$. Thus, a simple application of the theory above implies the following result, which, up to minor differences, coincides with [26, Theorem 4.5].

Lemma 4.4. *For each $\boldsymbol{\eta} = (\eta_1,\eta_2)\in\boldsymbol{\mathcal{M}}$, there exists a unique $(\boldsymbol{\varphi},\chi)\in \mathrm{X}_2\times\mathrm{M}_1$ solution to (4.3), and hence one can define $\widetilde{\mathrm{S}}(\boldsymbol{\eta}) := \boldsymbol{\varphi}\in\mathrm{X}_2$. Moreover, there exist positive constants $C_{\widetilde{\mathbf{S}}}$ and $\widetilde{C}_{\widetilde{\mathbf{S}}}$, which depend on ε, c_r (cf. (3.26)), $|\Omega|$, ρ, r, $\widetilde{\alpha}$, $\widetilde{\beta}_1$, and $\widetilde{\beta}_2$, such that*

$$\|\widetilde{\mathrm{S}}(\boldsymbol{\eta})\|_{\mathrm{X}_2} = \|\boldsymbol{\varphi}\|_{\mathrm{X}_2} \le C_{\widetilde{\mathbf{S}}}\left\{\|\chi_{\mathrm{D}}\|_{1/s,r;\Gamma} + \|\boldsymbol{\eta}\|_{0,\rho;\Omega}\right\}, \quad \textit{and} \tag{4.21a}$$

$$\|\chi\|_{\mathrm{M}_1} \le \widetilde{C}_{\widetilde{\mathbf{S}}}\left\{\|\chi_{\mathrm{D}}\|_{1/s,r;\Gamma} + \|\boldsymbol{\eta}\|_{0,\rho;\Omega}\right\}. \tag{4.21b}$$

Proof. We omit further details and just mention that the derivations of (4.21a) and (4.21b) make use of the boundedness of $F_{\boldsymbol{\eta}}$ (cf. (3.25)) and G (cf. (3.26)). □

4.4. Well-definedness of the operators $\mathbf{T}$ and $\boldsymbol{\Xi}$

In this section, we follow the approaches from [20, Section 3.2.2] and [31, Section 3.3] to prove that the operators $\mathbf{T}$ and $\boldsymbol{\Xi}$ are well-defined. More precisely, we first apply [7, Theorem 2.1] and [31, Theorem 3.2] to the formulation arising from (4.4) when $\mathcal{E}_{\boldsymbol{w},\boldsymbol{\phi}}$ and $\mathcal{D}_{\boldsymbol{w}}$ are dropped, and then employ the Banach–Nečas–Babuška theorem (cf. [27, Theorem 2.6]) to conclude that the full system (4.4) is well-posed for each $i\in\{1,2\}$. To this end, as announced above, and similarly as in Section 4.2, we let $\boldsymbol{\mathcal{A}} : (\boldsymbol{\mathcal{H}}\times\mathcal{M})\times(\boldsymbol{\mathcal{H}}\times\mathcal{M})\to\mathrm{R}$ be the bounded bilinear defined by the sum of the left-hand sides of (4.4), excluding $\mathcal{D}_{\boldsymbol{w}}$ and $\mathcal{E}_{\boldsymbol{w},\boldsymbol{\phi}}$, that is

$$\boldsymbol{\mathcal{A}}((\vec{\boldsymbol{r}}_i,\xi_i),(\vec{\boldsymbol{s}}_i,\eta_i)) := \mathcal{A}(\vec{\boldsymbol{r}}_i,\vec{\boldsymbol{s}}_i) + \mathcal{B}(\vec{\boldsymbol{s}}_i,\xi_i) + \mathcal{B}(\vec{\boldsymbol{r}}_i,\eta_i) - \mathcal{C}(\xi_i,\eta_i) \qquad \forall(\vec{\boldsymbol{r}}_i,\xi_i),(\vec{\boldsymbol{s}}_i,\eta_i)\in\boldsymbol{\mathcal{H}}\times\mathcal{M}, \tag{4.22}$$

and proceed to show next that $\boldsymbol{\mathcal{A}}$ is inf-sup stable with respect to its first and second components. Needless to say, the boundedness of $\boldsymbol{\mathcal{A}}$ follows from those of $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ (cf. (3.33)).

It follows from (4.22) that the aforementioned property for $\boldsymbol{\mathcal{A}}$ is equivalent to proving that the bilinear forms $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, satisfy the hypotheses of [31, Theorem 3.2], which is actually a slight improvement of the original result for perturbed saddle-point problems provided by [24, Theorem 3.4]. In this regard, we first notice from (3.32) that $\mathcal{A}$ and $\mathcal{C}$ are positive semi-definite, that is

$$\mathcal{A}(\vec{s}_i,\vec{s}_i) \geq \kappa_i \|s_i\|_{0,\Omega}^2 \geq 0 \qquad \forall\, \vec{s}_i \in \boldsymbol{\mathcal{H}}\,, \quad \text{and} \quad \mathcal{C}(\eta_i,\eta_i) = \|\eta_i\|_{0,\Omega}^2 \geq 0 \qquad \forall\, \eta_i \in \mathcal{M}\,.$$

In turn, it is readily seen that $\mathcal{C}$ is symmetric, and that the null space V of $\mathcal{B}$ is given by

$$V := \mathcal{H}_1 \times V_0\,, \quad \text{where} \quad V_0 := \Big\{ \boldsymbol{\tau}_i \in \mathbf{H}(\mathrm{div}_\varrho;\Omega): \quad \mathrm{div}(\boldsymbol{\tau}_i) = 0 \quad \text{in} \quad \Omega \Big\}. \tag{4.23}$$

In addition, $\mathcal{A}$ shows the matrix representation $\begin{pmatrix} A & B_1 \\ B_2 & \end{pmatrix}$, where $A: \mathcal{H}_1 \times \mathcal{H}_1 \to \mathrm{R}$, $B_1 : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathrm{R}$, and $B_2 : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathrm{R}$ are the bilinear forms defined as

$$\begin{aligned} A(\boldsymbol{t}_i,\boldsymbol{s}_i) &:= \kappa_i \int_\Omega \boldsymbol{t}_i \cdot \boldsymbol{s}_i \qquad && \forall\, \boldsymbol{t}_i,\boldsymbol{s}_i \in \mathcal{H}_1 \\ B_1(\boldsymbol{s}_i,\boldsymbol{\tau}_i) &:= -\int_\Omega \boldsymbol{\tau}_i \cdot \boldsymbol{s}_i \qquad && \forall\, (\boldsymbol{s}_i,\boldsymbol{\tau}_i) \in \mathcal{H}_1 \times \mathcal{H}_2\,, \\ B_2(\boldsymbol{s}_i,\boldsymbol{\tau}_i) &:= \int_\Omega \boldsymbol{\tau}_i \cdot \boldsymbol{s}_i \qquad && \forall\, (\boldsymbol{s}_i,\boldsymbol{\tau}_i) \in \mathcal{H}_1 \times \mathcal{H}_2\,. \end{aligned} \tag{4.24}$$

According to the above, and similarly as in Section 4.2, we deduce that in order for $\mathcal{A}$ to satisfy the inf-sup conditions specified in [31, eqns. (3.31) and (3.32)], we just need to prove that A, B_1, and B_2 verify the hypothesis of [7, Theorem 2.1]. In particular, it is easily seen that A is $\mathcal{H}_1$-elliptic since

$$A(\boldsymbol{s}_i,\boldsymbol{s}_i) = \kappa_i\, \|\boldsymbol{s}_i\|_{0,\Omega}^2 \quad \forall\, \boldsymbol{s}_i \in \mathcal{H}_1\,, \tag{4.25}$$

and hence A satisfies the assumptions of [7, Theorem 2.1, eqns. (2.8) and (2.9)]. Note that this holds irrespective of the conditions defining the kernels $\mathcal{K}_j$ of $B_j|_{\mathbf{L}^2(\Omega)\times V_0}$, $j \in \{1,2\}$, which, due to the fact that $B_1 = -B_2$, are given by

$$\mathcal{K}_1 = \mathcal{K}_2 = \mathcal{K} := \Big\{ \boldsymbol{s}_i \in \mathcal{H}_1 : \quad \int_\Omega \boldsymbol{s}_i \cdot \boldsymbol{\tau}_i = 0 \quad \forall\, \boldsymbol{\tau}_i \in V_0 \Big\}. \tag{4.26}$$

Indeed, all what is needed is that $\mathcal{K}$ be contained in $\mathcal{H}_1$. Furthermore, regarding the bilinear forms B_1 and B_2, we now consider $\boldsymbol{\tau}_i \in V_0$ (cf. (4.23)), that is $\boldsymbol{\tau}_i \in \mathbf{H}(\mathrm{div}_\varrho;\Omega)$ such that $\mathrm{div}(\boldsymbol{\tau}_i) = 0$ in Ω, and observe that, bounding by below with $\boldsymbol{s}_i = -\boldsymbol{\tau}_i$ (for B_1) and $\boldsymbol{s}_i = \boldsymbol{\tau}_i$ (for B_2), there holds for each $j \in \{1,2\}$

$$\sup_{\boldsymbol{s}_i \in \mathcal{H}_1 \backslash \{\mathbf{0}\}} \frac{B_j(\boldsymbol{s}_i,\boldsymbol{\tau}_i)}{\|\boldsymbol{s}_i\|_{\mathcal{H}_1}} = \sup_{\boldsymbol{s}_i \in \mathbf{L}^2(\Omega) \backslash \{\mathbf{0}\}} \frac{B_j(\boldsymbol{s}_i,\boldsymbol{\tau}_i)}{\|\boldsymbol{s}_i\|_{0,\Omega}} \geq \|\boldsymbol{\tau}_i\|_{0,\Omega} = \|\boldsymbol{\tau}_i\|_{\mathcal{H}_2} \qquad \forall\, \boldsymbol{\tau}_i \in V_0\,. \tag{4.27}$$

Hence, thanks to (4.25) and (4.27), we can apply [7, Theorem 2.1] to conclude that there exists a positive $\widehat{\alpha}$, depending only on κ_i, such that the whole bilinear form $\mathcal{A}$ satisfies

$$\sup_{\vec{s}_i \in V \backslash \{\mathbf{0}\}} \frac{\mathcal{A}(\vec{r}_i,\vec{s}_i)}{\|\vec{s}_i\|_{\boldsymbol{\mathcal{H}}}} \geq \widehat{\alpha}\, \|\vec{r}_i\|_{\boldsymbol{\mathcal{H}}} \qquad \forall\, \vec{r}_i \in V\,. \tag{4.28}$$

Moreover, exchanging the roles of B_1 and B_2, and applying again [7, Theorem 2.1], we conclude that

$$\sup_{\vec{r}_i \in V \backslash \{\mathbf{0}\}} \frac{\mathcal{A}(\vec{r}_i,\vec{s}_i)}{\|\vec{r}_i\|_{\boldsymbol{\mathcal{H}}}} \geq \widehat{\alpha}\, \|\vec{s}_i\|_{\boldsymbol{\mathcal{H}}} \qquad \forall\, \vec{s}_i \in V\,.$$

On the other hand, we know from [30, Lemma 2.9] (see also [20, eqn. (3.23)]) that $\mathcal{B}$ (cf. (3.32)) satisfies the required continuous inf-sup condition, which means that there exists a positive constant $\beta_{\mathcal{B}}$ such that

$$\sup_{\vec{s}_i \in \boldsymbol{\mathcal{H}} \backslash \{\mathbf{0}\}} \frac{\mathcal{B}(\vec{s}_i,\eta_i)}{\|\vec{s}_i\|_{\boldsymbol{\mathcal{H}}}} \geq \beta_{\mathcal{B}}\, \|\eta_i\|_{0,\rho,\Omega} \qquad \forall\, \eta_i \in \mathcal{M}\,.$$

Thus, having $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ satisfied the hypotheses of [31, Theorem 3.2], we deduce the existence of a positive constant $\boldsymbol{\alpha}_{\mathcal{A}}$, depending only on $\widehat{\alpha}$, $\beta_{\mathcal{B}}$, $\|\mathcal{A}\|$, and $\|\mathcal{C}\|$, such that

$$\sup_{\substack{(\vec{s}_i,\eta_i) \in \boldsymbol{\mathcal{H}} \times \mathcal{M} \\ (\vec{s}_i,\eta_i) \neq \mathbf{0}}} \frac{\boldsymbol{\mathcal{A}}((\vec{r}_i,\xi_i),(\vec{s}_i,\eta_i))}{\|(\vec{s}_i,\eta_i)\|_{\boldsymbol{\mathcal{H}} \times \mathcal{M}}} \geq \boldsymbol{\alpha}_{\mathcal{A}}\, \|(\vec{r}_i,\xi_i)\|_{\boldsymbol{\mathcal{H}} \times \mathcal{M}} \qquad \forall\, (\vec{r}_i,\xi_i) \in \boldsymbol{\mathcal{H}} \times \mathcal{M}\,, \quad \text{and} \tag{4.29a}$$

$$\sup_{\substack{(\vec{r}_i,\widetilde{\xi}_i)\in\mathcal{H}\times\mathcal{M}\\(\vec{r}_i,\xi_i)\neq\mathbf{0}}}\frac{\mathcal{A}((\vec{r}_i,\xi_i),(\vec{s}_i,\eta_i))}{\|(\vec{r}_i,\xi_i)\|_{\mathcal{H}\times\mathcal{M}}}\geq\alpha_{\mathcal{A}}\,\|(\vec{s}_i,\eta_i)\|_{\mathcal{H}\times\mathcal{M}}\qquad\forall(\vec{s}_i,\eta_i)\in\mathcal{H}\times\mathcal{M}. \tag{4.29b}$$

Going back to (4.4) with the given $(\boldsymbol{w},\boldsymbol{\phi})\in\mathbf{X}_2\times\mathrm{X}_2$, we let $\mathcal{A}_{\boldsymbol{w},\boldsymbol{\phi}}:(\mathcal{H}\times\mathcal{M})\times(\mathcal{H}\times\mathcal{M})\to\mathrm{R}$ be the bounded bilinear form arising after adding its left-hand sides, that is (cf. (4.22))

$$\mathcal{A}_{\boldsymbol{w},\boldsymbol{\phi}}((\vec{r}_i,\xi_i),(\vec{s}_i,\eta_i)) := \mathcal{A}((\vec{r}_i,\xi_i),(\vec{s}_i,\eta_i)) + \mathcal{E}_{\boldsymbol{w},\boldsymbol{\phi}}(\vec{s}_i,\xi_i) + \mathcal{D}_{\boldsymbol{w}}(\xi_i,\eta_i), \tag{4.30}$$

for all $(\vec{r}_i,\xi_i),(\vec{s}_i,\eta_i)\in\mathcal{H}\times\mathcal{M}$. In this way, (4.4) is rewritten as: Find $(\vec{t}_i,\xi_i)\in\mathcal{H}\times\mathcal{M}$ such that

$$\mathcal{A}_{\boldsymbol{w},\boldsymbol{\phi}}((\vec{t}_i,\xi_i),(\vec{s}_i,\eta_i)) = \mathcal{G}(\vec{s}_i) + \mathcal{F}(\eta_i)\qquad\forall(\vec{s}_i,\eta_i)\in\mathcal{H}\times\mathcal{M}. \tag{4.31}$$

Note that the boundedness of $\mathcal{A}$, $\mathcal{E}_{\boldsymbol{w},\boldsymbol{\phi}}$, and $\mathcal{D}_{\boldsymbol{w}}$ (cf. (3.33), (3.34), and (3.35)) guarantees that $\mathcal{A}_{\boldsymbol{w},\boldsymbol{\phi}}$ is bounded as well. In turn, bearing in mind (4.30), (4.29a), (4.29b), and again the boundedness of $\mathcal{E}_{\boldsymbol{w},\boldsymbol{\phi}}$ and $\mathcal{D}_{\boldsymbol{w}}$, and assuming that

$$\|(\boldsymbol{w},\boldsymbol{\phi})\|_{\mathbf{X}_2\times\mathrm{X}_2}\leq\mathtt{R} := \frac{\alpha_{\mathcal{A}}}{2\,\max\left\{\|\mathcal{D}\|,\|\mathcal{E}\|\right\}}, \tag{4.32}$$

we conclude that

$$\sup_{\substack{(\vec{s}_i,\eta_i)\in\mathcal{H}\times\mathcal{M}\\(\vec{s}_i,\eta_i)\neq\mathbf{0}}}\frac{\mathcal{A}_{\boldsymbol{w},\boldsymbol{\phi}}((\vec{r}_i,\xi_i),(\vec{s}_i,\eta_i))}{\|(\vec{s}_i,\eta_i)\|_{\mathcal{H}\times\mathcal{M}}}\geq\frac{\alpha_{\mathcal{A}}}{2}\|(\vec{r}_i,\xi_i)\|_{\mathcal{H}\times\mathcal{M}}\qquad\forall(\vec{r}_i,\xi_i)\in\mathcal{H}\times\mathcal{M},\quad\text{and} \tag{4.33}$$

$$\sup_{\substack{(\vec{r}_i,\xi_i)\in\mathcal{H}\times\mathcal{M}\\(\vec{r}_i,\xi_i)\neq\mathbf{0}}}\frac{\mathcal{A}_{\boldsymbol{w},\boldsymbol{\phi}}((\vec{r}_i,\xi_i),(\vec{s}_i,\eta_i))}{\|(\vec{r}_i,\widetilde{\xi}_i)\|_{\mathcal{H}\times\mathcal{M}}}\geq\frac{\alpha_{\mathcal{A}}}{2}\|(\vec{s}_i,\eta_i)\|_{\mathcal{H}\times\mathcal{M}}\qquad\forall(\vec{s}_i,\eta_i)\in\mathcal{H}\times\mathcal{M}. \tag{4.34}$$

The well-definedness of the components of $\mathbf{T}$ and $\boldsymbol{\Xi}$, and hence of themselves, can be stated now.

Lemma 4.5. *For each $i\in\{1,2\}$, and for each $(\boldsymbol{w},\boldsymbol{\phi})\in\mathbf{X}_2\times\mathrm{X}_2$ satisfying (4.32), there exists a unique $(\vec{t}_i,\xi_i)=\big((\boldsymbol{t}_i,\boldsymbol{\sigma}_i),\xi_i\big)\in\mathcal{H}\times\mathcal{M}$ solution of (4.4), and hence we can define $\mathrm{T}_i(\boldsymbol{w},\boldsymbol{\phi}):=\boldsymbol{t}_i\in\mathcal{H}_1$ and $\Xi_i(\boldsymbol{w},\boldsymbol{\phi}):=\xi_i\in\mathcal{M}$. Moreover, there exists a positive constant $C_{\mathbf{T}}$, independent of $(\boldsymbol{w},\boldsymbol{\phi})$, such that*

$$\begin{aligned}\|\mathrm{T}_i(\boldsymbol{w},\boldsymbol{\phi})\|_{\mathcal{H}_1} + \|\Xi_i(\boldsymbol{w},\boldsymbol{\phi})\|_{\mathcal{M}} &= \|\boldsymbol{t}_i\|_{\mathcal{H}_1} + \|\xi_i\|_{\mathcal{M}}\\ &\leq \|(\vec{t}_i,\xi_i)\|_{\mathcal{H}\times\mathcal{M}} \leq C_{\mathbf{T}}\left\{\|\xi_{i,\mathrm{D}}\|_{1/2,\Gamma} + \|f_i\|_{0,\varrho;\Omega}\right\}.\end{aligned} \tag{4.35}$$

Proof. Thanks to (4.33) and (4.34), the proof reduces to a direct application of [27, Theorem 2.6], where the derivation of the a priori estimate (4.35) makes use of the expressions for $\|\mathcal{G}\|$ and $\|\mathcal{F}\|$ given by (3.33). □

4.5. Solvability analysis of the fixed-point equation

Knowing that $\mathbf{S}$, $\widetilde{\mathbf{S}}$, $\mathbf{T}$, $\boldsymbol{\Xi}$ (and hence $\boldsymbol{\Pi}$ as well) are well-defined, we address the solvability of the fixed-point equation (4.5) by means of the Banach fixed-point theorem. First we define the ball

$$\mathrm{W}(\mathtt{R}) := \left\{(\boldsymbol{w},\boldsymbol{\phi})\in\mathbf{X}_2\times\mathrm{X}_2:\quad\|(\boldsymbol{w},\boldsymbol{\phi})\|_{\mathbf{X}_2\times\mathrm{X}_2}\leq\mathtt{R}\right\}, \tag{4.36}$$

where $\mathtt{R}>0$ is defined in (4.32), and provide next a condition on the data ensuring that $\boldsymbol{\Pi}$ maps $\mathrm{W}(\mathtt{R})$ into itself. In fact, bearing in mind the definition of $\boldsymbol{\Pi}$ (cf. (4.5)), and employing the a priori estimates for $\mathbf{S}$, $\widetilde{\mathbf{S}}$, $\mathbf{T}$, and $\boldsymbol{\Xi}$ (cf. (4.20), (4.21a), and (4.35), we deduce the existence of a positive constant $C(\mathtt{R})$, depending only on $C_{\mathbf{S}}$, $C_{\widetilde{\mathbf{S}}}$, $C_{\mathbf{T}}$, and $\mathtt{R}$, such that for each $(\boldsymbol{w},\boldsymbol{\phi})\in\mathrm{W}(\mathtt{R})$ there holds

$$\|\boldsymbol{\Pi}(\boldsymbol{w},\boldsymbol{\phi})\|_{\mathbf{X}_2\times\mathrm{X}_2}\leq C(\mathtt{R})\left\{\|\boldsymbol{f}\|_{0,\Omega} + \|g\|_{0,\Omega} + \|\chi_{\mathrm{D}}\|_{1/s,r;\Gamma} + \sum_{i=1}^{2}\Big(\|\xi_{i,\mathrm{D}}\|_{1/2,\Gamma} + \|f_i\|_{0,\varrho;\Omega}\Big)\right\}. \tag{4.37}$$

A straightforward consequence of (4.37) implies the following result.

Lemma 4.6. *Assume that the data are sufficiently small so that*

$$C(\mathtt{R})\left\{\|\boldsymbol{f}\|_{0,\Omega} + \|g\|_{0,\Omega} + \|\chi_{\mathrm{D}}\|_{1/s,r;\Gamma} + \sum_{i=1}^{2}\Big(\|\xi_{i,\mathrm{D}}\|_{1/2,\Gamma} + \|f_i\|_{0,\varrho;\Omega}\Big)\right\}\leq\mathtt{R}. \tag{4.38}$$

Then, $\boldsymbol{\Pi}\big(\mathrm{W}(\mathtt{R})\big)\subseteq\mathrm{W}(\mathtt{R})$.

Our following goal is to show that $\mathbf{\Pi}$ is Lipschitz-continuous, for which it suffices to show that $\mathbf{S}$, $\widetilde{\mathbf{S}}$, Ξ, and $\mathbf{T}$ satisfy suitable continuity properties. We begin with the corresponding result for $\mathbf{S}$.

Lemma 4.7. *There exists a positive constant $L_{\mathbf{S}}$, depending on ε, δ, and $\boldsymbol{\alpha}_{\mathbf{A}}$, such that*

$$\|\mathbf{S}(\boldsymbol{\varphi},\xi,\boldsymbol{t})-\mathbf{S}(\boldsymbol{\phi},\boldsymbol{\eta},\boldsymbol{r})\|_{\mathrm{X}_2} \le L_{\mathbf{S}}\left\{\|\xi\|_{\mathcal{M}}\|\boldsymbol{\varphi}-\boldsymbol{\phi}\|_{\mathrm{X}_2}+\|\boldsymbol{\phi}\|_{\mathrm{X}_2}\|\xi-\boldsymbol{\eta}\|_{\mathcal{M}}+\|\boldsymbol{t}-\boldsymbol{r}\|_{\mathcal{H}_1}\right\}, \tag{4.39}$$

for all $(\boldsymbol{\varphi},\xi,\boldsymbol{t})$, $(\boldsymbol{\phi},\boldsymbol{\eta},\boldsymbol{r})\in \mathrm{X}_2\times\mathcal{M}\times\mathcal{H}_1$.

Proof. It follows in the usual way, namely defining $\mathbf{S}(\boldsymbol{\varphi},\xi,\boldsymbol{t})$ and $\mathbf{S}(\boldsymbol{\phi},\boldsymbol{\eta},\boldsymbol{r})$ in terms of the respective solutions of (4.2), and then applying the global inf-sup condition (4.19a). We omit further details and refer to the preprint version of the present work (cf. [29, Lemma 4.7]). □

Next, we resort to a result proven in [26, Lemma 4.9] to establish the continuity of $\widetilde{\mathbf{S}}$.

Lemma 4.8. *There exists $L_{\widetilde{\mathbf{S}}}>0$, depending only on Ω, the inf-sup constants $\widetilde{\alpha}$ and $\widetilde{\beta}_2$ (cf. Section 4.3), and $\|a\|$ (cf. (3.24)), such that*

$$\|\widetilde{\mathbf{S}}(\xi)-\widetilde{\mathbf{S}}(\boldsymbol{\eta})\|_{\mathrm{X}_2} \le L_{\widetilde{\mathbf{S}}}\|\xi-\boldsymbol{\eta}\|_{\mathcal{M}} \qquad \forall\,\xi,\boldsymbol{\eta}\in\mathcal{M}. \tag{4.40}$$

Recalling that $\mathrm{W}(\mathrm{R})$ is the closed ball defined by (4.36), we now prove the continuity of $\mathbf{T}$ and Ξ.

Lemma 4.9. *There exists $L_{\mathbf{T}}>0$, depending only on $\boldsymbol{\alpha}_{\mathcal{A}}$, $C_{\mathbf{T}}$, ε, and κ_i, $i\in\{1,2\}$, such that*

$$\begin{aligned}&\|\mathbf{T}(z,\boldsymbol{\varphi})-\mathbf{T}(\boldsymbol{w},\boldsymbol{\phi})\|_{\mathcal{H}_1}+\|\Xi(z,\boldsymbol{\varphi})-\Xi(\boldsymbol{w},\boldsymbol{\phi})\|_{\mathcal{M}}\\ &\quad\le L_{\mathbf{T}}\sum_{i=1}^{2}\left\{\|\xi_{i,\mathrm{D}}\|_{1/2,\Gamma}+\|f_i\|_{0,\varrho;\Omega}\right\}\|(z,\boldsymbol{\varphi})-(\boldsymbol{w},\boldsymbol{\phi})\|_{\mathbf{X}_2\times\mathrm{X}_2},\end{aligned} \tag{4.41}$$

for all $(z,\boldsymbol{\varphi})$, $(\boldsymbol{w},\boldsymbol{\phi})\in\mathrm{W}(\mathrm{R})$.

Proof. Given $(z,\boldsymbol{\varphi})$, $(\boldsymbol{w},\boldsymbol{\phi})\in\mathrm{W}(\mathrm{R})$, we let for each $i\in\{1,2\}$

$$\mathrm{T}_i(z,\boldsymbol{\varphi}) := \boldsymbol{t}_i\in\mathcal{H}_1,\quad \Xi_i(z,\boldsymbol{\varphi}) := \xi_i\in\mathcal{M},\quad \mathrm{T}_i(\boldsymbol{w},\boldsymbol{\phi}) := \boldsymbol{r}_i\in\mathcal{H}_1,\quad\text{and}\quad \Xi_i(\boldsymbol{w},\boldsymbol{\phi}) := \varkappa_i\in\mathcal{M},$$

where $(\vec{\boldsymbol{t}}_i,\xi_i)=\big((\boldsymbol{t}_i,\boldsymbol{\sigma}_i),\xi_i\big)$, $(\vec{\boldsymbol{r}}_i,\varkappa_i)=\big((\boldsymbol{r}_i,\boldsymbol{\zeta}_i),\varkappa_i\big)\in\boldsymbol{\mathcal{H}}\times\mathcal{M}$ are the respective solutions of (4.31), that is

$$\boldsymbol{\mathcal{A}}_{z,\boldsymbol{\varphi}}\big((\vec{\boldsymbol{t}}_i,\xi_i),(\vec{\boldsymbol{s}}_i,\eta_i)\big)=\mathcal{G}(\vec{\boldsymbol{s}}_i)+\mathcal{F}(\eta_i)\quad\text{and}\quad\boldsymbol{\mathcal{A}}_{\boldsymbol{w},\boldsymbol{\phi}}((\vec{\boldsymbol{r}}_i,\varkappa_i),(\vec{\boldsymbol{s}}_i,\eta_i))=\mathcal{G}(\vec{\boldsymbol{s}}_i)+\mathcal{F}(\eta_i),$$

for all $(\vec{\boldsymbol{s}}_i,\eta_i)\in\boldsymbol{\mathcal{H}}\times\mathcal{M}$. It follows from the foregoing identities and the definitions of the bilinear forms $\boldsymbol{\mathcal{A}}_{\boldsymbol{w},\boldsymbol{\phi}}$ (cf. (4.22), (4.30)), and $\mathcal{D}_{\boldsymbol{w},\boldsymbol{\phi}}$ and $\mathcal{E}_{\boldsymbol{w}}$ (cf. (3.32)), that

$$\begin{aligned}\boldsymbol{\mathcal{A}}_{z,\boldsymbol{\varphi}}\big((\vec{\boldsymbol{t}}_i,\xi_i)-(\vec{\boldsymbol{r}}_i,\varkappa_i),(\vec{\boldsymbol{s}}_i,\eta_i)\big)&=\boldsymbol{\mathcal{A}}_{z,\boldsymbol{\varphi}}\big((\vec{\boldsymbol{t}}_i,\xi_i),(\vec{\boldsymbol{s}}_i,\eta_i)\big)-\boldsymbol{\mathcal{A}}_{z,\boldsymbol{\varphi}}\big((\vec{\boldsymbol{r}}_i,\varkappa_i),(\vec{\boldsymbol{s}}_i,\eta_i)\big)\\ &=\boldsymbol{\mathcal{A}}_{\boldsymbol{w},\boldsymbol{\phi}}\big((\vec{\boldsymbol{r}}_i,\varkappa_i),(\vec{\boldsymbol{s}}_i,\eta_i)\big)-\boldsymbol{\mathcal{A}}_{z,\boldsymbol{\varphi}}\big((\vec{\boldsymbol{r}}_i,\varkappa_i),(\vec{\boldsymbol{s}}_i,\eta_i)\big)\\ &=\mathcal{E}_{\boldsymbol{w}-z,\boldsymbol{\phi}-\boldsymbol{\varphi}}(\vec{\boldsymbol{s}}_i,\varkappa_i)+\mathcal{D}_{\boldsymbol{w}-z}(\varkappa_i,\eta_i).\end{aligned} \tag{4.42}$$

Hence, applying the global inf-sup condition (4.33) to the bilinear form $\boldsymbol{\mathcal{A}}_{z,\boldsymbol{\varphi}}$ and the vector $(\vec{\boldsymbol{t}}_i,\xi_i)-(\vec{\boldsymbol{r}}_i,\varkappa_i)$, and employing (4.42) and the boundedness of $\mathcal{E}_{\boldsymbol{w},\boldsymbol{\phi}}$ and $\mathcal{D}_{\boldsymbol{w}}$ (cf. (3.34)), we find that

$$\begin{aligned}\|(\vec{\boldsymbol{t}}_i,\xi_i)-(\vec{\boldsymbol{r}}_i,\varkappa_i)\|_{\boldsymbol{\mathcal{H}}\times\mathcal{M}}&\le\frac{2}{\boldsymbol{\alpha}_{\mathcal{A}}}\sup_{\substack{(\vec{\boldsymbol{s}}_i,\eta_i)\in\boldsymbol{\mathcal{H}}\times\mathcal{M}\\(\vec{\boldsymbol{s}}_i,\eta_i)\neq\mathbf{0}}}\frac{\mathcal{E}_{\boldsymbol{w}-z,\boldsymbol{\phi}-\boldsymbol{\varphi}}(\vec{\boldsymbol{s}}_i,\varkappa_i)+\mathcal{D}_{\boldsymbol{w}-z}(\varkappa_i,\eta_i)}{\|(\vec{\boldsymbol{s}}_i,\eta_i)\|_{\boldsymbol{\mathcal{H}}\times\mathcal{M}}}\\ &\le\frac{2\max\left\{\|\mathcal{E}\|,\|\mathcal{D}\|\right\}}{\boldsymbol{\alpha}_{\mathcal{A}}}\|\varkappa_i\|_{0,\rho,\Omega}\|(z,\boldsymbol{\varphi})-(\boldsymbol{w},\boldsymbol{\phi})\|_{\mathbf{X}_2\times\mathrm{X}_2},\end{aligned}$$

from which, along with the a priori estimate (4.35) for $\|\varkappa_i\|_{0,\varrho,\Omega}$, $i\in\{1,2\}$, and the expressions for $\|\mathcal{E}\|$ and $\|\mathcal{D}\|$ (cf. (3.35)), we conclude (4.41) with $L_{\mathbf{T}}$ as indicated. □

Given $(z,\boldsymbol{\varphi})$, $(\boldsymbol{w},\boldsymbol{\phi})\in\mathrm{W}(\mathrm{R})$, the Lipschitz-continuity of $\mathbf{S}$ (cf. (4.39)), $\widetilde{\mathbf{S}}$ (cf. (4.40)), $\mathbf{T}$ and Ξ (cf. (4.41)), the a priori estimate for $\|\Xi(z,\boldsymbol{\varphi})\|$ (cf. (4.35)), and the fact that $\|\boldsymbol{\phi}\|\le\mathrm{R}$, we deduce the existence of a positive constant $L_{\mathbf{\Pi}}$, depending only on $L_{\mathbf{S}}$, $C_{\mathbf{T}}$, $L_{\widetilde{\mathbf{S}}}$, $L_{\mathbf{T}}$, and R, such that

$$\|\mathbf{\Pi}(z,\boldsymbol{\varphi})-\mathbf{\Pi}(\boldsymbol{w},\boldsymbol{\phi})\|\le L_{\mathbf{\Pi}}\sum_{i=1}^{2}\left\{\|\xi_{i,\mathrm{D}}\|_{1/2;\Gamma}+\|f_i\|_{0,\varrho;\Omega}\right\}\|(z,\boldsymbol{\varphi})-(\boldsymbol{w},\boldsymbol{\phi})\|. \tag{4.43}$$

As a consequence of (4.43), we state next the main result of this section.

Theorem 4.10. *Besides* (4.18) *and* (4.38)*, assume that the data satisfy*

$$L_{\mathbf{\Pi}} \sum_{i=1}^{2} \left\{ \|\xi_{i,\mathrm{D}}\|_{1/2;\Gamma} + \|f_i\|_{0,\varrho;\Omega} \right\} < 1\,. \tag{4.44}$$

Then, the fixed-point equation (4.6) *has a unique solution* $(\boldsymbol{z},\boldsymbol{\varphi}) \in \mathrm{W}(\mathrm{R})$*. Equivalently, the coupled problem* (3.36) *has a unique solution* $\big((\boldsymbol{u},\theta),(\boldsymbol{z},p)\big) \in (\mathbf{X}\times\mathbf{Q})\times(\mathbf{X}_2\times\mathbf{Q}_1)$*,* $(\boldsymbol{\varphi},\chi)\in \mathrm{X}_2\times\mathrm{M}_1$*, and* $(\vec{\boldsymbol{t}}_i,\xi_i)\in\boldsymbol{\mathcal{H}}\times\mathcal{M}, i\in\{1,2\}$*. Moreover, the following a priori estimates hold true*

$$\begin{aligned}
\|(\boldsymbol{u},\theta)\|_{\mathbf{X}\times\mathbf{Q}} + \|(\boldsymbol{z},p)\|_{\mathbf{X}_2\times\mathbf{Q}_1} &\le \widetilde{C}_{\mathbf{S}}\left\{ \|\boldsymbol{f}\|_{0,\Omega} + \|g\|_{0,\Omega} + \sum_{i=1}^{2}\big(\|\xi_{i,\mathrm{D}}\|_{1/2;\Gamma} + \|f_i\|_{0,\varrho;\Omega}\big)\right\},\\
\|(\boldsymbol{\varphi},\chi)\|_{\mathrm{X}_2\times\mathrm{M}_1} &\le \widetilde{C}_{\widetilde{\mathbf{S}}}\left\{ \|\chi_{\mathrm{D}}\|_{1/s,r;\Gamma} + \sum_{i=1}^{2}\big(\|\xi_{i,\mathrm{D}}\|_{1/2;\Gamma} + \|f_i\|_{0,\varrho;\Omega}\big)\right\},\\
\|(\vec{\boldsymbol{t}}_i,\xi_i)\|_{\boldsymbol{\mathcal{H}}\times\mathcal{M}} &\le C_{\mathbf{T}}\left\{ \|\xi_{i,\mathrm{D}}\|_{1/2,\Gamma} + \|f_i\|_{0,\varrho;\Omega}\right\}, \quad i\in\{1,2\},
\end{aligned}$$

where $\widetilde{C}_{\mathbf{S}}$ *and* $\widetilde{C}_{\widetilde{\mathbf{S}}}$ *are positive constants depending only on* $C_{\mathbf{S}}$*,* $C_{\widetilde{\mathbf{S}}}$*,* $C_{\mathbf{T}}$*, and* R*.*

Proof. Lemma 4.6 guarantees that $\mathbf{\Pi}$ maps $\mathrm{W}(\mathrm{R})$ into itself. Hence, by virtue of the equivalence between (3.36) and (4.6), the Lipschitz-continuity of $\mathbf{\Pi}$ (cf. (4.43)) and the hypothesis (4.44), we have that Banach's fixed-point theorem implies the well-posedness of (4.6) and equivalently of (3.36). In addition, the a priori estimates follow straightforwardly from (4.20), (4.21a), (4.21b), (4.35), and bounding $\|\boldsymbol{\varphi}\|_{0,r;\Omega}$, which appears in the original version of the first estimate above (cf. (4.20)), by R. □

5. A Galerkin scheme

We now introduce a Galerkin scheme for (3.36), and analyze its well-posedness by means of the discrete analogue of the fixed-point approach developed in Section 4. In particular, for the discrete solvability associated with the decoupled problems from Sections 4.2, 4.3, and 4.4, we employ [9, Theorem 4.3.1], and the discrete versions of [27, Theorem 2.6], [7, Theorem 2.1, Corollary 2.1], and [24, Theorem 3.4], which are given by [27, Theorem 2.22], [7, Corollary 2.2], and [24, Theorem 3.5], respectively.

5.1. Preliminaries

Let us consider arbitrary finite element subspaces of the continuous spaces indicated as follows

$$\mathbf{X}_h \subseteq \mathbf{H}_0^1(\Omega),\quad \mathbf{Q}_h \subseteq \mathrm{L}^2(\Omega),\quad \mathbf{X}_{2,h} \subseteq \mathbf{H}_0^r(\mathrm{div};\Omega),\quad \mathbf{X}_{1,h} \subseteq \mathbf{H}_0^s(\mathrm{div}_s;\Omega),\quad \mathbf{Q}_{1,h} \subseteq \mathrm{L}_0^r(\Omega),\quad \mathbf{Q}_{2,h} \subseteq \mathrm{L}_0^2(\Omega),$$
$$\mathrm{X}_{2,h} \subseteq \mathrm{X}_2\,,\quad \mathrm{X}_{1,h} \subseteq \mathrm{X}_1\,,\quad \mathrm{M}_{1,h} \subseteq \mathrm{M}_1\,,\quad \mathrm{M}_{2,h} \subseteq \mathrm{M}_2,\quad \mathcal{H}_{1,h} \subseteq \mathcal{H}_1\,,\quad \mathcal{H}_{2,h} \subseteq \mathcal{H}_2\,,\quad \text{and}\quad \mathcal{M}_h \subseteq \mathcal{M}\,.$$

Hereafter, h stands for both the sub-index of each foregoing subspace and the size of a regular triangulation $\mathcal{T}_h$ of $\bar{\Omega}$ made up of triangles K (when $n=2$) or tetrahedra K (when $n=3$) of diameter h_K, i.e., $h := \max\left\{h_K : \quad K\in\mathcal{T}_h\right\}$. Specific finite element subspaces satisfying the stability conditions to be introduced along the analysis will be provided later on in Section 6. Then, setting the notation

$$\vec{\boldsymbol{t}}_{i,h} := (\boldsymbol{t}_{i,h},\boldsymbol{\sigma}_{i,h}),\quad \vec{\boldsymbol{r}}_{i,h} := (\boldsymbol{r}_{i,h},\boldsymbol{\zeta}_{i,h}),\quad \text{and}\quad \vec{\boldsymbol{s}}_{i,h} := (\boldsymbol{s}_{i,h},\boldsymbol{\tau}_{i,h}) \in \boldsymbol{\mathcal{H}}_h := \mathcal{H}_{1,h}\times\mathcal{H}_{2,h}\,,$$

the Galerkin scheme associated with (3.36) reads: Find $(\boldsymbol{u}_h,\theta_h)\in\mathbf{X}_h\times\mathbf{Q}_h$, $(\boldsymbol{z}_h,p_h)\in\mathbf{X}_{2,h}\times\mathbf{Q}_{1,h}$, $(\boldsymbol{\varphi}_h,\chi_h)\in\mathrm{X}_{2,h}\times\mathrm{M}_{1,h}$, and $(\vec{\boldsymbol{t}}_{i,h},\xi_{i,h})\in\boldsymbol{\mathcal{H}}_h\times\mathcal{M}_h$, $i\in\{1,2\}$, such that

$$\begin{array}{lllll}
\mathbf{a}_s(\boldsymbol{u}_h,\boldsymbol{v}_h) + \mathbf{b}_s(\boldsymbol{v}_h,\theta_h) & & & = & \mathbf{F}_{\boldsymbol{\varphi}_h,\xi_h,\boldsymbol{t}_h}(\boldsymbol{v}_h)\,,\\
\mathbf{b}_s(\boldsymbol{u}_h,\vartheta_h) - \mathbf{c}_s(\theta_h,\vartheta_h) & & +\,\mathbf{e}_s(p_h,\vartheta_h) & = & 0\,,\\
 & \mathbf{a}_f(\boldsymbol{z}_h,\boldsymbol{w}_h) + \mathbf{d}_1(\boldsymbol{w}_h,p_h) & & = & 0\,,\\
 & \mathbf{d}_2(\boldsymbol{z}_h,q_h) & +\,\mathbf{e}_f((\theta_h,p_h),q_h) & = & \mathbf{G}(q_h)\,,\\
a(\boldsymbol{\varphi}_h,\boldsymbol{\psi}_h) + b_1(\boldsymbol{\psi}_h,\chi_h) & & & = & G(\boldsymbol{\psi}_h)\,,\\
b_2(\boldsymbol{\varphi}_h,\gamma_h) & & & = & F_{\xi_h}(\gamma_h)\,,\\
\mathcal{A}(\vec{\boldsymbol{t}}_{i,h},\vec{\boldsymbol{s}}_{i,h}) + \mathcal{B}(\vec{\boldsymbol{s}}_{i,h},\xi_{i,h}) & & +\,\mathcal{E}_{\boldsymbol{z}_h,\boldsymbol{\varphi}_h}(\vec{\boldsymbol{s}}_{i,h},\xi_{i,h}) & = & \mathcal{G}(\vec{\boldsymbol{s}}_{i,h})\,,\\
\mathcal{B}(\vec{\boldsymbol{t}}_{i,h},\eta_{i,h}) - \mathcal{C}(\xi_{i,h},\eta_{i,h}) & & +\,\mathcal{D}_{\boldsymbol{z}_h}(\xi_{i,h},\eta_{i,h}) & = & \mathcal{F}(\eta_{i,h})\,,
\end{array} \tag{5.1}$$

for all $(\boldsymbol{v}_h,\vartheta_h)\in\mathbf{X}_h\times\mathbf{Q}_h$, $(\boldsymbol{w}_h,q_h)\in\mathbf{X}_{1,h}\times\mathbf{Q}_{2,h}$, $(\boldsymbol{\psi}_h,\gamma_h)\in\mathrm{X}_{1,h}\times\mathrm{M}_{2,h}$, and $(\vec{\boldsymbol{s}}_{i,h},\eta_{i,h})\in\boldsymbol{\mathcal{H}}_h\times\mathcal{M}_h$.

5.2. Discrete fixed-point approach

In order to analyze the solvability of (5.1), we introduce next the discrete version of the strategy employed in Section 4.1. We begin by adopting the notation

$$\boldsymbol{t}_h := (\boldsymbol{t}_{1,h}, \boldsymbol{t}_{2,h}), \quad \boldsymbol{r}_h := (\boldsymbol{r}_{1,h}, \boldsymbol{r}_{2,h}) \in \boldsymbol{\mathcal{H}}_{1,h} := \mathcal{H}_{1,h} \times \mathcal{H}_{1,h},$$
$$\boldsymbol{\xi}_h := (\xi_{1,h}, \xi_{2,h}), \quad \boldsymbol{\eta}_h := (\eta_{1,h}, \eta_{2,h}) \in \boldsymbol{\mathcal{M}}_h := \mathcal{M}_h \times \mathcal{M}_h,$$

and by letting $\mathbf{S}_h : \mathrm{X}_{2,h} \times \boldsymbol{\mathcal{M}}_h \times \boldsymbol{\mathcal{H}}_{1,h} \to \mathbf{X}_{2,h}$ be the operator defined by

$$\mathbf{S}_h(\boldsymbol{\phi}_h, \boldsymbol{\eta}_h, \boldsymbol{r}_h) := \boldsymbol{z}_h \qquad \forall (\boldsymbol{\phi}_h, \boldsymbol{\eta}_h, \boldsymbol{r}_h) \in \mathrm{X}_{2,h} \times \boldsymbol{\mathcal{M}}_h \times \boldsymbol{\mathcal{H}}_{1,h},$$

where $(\boldsymbol{u}_h, \theta_h) \in \mathbf{X}_h \times \mathbf{Q}_h$ and $(\boldsymbol{z}_h, p_h) \in \mathbf{X}_{2,h} \times \mathbf{Q}_{1,h}$ constitute the unique solution (to be confirmed) of the first four rows of (5.1) with $\mathbf{F}_{\boldsymbol{\phi}_h, \boldsymbol{\eta}_h, \boldsymbol{r}_h}$ instead of $\mathbf{F}_{\boldsymbol{\varphi}_h, \boldsymbol{\xi}_h, \boldsymbol{t}_h}$. Similarly, we define $\widetilde{\mathbf{S}}_h : \boldsymbol{\mathcal{M}}_h \to \mathrm{X}_{2,h}$ as

$$\widetilde{\mathbf{S}}_h(\boldsymbol{\eta}_h) := \boldsymbol{\varphi}_h \quad \forall \boldsymbol{\eta}_h \in \boldsymbol{\mathcal{M}}_h,$$

where $(\boldsymbol{\varphi}_h, \chi_h) \in \mathrm{X}_{2,h} \times \mathrm{M}_{1,h}$ is the unique solution (to be confirmed) of the fifth and sixth rows of (5.1) with $F_{\boldsymbol{\eta}_h}$ instead of $F_{\boldsymbol{\xi}_h}$. Furthermore, we let $\mathrm{T}_{i,h} : \mathbf{X}_{2,h} \times \mathrm{X}_{2,h} \to \mathcal{H}_{1,h}$ and $\Xi_{i,h} : \mathbf{X}_{2,h} \times \mathrm{X}_{2,h} \to \mathcal{M}_h$, $i \in \{1,2\}$, be the operators given for each $(\boldsymbol{w}_h, \boldsymbol{\phi}_h) \in \mathbf{X}_{2,h} \times \mathrm{X}_{2,h}$ by

$$\mathrm{T}_{i,h}(\boldsymbol{w}_h, \boldsymbol{\phi}_h) := \boldsymbol{t}_{i,h} \quad \text{and} \quad \Xi_{i,h}(\boldsymbol{w}_h, \boldsymbol{\phi}_h) := \xi_{i,h},$$

where $(\vec{\boldsymbol{t}}_{i,h}, \xi_{i,h}) = \big((\boldsymbol{t}_{i,h}, \boldsymbol{\sigma}_{i,h}), \xi_{i,h}\big) \in \boldsymbol{H}_h \times \mathrm{M}_h$ is the unique solution (to be confirmed) of the last two rows of (5.1) with $\mathcal{E}_{\boldsymbol{w}_h, \boldsymbol{\phi}_h}$ and $\mathcal{D}_{\boldsymbol{w}_h}$ instead of $\mathcal{E}_{\boldsymbol{z}_h, \boldsymbol{\varphi}_h}$ and $\mathcal{D}_{\boldsymbol{z}_h}$, respectively. Hence, we can set the operators $\mathbf{T}_h : \mathbf{X}_{2,h} \times \mathrm{X}_{2,h} \to \boldsymbol{\mathcal{H}}_{1,h}$ and $\boldsymbol{\Xi}_h : \mathbf{X}_{2,h} \times \mathrm{X}_{2,h} \to \boldsymbol{\mathcal{M}}_h$ as

$$\mathbf{T}_h(\boldsymbol{w}_h, \boldsymbol{\phi}_h) := (\mathrm{T}_{1,h}(\boldsymbol{w}_h, \boldsymbol{\phi}_h), \mathrm{T}_{2,h}(\boldsymbol{w}_h, \boldsymbol{\phi}_h)) = \boldsymbol{t}_h, \quad \text{and}$$
$$\boldsymbol{\Xi}_h(\boldsymbol{w}_h, \boldsymbol{\phi}_h) := (\Xi_{1,h}(\boldsymbol{w}_h, \boldsymbol{\phi}_h), \Xi_{2,h}(\boldsymbol{w}_h, \boldsymbol{\phi}_h)) = \boldsymbol{\xi}_h,$$

for all $(\boldsymbol{w}_h, \boldsymbol{\phi}_h) \in \mathbf{X}_{2,h} \times \mathrm{X}_{2,h}$. Finally, introducing the operator $\boldsymbol{\Pi}_h : \mathbf{X}_{2,h} \times \mathrm{X}_{2,h} \to \mathbf{X}_{2,h} \times \mathrm{X}_{2,h}$ defined as

$$\boldsymbol{\Pi}_h(\boldsymbol{w}_h, \boldsymbol{\phi}_h) := \big(\mathbf{S}_h(\boldsymbol{\phi}_h, \boldsymbol{\Xi}_h(\boldsymbol{w}_h, \boldsymbol{\phi}_h), \mathbf{T}_h(\boldsymbol{w}_h, \boldsymbol{\phi}_h)), \widetilde{\mathbf{S}}_h(\boldsymbol{\Xi}_h(\boldsymbol{w}_h, \boldsymbol{\phi}_h))\big) \quad \forall (\boldsymbol{w}_h, \boldsymbol{\phi}_h) \in \mathbf{X}_{2,h} \times \mathrm{X}_{2,h},$$

we see that solving (5.1) is equivalent to finding a fixed-point of $\boldsymbol{\Pi}_h$, i.e., $(\boldsymbol{z}_h, \boldsymbol{\varphi}_h) \in \mathbf{X}_{2,h} \times \mathrm{X}_{2,h}$ such that

$$\boldsymbol{\Pi}_h(\boldsymbol{z}_h, \boldsymbol{\varphi}_h) = (\boldsymbol{z}_h, \boldsymbol{\varphi}_h). \tag{5.2}$$

5.3. Well-definedness of the operator $\mathbf{S}_h$

In what follows we proceed as in Section 4.2. In fact, we first observe that the symmetry, positive semi-definiteness, and ellipticity of $\mathbf{a}_s$ and $\mathbf{c}_s$ remain valid in the discrete context. In particular, $\mathbf{a}_s$ is $\mathbf{X}_h$-elliptic with the same constant $\alpha_s := \mu C_P$ (cf. (4.13)). Next, in order to continue the analysis, we need to assume the discrete version of (4.14):
(H.1) there exists a positive constant $\beta_{s,\mathrm{d}}$, independent of h, such that

$$\sup_{\substack{\boldsymbol{v}_h \in \mathbf{X}_h \\ \boldsymbol{v}_h \neq \mathbf{0}}} \frac{\mathbf{b}_s(\boldsymbol{v}_h, \vartheta_h)}{\|\boldsymbol{v}_h\|_{\mathbf{Q}}} \geq \beta_{s,\mathrm{d}} \|\vartheta_h\|_{\mathbf{Q}} \qquad \forall \vartheta_h \in \mathbf{Q}_h.$$

Thanks to this, we can apply again [9, Theorem 4.3.1] to deduce the discrete analogue of (4.12a) with a positive constant $\boldsymbol{\alpha}_{s,\mathrm{d}}$, depending on $\|\mathbf{a}_s\|$, $\|\mathbf{c}_s\|$, α_s, and $\beta_{s,\mathrm{d}}$.

On the other hand, we now introduce the discrete kernels of $\mathbf{d}_i$, $i \in \{1,2\}$, namely

$$\mathbf{K}_{1,h} := \Big\{\boldsymbol{w}_h \in \mathbf{X}_{1,h} : \quad \mathbf{d}_1(\boldsymbol{w}_h, q_h) = 0 \quad \forall q_h \in \mathbf{Q}_{1,h}\Big\} \quad \text{and}$$
$$\mathbf{K}_{2,h} := \Big\{\boldsymbol{w}_h \in \mathbf{X}_{2,h} : \quad \mathbf{d}_2(\boldsymbol{w}_h, q_h) = 0 \quad \forall q_h \in \mathbf{Q}_{2,h}\Big\},$$

and consider the following additional hypotheses:
(H.2) there exists a positive constant $\alpha_{f,\mathrm{d}}$, independent of h, such that

$$\sup_{\substack{\boldsymbol{w}_h \in \mathbf{K}_{1,h} \\ \boldsymbol{w}_h \neq \mathbf{0}}} \frac{\mathbf{a}_f(\boldsymbol{z}_h, \boldsymbol{w}_h)}{\|\boldsymbol{w}_h\|_{\mathbf{X}_1}} \geq \alpha_{f,\mathrm{d}} \|\boldsymbol{z}_h\|_{\mathbf{X}_2} \quad \forall \boldsymbol{z}_h \in \mathbf{K}_{2,h}, \quad \sup_{\boldsymbol{z}_h \in \mathbf{K}_{2,h}} \mathbf{a}_f(\boldsymbol{z}_h, \boldsymbol{w}_h) > 0 \quad \forall \boldsymbol{w}_h \in \mathbf{K}_{1,h} \setminus \{\mathbf{0}\}, \quad \text{and}$$

(H.3) for each $i \in \{1,2\}$ there exists a positive constant $\beta_{i,\mathrm{d}}$, independent of h, such that

$$\sup_{\substack{\boldsymbol{w}_h\in\mathbf{X}_{i,h}\\ \boldsymbol{w}_h\neq\mathbf{0}}} \frac{\mathbf{d}_i(\boldsymbol{w}_h,q_h)}{\|\boldsymbol{w}_h\|_{\mathbf{X}_i}} \geq \beta_{i,\mathrm{d}}\|q_h\|_{\mathbf{Q}_i} \quad \forall\, q_h\in\mathbf{Q}_{i,h}\,.$$

These assumptions and [7, Corollary 2.2, Section 2.2] yield the discrete version of (4.12b) with a constant $\boldsymbol{\alpha}_{f,\mathrm{d}}>0$ depending only on $\|\mathbf{a}_f\|$, $\alpha_{f,\mathrm{d}}$, $\beta_{1,\mathrm{d}}$, and $\beta_{2,\mathrm{d}}$ Then, a direct application of [27, Proposition 2.42], along with the discrete version of (4.11), imply, in turn, the discrete version of (4.9a) with constant $\boldsymbol{\alpha}_{\mathbf{A},\mathrm{d}} := \frac{1}{2}\min\left\{\boldsymbol{\alpha}_{s,\mathrm{d}},\boldsymbol{\alpha}_{f,\mathrm{d}}\right\}$. Moreover, using again the symmetry of $\mathbf{A}_s$ and $\mathbf{A}_f^{\mathrm{t}}$ (as in the continuous analysis), we can prove the discrete analogue of (4.9b). Consequently, under the discrete counterpart of (4.18), i.e.,

$$\max\left\{\|\mathbf{e}_s\|,\|\mathbf{e}_f\|\right\} := C_r(\Omega)\,\max\left\{c_0+\frac{\alpha^2}{\lambda},\frac{\alpha}{\lambda}\right\} \leq \frac{\boldsymbol{\alpha}_{\mathbf{A},\mathrm{d}}}{2}\,, \tag{5.3}$$

we arrive at the discrete versions of (4.19a) and (4.19b), and hence we can state the following result.

Lemma 5.1. *Assume that the data satisfy* (5.3). *Then, for each* $(\boldsymbol{\phi}_h,\boldsymbol{\eta}_h,\boldsymbol{r}_h)\in \mathrm{X}_{2,h}\times\boldsymbol{\mathcal{M}}_h\times\boldsymbol{\mathcal{H}}_{1,h}$, *there exists a unique* $((\boldsymbol{u}_h,\theta_h),(\boldsymbol{z}_h,p_h))\in(\mathbf{X}_h\times\mathbf{Q}_h)\times(\mathbf{X}_{2,h}\times\mathbf{Q}_{1,h})$ *solution of the first four rows of* (5.1), *and hence one can define* $\mathbf{S}_h(\boldsymbol{\phi}_h,\boldsymbol{\eta}_h,\boldsymbol{r}_h):=\boldsymbol{z}_h\in\mathbf{X}_{2,h}$. *Moreover, there exists a positive constant* $C_{\mathbf{S},\mathrm{d}}$, *depending only on* $\boldsymbol{\alpha}_{\mathbf{A},\mathrm{d}}$, ε, *and* δ, *such that*

$$\begin{aligned}\|\mathbf{S}_h(\boldsymbol{\phi}_h,\boldsymbol{\eta}_h,\boldsymbol{r}_h)\|_{\mathbf{X}_2} &= \|\boldsymbol{z}_h\|_{\mathbf{X}_2} \leq \|(\boldsymbol{u}_h,\theta_h)\|_{\mathbf{X}\times\mathbf{Q}} + \|(\boldsymbol{z}_h,p_h)\|_{\mathbf{X}_2\times\mathbf{Q}_1}\\ &\leq C_{\mathbf{S},\mathrm{d}}\left\{\|\boldsymbol{f}\|_{0,\Omega}+\|g\|_{0,\Omega}+\|\boldsymbol{\eta}_h\|_{\boldsymbol{\mathcal{M}}}\,\|\boldsymbol{\phi}_h\|_{0,r;\Omega}+\|\boldsymbol{r}_h\|_{\boldsymbol{\mathcal{H}}_1}\right\}.\end{aligned} \tag{5.4}$$

Proof. Similarly to the proof of Lemma 4.3, the result follows as a direct application of the discrete Banach–Nečas–Babuška Theorem (cf. [27, Theorem 2.22]). □

5.4. Well-definedness of the operator $\widetilde{\mathbf{S}}_h$

We begin by letting $K_{i,h}$ be the discrete kernel of b_i, $i\in\{1,2\}$, that is

$$K_{i,h} := \left\{\psi_h\in \mathrm{X}_{i,h}:\quad b_i(\psi_h,\gamma_h)=0 \quad \forall\,\gamma\in \mathrm{M}_{i,h}\right\},$$

and by assuming the following hypotheses

(H.4) there exists a positive constant $\widetilde{\alpha}_{\mathrm{d}}$, independent of h, such that

$$\sup_{\substack{\psi_h\in K_{1,h}\\ \psi_h\neq\mathbf{0}}} \frac{a(\phi_h,\psi_h)}{\|\psi_h\|_{\mathrm{X}_1}} \geq \widetilde{\alpha}_{\mathrm{d}}\,\|\phi_h\|_{\mathrm{X}_2} \qquad \forall\,\phi_h\in K_{2,h}\,,$$

$$\sup_{\phi_h\in K_{2,h}} a(\phi_h,\psi_h) > 0 \qquad \forall\,\psi_h\in K_{1,h}\,,\quad \psi_h\neq\mathbf{0}\,,\quad \text{and}$$

(H.5) for each $i\in\{1,2\}$ there exists a positive constant $\widetilde{\beta}_{i,\mathrm{d}}$, independent of h, such that

$$\sup_{\substack{\psi_h\in \mathrm{X}_{i,h}\\ \psi_h\neq\mathbf{0}}} \frac{b_i(\psi_h,\gamma_h)}{\|\psi_h\|_{\mathrm{X}_i}} \geq \widetilde{\beta}_{i,\mathrm{d}}\,\|\gamma_h\|_{\mathrm{M}_i} \qquad \forall\,\gamma_h\in \mathrm{M}_{i,h}\,.$$

As a consequence of **(H.4)** and **(H.5)** we are able to state now the discrete version of Lemma 4.4

Lemma 5.2. *For each* $\boldsymbol{\eta}_h=(\eta_{1,h},\eta_{2,h})\in\boldsymbol{\mathcal{M}}_h$, *there exists a unique* $(\boldsymbol{\varphi}_h,\chi_h)\in \mathrm{X}_{2,h}\times \mathrm{M}_{1,h}$ *solution of the fifth and sixth rows of* (5.1), *and hence one can define* $\widetilde{\mathbf{S}}_h(\boldsymbol{\eta}_h):=\boldsymbol{\varphi}_h\in \mathrm{X}_{2,h}$. *Moreover, there exist positive constants* $C_{\widetilde{\mathbf{S}},\mathrm{d}}$ *and* $\widetilde{C}_{\widetilde{\mathbf{S}},\mathrm{d}}$, *which depend on* ε, c_r *(cf.* (3.26)), $|\Omega|$, ρ, r, $\widetilde{\alpha}_{\mathrm{d}}$, $\widetilde{\beta}_{1,\mathrm{d}}$, *and* $\widetilde{\beta}_{2,\mathrm{d}}$, *such that*

$$\|\widetilde{\mathbf{S}}_h(\boldsymbol{\eta}_h)\|_{\mathrm{X}_2} = \|\boldsymbol{\varphi}_h\|_{\mathrm{X}_2} \leq C_{\widetilde{\mathbf{S}},\mathrm{d}}\left\{\|\chi_{\mathrm{D}}\|_{1/s,r;\Gamma}+\|\boldsymbol{\eta}_h\|_{0,\rho;\Omega}\right\}, \quad \textit{and} \tag{5.5a}$$

$$\|\chi_h\|_{\mathrm{M}_1} \leq \widetilde{C}_{\widetilde{\mathbf{S}},\mathrm{d}}\left\{\|\chi_{\mathrm{D}}\|_{1/s,r;\Gamma}+\|\boldsymbol{\eta}_h\|_{0,\rho;\Omega}\right\}. \tag{5.5b}$$

Proof. It reduces to a direct application of [7, Corollary 2.2, eqns. (2.24), (2.25)]. □

5.5. Well-definedness of the operators $\mathbf{T}_h$ *and* $\boldsymbol{\Xi}_h$

We proceed as in Section 4.3. Firstly, the positive semi-definiteness and symmetry of $\mathcal{A}$ and $\mathcal{C}$ are also valid in the discrete context. In turn, it is easily seen that the discrete Kernel V_h of $\mathcal{B}$ is given by

$$V_h := \mathcal{H}_{1,h} \times V_{0,h}, \quad \text{where} \quad V_{0,h} := \left\{ \boldsymbol{\tau}_{i,h} \in \mathcal{H}_{2,h} : \int_\Omega \eta_{i,h} \operatorname{div}(\boldsymbol{\tau}_{i,h}) = 0 \quad \forall\, \eta_{i,h} \in \mathcal{M}_h \right\}.$$

Thus, assuming the hypothesis
(H.6) $\operatorname{div}(\mathcal{H}_{2,h}) \subseteq \mathcal{M}_h$,
we readily deduce that $V_{0,h} := \left\{ \boldsymbol{\tau}_{i,h} \in \mathcal{H}_{2,h} : \operatorname{div}(\boldsymbol{\tau}_{i,h}) = 0 \quad \text{in} \quad \Omega \right\}$, which constitutes the discrete version of (4.23). In addition, since the $\mathcal{H}_1$-ellipticity of A (cf. (4.25)) is naturally inherited by the subspace $\mathcal{H}_{1,h}$, we conclude that the required discrete inf-sup conditions are clearly satisfied by A. Next, we assume that
(H.7) $V_{0,h} \subseteq \mathcal{H}_{1,h}$,
which allows to prove that B_1 and B_2 satisfy the discrete inf-sup condition specified in [7, eqn. (2.22)]. It remains to assume the discrete inf-sup condition for $\mathcal{B}$, namely
(H.8) there exists a positive constant $\beta_{\mathcal{B},\mathrm{d}}$, independent of h, such that

$$\sup_{\substack{\vec{\boldsymbol{s}}_{i,h} \in \mathbf{H}_h \\ \vec{\boldsymbol{s}}_{i,h} \neq \mathbf{0}}} \frac{\mathcal{B}(\vec{\boldsymbol{s}}_{i,h}, \xi_{i,h})}{\|\vec{\boldsymbol{s}}_{i,h}\|_{\boldsymbol{\mathcal{H}}}} \geq \beta_{\mathcal{B},\mathrm{d}} \|\xi_{i,h}\|_{\mathcal{M}} \qquad \forall\, \xi_{i,h} \in \mathcal{M}_h .$$

Since A, B_1 and B_2 satisfy the hypotheses of [7, Corollary 2.2], we conclude the discrete analogue of (4.28) for $\mathcal{A}$ with the same constant $\widehat{\alpha}$. This inequality, along with **(H.8)**, imply the discrete inf-sup condition for $\boldsymbol{\mathcal{A}}_{\boldsymbol{w}_h,\boldsymbol{\phi}_h}$, which is satisfied by each $(\boldsymbol{w}_h, \boldsymbol{\phi}_h) \in \mathbf{X}_{2,h} \times \mathrm{X}_{2,h}$ such that

$$\|(\boldsymbol{w}_h, \boldsymbol{\phi}_h)\|_{\mathbf{X}_2 \times \mathrm{X}_2} \leq \mathrm{R}_\mathrm{d} := \frac{\boldsymbol{\alpha}_{\mathcal{A},\mathrm{d}}}{2 \max\{ \|\mathcal{D}\|, \|\mathcal{E}\| \}}, \tag{5.6}$$

where $\boldsymbol{\alpha}_{\mathcal{A},\mathrm{d}}$ is a positive constant depending only on $\widehat{\alpha}$, $\beta_{\mathcal{B},\mathrm{d}}$, $\|\mathcal{A}\|$, and $\|\mathcal{C}\|$.
Consequently, we can state the well-definedness of the components of $\mathbf{T}_h$ and $\boldsymbol{\Xi}_h$ as follows.

Lemma 5.3. *For each $i \in \{1,2\}$, and for each $(\boldsymbol{w}_h, \boldsymbol{\phi}_h) \in \mathbf{X}_{2,h} \times \mathrm{X}_{2,h}$ satisfying (5.6), there exists a unique $(\vec{\boldsymbol{t}}_{i,h}, \xi_{i,h}) = \big((\boldsymbol{t}_{i,h}, \boldsymbol{\sigma}_{i,h}), \xi_{i,h}\big) \in \boldsymbol{\mathcal{H}}_h \times \mathcal{M}_h$ solution of the seventh and eighth rows of (5.1), and hence we can define $\mathrm{T}_{i,h}(\boldsymbol{w}_h, \boldsymbol{\phi}_h) := \boldsymbol{t}_{i,h} \in \mathcal{H}_{1,h}$ and $\Xi_{i,h}(\boldsymbol{w}_h, \boldsymbol{\phi}_h) := \xi_{i,h} \in \mathcal{M}_h$. Moreover, there exists a positive constant $C_{\mathbf{T},\mathrm{d}}$, independent of $(\boldsymbol{w}_h, \boldsymbol{\phi}_h)$, such that*

$$\begin{aligned} \|\mathrm{T}_{i,h}(\boldsymbol{w}_h, \boldsymbol{\phi}_h)\|_{\mathcal{H}_1} + \|\Xi_{i,h}(\boldsymbol{w}_h, \boldsymbol{\phi}_h)\|_{\mathcal{M}} &= \|\boldsymbol{t}_{i,h}\|_{\mathcal{H}_1} + \|\xi_{i,h}\|_{\mathcal{M}} \\ &\leq \|(\vec{\boldsymbol{t}}_{i,h}, \xi_{i,h})\|_{\boldsymbol{\mathcal{H}} \times \mathcal{M}} \leq C_{\mathbf{T},\mathrm{d}} \left\{ \|\xi_{i,\mathrm{D}}\|_{1/2,\Gamma} + \|f_i\|_{0,\varrho;\Omega} \right\}. \end{aligned} \tag{5.7}$$

Proof. It is a straightforward application of [27, Theorem 2.22]. □

5.6. Solvability analysis of the discrete fixed-point equation

Let us introduce the ball

$$\mathrm{W}(\mathrm{R}_\mathrm{d}) := \left\{ (\boldsymbol{w}_h, \boldsymbol{\phi}_h) \in \mathbf{X}_{2,h} \times \mathrm{X}_{2,h} : \quad \|(\boldsymbol{w}_h, \boldsymbol{\phi}_h)\|_{\mathbf{X}_2 \times \mathrm{X}_2} \leq \mathrm{R}_\mathrm{d} \right\}.$$

Then, analogously to the derivation of Lemma 4.6, we deduce that $\boldsymbol{\Pi}_h$ maps $\mathrm{W}(\mathrm{R}_\mathrm{d})$ into itself under the same assumption (4.38), except that $C(\mathrm{R})$ is replaced by a constant $C(\mathrm{R}_\mathrm{d})$ depending on $C_{\mathbf{S},\mathrm{d}}$, $C_{\widetilde{\mathbf{S}},\mathrm{d}}$, $C_{\mathbf{T},\mathrm{d}}$ and R_d. Moreover, as in the continuous case, we are able to prove the continuity of $\mathbf{S}_h$, $\widetilde{\mathbf{S}}_h$, $\mathbf{T}_h$, and $\boldsymbol{\Xi}_h$, with corresponding constants denoted by $L_{\mathbf{S},\mathrm{d}}$, $L_{\widetilde{\mathbf{S}},\mathrm{d}}$, and $L_{\mathbf{T},\mathrm{d}}$, respectively. Hence, there exists a positive constant $L_{\boldsymbol{\Pi},\mathrm{d}}$, depending only on $L_{\mathbf{S},\mathrm{d}}$, $L_{\widetilde{\mathbf{S}},\mathrm{d}}$, $L_{\mathbf{T},\mathrm{d}}$, and R_d, such that

$$\|\boldsymbol{\Pi}_h(\boldsymbol{z}_h, \boldsymbol{\varphi}_h) - \boldsymbol{\Pi}_h(\boldsymbol{w}_h, \boldsymbol{\phi}_h)\| \leq L_{\boldsymbol{\Pi},\mathrm{d}} \sum_{i=1}^{2} \left\{ \|\xi_{i,\mathrm{D}}\|_{1/2;\Gamma} + \|f_i\|_{0,\varrho;\Omega} \right\} \|(\boldsymbol{z}_h, \boldsymbol{\varphi}_h) - (\boldsymbol{w}_h, \boldsymbol{\phi}_h)\|. \tag{5.8}$$

for all $(\boldsymbol{z}_h, \boldsymbol{\varphi}_h)$, $(\boldsymbol{w}_h, \boldsymbol{\phi}_h) \in \mathrm{W}(\mathrm{R}_\mathrm{d})$.
According to the above, the main result of this section is established as follows.

Theorem 5.4. *Assume that the data satisfy (5.3) and the discrete version of (4.38), that is*

$$C(\mathrm{R}_\mathrm{d}) \left\{ \|\boldsymbol{f}\|_{0,\Omega} + \|g\|_{0,\Omega} + \|\chi_\mathrm{D}\|_{1/s,r;\Gamma} + \sum_{i=1}^{2} \Big(\|\xi_{i,\mathrm{D}}\|_{1/2,\Gamma} + \|f_i\|_{0,\varrho;\Omega} \Big) \right\} \leq \mathrm{R}_\mathrm{d}. \tag{5.9}$$

In addition, assume that

$$L_{\boldsymbol{\Pi},\mathrm{d}} \sum_{i=1}^{2} \left\{ \|\xi_{i,\mathrm{D}}\|_{1/2;\Gamma} + \|f_i\|_{0,\varrho;\Omega} \right\} < 1. \tag{5.10}$$

Then, the discrete fixed point equation (5.2) *has a unique solution* $(\boldsymbol{z}_h, \boldsymbol{\varphi}_h) \in \mathrm{W}(\mathrm{R}_\mathrm{d})$. *Equivalently, the coupled problem* (5.1) *has a unique solution* $\big((\boldsymbol{u}_h,\theta_h),(\boldsymbol{z}_h,p_h)\big) \in (\mathbf{X}_h \times \mathbf{Q}_h)\times(\mathbf{X}_{2,h}\times\mathbf{Q}_{1,h})$, $(\boldsymbol{\varphi}_h,\xi_h)\in \mathrm{X}_{2,h}\times \mathrm{M}_{1,h}$, *and* $(\vec{\boldsymbol{t}}_i,\xi_i)\in \boldsymbol{\mathcal{H}}_h\times\mathcal{M}_h$, $i\in\{1,2\}$. *Moreover, there hold the following a priori estimates*

$$
\begin{aligned}
\|(\boldsymbol{u}_h,\theta_h)\|_{\mathbf{X}\times\mathbf{Q}} + \|(\boldsymbol{z}_h,p_h)\|_{\mathbf{X}_2\times\mathbf{Q}_1} &\le \widetilde{C}_{\mathbf{S},\mathrm{d}}\Big\{\|\boldsymbol{f}\|_{0,\Omega} + \|g\|_{0,\Omega} + \sum_{i=1}^{2}\big(\|\xi_{i,\mathrm{D}}\|_{1/2;\Gamma} + \|f_i\|_{0,\rho;\Omega}\big)\Big\},\\
\|(\boldsymbol{\varphi}_h,\xi_h)\|_{\mathrm{X}_2\times\mathrm{M}_1} &\le \widetilde{C}_{\widetilde{\mathbf{S}},\mathrm{d}}\Big\{\|\chi_\mathrm{D}\|_{1/s,r;\Gamma} + \sum_{i=1}^{2}\big(\|\xi_{i,\mathrm{D}}\|_{1/2;\Gamma} + \|f_i\|_{0,\rho;\Omega}\big)\Big\},\\
\|(\vec{\boldsymbol{t}}_i,\xi_i)\|_{\boldsymbol{\mathcal{H}}\times\mathcal{M}} &\le C_{\mathbf{T},\mathrm{d}}\Big\{\|\xi_{i,\mathrm{D}}\|_{1/2,\Gamma} + \|f_i\|_{0,\rho;\Omega}\Big\},\quad i\in\{1,2\},
\end{aligned}
$$

where $\widetilde{C}_{\mathbf{S},\mathrm{d}}$ *and* $\widetilde{C}_{\widetilde{\mathbf{S}},\mathrm{d}}$ *are positive constants depending only on* $C_{\mathbf{S},\mathrm{d}}$, $C_{\widetilde{\mathbf{S}},\mathrm{d}}$, $C_{\mathbf{T},\mathrm{d}}$, *and* R_d.

Proof. We recall that (5.9) guarantees that $\boldsymbol{\Pi}_h$ maps $\mathrm{W}(\mathrm{R}_\mathrm{d})$ into itself, and knowing from (5.8) and (5.10) that $\boldsymbol{\Pi}_h : \mathrm{W}(\mathrm{R}_\mathrm{d}) \to \mathrm{W}(\mathrm{R}_\mathrm{d})$ is a contraction, a straightforward application of the Banach fixed-point theorem yields the unique solvability of (5.2) and of (5.1). Finally, the a priori estimates are consequence of (5.4), (5.5a), (5.5b), (5.7), and the fact that $\|\boldsymbol{\varphi}_h\|_{0,r;\Omega} \le \mathrm{R}_\mathrm{d}$. □

We end the section by stressing that the assumption (5.10) could be dropped from the statement of Theorem 5.4, in which case Brouwer's fixed-point theorem (cf. [22, Theorem 9.9-2]) would imply only existence of solution of (5.2) (and hence of (5.1)).

5.7. A priori error analysis

In this section, we derive an a priori error estimate for the Galerkin scheme (5.1) with arbitrary finite element subspaces satisfying the hypotheses introduced in Sections 5.3, 5.4, and 5.5. More precisely, recalling that $\big((\boldsymbol{u},\theta),(\boldsymbol{z},p)\big)\in(\mathbf{X}\times\mathbf{Q})\times(\mathbf{X}_2\times\mathbf{Q}_1)$, $(\boldsymbol{\varphi},\chi)\in\mathrm{X}_2\times\mathrm{M}_1$, and $(\vec{\boldsymbol{t}}_i,\xi_i)\in\boldsymbol{\mathcal{H}}\times\mathcal{M}, i\in\{1,2\}$, with $(\boldsymbol{z},\boldsymbol{\varphi})\in\mathrm{W}(\mathrm{R})$, constitute the unique solution of (3.36), and that, in turn, $\big((\boldsymbol{u}_h,\theta_h),(\boldsymbol{z}_h,p_h)\big)\in(\mathbf{X}_h\times\mathbf{Q}_h)\times(\mathbf{X}_{2,h}\times\mathbf{Q}_{1,h})$, $(\boldsymbol{\varphi}_h,\xi_h)\in\mathrm{X}_{2,h}\times\mathrm{M}_{1,h}$, and $(\vec{\boldsymbol{t}}_i,\xi_i)\in\boldsymbol{\mathcal{H}}_h\times\mathcal{M}_h$, $i\in\{1,2\}$, with $(\boldsymbol{z}_h,\boldsymbol{\varphi}_h)\in\mathrm{W}(\mathrm{R}_\mathrm{d})$, is the unique solution of (5.1), we establish a Céa estimate for the global error split as

$$
\mathrm{E} := \mathrm{E}_1 + \mathrm{E}_2 + \mathrm{E}_3,
$$

where

$$
\begin{aligned}
\mathrm{E}_1 &:= \|\boldsymbol{u}-\boldsymbol{u}_h\|_{\mathbf{X}} + \|\theta-\theta_h\|_{\mathbf{Q}} + \|\boldsymbol{z}-\boldsymbol{z}_h\|_{\mathbf{X}_2} + \|p-p_h\|_{\mathbf{Q}_1},\\
\mathrm{E}_2 &:= \|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{\mathrm{X}_2} + \|\chi-\chi_h\|_{\mathrm{M}_1},\quad\text{and}\quad \mathrm{E}_3 := \sum_{i=1}^{2}\Big\{\|\vec{\boldsymbol{t}}_i-\vec{\boldsymbol{t}}_{i,h}\|_{\boldsymbol{\mathcal{H}}} + \|\xi_i-\xi_{i,h}\|_{\mathcal{M}}\Big\}.
\end{aligned}
$$

In what follows, given a subspace Z_h of a generic Banach space $(Z,\|\cdot\|_Z)$, we set

$$
\mathrm{dist}(z,Z_h) := \inf_{z_h\in Z_h}\|z-z_h\|_Z \qquad \forall\, z\in Z.
$$

We begin the analysis by applying the Strang estimate from [27, Lemma 2.27] to the first four rows of equations (3.36) and (5.1). As a consequence, we obtain that there exists a positive constant $\widetilde{C}_1(\mathrm{E})$, depending on $\boldsymbol{\alpha}_{\mathbf{A},\mathrm{d}}$, $\|\mathbf{A}\|$ (cf. (4.7)), $\|\mathbf{e}_s\|$, and $\|\mathbf{e}_f\|$ (cf. (3.20)), such that there holds

$$
\mathrm{E}_1 \le \widetilde{C}_1(\mathrm{E})\Big\{\mathrm{dist}((\boldsymbol{u},\theta),\mathbf{X}_h\times\mathbf{Q}_h) + \mathrm{dist}((\boldsymbol{z},p),\mathbf{X}_{2,h}\times\mathbf{Q}_{1,h}) + \|\mathbf{F}_{\boldsymbol{\varphi},\xi,\boldsymbol{t}} - \mathbf{F}_{\boldsymbol{\varphi}_h,\xi_h,\boldsymbol{t}_h}\|_{\mathbf{X}'_h}\Big\}. \tag{5.11}
$$

Then, bearing in mind the definition of $\mathbf{F}_{\boldsymbol{\phi},\boldsymbol{\eta},\boldsymbol{r}}$ (cf. (3.16)), and proceeding as in the proof of Lemma 4.7 (cf. [29, eqs. (4.43), (4.44), and (4.45) in proof of Lemma 4.7]), we find that

$$
\|\mathbf{F}_{\boldsymbol{\varphi},\xi,\boldsymbol{t}} - \mathbf{F}_{\boldsymbol{\varphi}_h,\xi_h,\boldsymbol{t}_h}\|_{\mathbf{X}'_h} \le \max\{\varepsilon^{-1},\delta\}\Big\{\|\xi\|_{\mathcal{M}}\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{\mathrm{X}_2} + \|\boldsymbol{\varphi}_h\|_{\mathrm{X}_2}\|\xi-\xi_h\|_{\mathcal{M}} + \|\boldsymbol{t}-\boldsymbol{t}_h\|_{\mathcal{H}_1}\Big\},
$$

which, replaced back into (5.11), yields

$$
\begin{aligned}
\mathrm{E}_1 \le C_1(\mathrm{E})\Big\{&\mathrm{dist}((\boldsymbol{u},\theta),\mathbf{X}_h\times\mathbf{Q}_h) + \mathrm{dist}((\boldsymbol{z},p),\mathbf{X}_{2,h}\times\mathbf{Q}_{1,h})\\
&+ \|\xi\|_{\mathcal{M}}\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{\mathrm{X}_2} + \|\boldsymbol{\varphi}_h\|_{\mathrm{X}_2}\|\xi-\xi_h\|_{\mathcal{M}} + \|\boldsymbol{t}-\boldsymbol{t}_h\|_{\mathcal{H}_1}\Big\},
\end{aligned} \tag{5.12}
$$

with $C_1(\mathrm{E}) := \widetilde{C}_1(\mathrm{E})\max\{1,\varepsilon^{-1},\delta\}$. Next, applying again [27, Lemma 2.27], but now to the fifth and sixth rows of equations (3.36) and (5.1), and using that (cf. (3.25), see also [26, eqn. (95)])

$$
\|F_\xi - F_{\xi_h}\|_{\mathrm{M}'_2} = \|F_{\xi-\xi_h}\|_{\mathrm{M}'_2} \le |\Omega|^{(\rho-r)/\rho r}\|\xi-\xi_h\|_{\mathcal{M}},
$$

we arrive at

$$\mathrm{E}_2 \,\le\, C_2(\mathrm{E}) \left\{ \mathrm{dist}((\boldsymbol{\varphi},\chi), \mathrm{X}_{2,h} \times \mathrm{M}_{1,h}) \,+\, \|\xi - \xi_h\|_{\mathcal{M}} \right\}, \tag{5.13}$$

with a positive constant $C_2(\mathrm{E})$ depending only on ε, $\widetilde{\alpha}_{\mathrm{d}}$, $\widetilde{\beta}_{1,\mathrm{d}}$, $\widetilde{\beta}_{2,\mathrm{d}}$, $|\Omega|$, ρ, and r. For the last two rows of (3.36) and (5.1), we employ the same Strang estimate from [27, Lemma 2.27] to conclude the existence of a positive constant $\widetilde{C}_3(\mathrm{E})$, depending only on $\boldsymbol{\alpha}_{\mathcal{A}}$, $\|\mathcal{A}\|$, $\|\mathcal{B}\|$, and $\|\mathcal{C}\|$ (cf. (3.33)), such that

$$\begin{aligned}\mathrm{E}_3 \,\le\, \widetilde{C}_3(\mathrm{E}) \sum_{i=1}^{2} \Big\{ &\mathrm{dist}((\vec{\boldsymbol{t}}_i,\xi_i), \boldsymbol{\mathcal{H}}_h \times \mathcal{M}_h) \,+\, \|\mathcal{E}_{\boldsymbol{z},\boldsymbol{\varphi}}(\cdot,\xi_i) - \mathcal{E}_{\boldsymbol{z}_h,\boldsymbol{\varphi}_h}(\cdot,\xi_{i,h})\|_{\boldsymbol{\mathcal{H}}'_h} \\ &+\, \|\mathcal{D}_{\boldsymbol{z}}(\xi_i,\cdot) - \mathcal{D}_{\boldsymbol{z}_h}(\xi_{i,h},\cdot)\|_{\mathcal{M}'_h} \Big\}. \end{aligned} \tag{5.14}$$

In turn, from the definitions of $\mathcal{E}_{\boldsymbol{z},\boldsymbol{\varphi}}$ and $\mathcal{D}_{\boldsymbol{z}}$ (cf. (3.32)), we readily get that

$$\begin{aligned}\|\mathcal{E}_{\boldsymbol{z},\boldsymbol{\varphi}}(\cdot,\xi_i) - \mathcal{E}_{\boldsymbol{z}_h,\boldsymbol{\varphi}_h}(\cdot,\xi_{i,h})\|_{\boldsymbol{\mathcal{H}}'_h} \,\le\, \|\mathcal{E}\| \Big\{ &\|\xi_{i,h}\|_{0,\rho;\Omega} \left(\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{r,\mathrm{div}_r;\Omega} \,+\, \|\boldsymbol{z} - \boldsymbol{z}_h\|_{0,r;\Omega} \right) \\ &+\, \left(\|\boldsymbol{z}\|_{0,r;\Omega} \,+\, \|\boldsymbol{\varphi}\|_{r,\mathrm{div}_r;\Omega} \right) \|\xi_i - \xi_{i,h}\|_{0,\rho;\Omega} \Big\}, \end{aligned}$$

and

$$\|\mathcal{D}_{\boldsymbol{z}}(\xi_i,\cdot) - \mathcal{D}_{\boldsymbol{z}_h}(\xi_{i,h},\cdot)\|_{\mathcal{M}'_h} \,\le\, \|\mathcal{D}\| \left\{ \|\mathrm{div}(\boldsymbol{z})\|_{0,\Omega} \|\xi_i - \xi_{i,h}\|_{0,\rho;\Omega} \,+\, \|\xi_{i,h}\|_{0,\rho;\Omega} \|\mathrm{div}(\boldsymbol{z}) - \mathrm{div}(\boldsymbol{z}_h)\|_{0,\Omega} \right\},$$

which, jointly with (5.14), imply

$$\begin{aligned}\mathrm{E}_3 \,\le\, C_3(\mathrm{E}) \sum_{i=1}^{2} \Big\{ &\mathrm{dist}((\vec{\boldsymbol{t}}_i,\xi_i), \boldsymbol{\mathcal{H}}_h \times \mathcal{M}_h) + \|\xi_{i,h}\|_{\mathcal{M}} \left(\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{\mathrm{X}_2} + \|\boldsymbol{z} - \boldsymbol{z}_h\|_{\mathbf{X}_2} \right) \\ &+\, \left(\|\boldsymbol{z}\|_{\mathbf{X}_2} \,+\, \|\boldsymbol{\varphi}\|_{\mathrm{X}_2} \right) \|\xi_i - \xi_{i,h}\|_{\mathcal{M}} \Big\}, \end{aligned} \tag{5.15}$$

with a positive constant $C_3(\mathrm{E})$ depending only on $\widetilde{C}_3(\mathrm{E})$, $\|\mathcal{E}\|$, and $\|\mathcal{D}\|$. Consequently, adding (5.12), (5.13), and (5.15), performing algebraic manipulations, and employing the bounds for $\|\boldsymbol{z}\|_{\mathbf{X}_2}$, $\|\boldsymbol{\varphi}\|_{\mathrm{X}_2}$, $\|\xi\|_{\mathcal{M}}$, $\|\boldsymbol{\varphi}_h\|_{\mathrm{X}_2}$, and $\|\xi_{i,h}\|_{\mathcal{M}}$ provided by Theorems 4.10 and 5.4, we deduce the existence of a positive constant $\widetilde{C}(\mathrm{E})$, depending only on $\widetilde{C}_{\mathbf{S}}$, $\widetilde{C}_{\widetilde{\mathbf{S}}}$, $C_{\mathbf{T}}$, $\widetilde{C}_{\mathbf{S},\mathrm{d}}$, $\widetilde{C}_{\widetilde{\mathbf{S}},\mathrm{d}}$, and $C_{\mathbf{T},\mathrm{d}}$, and hence independent of h, such that

$$\begin{aligned}\mathrm{E} \le \widetilde{C}(\mathrm{E}) \Big\{ &\mathrm{dist}((\boldsymbol{u},\theta), \mathbf{X}_h \times \mathbf{Q}_h) \,+\, \mathrm{dist}((\boldsymbol{z},p), \mathbf{X}_{2,h} \times \mathbf{Q}_{1,h}) \\ &+ \mathrm{dist}((\boldsymbol{\varphi},\chi), \mathrm{X}_{2,h} \times \mathrm{M}_{1,h}) + \sum_{i=1}^{2} \mathrm{dist}((\vec{\boldsymbol{t}}_i,\xi_i), \boldsymbol{\mathcal{H}}_h \times \mathcal{M}_h) \Big\} \\ &+ \widetilde{C}(\mathrm{E}) \left\{ \|\boldsymbol{f}\|_{0,\Omega} + \|g\|_{0,\Omega} + \|\chi_{\mathrm{D}}\|_{1/s,r;\Gamma} + \sum_{i=1}^{2} \left(\|\xi_{i,\mathrm{D}}\|_{1/2,\Gamma} + \|f_i\|_{0,\rho;\Omega} \right) \right\} \mathrm{E}. \end{aligned} \tag{5.16}$$

We summarize our findings with the next result, following straightforwardly from (5.16) and (5.17).

Theorem 5.5. *In addition to the hypotheses of Theorems 4.10 and 5.4, assume that*

$$\widetilde{C}(\mathrm{E}) \left\{ \|\boldsymbol{f}\|_{0,\Omega} + \|g\|_{0,\Omega} + \|\chi_{\mathrm{D}}\|_{1/s,r;\Gamma} + \sum_{i=1}^{2} \left(\|\xi_{i,\mathrm{D}}\|_{1/2,\Gamma} + \|f_i\|_{0,\rho;\Omega} \right) \right\} \le \frac{1}{2}. \tag{5.17}$$

Then, letting $C(\mathrm{E}) := 2\,\widetilde{C}(\mathrm{E})$, there holds

$$\begin{aligned}\mathrm{E} \le C(\mathrm{E}) \Big\{ &\mathrm{dist}((\boldsymbol{u},\theta), \mathbf{X}_h \times \mathbf{Q}_h) \,+\, \mathrm{dist}((\boldsymbol{z},p), \mathbf{X}_{2,h} \times \mathbf{Q}_{1,h}) \\ &+ \mathrm{dist}((\boldsymbol{\varphi},\chi), \mathrm{X}_{2,h} \times \mathrm{M}_{1,h}) \,+\, \sum_{i=1}^{2} \mathrm{dist}((\vec{\boldsymbol{t}}_i,\xi_i), \boldsymbol{\mathcal{H}}_h \times \mathcal{M}_h) \Big\}. \end{aligned} \tag{5.18}$$

6. Specific finite element subspaces

In this section, we define specific finite element subspaces satisfying the conditions **(H.1)** - **(H.8)** introduced in Sections 5.3, 5.4, and 5.5, collect their respective approximation properties, and provide the associated rates of convergence of the resulting method.

6.1. Preliminaries

Given an integer $\ell \ge 0$ and $K \in \mathcal{T}_h$, we let $\mathrm{P}_\ell(K)$ (resp. $\widetilde{\mathrm{P}}_k(K)$) be the space of polynomials of degree $\le k$ (resp. $= k$) defined on K, and denote its vector version by $\mathbf{P}_\ell(K)$. In addition, we let $\mathbf{RT}_\ell(K) = \mathbf{P}_\ell(K) + \widetilde{\mathrm{P}}_\ell(K)\mathbf{x}$ be the local Raviart–Thomas space of order ℓ defined on K, where $\mathbf{x}$ stands for a generic vector in R^d. In turn, we let $\mathrm{P}_\ell(\mathcal{T}_h)$, $\mathbf{P}_\ell(\mathcal{T}_h)$, and $\mathbf{RT}_\ell(\mathcal{T}_h)$ be the corresponding global versions of $\mathrm{P}_\ell(K)$, $\mathbf{P}_\ell(K)$ and $\mathbf{RT}_\ell(K)$, respectively, that is

$$\begin{aligned}\mathrm{P}_\ell(\mathcal{T}_h) &:= \Big\{\theta_h \in \mathrm{L}^2(\Omega): \quad \theta_h|_K \in \mathrm{P}_\ell(K) \quad \forall\, K \in \mathcal{T}_h\Big\},\\ \mathbf{P}_\ell(\mathcal{T}_h) &:= \Big\{\boldsymbol{v}_h \in \mathbf{L}^2(\Omega): \quad \boldsymbol{v}_h|_K \in \mathbf{P}_\ell(K) \quad \forall\, K \in \mathcal{T}_h\Big\}, \quad \text{and}\\ \mathbf{RT}_\ell(\mathcal{T}_h) &:= \Big\{\boldsymbol{q}_h \in \mathbf{H}(\mathrm{div};\Omega): \quad \boldsymbol{q}_h|_K \in \mathbf{RT}_\ell(K) \quad \forall\, K \in \mathcal{T}_h\Big\}.\end{aligned}$$

For each $t \in (1, +\infty)$, the inclusions $\mathrm{P}_\ell(\mathcal{T}_h) \subseteq \mathrm{L}^t(\Omega)$, $\mathbf{P}_\ell(\mathcal{T}_h) \subseteq \mathbf{H}^1(\Omega)$, $\mathbf{RT}_k(\mathcal{T}_h) \subseteq \mathbf{H}(\mathrm{div}_t;\Omega)$, $\mathbf{RT}_\ell(\mathcal{T}_h) \subseteq \mathbf{H}^t(\mathrm{div};\Omega)$, and $\mathbf{RT}_\ell(\mathcal{T}_h) \subseteq \mathbf{H}^t(\mathrm{div}_t;\Omega)$, are implicitly utilized below. Indeed, bearing in mind the notation from Section 5.1, and given an integer $k \geq 0$, we now define

$$\begin{aligned}&\mathbf{X}_h := \mathbf{P}_{k+2}(\mathcal{T}_h) \cap \mathbf{H}^1_0(\Omega), \quad \mathbf{Q}_h := \mathrm{P}_{k+1}(\mathcal{T}_h) \cap C(\Omega),\\ &\mathbf{X}_{2,h} := \mathbf{RT}_k(\mathcal{T}_h) \cap \mathbf{H}^r_0(\mathrm{div};\Omega), \quad \mathbf{X}_{1,h} := \mathbf{RT}_k(\mathcal{T}_h) \cap \mathbf{H}^s_0(\mathrm{div}_s;\Omega),\\ &\mathbf{Q}_{1,h} := \mathrm{P}_k(\mathcal{T}_h) \cap \mathrm{L}^r_0(\Omega), \quad \mathbf{Q}_{2,h} := \mathrm{P}_k(\mathcal{T}_h) \cap \mathrm{L}^2_0(\Omega),\\ &\mathrm{X}_{2,h} = \mathrm{X}_{1,h} := \mathbf{RT}_k(\mathcal{T}_h), \quad \mathrm{M}_{1,h} = \mathrm{M}_{2,h} := \mathrm{P}_k(\mathcal{T}_h),\\ &\mathcal{H}_{1,h} := \mathbf{P}_k(\mathcal{T}_h), \quad \mathcal{H}_{2,h} := \mathbf{RT}_k(\mathcal{T}_h), \quad \text{and} \quad \mathcal{M}_h := \mathrm{P}_k(\mathcal{T}_h).\end{aligned} \tag{6.1}$$

6.2. Verification of the stability conditions

In what follows we verify that the spaces (6.1) satisfy **(H.1)** - **(H.8)**. The inf-sup condition **(H.1)** for $(\mathbf{X}_h, \mathbf{Q}_h)$ has been proved in [8] for the two and three-dimensional cases. In turn, the proof of **(H.2)** for 2D was established in [30, Lemma 4.3], thanks to the boundedness of the L^2-type projector onto the discrete kernel of the bilinear forms $\mathbf{d}_1$ and $\mathbf{d}_2$. Whether this boundedness holds in 3D, remains still an open question up to the authors' knowledge. As we will see in what follows, all other hypotheses hold in both dimensions. Regarding **(H.3)**, the discrete inf-sup condition for $\mathbf{d}_1$ is available in [30, Lemma 4.4], whereas that for $\mathbf{d}_2$, which is stated next, is proved employing some results provided in [20, Appendix A]. We omit its proof here and refer for details to the preprint version of the present work (cf. [29, Lemma 6.1]).

Lemma 6.1. *There exists a positive constant $\beta_{2,\mathrm{d}}$, independent of h, such that*

$$\sup_{\substack{\boldsymbol{z}_h \in \mathbf{X}_{2,h}\\ \boldsymbol{z}_h \neq \mathbf{0}}} \frac{\mathbf{d}_2(\boldsymbol{z}_h, q_h)}{\|\boldsymbol{z}_h\|_{\mathbf{X}_{2,h}}} \geq \beta_{2,\mathrm{d}} \|q_h\|_{\mathbf{Q}_{2,h}} \qquad \forall\, q_h \in \mathbf{Q}_{2,h}. \tag{6.2}$$

For **(H.4)** we refer to [21, Lemma 5.2] (the preprint version of [20]). That proof follows analogously to that of [30, Lemma 4.3], with the exception that the operator is instead defined as a slight modification of the one derived in [20, Lemma 3.3]. In turn, the proofs of the discrete inf-sup conditions required by **(H.5)**, which adapt the continuous analysis from [26, Lemma 4.4] to the present discrete setting, reduce basically to slight modifications of those of [30, Lemma 4.5] (or [20, Lemma 5.3]). On the other hand, we readily observe from (6.1) that $\mathrm{div}(\mathcal{H}_{2,h}) \subseteq \mathcal{M}_h$ and $V_{0,h} \subseteq \mathcal{H}_{1,h}$, which confirms the verification of **(H.6)** and **(H.7)**. Finally, we notice that **(H.8)** is proved in [30, Lemma 4.5].

6.3. Rates of convergence

Here we provide the rates of convergence of the Galerkin schemes (5.1) with the specific finite element subspaces introduced in Section 6.1.

Theorem 6.2. *Let $((\boldsymbol{u},\theta),(\boldsymbol{z},p)) \in (\mathbf{X} \times \mathbf{Q}) \times (\mathbf{X}_2 \times \mathbf{Q}_1)$, $(\boldsymbol{\varphi},\chi) \in \mathrm{X}_2 \times \mathrm{M}_1$, and $(\vec{\boldsymbol{t}}_i,\xi_i) \in \boldsymbol{\mathcal{H}} \times \mathcal{M}$, be the unique solution of (3.36), with $(\boldsymbol{z},\boldsymbol{\varphi}) \in \mathrm{W}(\mathrm{R})$, and let $((\boldsymbol{u}_h,\theta_h),(\boldsymbol{z}_h,p_h)) \in (\mathbf{X}_h \times \mathbf{Q}_h) \times (\mathbf{X}_{2,h} \times \mathbf{Q}_{1,h})$, $(\boldsymbol{\varphi}_h,\xi_h) \in \mathrm{X}_{2,h} \times \mathrm{M}_{1,h}$, and $(\vec{\boldsymbol{t}}_{i,h},\xi_{i,h}) \in \boldsymbol{\mathcal{H}}_h \times \mathcal{M}_h$, be the unique solution of (5.1), with $(\boldsymbol{z}_h,\boldsymbol{\varphi}_h) \in \mathrm{W}(\mathrm{R}_{\mathrm{d}})$, which is guaranteed by Theorems 4.10 and 5.4, respectively. Assume the hypotheses of Theorem 5.5, and that there exist $s,\, l \in [1, k+1]$, such that $\boldsymbol{u} \in \mathbf{H}^{s+2}(\Omega)$, $\theta \in \mathrm{H}^{s+1}(\Omega)$ $\boldsymbol{z} \in \mathbf{W}^{l,r}(\Omega)$, $\mathrm{div}(\boldsymbol{z}) \in \mathrm{H}^l(\Omega)$, $p \in \mathrm{H}^l(\Omega)$, $\boldsymbol{\varphi} \in \mathbf{W}^{l,r}(\Omega)$, $\mathrm{div}(\boldsymbol{\varphi}) \in \mathrm{W}^{l,r}(\Omega)$, $\chi \in \mathrm{W}^{l,r}(\Omega)$, $\boldsymbol{t}_i \in \mathbf{H}^l(\Omega)$, $\boldsymbol{\sigma}_i \in \mathbf{H}^l(\Omega)$, $\mathrm{div}(\boldsymbol{\sigma}_i) \in \mathrm{W}^{l,\varrho}(\Omega)$, and $\xi_i \in \mathrm{W}^{l,\rho}(\Omega)$, $i \in \{1,2\}$. Then, there exists a positive constant C, independent of h, such that*

$$\begin{aligned}\mathrm{E} \leq h^{\min\{s+1,l\}} \Big\{ &\|\boldsymbol{u}\|_{s+2,\Omega} + \|\theta\|_{s+1,\Omega} + \|\boldsymbol{z}\|_{l,r;\Omega} + \|\mathrm{div}(\boldsymbol{z})\|_{l,\Omega} + \|p\|_{l,r;\Omega} + \|\boldsymbol{\varphi}\|_{l,r;\Omega}\\ &+ \|\mathrm{div}(\boldsymbol{\varphi})\|_{l,r;\Omega} + \|\chi\|_{l,r;\Omega} + \sum_{i=1}^{2} \Big(\|\boldsymbol{t}_i\|_{l,\Omega} + \|\boldsymbol{\sigma}_i\|_{l,\Omega} + \|\mathrm{div}(\boldsymbol{\sigma}_i)\|_{l,\varrho;\Omega} + \|\xi_i\|_{l,\rho;\Omega} \Big) \Big\}.\end{aligned}$$

Table 7.1

Example 1. Error history for the primal-mixed scheme in 2D, showing here the Biot, mixed Poisson, and PNP unknowns (while DoF refers to the total number of degrees of freedom). The last block of the table shows the total error (exactly the same for both Picard and Newton–Raphson schemes), experimental convergence rate, and number of nonlinear solver iterations required to reach the prescribed tolerance set to each approach.

Biot unknowns

DoF	h	$e(\boldsymbol{u})$	$r(\boldsymbol{u})$	$e(\theta)$	$r(\theta)$	$e(\boldsymbol{z})$	$r(\boldsymbol{z})$	$e(p)$	$r(p)$
155	0.7071	1.09e+0	⋆	9.75e-01	⋆	6.52e+0	⋆	3.54e-01	⋆
605	0.3536	2.76e-01	1.983	2.26e-01	2.109	3.48e+0	0.907	1.60e-01	1.141
2393	0.1768	6.83e-02	2.014	5.56e-02	2.024	1.76e+0	0.979	7.78e-02	1.042
9521	0.0884	1.70e-02	2.008	1.41e-02	1.982	8.85e-01	0.995	3.86e-02	1.011
37985	0.0442	4.26e-03	1.994	3.70e-03	1.928	4.43e-01	0.999	1.93e-02	1.003
151745	0.0221	1.08e-03	1.976	1.02e-03	1.965	2.22e-01	1.000	9.63e-03	1.001

Mixed Poisson unknowns

DoF	h	$e(\boldsymbol{\varphi})$	$r(\boldsymbol{\varphi})$	$e(\chi)$	$r(\chi)$
155	0.7071	6.60e+0	⋆	2.79e-01	⋆
605	0.3536	3.56e+0	0.890	1.51e-01	0.888
2393	0.1768	1.81e+0	0.978	7.66e-02	0.978
9521	0.0884	9.08e-01	0.994	3.85e-02	0.994
37985	0.0442	4.54e-01	0.999	1.93e-02	0.999
151745	0.0221	2.27e-01	1.000	9.63e-03	1.000

Nernst–Planck unknowns

DoF	h	$e(\boldsymbol{t}_1)$	$r(\boldsymbol{t}_1)$	$e(\boldsymbol{t}_2)$	$r(\boldsymbol{t}_2)$	$e(\boldsymbol{\sigma}_1)$	$r(\boldsymbol{\sigma}_1)$	$e(\boldsymbol{\sigma}_2)$	$r(\boldsymbol{\sigma}_2)$	$e(\xi_1)$	$r(\xi_1)$	$e(\xi_2)$	$r(\xi_2)$
155	0.7071	1.54e+0	⋆	1.38e+0	⋆	1.17e+1	⋆	1.03e+1	⋆	4.21e-01	⋆	3.56e-01	⋆
605	0.3536	7.65e-01	1.012	7.83e-01	0.812	5.33e+0	1.134	6.53e+0	0.659	2.18e-01	0.950	2.13e-01	0.741
2393	0.1768	3.84e-01	0.995	3.91e-01	1.002	2.65e+0	1.009	3.24e+0	1.010	1.09e-01	0.994	1.09e-01	0.969
9521	0.0884	1.92e-01	0.998	1.95e-01	1.001	1.32e+0	1.001	1.63e+0	0.991	5.48e-02	0.998	5.47e-02	0.992
37985	0.0442	9.62e-02	0.999	9.77e-02	1.000	6.62e-01	1.000	8.16e-01	1.000	2.74e-02	0.999	2.74e-02	0.998
151745	0.0221	4.81e-02	1.000	4.88e-02	1.000	3.31e-01	1.000	4.08e-01	1.000	1.37e-02	1.000	1.37e-02	0.999

Total error and nonlinear solver iteration count

DoF	h	e_{total}	r_{total}	$\mathtt{it}_{\text{Newton}}$	$\mathtt{it}_{\text{Picard}}$
155	0.7071	6.08e+1	⋆	4	10
605	0.3536	3.04e+1	0.996	5	12
2393	0.1768	1.54e+1	0.995	4	13
9521	0.0884	7.65e+0	1.016	4	14
37985	0.0442	3.82e+0	1.003	4	14
151745	0.0221	1.91e+0	1.000	4	14

Proof. It follows directly from the (5.18), properties of the Raviart–Thomas interpolator (see, e.g., [30, Section 4.1, eqs. (4.6) and (4.7)] and [20, Appendix A]), the scalar and vector L^2-type projector onto piecewise polynomials ([27, Proposition 1.135]), and interpolation estimates of Sobolev spaces. □

7. Numerical tests

For the computational results that verify the error estimates from Section 6 we employ the open source finite element library `GridapDistributed` [5]. We solve numerically the coupled systems using two approaches: a) separating the coupled problem by fixed-point iterations between three subproblems, and b) solving the nonlinear algebraic system (5.1) with Newton–Raphson's method with exact Jacobian. For the Picard fixed-point scheme we set a tolerance of 10^{-8} on the norm of the incremental solutions (monitoring only the coupling variables $\boldsymbol{z}, \boldsymbol{\varphi}, \boldsymbol{t}_i, \xi_i$), whereas for the Newton method we set a tolerance of 10^{-8} on either the ℓ^∞-norm of the nonlinear residual or the ℓ^2-norm of the incremental solution vector. All resulting linear systems are solved with the unsymmetric multifrontal direct method for sparse matrices `UMFPACK`.

Remark. Following the approach in [15, Section 5.2], the convergence of a Newton–Raphson method for (5.1) can be established using the Kantorovich Theorem (cf. [4, Theorem 5]). The arguments use again data smallness assumptions similarly to Theorem 5.4. Further details are presented in Appendix 6.2.

Example 1. We carry out the error history associated with the family of discretizations specified in Section 6.1, using the lowest order polynomial degree (additional tests conducted with higher polynomial degrees, and using the Stokes inf-sup pair $\mathbf{P}_{k+2} - \mathrm{P}_0$ elements for displacement and total pressure, Brezzi–Douglas–Marini elements for the Darcy flux, electric field and ionic fluxes, not shown here, showed the same qualitative behavior as the one reported here). We take the unit square and unit cube domains $\Omega = (0,1)^d$ ($d = 2, 3$), with unity model parameters. The Lebesgue exponents in (3.5) are chosen as $r = 3$, $s = \frac{3}{2}$, $\rho = 6$, $\varrho = \frac{6}{5}$ (and they are valid for both 2D and 3D cases). We manufacture the right-hand side and non-homogeneous boundary data $\boldsymbol{f}, g, f_i, \chi_{\mathrm{D}}, \xi_{i,\mathrm{D}}$ in such a way that the governing equations have the following smooth exact solutions to the primal strong form (2.1)-(2.3)

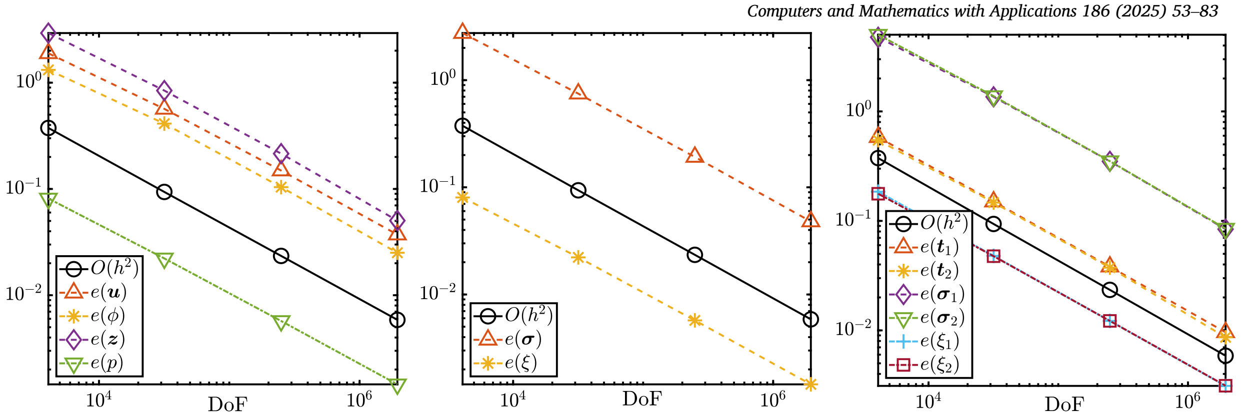

Fig. 7.1. Error history for the primal-mixed scheme in 3D, showing the convergence of all individual errors for the Biot, mixed Poisson, and Nernst–Planck sub-systems (left, center, and right panels, respectively).

$$\text{in 2D: } \begin{cases} \boldsymbol{u}_{\rm ex}(x,y) = \begin{pmatrix} \sin(\pi[x+y]) \\ \cos(\pi[x^2+y^2]) \end{pmatrix}, \quad p_{\rm ex}(x,y) = \sin(\pi x)\sin(\pi y), \quad \chi_{\rm ex}(x,y) = \cos(\pi x)\cos(\pi y), \\ \xi_{1,\rm ex}(x,y) = \cos(\pi[x+y]), \quad \xi_{2,\rm ex}(x,y) = \sin(\pi[x+y]), \end{cases}$$

$$\text{in 3D: } \begin{cases} \boldsymbol{u}_{\rm ex}(x,y,z) = \begin{pmatrix} \sin(\pi[x+y+z]) \\ \cos(\pi[x^2+y^2+z^2]) \\ \cos(\pi[x+y+z]) \end{pmatrix}, \quad p_{\rm ex}(x,y,z) = \sin(\pi x)\sin(\pi y)\sin(\pi z), \\ \chi_{\rm ex}(x,y,z) = \cos(\pi x)\cos(\pi y)\cos(\pi z), \quad \xi_{1,\rm ex}(x,y,z) = \cos(\pi[x+y+z]), \\ \xi_{2,\rm ex}(x,y,z) = \sin(\pi[x+y+z]), \end{cases}$$

and the exact values of the mixed variables are assigned from the primal ones as

$$\theta_{\rm ex} = \alpha\, p_{\rm ex} - \lambda \operatorname{div} \boldsymbol{u}_{\rm ex}, \quad \boldsymbol{z}_{\rm ex} = -\frac{\kappa}{\nu}\nabla p_{\rm ex}, \quad \boldsymbol{\varphi}_{\rm ex} = \varepsilon \nabla \chi_{\rm ex}, \quad \boldsymbol{t}_i = \nabla \xi_{i,\rm ex},$$
$$\boldsymbol{\sigma}_{i,\rm ex} = \kappa_i \nabla \xi_{i,\rm ex} + q_i\, \kappa_i \nabla \chi_{\rm ex} + \frac{\kappa}{\nu} \xi_{i,\rm ex} \nabla p_{\rm ex}.$$

For the convergence tests we use (possibly non-homogeneous) Dirichlet boundary conditions (2.8)-(2.9). We construct a sequence of six successively refined structured grids $l = 0, 1, \ldots$ of maximum mesh size $h_l = 2^{-l}\sqrt{2}$ (in 2D) on which we generate approximate solutions, and we compute errors for each unknown $e_l(\cdot)$ and experimental orders of convergence

$$r_{l+1}(\cdot) = \log(e_{l+1}(\cdot)/e_l(\cdot))[\log(h_{l+1}/h_l)]^{-1}, \quad l = 0, 1, \ldots$$

Table 7.1 portrays the error history in 2D (we break it into Biot, mixed Poisson, and Nernst–Planck unknowns), from which we can readily confirm a convergence of $O(h^{k+2})$ for the Biot unknowns and $O(h^{k+1})$ for all remaining field variables. The symbol $\star$ in the first refinement level indicates that no convergence rate is computed. For every mesh refinement and polynomial degree, the Newton–Raphson (resp. Picard) algorithm has taken around four (resp. fourteen) iterations to achieve the desired converge criterion. Both cases produce the same individual errors. We can also observe that the error associated with the Raviart–Thomas vector fields of Darcy flux, electric field, and ionic fluxes $(\boldsymbol{z}, \boldsymbol{\varphi}, \boldsymbol{\sigma}_i)$ are slightly higher than that in the remaining unknowns. Convergence results are also optimal in the 3D case, which we report in Fig. 7.1 for the second-order method (using again Taylor–Hood elements for displacement-total pressure pair), where we see agreement with Theorem 6.2. Sample approximate solutions are depicted in Fig. 7.2.

Example 2. After the numerical verification of optimal convergence rates we address the simulation of electrochemically coupled poroelasticity in radially unconfined compression. This type of tests are typical in poromechanics [3,19,42], and have also been used for coupling with PNP equations in [39,46,50] (where that model includes additional mechanical nonlinearities). The domain is the 2D cut of a disk of cartilage tissue confined between two impermeable rigid plates, giving $\Omega = (0, 1.5) \times (0, 0.5)$ mm^2. Differently than in the previous example, here we employ mixed boundary conditions (which were not analyzed in the paper) since this is the configuration usually employed in poroelastic benchmarks. On the radial surface (the right edge of the 2D domain) we set zero fluid pressure, zero electrostatic potential, prescribe the potential and ionic concentrations, as well as zero normal total stress. This allows free flow of fluid and current along that boundary. On the left edge we impose zero normal displacement, zero tangential total stress, and zero ionic fluxes and electric field. On the bottom plate we set zero normal displacement, zero tangential stress, and zero fluxes, whereas on the top plate we prescribe a given normal traction $\boldsymbol{\sigma n} = (0, -M)^{\rm t}$ with $M = 0.1$ N/mm^2, together with zero fluxes. The model parameters are as follows $E_{\rm Y} = 0.5$ N/mm^2, $\nu_{\rm P} = 0.1$ (Young modulus and Poisson ratio), $\kappa = 10^{-9}$ mm^2 (permeability), $\kappa_1 = 1.28 \times 10^{-2}$ mm^2/s, $\kappa_1 = 1.77 \times 10^{-2}$ mm^2/s (ionic diffusivities), $\alpha = 0.8$ (Biot–Willis coefficient), $c_0 = 4 \times 10^{-4}$ 1/(N/mm^2) (storativity), $\nu = 10^{-4}$ N/mm^2 s (fluid viscosity). As outputs, in Fig. 7.3 we report on the total stress tensor magnitude, cation and

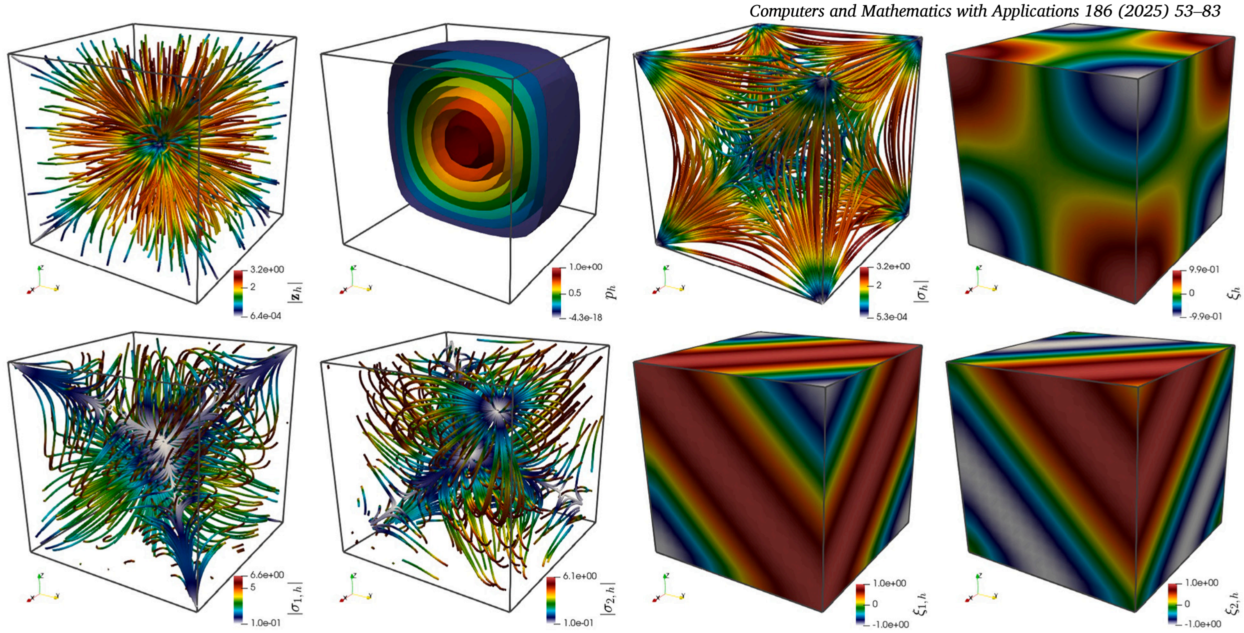

Fig. 7.2. Example 1. Sample of approximate solutions for the convergence test in 3D.

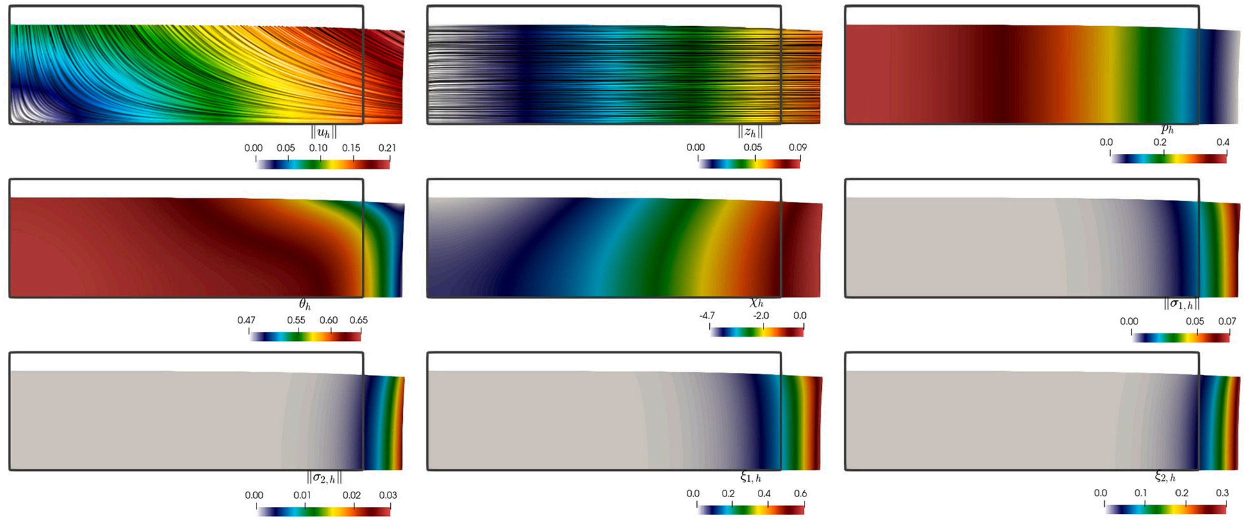

Fig. 7.3. Example 2. Unconfined compression of poroelastic material between impermeable plates. Sample of approximate solutions (displacement, Darcy flux, fluid pressure, electrostatic potential, cation and anion fluxes, and cation and anion concentration shown on the deformed configuration).

anion fluxes, and electric field. All quantities are plotted on the deformed domain. We see the typical deformation of the rightmost part of the domain and the Darcy flux moving in the horizontal direction. For this test we have used the second-order scheme (setting $k = 1$).

Example 3. Finally, we simulate the ion spreading and the poromechanical response of a fully saturated deformable porous structure. For this we adapt the configuration in [25] and [40, Section 5.2] to the poroelastic regime and use the domain $\Omega = (0,1)\times(0,2)$, which we discretize into a structured mesh of 10’000 triangles. The boundary conditions are as follows: for the solid phase we set clamped conditions $\boldsymbol{u} = \mathbf{0}$ on the left boundary ($x = 0$) for the fluid phase we impose slip conditions $\boldsymbol{z}\cdot\boldsymbol{n} = 0$ everywhere on the boundary. For the chemical species we assume that the normal trace of the total fluxes is zero everywhere on the boundary $\boldsymbol{\sigma}_i \cdot \boldsymbol{n} = 0$ (that is, the boundary is considered impenetrable for the ionic quantities), which is imposed essentially. For the electrostatic sub-system we consider two separate sub-boundaries: on the top segment ($y = 2$) we prescribe a given potential χ_0 (representing a ground condition, imposed naturally), on the vertical walls of the reservoir we set zero normal trace of the electric field $\boldsymbol{\varphi}\cdot\boldsymbol{n} = 0$, and the bottom segment is regarded as a positively charged surface $\boldsymbol{\varphi}\cdot\boldsymbol{n} = s_E$ (the two last conditions are imposed essentially).

Note that just for this test, the drag due to electric field and concentration difference is considered as a right-hand side of the Darcy momentum equation. Also, for this test we consider the time-dependent version of the equations and so we include the term

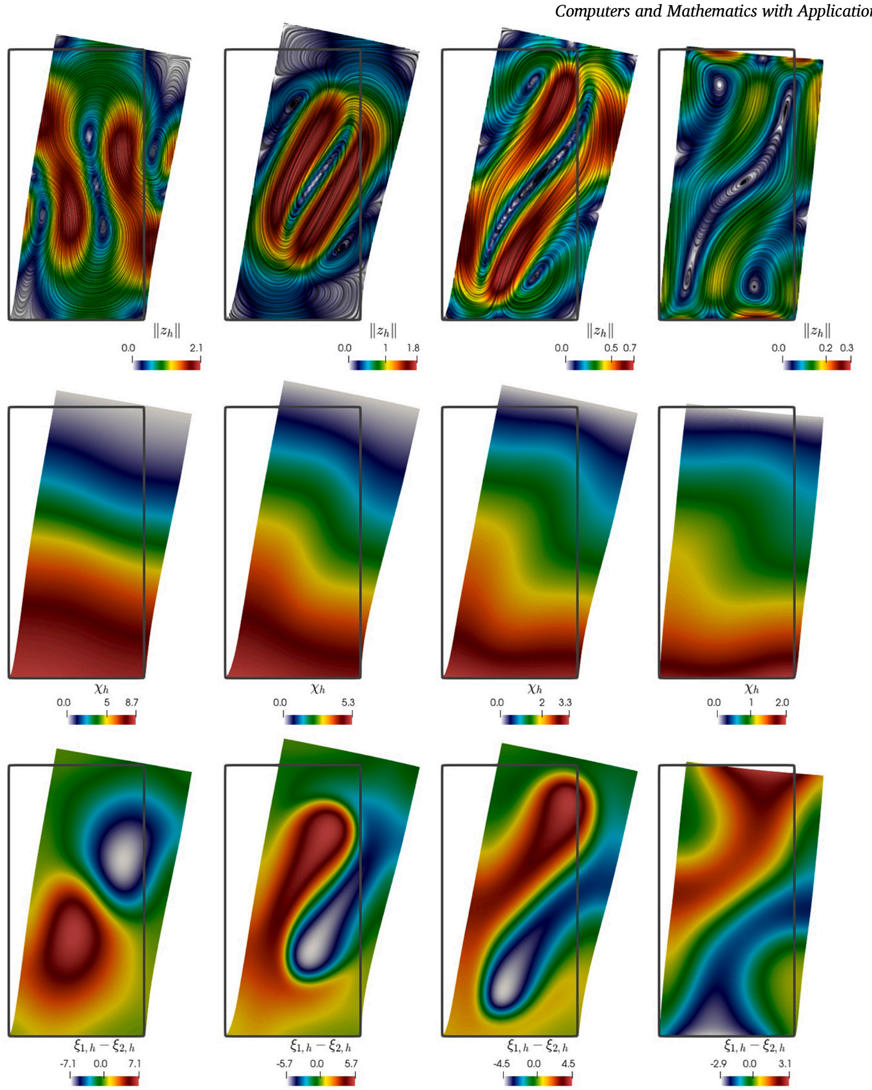

Fig. 7.4. Example 3. Ion spreading in a charged deformable cell. Sample of approximate solutions at times $t = 0.1, 0.4, 0.8, 2$ (from left to right). We display the Darcy flux, electrostatic potential, and relative concentration on the deformed configuration.

$\frac{1}{\Delta t}\frac{\alpha}{\lambda}(\theta^{m+1} - \theta^m) - \frac{1}{\Delta t}(c_0 + \frac{\alpha^2}{\lambda})(c^{m+1} - c^m)$ in the mass conservation equation and the terms $-\frac{1}{\Delta t}(\xi_1^{m+1} - \xi_1^m) - \frac{1}{\Delta t}(\xi_2^{m+1} - \xi_2^m)$ in the two ion conservation equations, where the superscripts $m, m+1$ denote approximations at time instants t^m, t^{m+1} using backward Euler's method. For this we take a constant time step $\Delta t = 0.01$ and run the system until the final time $t = 2$. The initial pressure and total pressure are zero and the initial concentrations of positively and negatively charged particles are as follows

$$\xi_{i,0}(\boldsymbol{x}) = \frac{\hat{\xi}_0}{2\pi R^2}\exp\left\{-\frac{(x - \frac{1}{2} + \frac{q_i}{8})^2 + (y - 1 + \frac{q_i}{2})^2}{2R^2}\right\},$$

and the remaining parameters adopt the values (all adimensional) $c_0 = 0.01$, $\alpha = 0.9$, $\mu = 10$, $\lambda = 1000$, $\varepsilon = 0.5$, $\nu = 0.08$, $\kappa_1 = \kappa_2 = 0.01$, $s_E = 1$, $\chi_0 = 0$, $\hat{\xi}_0 = 3$, $R = \frac{1}{4}$.

Snapshots of the approximate solutions, computed using the lowest order method, and taken at four time instants are shown in Fig. 7.4. We plot the net charge (difference between concentrations of ionic concentrations), the line integral convolution of the relative fluid velocity, and the electrostatic potential.

We end this section by mentioning that the numerical experiments included data that we could not guarantee will satisfy the hypotheses on small data (for either the Banach fixed-point or Newton-Kantorovich arguments). Nevertheless we do observe optimal convergence in all cases. A more general theory for large data would be desirable such as Leray-Schauder schemes with homotopy arguments or weak compactness arguments and passage to the limit, but they unfortunately do not fit the present framework in general Banach spaces.

Appendix A. Newton's method

In this appendix, we define a Newton's method to solve the discrete scheme (5.1). To guarantee the convergence of this method, we establish sufficient conditions on the problem data and invoke the Kantorovich Theorem (cf. [4, Theorem 5]), following [15, Section 5.2]. Let X and Y be two Banach spaces, and let $x_0 \in X$. Additionally, define the bounded sets

$$B_R := \left\{ x \in X : \quad \|x - x_0\|_X < R \right\} \quad \text{and} \quad \overline{B}_r := \left\{ x \in X : \quad \|x - x_0\|_X < r \right\},$$

where R and r are positive constants satisfying $0 < r < R$. Assume that B_R contains a zero of an operator $P : B_R \subset X \to Y$ and that P has a continuous derivative in B_R.

Theorem A.1 *(Kantorovich). Let P be defined on $B_R \subset X$ with continuous second derivative in $\overline{B}_r$. Moreover assume that*

i) the continuous linear operator $[P'(x^0)]^{-1}$ exists,
ii) there exists a constant $K_1 > 0$ such that $\|[P'(x^0)]^{-1}](P(x^0))\| \leq K_1$,
iii) there exists a constant $K_2 > 0$ such that $\|[P'(x^0)]^{-1}\, P''(x)\| \leq K_2$, for all $x \in \overline{B}_r$.

If $K_3 = K_1\, K_2 \leq \frac{1}{2}$, and the radius r of $\overline{B}_r$ satisfies

$$r \geq r_0 := \frac{1 - \sqrt{1 - 2K_3}}{K_3} K_1\,, \tag{A.1}$$

then there exists a zero x^* of P to which Newton's iteration converges. And in this case, $\|x - x^*\|_X \leq r_0$. Furthermore, if for $K_3 < 1/2$,

$$r < r_1 := \frac{1 - \sqrt{1 - 2K_3}}{K_3} K_1\,, \tag{A.2}$$

or for $K_3 = 1/2$, $r \leq r_1$, then the solution x^* is unique in the set $\overline{B}_r$, and the rate of convergence is

$$\|x^* - x^m\|_X \leq \frac{1}{2^m}(2K_3)^{2^m}\frac{K_1}{K_3}\,, \quad m \geq 0\,.$$

Now we present Newton's method associated with (5.1). For simplicity of notation, we focus on the continuous case (3.36) (the same analysis directly applies to the discrete setting). First we set

$$\underline{\boldsymbol{u}} := (\vec{\boldsymbol{u}}, (\boldsymbol{\varphi}, \chi), (\vec{\boldsymbol{t}}_1, \xi_1), (\vec{\boldsymbol{t}}_2, \xi_2)) \in \mathbf{H} := \mathbb{X} \times (\mathrm{X}_2 \times \mathrm{M}_1) \times (\boldsymbol{\mathcal{H}} \times \boldsymbol{\mathcal{M}}) \times (\boldsymbol{\mathcal{H}} \times \boldsymbol{\mathcal{M}}),$$
$$\underline{\boldsymbol{v}} := (\vec{\boldsymbol{v}}, (\boldsymbol{\psi}, \gamma), (\vec{\boldsymbol{s}}_1, \eta_1), (\vec{\boldsymbol{s}}_2, \eta_2)) \in \mathbf{M} := \mathbb{Q} \times (\mathrm{X}_1 \times \mathrm{M}_2) \times (\boldsymbol{\mathcal{H}} \times \boldsymbol{\mathcal{M}}) \times (\boldsymbol{\mathcal{H}} \times \boldsymbol{\mathcal{M}}).$$

Next, given $\underline{\boldsymbol{u}}^0 \in \mathbf{H}$, we introduce the linear operator $\mathscr{N}(\underline{\boldsymbol{u}}^0, \cdot) : \mathbf{H} \to \mathbf{M}'$, defined as

$$\mathscr{N}(\underline{\boldsymbol{u}}^0, \underline{\boldsymbol{u}})(\underline{\boldsymbol{v}}) := \mathscr{A}(\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}) + \mathscr{C}(\underline{\boldsymbol{u}}^0; \underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}) \qquad \forall\, (\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}) \in \mathbf{H} \times \mathbf{M},$$

where the bilinear form $\mathscr{A} : \mathbf{H} \times \mathbf{M} \to \mathrm{R}$ is defined as the sum of each bilinear form in the left-hand side of (3.36). Furthermore, given $\underline{\boldsymbol{u}}^0 \in \mathbf{H}$, the bilinear form $\mathscr{C}(\underline{\boldsymbol{u}}^0, \cdot, \cdot) : \mathbf{H} \times \mathbf{M} \to \mathrm{R}$ is defined as

$$\mathscr{C}(\underline{\boldsymbol{u}}^0; \underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}) := \varepsilon^{-1} \int_\Omega (\xi_1 - \xi_2)\boldsymbol{\varphi}^0 \cdot \boldsymbol{v} + \sum_{i=1}^{2} \Big(\mathcal{D}_{z^0}(\xi_i, \eta_i) + \mathcal{E}_{z^0, \boldsymbol{\varphi}^0}(\vec{\boldsymbol{s}}_i, \xi_i) \Big) \qquad \forall\, (\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}) \in \mathbf{H} \times \mathbf{M}.$$

Additionally, given $\underline{\boldsymbol{u}}^0 \in \mathbf{H}$, we define the functional $\mathscr{F}_{\underline{\boldsymbol{u}}^0} \in \mathbf{M}'$ as

$$\mathscr{F}_{\underline{\boldsymbol{u}}^0}(\underline{\boldsymbol{v}}) := \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} - \delta \int_\Omega (\boldsymbol{t}_1^0 - \boldsymbol{t}_2^0) \cdot \boldsymbol{v} + \mathbf{G}(q) + G(\boldsymbol{\psi}) + F_{\xi^0}(\gamma) + \sum_{i=1}^{2} \Big(\mathcal{G}(\vec{\boldsymbol{s}}_i) + \mathcal{F}(\eta_i) \Big) \qquad \forall\, \underline{\boldsymbol{v}} \in \mathbf{M}.$$

Note that the second and fourth terms in the definition of $\mathscr{F}_{\underline{\boldsymbol{u}}^0}$ is crucial for the forthcoming analysis in establishing the bijectivity of $\mathscr{A}$. Given $\underline{\boldsymbol{u}}^0 \in \mathbf{H}$, defining the nonlinear operator $\mathscr{P}_{\underline{\boldsymbol{u}}^0} : \mathbf{H} \to \mathbf{M}'$ as

$$\mathscr{P}_{\underline{\boldsymbol{u}}^0}(\underline{\boldsymbol{u}}) := \mathscr{N}(\underline{\boldsymbol{u}}, \underline{\boldsymbol{u}}) - \mathscr{F}_{\underline{\boldsymbol{u}}^0} \qquad \forall\, \underline{\boldsymbol{u}} \in \mathbf{H}',$$

we notice that $\underline{\boldsymbol{u}} \in \mathbf{H}$ is the unique solution of (3.36), if and only if $\mathscr{P}_{\underline{\boldsymbol{u}}}(\underline{\boldsymbol{u}}) = \mathbf{0} \in \mathbf{M}'$. After simple manipulations, we obtain that the Gâteaux derivative of $\mathscr{P}_{\underline{\boldsymbol{u}}^0}$, $\mathscr{P}' : \mathbf{H} \to \mathcal{L}(\mathbf{H}, \mathbf{M}')$ is

$$(\mathscr{P}'(\underline{\boldsymbol{u}}^0)(\underline{\boldsymbol{u}}))(\underline{\boldsymbol{v}}) := \mathscr{A}(\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}) + \mathscr{C}(\underline{\boldsymbol{u}}; \underline{\boldsymbol{u}}^0, \underline{\boldsymbol{v}}) + \mathscr{C}(\underline{\boldsymbol{u}}^0; \underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}) \qquad \forall\, (\underline{\boldsymbol{u}}^0, \underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}) \in \mathbf{H} \times \mathbf{H} \times \mathbf{M},$$

where $\mathcal{L}(\mathbf{H}, \mathbf{M}')$ is the set of linear and bounded operators from $\mathbf{H}$ to $\mathbf{M}'$.

According to the above, Newton's method reads: Given $\underline{\boldsymbol{u}}^0 \in \mathbf{H}$, for $m \geq 0$, find $\underline{\boldsymbol{u}}^{m+1} \in \mathbf{H}$ such that

$$(\mathscr{P}'(\underline{\boldsymbol{u}}^m)(\underline{\boldsymbol{u}}^{m+1} - \underline{\boldsymbol{u}}^m))(\underline{\boldsymbol{v}}) = \mathscr{P}_{\underline{\boldsymbol{u}}^m}(\underline{\boldsymbol{u}}^m)(\underline{\boldsymbol{v}}) \qquad \forall\, \underline{\boldsymbol{v}} \in \mathbf{M}. \tag{A.3}$$

Theorem A.2. *Assume that, besides* (4.18)*, the data satisfy*

$$C \left\{ \|\mathbf{f}\|_{0,\Omega} + \|g\|_{0,\Omega} + \|\chi_{\mathrm{D}}\|_{1/s,r;\Gamma} + \sum_{i=1}^{2} \left(\|\xi_{i,\mathrm{D}}\|_{1/2;\Gamma} + \|f_i\|_{0,\varrho;\Omega} \right) \right\} < \frac{1}{2}, \tag{A.4}$$

where C is a positive constant to be specified below. Then, for the initial guess $\underline{\boldsymbol{u}}^0 = \mathbf{0}$, the sequence $\{\underline{\boldsymbol{u}}^m\}_{m\in\mathbb{N}}$ computed by (A.3) *converges to a unique solution $\underline{\boldsymbol{u}} \in \overline{B}_r := \left\{ \overline{\boldsymbol{v}} \in \mathbf{H} : \quad \|\underline{\boldsymbol{v}}\|_{\mathbf{H}} \leq r \right\}$, where r satisfies* (A.1) *and* (A.2)*. In addition,*

$$\|\underline{\boldsymbol{u}} - \underline{\boldsymbol{u}}^m\|_{\mathbf{H}} \leq \frac{1}{2^m}(2K_3)^{2^m} \frac{K_1}{K_3}, \quad m \geq 0.$$

Proof. We proceed to verify the hypotheses of Theorem A.1. Considering $\underline{\boldsymbol{u}}^0 = \mathbf{0}$, it is easy to see that there exists a positive constant $\widetilde{C}$, depending on 1, c_r, and $\|i_\rho\|$ (cf. (3.26) and (3.33)), such that

$$\|\mathscr{P}_{\underline{\boldsymbol{u}}^0}(\underline{\boldsymbol{u}}^0)\|_{\mathbf{M}'} \leq \widetilde{C} \left\{ \|\mathbf{f}\|_{0,\Omega} + \|g\|_{0,\Omega} + \|\chi_{\mathrm{D}}\|_{1/s,r;\Gamma} + \sum_{i=1}^{2} \left(\|\xi_{i,\mathrm{D}}\|_{1/2;\Gamma} + \|f_i\|_{0,\varrho;\Omega} \right) \right\}. \tag{A.5}$$

Next, we note that the bilinear forms a, b_1 and b_2 (cf. Section 3.3), satisfy the hypotheses of [7, Theorem 2.1, Section 2.1] (cf. [26, Lemmas 4.3 and 4.4]) with constants $\widetilde{\alpha}$, $\widetilde{\beta}_1$, and $\widetilde{\beta}_2$. Thus, there exists $\boldsymbol{\alpha}$, depending on these constants, as well as $\|a\|$, $\|b_1\|$, and $\|b_2\|$, such that

$$\sup_{\substack{(\boldsymbol{\varphi},\chi)\in \mathrm{X}_2\times\mathrm{M}_1 \\ (\boldsymbol{\varphi},\chi)\neq\mathbf{0}}} \frac{a(\boldsymbol{\varphi},\boldsymbol{\psi}) + b_1(\boldsymbol{\psi},\chi) + b_2(\boldsymbol{\varphi},\gamma)}{\|(\boldsymbol{\varphi},\chi)\|_{\mathrm{X}_2\times\mathrm{M}_1}} \geq \boldsymbol{\alpha}\|(\boldsymbol{\psi},\gamma)\|_{\mathrm{X}_1\times\mathrm{M}_2} \qquad \forall\,(\boldsymbol{\psi},\gamma)\in \mathrm{X}_1\times\mathrm{M}_2, \quad \text{and}$$

$$\sup_{\substack{(\boldsymbol{\psi},\gamma)\in \mathrm{X}_1\times\mathrm{M}_2 \\ (\boldsymbol{\psi},\gamma)\neq\mathbf{0}}} \frac{a(\boldsymbol{\varphi},\boldsymbol{\psi}) + b_1(\boldsymbol{\psi},\chi) + b_2(\boldsymbol{\varphi},\gamma)}{\|(\boldsymbol{\psi},\gamma)\|_{\mathrm{X}_1\times\mathrm{M}_2}} \geq \boldsymbol{\alpha}\|(\boldsymbol{\varphi},\chi)\|_{\mathrm{X}_2\times\mathrm{M}_1} \qquad \forall\,(\boldsymbol{\varphi},\chi)\in \mathrm{X}_2\times\mathrm{M}_1.$$

In consequence, observing that $\mathscr{P}'(\underline{\boldsymbol{u}}^0)(\underline{\boldsymbol{u}})(\underline{\boldsymbol{v}}) = \mathscr{A}(\underline{\boldsymbol{u}},\underline{\boldsymbol{v}})$, and using previous inf-sup conditions, as well as (4.9a)-(4.9b) and (4.29a)-(4.29b), we can state the following

$$\sup_{\underline{\boldsymbol{u}}\in\mathbf{H}\setminus\{\mathbf{0}\}} \frac{\mathscr{A}(\underline{\boldsymbol{u}},\underline{\boldsymbol{v}})}{\|\underline{\boldsymbol{u}}\|_{\mathbf{H}}} \geq \alpha_{\mathscr{A}}\|\underline{\boldsymbol{v}}\|_{\mathbf{M}} \quad \forall\,\underline{\boldsymbol{v}}\in\mathbf{M}, \quad \text{and} \quad \sup_{\underline{\boldsymbol{v}}\in\mathbf{Q}\setminus\{\mathbf{0}\}} \frac{\mathscr{A}(\underline{\boldsymbol{u}},\underline{\boldsymbol{v}})}{\|\underline{\boldsymbol{v}}\|_{\mathbf{M}}} \geq \alpha_{\mathscr{A}}\|\underline{\boldsymbol{u}}\|_{\mathbf{H}} \quad \forall\,\underline{\boldsymbol{u}}\in\mathbf{H}, \tag{A.6}$$

where $\alpha_{\mathscr{A}} := \frac{1}{3}(\boldsymbol{\alpha}_{\mathbf{A}} + \boldsymbol{\alpha} + \boldsymbol{\alpha}_{\mathcal{A}})$. Then, (A.6) implies that $\mathscr{P}'(\underline{\boldsymbol{u}}^0) : \mathbf{H} \to \mathbf{M}'$ is bijective operator. Hence, *i*) is established, that is, there exists $[\mathscr{P}'(\underline{\boldsymbol{u}}^0)]^{-1}$ and

$$\|[\mathscr{P}'(\underline{\boldsymbol{u}}^0)]^{-1}\|_{\mathcal{L}(\mathbf{M}',\mathbf{H})} \leq \frac{1}{\alpha_{\mathscr{A}}}. \tag{A.7}$$

Thus, thanks to (A.5) and (A.7), *ii*) is satisfied with

$$K_1 := \frac{\widetilde{C}}{\alpha_{\mathscr{A}}} \left\{ \|\mathbf{f}\|_{0,\Omega} + \|g\|_{0,\Omega} + \|\chi_{\mathrm{D}}\|_{1/s,r;\Gamma} + \sum_{i=1}^{2} \left(\|\xi_{i,\mathrm{D}}\|_{1/2;\Gamma} + \|f_i\|_{0,\varrho;\Omega} \right) \right\}.$$

Furthermore, since the second derivative of $\mathscr{P}_{\underline{\boldsymbol{u}}^0}$ is given by

$$(\mathscr{P}''(\underline{\widetilde{\boldsymbol{u}}})(\underline{\boldsymbol{u}}))(\underline{\boldsymbol{w}})(\underline{\boldsymbol{v}}) = \mathscr{C}(\underline{\boldsymbol{w}};\underline{\boldsymbol{u}},\underline{\boldsymbol{v}}) + \mathscr{C}(\underline{\boldsymbol{u}};\underline{\boldsymbol{w}},\underline{\boldsymbol{v}}) \quad \forall\,\underline{\widetilde{\boldsymbol{u}}},\underline{\boldsymbol{u}},\underline{\boldsymbol{w}}\in\mathbf{H}, \quad \forall\,\underline{\boldsymbol{v}}\in\mathbf{M},$$

it is straightforward to show that

$$\|\mathscr{P}''(\underline{\widetilde{\boldsymbol{u}}})\|_{\mathcal{L}(\mathbf{H},\mathcal{L}(\mathbf{H},\mathbf{M}'))} \leq 2\max\left\{\varepsilon^{-1}, \|\mathcal{D}\|, \|\mathcal{E}\|\right\}.$$

Thus, defining $K_2 := \frac{2}{\alpha_{\mathscr{A}}}\max\{\varepsilon^{-1}, \|\mathcal{D}\|, \|\mathcal{E}\|\}$, we conclude *iii*). Hence, having verified the hypothesis of Theorem A.1, and assuming (A.4) with $C := \frac{\widetilde{C}}{\alpha_{\mathscr{A}}}K_2$, the proof is complete. □

Data availability

Data will be made available on request.

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