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Banach spaces-based analysis of a fully-mixed finite element method for the steady-state model of fluidized beds[☆]

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ABSTRACT

In this paper we propose and analyze a fully-mixed finite element method for the steady-state model of fluidized beds. This numerical technique, which arises from the use of a dual-mixed approach in each phase, is motivated by a methodology previously applied to the stationary Navier–Stokes equations and related models. More precisely, we modify the stress tensors of the fluid and solid phases by defining pseudostresses as phasic stresses that include shear, pressure, and convective effects. Next, we eliminate the pressures from the equations and derive constitutive relations depending only on the aforementioned pseudostresses and the velocities of the fluid and the particles. In this way, these variables, together with the skew-symmetric parts of the velocity gradients, also named vorticities, become the only unknowns of our variational formulation. As usual, the latter is obtained by testing against suitable functions, and then integrating and integrating by parts, respectively, the equilibrium and the constitutive equations. The particle pressure, a known function of the concentration, is given as a datum, and the fluid pressure is computed afterwards via a postprocessing formula. The continuous setting, lying in a Banach spaces framework rather than in a Hilbertian one, is rewritten as an equivalent fixed-point equation, and hence the well-posedness analysis is carried out by combining the Babuška–Brezzi theory, the Banach–Nečas–Babuška Theorem, and the classical Banach fixed-point Theorem. Thus, existence of a unique solution in a closed ball is guaranteed for sufficiently small data. In turn, the associated Galerkin scheme is introduced and analyzed analogously, so that, under suitable assumptions on generic finite element subspaces, and for sufficiently small data as well, the Brouwer and Banach fixed-point Theorems allow to conclude existence and uniqueness of solution, respectively. Specific finite element subspaces satisfying the required hypotheses are described, and optimal a priori error estimates are derived. Finally, several numerical examples illustrating the performance of the method and confirming the theoretical rates of convergence, are reported.

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1. Introduction

We begin this section by explaining the physical origin of the fluidized bed concept, for which we consider a set of solid particles in a reservoir through which there is an upward flow of a fluid. When the flow rate is small, the fluid flows through the set of particles as if it was a porous medium. When the flow rate increases and reaches a level at which the fluid drag experienced by the particles is such that it balances their net weight, a few particles become mobile and a small expansion of the region occupied by the particles is observed. Any further increase on the flow rate causes the particles to become fully mobile and to occupy a larger region of the reservoir. At this stage, the particles are said to be fluidized, and the system is usually referred to as a fluidized bed. The name fluidized bed is due to the fact that the particles in this condition can be stirred and poured as a fluid [1].

Fluidized beds are extensively used as chemical reactors in industrial scale due to the high levels of interaction between the fluid and the particles that can be achieved in these flows [2]. Higher efficiencies in heat and mass transfer are obtained in fluidized systems, when compared to fixed bed systems. In addition, the fact that particles behave as a fluid allows for a continuous operation of the reactor, with old (used) particles being removed and new particles being fed in as necessary, without the need to interrupt the operation of the system. Therefore, there is a strong industrial drive to understand the dynamics of these flows, and mathematical and numerical modeling play a crucial role in this task.

The pioneering work in the mathematical modeling of fluidized beds was developed by Anderson & Jackson in [3]. In this model, a volume averaging procedure is used to treat the fluidized particles as a continuum phase interpenetrating the fluid, for which the balance equations of continuum mechanics for mass, momentum, and eventually energy, could be written in terms of field quantities such as velocity and particle concentration, rather than in terms of the properties of the individual particles. This model is often referred to as the two-fluid model of fluidized beds [1]. Despite the advantage of not having to track individual particles, the drawback of this continuum approach is that unknown terms appear in the averaged equations of conservation. Constitutive laws must be proposed to account for these terms, namely the fluid–particle interaction f orce and the particle phase stress tensor. There are several constitutive models discussed in the literature and there seems to be a general agreement that the particle stress tensor can be modeled very similarly to that of a Newtonian fluid stress tensor, but with a particle pressure and a particle viscosity that depend on the local particle concentration of particles [3–5], and potentially also on the particle velocity fluctuations, for which another conservation equation has to be written [6,7].

There are several examples on the literature that have presented results of numerical simulations of flows in fluidized beds, based on different constitutive models and solved with different numerical schemes. For example, the evolution of small amplitude disturbances in both liquid– and gas–solid fluidized beds to finite amplitude structures was investigated with a two-fluid standard model in [4,8]. An industrial circulating fluidized bed was investigated in detail with a two-fluid model that used kinetic theory equations to account for particle stresses in [9]. In [10], a steady-state model based on a two-fluid model was used to study the effect of turbulence on axisymmetrical fluidized beds. More recently, the problem involving the determination of the particle stress tensor was avoided by coupling the fluid phase equations derived in [3] with the Discrete Element Method to solve the motion of individual particles [11–14], which is responsible to feed the concentration and the velocities of the particles to the continuum fluid phase equation. Although several types of discretizations were used in these works, in neither of them a rigorous study of the numerical scheme, nor an a priori error analysis, were carried out and, to the best knowledge of the authors, contributions in this direction do not seem to be available in the literature. In particular, the use of the finite element methodology, and even more interestingly, the derivation of mixed finite element methods taking advantages of the main features of the corresponding constitutive and momentum equations, have not been considered at all so far.

Due to the aforementioned lack of utilization of finite element techniques, and motivated by the increasing development during the last decade of new mixed finite element methods for solving diverse nonlinear models in continuum mechanics, we aim here to extend the applicability of this approach to fluidized beds. More precisely, since the nonlinearities involved in this model are similar to those from the Navier–Stokes and related equations, and rather than using a classical Hilbertian framework, we adapt to our present model the Banach spaces-based approach employed in several recent works (see, e.g., [15–19]), and which has shown to be very suitable to solve fluid-flow problems via dual-mixed formulations and the resulting mixed finite element schemes. Indeed, one of its main advantages is the fact that it does not need to make use of any augmentation procedure thus leaving the variational formulations as simple as possible and employing the natural spaces arising from the equations for their respective settings. Furthermore, it allows, on one hand, to derive momentum conservative numerical schemes, and on the other hand, to obtain direct approximations of further variables of interest, some of them through their incorporation as unknowns of the formulation, and others through postprocessing formulae defined in terms of the discrete solution.

In order to provide further details on the above discussion, we begin by referring to the numerical method introduced in [16] for the stationary Navier–Stokes problem. There, the system is rewritten in terms of the velocity and a suitable pseudostress tensor relating the gradient of the velocity, the pressure and the convective term, leading to a dual-mixed momentum conservative scheme where both unknowns, velocity and psuedostress, are set in Banach spaces. The latter allows to prove existence and uniqueness of solution by means of a fixed-point argument and the well-known Banach–Nečas–Babuška Theorem. In addition, the pressure, as well as the velocity gradient and the vorticity, can be obtained through a simple postprocessing of the solution without applying any numerical differentiation, thus avoiding further sources of error. This technique has also been successfully applied to the Boussinesq system (see [17,19]), magneto-hydrodynamics (see [18]) and flow-transport problems (see [15]), among others. For instance, the approach employed in [19] to deal with the fluid part of the model is extended in [17] to the associated heat equation. In this way, a modified mixed formulation is utilized in the latter, which is based on the introduction of the gradient of temperature and a vector version of the Bernoulli tensor as auxiliary unknowns. As a consequence, the same Banach saddle-point structure arises for both the fluid and energy equations. The analysis from [17] was later on adapted to the Oberbeck–Boussinesq system in [20], where analogue results were obtained.

Consequently, in this work we introduce and analyze a fully-mixed finite element method for numerically solving the steady-state model of fluidized beds. The rest of the paper is organized as follows. In Section 2 we introduce the problem of interest. More precisely, after collecting some preliminary notations and defining the evolutive fluidized bed model, its steady-state version is described there in terms of a dual-mixed approach in each phase. As a consequence, the pseudostress and vorticity tensors in the fluid and solid parts, together with the corresponding velocity vector fields, become the respective unknowns. Then, coherently with the above, the associated fully-mixed variational formulation is derived and analyzed in Section 3 within a Banach framework. Indeed, besides providing the boundedness properties of all the forms involved, the equivalence of the continuous formulation with a fixed-point equation is established, and the well-definedness of the corresponding operator is proved. Finally, the Banach fixed-point Theorem is applied to conclude the existence of a unique solution. In Section 4 we apply the same procedure from Section 3 to introduce and analyze a generic Galerkin scheme. In this way, under suitable assumptions on the finite element subspaces, and employing again fixed-point arguments, we are able to prove existence and then uniqueness of the discrete solution by applying the Brouwer and Banach Theorems, respectively. In addition, it is shown that basically any stable triplet for the Hilbertian framework of mixed linear elasticity is also stable for our present Banach framework of the fluidized bed model. Next, in Section 5 we develop the a priori error analysis of the Galerkin scheme and provide the associated rates of convergence. Finally, several illustrative numerical results are presented in Section 6.

2. The model problem

2.1. Preliminaries

Let us denote by $\Omega \subseteq \mathrm{R}^n$, $n \in \{2,3\}$ a given bounded domain with polyhedral boundary Γ, and denote by $\mathbf{n}$ the outward unit normal vector on Γ. Standard notations will be adopted for Lebesgue spaces $\mathrm{L}^p(\Omega)$, with $p \in [1,\infty]$ and Sobolev spaces $\mathrm{W}^{r,p}(\Omega)$ with $r \geq 0$, endowed with the norms $\|\cdot\|_{0,p;\Omega}$ and $\|\cdot\|_{r,p;\Omega}$, respectively, whose vectorial and tensorial versions are denoted in the same way. Note that $\mathrm{W}^{0,p}(\Omega) = \mathrm{L}^p(\Omega)$ and if $p = 2$, we write $\mathrm{H}^r(\Omega)$ in place of $\mathrm{W}^{r,2}(\Omega)$, with the corresponding Lebesgue and Sobolev norms denoted by $\|\cdot\|_{0,\Omega}$ and $\|\cdot\|_{r,\Omega}$, respectively. We also write $|\cdot|_{r,\Omega}$ for the H^r-seminorm. In addition, $\mathrm{H}^{1/2}(\Gamma)$ is the spaces of traces of functions of $\mathrm{H}^1(\Omega)$ and $\mathrm{H}^{-1/2}(\Gamma)$ denotes its dual. With $\langle\cdot,\cdot\rangle$ we denote the corresponding product of duality between $\mathrm{H}^{1/2}(\Gamma)$ and $\mathrm{H}^{-1/2}(\Gamma)$. By $\mathbf{S}$ and $\mathbb{S}$ we will denote the corresponding vectorial and tensorial counterparts, respectively, of the generic scalar functional space S. In turn, for any vector fields $\mathbf{v} = (v_i)_{i=1,n}$ and $\mathbf{w} = (w_i)_{i=1,n}$ we set the gradient, symmetric part of the gradient, divergence, and tensor product operators, as

$$\nabla \mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,n}, \qquad \mathbf{e}(\mathbf{v}) := \frac{1}{2}\left\{(\nabla\mathbf{v}) + (\nabla\mathbf{v})^{\mathrm{t}}\right\},$$
$$\operatorname{div}\mathbf{v} := \sum_{j=1}^{n}\frac{\partial v_j}{\partial x_j}, \qquad \text{and} \quad \mathbf{v}\otimes\mathbf{w} := (v_i w_j)_{i,j=1,n}.$$

In addition, for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,n}$, we let $\mathbf{div}(\boldsymbol{\tau})$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the tensor inner product, and the deviatoric tensor, respectively, as

$$\boldsymbol{\tau}^{\mathrm{t}} := (\boldsymbol{\tau}_{ji})_{i,j=1,n}, \quad \operatorname{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{n}\tau_{ii}, \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{n}\tau_{ij}\zeta_{ij}, \quad \text{and } \boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{n}\operatorname{tr}(\boldsymbol{\tau})\mathbb{I},$$

where $\mathbb{I}$ is the identity tensor in $\mathrm{R}^{n\times n}$. For simplicity, in what follows we denote

$$(v,w)_\Omega := \int_\Omega vw, \quad (\mathbf{v},\mathbf{w})_\Omega := \int_\Omega \mathbf{v}\cdot\mathbf{w}, \quad (\mathbf{v},\mathbf{w})_\Gamma := \int_\Gamma \mathbf{u}\cdot\mathbf{v} \quad \text{and} \quad (\boldsymbol{\tau},\boldsymbol{\zeta})_\Omega := \int_\Omega \boldsymbol{\tau}:\boldsymbol{\zeta}.$$

Furthermore, we recall that the Hilbert space

$$\mathbb{H}(\mathbf{div};\Omega) := \left\{\boldsymbol{\tau}\in\mathbb{L}^2(\Omega): \quad \mathbf{div}(\boldsymbol{\tau})\in\mathbf{L}^2(\Omega)\right\}, \tag{2.1}$$

equipped with the usual norm $\|\boldsymbol{\tau}\|^2_{\mathbf{div};\Omega} := \|\boldsymbol{\tau}\|^2_{0,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|^2_{0,\Omega}$ is standard in the realm of mixed problems. In turn, given $p \geq \dfrac{2n}{n+2}$, in what follows we will also employ the Banach space $\mathbb{H}(\mathbf{div}_p;\Omega)$ defined by

$$\mathbb{H}(\mathbf{div}_p;\Omega) := \left\{ \boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \quad \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^p(\Omega) \right\}, \tag{2.2}$$

endowed with the norm $\|\boldsymbol{\tau}\|_{\mathbf{div}_p;\Omega} := \left(\|\boldsymbol{\tau}\|^2_{0,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|^2_{0,p;\Omega} \right)^{1/2}$.

2.2. The fluidized bed model

We assume that the domain Ω is the region in which a large number of solid particles is suspended by an upwards fluid flow of either a liquid or a gas. In the following, we shall focus our attention on the models used in [4] and, more recently, in [8]. Therefore, letting $\mathbf{g}$ be the (constant) acceleration of gravity and denoting the fluid viscosity by μ_f, the fluid density by ρ_f, the density of the particles by ρ_s, and a final time by $\mathbf{T}$, we are interested in the model problem described by the following set of equations:

$$\begin{gathered} \rho_f\, \varepsilon \left(\frac{\partial \mathbf{u}_f}{\partial t} + (\nabla \mathbf{u}_f)\, \mathbf{u}_f \right) = \mathbf{div}\, T_f - F(\mathbf{u}_f, \mathbf{u}_s) + \varepsilon \rho_f\, \mathbf{g} \quad \text{in} \quad \Omega \times (0, \mathbf{T}]\,, \\ T_f = -p_f\, \mathbb{I} + 2\mu_f\, \mathbf{e}(\mathbf{u}_f)^{\mathrm{d}} \quad \text{in} \quad \Omega \times (0,\mathbf{T}]\,, \quad \frac{\partial \varepsilon}{\partial t} + \mathrm{div}(\varepsilon \mathbf{u}_f) = 0 \quad \text{in} \quad \Omega \times (0, \mathbf{T}]\,, \end{gathered} \tag{2.3}$$

$$\begin{gathered} \rho_s\, \phi \left(\frac{\partial \mathbf{u}_s}{\partial t} + (\nabla \mathbf{u}_s)\, \mathbf{u}_s \right) = \mathbf{div}\,(T_s - T_f) + F(\mathbf{u}_f, \mathbf{u}_s) + \phi \rho_s\, \mathbf{g} \quad \text{in} \quad \Omega \times (0, \mathbf{T}]\,, \\ T_s = -p_s(\phi)\, \mathbb{I} + 2\mu_s(\phi)\, \mathbf{e}(\mathbf{u}_s)^{\mathrm{d}} \quad \text{in} \quad \Omega \times (0,\mathbf{T}]\,, \quad \frac{\partial \phi}{\partial t} + \mathrm{div}(\phi \mathbf{u}_s) = 0 \quad \text{in} \quad \Omega \times (0, \mathbf{T}]\,, \end{gathered} \tag{2.4}$$

where the unknowns $\mathbf{u}_f$, $\mathbf{u}_s$, p_f, ϕ and ε represent, respectively, the velocity of the fluid, the velocity of the particles, the pressure on the fluid phase, the concentration of particles and the void fraction. Note that the concentration of particles ϕ and the void fraction ε satisfy the identity

$$\phi + \varepsilon = 1 \quad \text{in} \quad \Omega\,. \tag{2.5}$$

The stress tensor of the fluid phase is denoted by T_f and that of the solid phase by T_s. The particle pressure $p_s : \mathrm{R} \to \mathrm{R}$ is a function of the particle concentration ϕ given by [4,8]:

$$p_s(\phi) := P\phi^3 \exp\left(\frac{r\phi}{\phi_p - \phi} \right), \tag{2.6}$$

where P, r are constants that allow for changes in the intensity and the slope of the particle pressure, and ϕ_p is the maximum close random packing of the spheres, usually taken as $\phi_p = 0.64$. The particle viscosity $\mu_s : \mathrm{R} \to \mathrm{R}$ is given by [4]:

$$\mu_s(\phi) := \frac{M\phi}{1 - \left(\dfrac{\phi}{\phi_p} \right)^{1/3}}, \tag{2.7}$$

where the constant M is also used to set the range of values of the particle viscosity. Finally, the fluid–particle interaction force $F : \mathrm{R}^n \times \mathrm{R}^n \to \mathrm{R}^n$ is a function of ϕ, $\mathbf{u}_f$ and $\mathbf{u}_s$, which usually takes the form of a viscous drag given by [4]:

$$F(\mathbf{u}_f, \mathbf{u}_s) := \delta(\phi)(\mathbf{u}_f - \mathbf{u}_s)\,, \tag{2.8}$$

with $\delta : \mathrm{R} \to \mathrm{R}$ denoting the drag coefficient based on the Richardson & Zaki correlation [21]:

$$\delta(\phi) := \frac{(\rho_s - \rho_f)\mathrm{g}}{v_t} \frac{\phi}{(1-\phi)^{m-1}}. \tag{2.9}$$

The experimental coefficient m is normally taken on the range $3 \leq m \leq 5$ [21].

2.3. The steady-state model

In what follows we consider the uncoupling between (ϕ, ε) and $(\mathbf{u}_f, \mathbf{u}_s, p_f)$ resulting from the steady-state counterpart of (2.3)–(2.5), that is, given ϕ and ε such that $\phi + \varepsilon = 1$ in Ω, we seek $\mathbf{u}_f$, $\mathbf{u}_s$, and p_f in suitable spaces such that

$$\begin{aligned} \rho_f\, \varepsilon\, (\nabla \mathbf{u}_f)\, \mathbf{u}_f &= \mathbf{div}\, T_f - F(\mathbf{u}_f, \mathbf{u}_s) + \varepsilon \rho_f\, \mathbf{g} \quad \text{in} \quad \Omega\,, \\ T_f &= -p_f\, \mathbb{I} + 2\mu_f\, \mathbf{e}(\mathbf{u}_f)^{\mathrm{d}} \quad \text{in} \quad \Omega\,, \quad \mathrm{div}(\varepsilon \mathbf{u}_f) = 0 \quad \text{in} \quad \Omega\,, \\ \rho_s\, \phi\, (\nabla \mathbf{u}_s)\, \mathbf{u}_s &= \mathbf{div}\,(T_s - T_f) + F(\mathbf{u}_f, \mathbf{u}_s) + \phi \rho_s\, \mathbf{g} \quad \text{in} \quad \Omega\,, \\ T_s &= -p_s(\phi)\, \mathbb{I} + 2\mu_s(\phi)\, \mathbf{e}(\mathbf{u}_s)^{\mathrm{d}} \quad \text{in} \quad \Omega\,, \quad \text{and} \quad \mathrm{div}(\phi \mathbf{u}_s) = 0 \quad \text{in} \quad \Omega\,. \end{aligned} \tag{2.10}$$

We first observe, thanks to the divergence-free property for $\varepsilon \mathbf{u}_f$ and $\phi \mathbf{u}_s$ (cf. second and fourth rows of (2.10)), that there hold

$$\mathbf{div}\big((\varepsilon \mathbf{u}_f) \otimes \mathbf{u}_f\big) = \varepsilon \left(\nabla \mathbf{u}_f\right) \mathbf{u}_f \quad \text{and} \quad \mathbf{div}\big((\phi \mathbf{u}_s) \otimes \mathbf{u}_s\big) = \phi \left(\nabla \mathbf{u}_s\right) \mathbf{u}_s \quad \text{in} \quad \Omega .$$

Then, bearing in mind the expressions of T_f and T_s, we now introduce the pseudostress tensors

$$\begin{aligned} \boldsymbol{\sigma}_f &:= 2\mu_f\, \mathbf{e}(\mathbf{u}_f)^{\mathrm{d}} - \rho_f(\varepsilon \mathbf{u}_f) \otimes \mathbf{u}_f - p_f\, \mathbb{I} \quad \text{in} \quad \Omega, \quad \text{and} \\ \boldsymbol{\sigma}_s &:= 2\mu_s(\phi)\, \mathbf{e}(\mathbf{u}_s)^{\mathrm{d}} - \rho_s(\phi \mathbf{u}_s) \otimes \mathbf{u}_s - \rho_f(\varepsilon \mathbf{u}_f) \otimes \mathbf{u}_f - p_s(\phi)\mathbb{I} \quad \text{in} \quad \Omega\,, \end{aligned} \tag{2.11}$$

whence the first and third rows of (2.10) can be rewritten, respectively, as follows

$$\begin{aligned} \mathbf{div}(\boldsymbol{\sigma}_f) - F(\mathbf{u}_f, \mathbf{u}_s) &= -\varepsilon \rho_f \mathbf{g} \quad \text{in} \quad \Omega\,, \quad \text{and} \\ \mathbf{div}(\boldsymbol{\sigma}_s) + F(\mathbf{u}_f, \mathbf{u}_s) &= \mathbf{div}(\boldsymbol{\sigma}_f) - \phi \rho_s \mathbf{g} \quad \text{in} \quad \Omega\,. \end{aligned} \tag{2.12}$$

Equivalently, replacing $\mathbf{div}(\boldsymbol{\sigma}_f)$ from the first equation of (2.12) into the second one, and keeping the former as it is, we arrive at

$$\begin{aligned} \mathbf{div}(\boldsymbol{\sigma}_f) - F(\mathbf{u}_f, \mathbf{u}_s) &= -\varepsilon \rho_f \mathbf{g} \quad \text{in} \quad \Omega\,, \quad \text{and} \\ \mathbf{div}(\boldsymbol{\sigma}_s) &= -(\varepsilon \rho_f + \phi \rho_s)\mathbf{g} \quad \text{in} \quad \Omega\,. \end{aligned} \tag{2.13}$$

In addition, it also follows from (2.11) that

$$\begin{aligned} \mathrm{tr}(\boldsymbol{\sigma}_f) &= -\rho_f \mathrm{tr}\big((\varepsilon \mathbf{u}_f) \otimes \mathbf{u}_f\big) - n p_f \quad \text{in} \quad \Omega\,, \quad \text{and} \\ \mathrm{tr}(\boldsymbol{\sigma}_s) &= -\mathrm{tr}\big(\rho_s(\phi \mathbf{u}_s) \otimes \mathbf{u}_s + \rho_f(\varepsilon \mathbf{u}_f) \otimes \mathbf{u}_f\big) - n p_s(\phi) \quad \text{in} \quad \Omega\,, \end{aligned}$$

from which we deduce that

$$\begin{aligned} p_f &= -\frac{1}{n} \mathrm{tr}\big(\boldsymbol{\sigma}_f + \rho_f(\varepsilon \mathbf{u}_f) \otimes \mathbf{u}_f\big) \quad \text{in} \quad \Omega\,, \quad \text{and} \\ p_s(\phi) &= -\frac{1}{n} \mathrm{tr}\big(\boldsymbol{\sigma}_s + \rho_s(\phi \mathbf{u}_s) \otimes \mathbf{u}_s + \rho_f(\varepsilon \mathbf{u}_f) \otimes \mathbf{u}_f\big) \quad \text{in} \quad \Omega\,. \end{aligned} \tag{2.14}$$

In this way, replacing the foregoing expressions for p_f and $p_s(\phi)$ back into (2.11), and recalling that $\mathbf{e}(\mathbf{u}_f)^{\mathrm{d}} = \mathbf{e}(\mathbf{u}_f) - \frac{1}{n}\, \mathrm{tr}\big(\mathbf{e}(\mathbf{u}_f)\big)\, \mathbb{I} = \mathbf{e}(\mathbf{u}_f) - \frac{1}{n}\, \mathrm{div}(\mathbf{u}_f)\, \mathbb{I}$, and similarly for $\mathbf{e}(\mathbf{u}_s)^{\mathrm{d}}$, we find that

$$\begin{aligned} \boldsymbol{\sigma}_f^{\mathrm{d}} &= 2\mu_f \mathbf{e}(\mathbf{u}_f) - \rho_f\big((\varepsilon \mathbf{u}_f) \otimes \mathbf{u}_f\big)^{\mathrm{d}} - \frac{2\mu_f}{n} \mathrm{div}(\mathbf{u}_f)\, \mathbb{I} \quad \text{in} \quad \Omega, \quad \text{and} \\ \boldsymbol{\sigma}_s^{\mathrm{d}} &= 2\mu_s(\phi)\mathbf{e}(\mathbf{u}_s) - \rho_s\big((\phi \mathbf{u}_s) \otimes \mathbf{u}_s\big)^{\mathrm{d}} - \rho_f\big((\varepsilon \mathbf{u}_f) \otimes \mathbf{u}_f\big)^{\mathrm{d}} - \frac{2\mu_s(\phi)}{n} \mathrm{div}(\mathbf{u}_s)\, \mathbb{I} \quad \text{in} \quad \Omega. \end{aligned} \tag{2.15}$$

At this point we notice that, similarly to [22], and employing again the incompressibility conditions from (2.10), one easily finds that the divergence terms of the foregoing equations can be replaced as follows

$$\mathrm{div}(\mathbf{u}_f) = -\frac{\nabla \varepsilon}{\varepsilon} \cdot \mathbf{u}_f \quad \text{and} \quad \mathrm{div}(\mathbf{u}_s) = -\frac{\nabla \phi}{\phi} \cdot \mathbf{u}_s \quad \text{in} \quad \Omega\,. \tag{2.16}$$

Furthermore, for sake of uniqueness of the pressure solution p_f, we impose the condition

$$\int_\Omega p_f = 0,$$

which, according to the first equation in (2.14), is equivalent to establishing

$$\int_\Omega \mathrm{tr}(\boldsymbol{\sigma}_f) = -\int_\Omega \mathrm{tr}\big(\rho_f(\varepsilon \mathbf{u}_f) \otimes \mathbf{u}_f\big)\,. \tag{2.17}$$

In turn, since $p_s(\phi)$ is explicitly known in terms of ϕ (cf. (2.6)), we derive from the second equation in (2.14) that

$$\int_\Omega \mathrm{tr}(\boldsymbol{\sigma}_s) = -\int_\Omega \Big\{ n p_s(\phi) + \mathrm{tr}\big(\rho_s(\phi \mathbf{u}_s) \otimes \mathbf{u}_s + \rho_f(\varepsilon \mathbf{u}_f) \otimes \mathbf{u}_f\big) \Big\}\,. \tag{2.18}$$

We remark that the identities (2.17) and (2.18) are crucial to solve later on for $\boldsymbol{\sigma}_f$ and $\boldsymbol{\sigma}_s$. The description of our model continues with the introduction of the skew-symmetric tensors

$$\boldsymbol{\gamma}_f := \frac{1}{2}\Big\{\nabla \mathbf{u}_f - (\nabla \mathbf{u}_f)^{\mathrm{t}}\Big\} \quad \text{and} \quad \boldsymbol{\gamma}_s := \frac{1}{2}\Big\{\nabla \mathbf{u}_s - (\nabla \mathbf{u}_s)^{\mathrm{t}}\Big\},$$

so that the strain tensors $\mathbf{e}(\mathbf{u}_f)$ and $\mathbf{e}(\mathbf{u}_s)$ can be decomposed as

$$\mathbf{e}(\mathbf{u}_f) = \nabla \mathbf{u}_f - \boldsymbol{\gamma}_f \quad \text{and} \quad \mathbf{e}(\mathbf{u}_s) = \nabla \mathbf{u}_s - \boldsymbol{\gamma}_s\,. \tag{2.19}$$

Finally, given $\mathbf{u}_{D,f},\ \mathbf{u}_{D,s} \in \mathbf{H}^{1/2}(\Gamma)$, we consider the Dirichlet boundary conditions for $\mathbf{u}_f$ and $\mathbf{u}_s$ given by

$$\mathbf{u}_f = \mathbf{u}_{D,f} \quad \text{and} \quad \mathbf{u}_s = \mathbf{u}_{D,s} \quad \text{on} \quad \Gamma\,. \tag{2.20}$$

We stress here that (2.20) makes sense under the assumption that $\mathbf{u}_s$ and $\mathbf{u}_f$ are sought originally in $\mathbf{H}^1(\Omega)$, which, in turn, implies that $\boldsymbol{\gamma}_s$ and $\boldsymbol{\gamma}_f$ belong to $\mathbb{L}^2_{\mathtt{skew}}(\Omega)$, where

$$\mathbb{L}^2_{\mathtt{skew}}(\Omega) := \Big\{ \boldsymbol{\eta} \in \mathbb{L}^2(\Omega) : \quad \boldsymbol{\eta}^{\mathtt{t}} = -\boldsymbol{\eta} \Big\}.$$

Summarizing, the steady-state model (2.10) is now reformulated in terms of Eqs. (2.13), (2.15), (2.17), (2.18), (2.19), and (2.20). The unknowns of the global system are the tensors $\boldsymbol{\sigma}_f$ and $\boldsymbol{\sigma}_s$, the vorticity tensors $\boldsymbol{\gamma}_s$ and $\boldsymbol{\gamma}_f$, and the velocity vector fields $\mathbf{u}_f$ and $\mathbf{u}_s$, whereas the pressure scalar field p_f is easily computed by using the postprocessing formula given by the first equation of (2.14).

3. The variational formulation

In this section we derive the variational setting of the aforementioned reformulation of the steady-state model (2.10), and then we analyze its solvability.

3.1. A fully-mixed approach

We begin by observing, thanks to the Cauchy–Schwarz inequality and the uniform boundedness of ε and ϕ by 1, that the tensors $\boldsymbol{\sigma}_f^{\mathrm{d}}$, $\boldsymbol{\sigma}_s^{\mathrm{d}}$, $\big((\varepsilon\mathbf{u}_f)\otimes\mathbf{u}_f\big)^{\mathrm{d}}$, and $\big((\phi\mathbf{u}_s)\otimes\mathbf{u}_s\big)^{\mathrm{d}}$ appearing in (2.15), are integrable against $\boldsymbol{\tau} \in \mathbb{L}^2(\Omega)$, if the pairs $(\boldsymbol{\sigma}_f, \boldsymbol{\sigma}_s)$ and $(\mathbf{u}_f, \mathbf{u}_s)$ are assumed to live in $\mathbb{L}^2(\Omega) \times \mathbb{L}^2(\Omega)$ and $\mathbf{L}^4(\Omega) \times \mathbf{L}^4(\Omega)$, respectively. Similarly, we deduce, using now the Hölder inequality, that the terms in (2.13) involving the divergence operator $\mathbf{div}$ are integrable against corresponding test functions in $\mathbf{L}^4(\Omega)$ if both $\mathbf{div}(\boldsymbol{\sigma}_f)$ and $\mathbf{div}(\boldsymbol{\sigma}_s)$ belong to $\mathbf{L}^{4/3}(\Omega)$. The above suggests to look for the unknowns $\boldsymbol{\sigma}_f$ and $\boldsymbol{\sigma}_s$ in $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$, where, according to (2.2), we set

$$\mathbb{H}(\mathbf{div}_{4/3};\Omega) := \Big\{ \boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \quad \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^{4/3}(\Omega) \Big\}.$$

Then, we notice that there holds

$$\mathbb{H}(\mathbf{div}_{4/3};\Omega) := \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \oplus \mathrm{R}\,\mathbb{I}, \tag{3.1}$$

where

$$\mathbb{H}_0(\mathbf{div}_{4/3};\Omega) := \Big\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega) : \quad \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) = 0 \Big\}, \tag{3.2}$$

which means that for each tensor $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega)$ there exist unique $\boldsymbol{\tau}_0 \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$ and $d_0 := \frac{1}{n|\Omega|}\int_\Omega \mathrm{tr}(\boldsymbol{\tau}) \in \mathrm{R}$, such that $\boldsymbol{\tau} = \boldsymbol{\tau}_0 + d_0\mathbb{I}$. In particular, we have the decompositions

$$\boldsymbol{\sigma}_f = \boldsymbol{\sigma}_{f,0} + d_{f,0}\,\mathbb{I} \quad \text{and} \quad \boldsymbol{\sigma}_s = \boldsymbol{\sigma}_{s,0} + d_{s,0}\,\mathbb{I},$$

where $\boldsymbol{\sigma}_{f,0}$, $\boldsymbol{\sigma}_{s,0} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, and the constants $d_{f,0}$ and $d_{s,0}$ are computed according to the foregoing definition of the generic constant d_0, and employing (2.17) and (2.18), respectively, which gives

$$d_{f,0} := -\frac{1}{n|\Omega|}\int_\Omega \mathrm{tr}\big(\rho_f(\varepsilon\mathbf{u}_f)\otimes\mathbf{u}_f\big)$$

and

$$d_{s,0} := -\frac{1}{n|\Omega|}\int_\Omega \Big\{ np_s(\phi) + \mathrm{tr}\Big(\rho_s(\phi\mathbf{u}_s)\otimes\mathbf{u}_s + \rho_f(\varepsilon\mathbf{u}_f)\otimes\mathbf{u}_f\Big)\Big\}.$$

As a consequence, and regarding the unknowns $\boldsymbol{\sigma}_f$ and $\boldsymbol{\sigma}_s$, it only remains to find their $\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$-components $\boldsymbol{\sigma}_{f,0}$ and $\boldsymbol{\sigma}_{s,0}$, which, because of the constant tensorial components given by $d_{f,0}\,\mathbb{I}$ and $d_{s,0}\,\mathbb{I}$, are easily shown to satisfy exactly the same Eqs. (2.13) and (2.15) satisfied by $\boldsymbol{\sigma}_f$ and $\boldsymbol{\sigma}_s$. In this way, from now on we denote $\boldsymbol{\sigma}_{f,0}$ and $\boldsymbol{\sigma}_{s,0}$ by simply $\boldsymbol{\sigma}_f$ and $\boldsymbol{\sigma}_s$, and look for them in $\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, and satisfying the aforementioned equations. In this regard, we now notice that there is no need to explicitly impose the testing of (2.15) with multiples of $\mathbb{I}$, since, in doing so, both sides of the equations are nullified, which means that (2.15) is implicitly satisfied.

According to the above discussion, and bearing in mind (3.1), we now proceed to test the equations of (2.15) with functions in $\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$. Indeed, multiplying the first equation of (2.15) by $\boldsymbol{\tau}_f \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, dividing by $2\mu_f$, replacing $\mathbf{e}(\mathbf{u}_f)$ by its decomposition from (2.19), integrating by parts, and utilizing the first identity of (2.16) and the Dirichlet boundary condition for $\mathbf{u}_f$, we obtain

$$\mathbf{a}_f(\boldsymbol{\sigma}_f,\boldsymbol{\tau}_f) + \mathbf{b}\big(\boldsymbol{\tau}_f,(\mathbf{u}_f,\boldsymbol{\gamma}_f)\big) + \mathbf{c}_f(\mathbf{u}_f,\boldsymbol{\tau}_f) + \mathbf{d}_f(\mathbf{u}_f;\mathbf{u}_f,\boldsymbol{\tau}_f) = \mathbf{F}_f(\boldsymbol{\tau}_f), \tag{3.3}$$

for all $\boldsymbol{\tau}_f \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, where the bilinear forms $\mathbf{a}_f$, $\mathbf{b}$, and $\mathbf{c}_f$, the trilinear form $\mathbf{d}_f$, and the linear functional $\mathbf{F}_f$ are defined by

$$\begin{aligned}
\mathbf{a}_f(\boldsymbol{\zeta}_f,\boldsymbol{\tau}_f) &:= \frac{1}{2\mu_f}\int_\Omega \boldsymbol{\zeta}_f^{\mathrm{d}} : \boldsymbol{\tau}_f^{\mathrm{d}}\,,\\
\mathbf{b}\big(\boldsymbol{\tau}_f,(\mathbf{v}_f,\boldsymbol{\eta}_f)\big) &:= \int_\Omega \mathbf{v}_f\cdot\mathbf{div}(\boldsymbol{\tau}_f) + \int_\Omega \boldsymbol{\eta}_f : \boldsymbol{\tau}_f\,,\\
\mathbf{c}_f(\mathbf{v}_f,\boldsymbol{\tau}_f) &:= -\frac{1}{n}\int_\Omega \Big(\frac{\nabla\varepsilon}{\varepsilon}\cdot\mathbf{v}_f\Big)\,\mathrm{tr}(\boldsymbol{\tau}_f)\,,\\
\mathbf{d}_f(\mathbf{w}_f;\mathbf{v}_f,\boldsymbol{\tau}_f) &:= \frac{\rho_f}{2\mu_f}\int_\Omega \big((\varepsilon\mathbf{w}_f)\otimes\mathbf{v}_f\big)^{\mathrm{d}} : \boldsymbol{\tau}_f\,,
\end{aligned} \tag{3.4}$$

and

$$\mathbf{F}_f(\boldsymbol{\tau}_f) := \big\langle \boldsymbol{\tau}_f\mathbf{n}, \mathbf{u}_{D,f}\big\rangle\,, \tag{3.5}$$

for all $\boldsymbol{\zeta}_f,\ \boldsymbol{\tau}_f \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, for all $\mathbf{v}_f,\ \mathbf{w}_f \in \mathbf{L}^4(\Omega)$, and for all $\boldsymbol{\eta}_f \in \mathbb{L}^2_{\mathtt{skew}}(\Omega)$. Similarly, multiplying now the second equation of (2.15) by $\boldsymbol{\tau}_s \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, dividing by $2\mu_s(\phi)$, replacing $\mathbf{e}(\mathbf{u}_s)$ by its decomposition from (2.19), integrating by parts, utilizing the second identity of (2.16) and the Dirichlet boundary condition for $\mathbf{u}_s$, and denoting from now on $\mathbf{u} := (\mathbf{u}_f,\mathbf{u}_s)$, we obtain

$$\mathbf{a}_s(\boldsymbol{\sigma}_s,\boldsymbol{\tau}_s) + \mathbf{b}\big(\boldsymbol{\tau}_s,(\mathbf{u}_s,\boldsymbol{\gamma}_s)\big) + \mathbf{c}_s(\mathbf{u}_s,\boldsymbol{\tau}_s) + \mathbf{d}_s(\mathbf{u}_s;\mathbf{u}_s,\boldsymbol{\tau}_s) = \mathbf{F}_s^{\mathbf{u}}(\boldsymbol{\tau}_s)\,, \tag{3.6}$$

for all $\boldsymbol{\tau}_s \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, where the bilinear forms $\mathbf{a}_s$ and $\mathbf{c}_s$, the trilinear form $\mathbf{d}_s$, and the linear functional $\mathbf{F}_s^{\mathbf{u}}$ are defined by

$$\begin{aligned}
\mathbf{a}_s(\boldsymbol{\zeta}_s,\boldsymbol{\tau}_s) &:= \int_\Omega \frac{1}{2\mu_s(\phi)}\,\boldsymbol{\zeta}_s^{\mathrm{d}} : \boldsymbol{\tau}_s^{\mathrm{d}}\,,\\
\mathbf{c}_s(\mathbf{v}_s,\boldsymbol{\tau}_s) &:= -\frac{1}{n}\int_\Omega \Big(\frac{\nabla\phi}{\phi}\cdot\mathbf{v}_s\Big)\,\mathrm{tr}(\boldsymbol{\tau}_s)\,,\\
\mathbf{d}_s(\mathbf{w}_s;\mathbf{v}_s,\boldsymbol{\tau}_s) &:= \int_\Omega \frac{\rho_s}{2\mu_s(\phi)}\big((\phi\mathbf{w}_s)\otimes\mathbf{v}_s\big)^{\mathrm{d}} : \boldsymbol{\tau}_s\,,
\end{aligned} \tag{3.7}$$

and

$$\mathbf{F}_s^{\mathbf{u}}(\boldsymbol{\tau}_s) := \big\langle \boldsymbol{\tau}_s\mathbf{n}, \mathbf{u}_{D,s}\big\rangle - \int_\Omega \frac{\rho_f}{2\mu_s(\phi)}\big((\varepsilon\mathbf{u}_f)\otimes\mathbf{u}_f\big)^{\mathrm{d}} : \boldsymbol{\tau}_s\,, \tag{3.8}$$

for all $\boldsymbol{\zeta}_s,\ \boldsymbol{\tau}_s \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, and for all $\mathbf{v}_s,\ \mathbf{w}_s \in \mathbf{L}^4(\Omega)$. Note that $\mathbf{F}_s^{\mathbf{u}}$ is denoted in this way irrespective of the fact that it only depends on the first component $\mathbf{u}_f$ of $\mathbf{u}$. Next, testing the equations of (2.13) against $\mathbf{v}_f \in \mathbf{L}^4(\Omega)$ and $\mathbf{v}_s \in \mathbf{L}^4(\Omega)$, respectively, we obtain

$$\int_\Omega \mathbf{v}_f\cdot\mathbf{div}(\boldsymbol{\sigma}_f) - \int_\Omega F(\mathbf{u})\cdot\mathbf{v}_f = -\int_\Omega \varepsilon\rho_f\mathbf{g}\cdot\mathbf{v}_f \qquad \forall\,\mathbf{v}_f \in \mathbf{L}^4(\Omega)\,, \tag{3.9}$$

and

$$\int_\Omega \mathbf{v}_s\cdot\mathbf{div}(\boldsymbol{\sigma}_s) = -\int_\Omega (\varepsilon\rho_f + \phi\rho_s)\mathbf{g}\cdot\mathbf{v}_s \qquad \forall\,\mathbf{v}_s \in \mathbf{L}^4(\Omega)\,. \tag{3.10}$$

Finally, the symmetries of $\boldsymbol{\sigma}_f$ and $\boldsymbol{\sigma}_s$ are imposed weakly as

$$\int_\Omega \boldsymbol{\sigma}_f : \boldsymbol{\eta}_f = 0 \qquad \forall\,\boldsymbol{\eta}_f \in \mathbb{L}^2_{\mathtt{skew}}(\Omega) \tag{3.11}$$

and

$$\int_\Omega \boldsymbol{\sigma}_s : \boldsymbol{\eta}_s = 0 \qquad \forall\,\boldsymbol{\eta}_s \in \mathbb{L}^2_{\mathtt{skew}}(\Omega)\,, \tag{3.12}$$

so that after adding (3.11) and (3.12) to (3.9) and (3.10), respectively, we end up with

$$\mathbf{b}\big(\boldsymbol{\sigma}_f,(\mathbf{v}_f,\boldsymbol{\eta}_f)\big) = \mathbf{G}_f^{\mathbf{u}}(\mathbf{v}_f,\boldsymbol{\eta}_f) \quad \forall\,(\mathbf{v}_f,\boldsymbol{\eta}_f) \in \mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\mathtt{skew}}(\Omega) \tag{3.13}$$

and

$$\mathbf{b}\big(\boldsymbol{\sigma}_s,(\mathbf{v}_s,\boldsymbol{\eta}_s)\big) = \mathbf{G}_s(\mathbf{v}_s,\boldsymbol{\eta}_s) \quad \forall\,(\mathbf{v}_s,\boldsymbol{\eta}_s) \in \mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\mathtt{skew}}(\Omega)\,, \tag{3.14}$$

where

$$\mathbf{G}_f^{\mathbf{u}}(\mathbf{v}_f,\boldsymbol{\eta}_f) := \int_\Omega F(\mathbf{u})\cdot\mathbf{v}_f - \int_\Omega \varepsilon\rho_f\mathbf{g}\cdot\mathbf{v}_f\,, \tag{3.15}$$

and

$$\mathbf{G}_s(\mathbf{v}_s,\boldsymbol{\eta}_s) := -\int_\Omega (\varepsilon\rho_f + \phi\rho_s)\mathbf{g}\cdot\mathbf{v}_s\,. \tag{3.16}$$

In this way, the fully-mixed variational formulation of (2.10) reduces basically to Eqs. (3.3), (3.6), (3.13), and (3.14). More precisely, introducing the spaces

$$\mathrm{H} := \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \quad\text{and}\quad \mathrm{Q} := \mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\mathtt{skew}}(\Omega)\,, \tag{3.17}$$

with norms $\|\boldsymbol{\tau}\|_{\mathrm{H}} := \|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega}$ for all $\boldsymbol{\tau}\in\mathrm{H}$, and $\|(\mathbf{v},\boldsymbol{\eta})\|_{\mathrm{Q}} := \Big\{\|\mathbf{v}\|^2_{0,4;\Omega} + \|\boldsymbol{\eta}\|_{0,\Omega}\Big\}^{1/2}$ for all $(\mathbf{v},\boldsymbol{\eta})\in\mathrm{Q}$, we seek $\big(\boldsymbol{\sigma}_f,(\mathbf{u}_f,\boldsymbol{\gamma}_f)\big)\in\mathrm{H}\times\mathrm{Q}$ and $\big(\boldsymbol{\sigma}_s,(\mathbf{u}_s,\boldsymbol{\gamma}_s)\big)\in\mathrm{H}\times Q$ such that

$$\begin{aligned}
\mathbf{a}_f(\boldsymbol{\sigma}_f,\boldsymbol{\tau}_f) + \mathbf{b}\big(\boldsymbol{\tau}_f,(\mathbf{u}_f,\boldsymbol{\gamma}_f)\big) + \mathbf{c}_f(\mathbf{u}_f,\boldsymbol{\tau}_f) + \mathbf{d}_f(\mathbf{u}_f;\mathbf{u}_f,\boldsymbol{\tau}_f) &= \mathbf{F}_f(\boldsymbol{\tau}_f)\,,\\
\mathbf{b}\big(\boldsymbol{\sigma}_f,(\mathbf{v}_f,\boldsymbol{\eta}_f)\big) &= \mathbf{G}^{\mathbf{u}}_f(\mathbf{v}_f,\boldsymbol{\eta}_f)\,,\\
\mathbf{a}_s(\boldsymbol{\sigma}_s,\boldsymbol{\tau}_s) + \mathbf{b}\big(\boldsymbol{\tau}_s,(\mathbf{u}_s,\boldsymbol{\gamma}_s)\big) + \mathbf{c}_s(\mathbf{u}_s,\boldsymbol{\tau}_s) + \mathbf{d}_s(\mathbf{u}_s;\mathbf{u}_s,\boldsymbol{\tau}_s) &= \mathbf{F}^{\mathbf{u}}_s(\boldsymbol{\tau}_s)\,,\\
\mathbf{b}\big(\boldsymbol{\sigma}_s,(\mathbf{v}_s,\boldsymbol{\eta}_s)\big) &= \mathbf{G}_s(\mathbf{v}_s,\boldsymbol{\eta}_s)\,,
\end{aligned} \tag{3.18}$$

for all $\big(\boldsymbol{\tau}_f,(\mathbf{v}_f,\boldsymbol{\eta}_f)\big)\in\mathrm{H}\times\mathrm{Q}$ and for all $\big(\boldsymbol{\tau}_s,(\mathbf{v}_s,\boldsymbol{\eta}_s)\big)\in\mathrm{H}\times Q$.

We end this section by establishing the boundedness properties of all the forms involved in (3.18). Firstly, regarding $\mathbf{a}_f$, $\mathbf{b}$, $\mathbf{c}_f$, $\mathbf{d}_f$, $\mathbf{F}_f$, and $\mathbf{G}^{\mathbf{u}}_f$, we notice from (3.4), (3.5), and (3.15), that direct applications of the Cauchy–Schwarz and Hölder inequalities, combined with the boundedness of the normal trace operator in $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$, and the expression for $F(\mathbf{u})$ given by (2.8), yield

$$|\mathbf{a}_f(\boldsymbol{\zeta}_f,\boldsymbol{\tau}_f)| \le \|\mathbf{a}_f\|\,\|\boldsymbol{\zeta}_f\|_{0,\Omega}\,\|\boldsymbol{\tau}_f\|_{0,\Omega}\,, \tag{3.19}$$

$$\big|\mathbf{b}\big(\boldsymbol{\tau}_f,(\mathbf{v}_f,\boldsymbol{\eta}_f)\big)\big| \le \|\mathbf{b}\|\,\|\boldsymbol{\tau}_f\|_{\mathrm{H}}\,\|(\mathbf{v}_f,\boldsymbol{\eta}_f)\|_{\mathrm{Q}}\,, \tag{3.20}$$

$$|\mathbf{c}_f(\mathbf{v}_f,\boldsymbol{\tau}_f)| \le \|\mathbf{c}_f\|\,\|\mathbf{v}_f\|_{0,4;\Omega}\,\|\boldsymbol{\tau}_f\|_{0,\Omega}\,, \tag{3.21}$$

$$|\mathbf{d}_f(\mathbf{w}_f;\mathbf{v}_f,\boldsymbol{\tau}_f)| \le \|\mathbf{d}_f\|\,\|\mathbf{w}_f\|_{0,4;\Omega}\,\|\mathbf{v}_f\|_{0,4;\Omega}\,\|\boldsymbol{\tau}_f\|_{0,\Omega}\,, \tag{3.22}$$

$$|\mathbf{F}_f(\boldsymbol{\tau}_f)| \le \|\mathbf{F}_f\|\,\|\boldsymbol{\tau}_f\|_{\mathrm{H}}\,,\quad\text{and} \tag{3.23}$$

$$|\mathbf{G}^{\mathbf{u}}_f(\mathbf{v}_f,\boldsymbol{\eta}_f)| \le \|\mathbf{G}^{\mathbf{u}}_f\|\,\|\mathbf{v}_f\|_{0,4;\Omega}\,, \tag{3.24}$$

where

$$\begin{gathered}
\|\mathbf{a}_f\| = \frac{1}{2\mu_f}\,,\qquad \|\mathbf{b}\| = 1\,,\qquad \|\mathbf{c}_f\| = \frac{1}{\sqrt{n}}\left\|\frac{\nabla\varepsilon}{\varepsilon}\right\|_{0,4;\Omega}\,,\\
\|\mathbf{d}_f\| = \frac{\rho_f}{2\mu_f}\,\|\varepsilon\|_{0,\infty;\Omega}\,,\qquad \|\mathbf{F}_f\| = \|\mathbf{u}_{D,f}\|_{1/2,\Gamma}\,,\quad\text{and}\\
\|\mathbf{G}^{\mathbf{u}}_f\| = \|\delta(\phi)\|_{0,\Omega}\,\|\mathbf{u}_f-\mathbf{u}_s\|_{0,4;\Omega} + |\Omega|^{3/4}\,\rho_f\,\mathbf{g}\,\|\varepsilon\|_{0,\infty;\Omega}\,.
\end{gathered} \tag{3.25}$$

In turn, in order to derive the respective bounds for $\mathbf{a}_s$, $\mathbf{c}_s$, $\mathbf{d}_s$, $\mathbf{F}^{\mathbf{u}}_s$, and $\mathbf{G}^{\mathbf{u}}_s$, we assume from now on that $\mu_s(\phi)$ is bounded above and below, which means that there exist positive constants μ_1 and μ_2, independent of the given ϕ, such that

$$0 < \mu_1 \le \mu_s(\phi) \le \mu_2\,. \tag{3.26}$$

Equivalently, and according to (2.7), the above means that ϕ is assumed to remain bounded away from its lower and upper bounds given by 0 and ϕ_p, respectively. Needless to say, this is precisely the case of fluidized beds. Then, proceeding as for (3.19)–(3.24), we find that

$$|\mathbf{a}_s(\boldsymbol{\zeta}_s,\boldsymbol{\tau}_s)| \le \|\mathbf{a}_s\|\,\|\boldsymbol{\zeta}_s\|_{0,\Omega}\,\|\boldsymbol{\tau}_s\|_{0,\Omega}\,, \tag{3.27}$$

$$|\mathbf{c}_s(\mathbf{v}_s,\boldsymbol{\tau}_s)| \le \|\mathbf{c}_s\|\,\|\mathbf{v}_s\|_{0,4;\Omega}\,\|\boldsymbol{\tau}_s\|_{0,\Omega}\,, \tag{3.28}$$

$$|\mathbf{d}_s(\mathbf{w}_s;\mathbf{v}_s,\boldsymbol{\tau}_s)| \le \|\mathbf{d}_s\|\,\|\mathbf{w}_s\|_{0,4;\Omega}\,\|\mathbf{v}_s\|_{0,4;\Omega}\,\|\boldsymbol{\tau}_s\|_{0,\Omega}\,, \tag{3.29}$$

$$|\mathbf{F}^{\mathbf{u}}_s(\boldsymbol{\tau}_s)| \le \|\mathbf{F}^{\mathbf{u}}_s\|\,\|\boldsymbol{\tau}_s\|_{\mathrm{H}}\,,\quad\text{and} \tag{3.30}$$

$$|\mathbf{G}_s(\mathbf{v}_s,\boldsymbol{\eta}_s)| \le \|\mathbf{G}_s\|\,\|\mathbf{v}_s\|_{0,4;\Omega}\,, \tag{3.31}$$

where

$$\|\mathbf{a}_s\| \,=\, \frac{1}{2\mu_1}\,, \quad \|\mathbf{c}_s\| \,=\, \frac{1}{\sqrt{n}}\left\|\frac{\nabla\phi}{\phi}\right\|_{0,4;\Omega}\,, \quad \|\mathbf{d}_s\| \,=\, \frac{\rho_s}{2\mu_1}\,\|\phi\|_{0,\infty;\Omega}\,,$$
$$\|\mathbf{F}_s^{\mathbf{u}}\| \,=\, \|\mathbf{u}_{D,s}\|_{1/2,\Gamma} \,+\, \frac{\rho_f}{2\mu_1}\,\|\varepsilon\|_{0,\infty;\Omega}\,\|\mathbf{u}_f\|_{0,4;\Omega}^2\,, \quad \text{and}$$
$$\|\mathbf{G}_s\| \,=\, |\Omega|^{3/4}\,\mathbf{g}\,\|\varepsilon\rho_f + \phi\rho_s\|_{0,\infty;\Omega}\,. \tag{3.32}$$

3.2. A fixed-point approach

In what follows we proceed as in related works (see, e.g. [15,17,23–27]), and [28], and introduce fixed-point strategies to analyze the solvability of (3.18). To this end, we first define the operator $\Theta_f : \mathbf{L}^4(\Omega)\times\mathbf{L}^4(\Omega)\to\mathbf{L}^4(\Omega)$ as

$$\Theta_f(\mathbf{w}) \,:=\, \widehat{\mathbf{u}}_f \qquad \forall\,\mathbf{w} := (\mathbf{w}_f,\mathbf{w}_s)\in\mathbf{L}^4(\Omega)\times\mathbf{L}^4(\Omega)\,, \tag{3.33}$$

where $\big(\widehat{\boldsymbol{\sigma}}_f,(\widehat{\mathbf{u}}_f,\widehat{\boldsymbol{\gamma}}_f)\big)\in \mathrm{H}\times\mathrm{Q}$ is the unique solution (to be confirmed below) of the first two equations of (3.18) when the first component $\mathbf{u}_f$ of $\mathbf{d}_f$ and the superscript $\mathbf{u}$ of $\mathbf{G}_f^{\mathbf{u}}$ are replaced by $\mathbf{w}_f$ and $\mathbf{w}$, respectively, that is

$$\begin{aligned} \mathbf{a}_f(\widehat{\boldsymbol{\sigma}}_f,\boldsymbol{\tau}_f)+\mathbf{b}\big(\boldsymbol{\tau}_f,(\widehat{\mathbf{u}}_f,\widehat{\boldsymbol{\gamma}}_f)\big)+\mathbf{c}_f(\widehat{\mathbf{u}}_f,\boldsymbol{\tau}_f)+\mathbf{d}_f(\mathbf{w}_f;\widehat{\mathbf{u}}_f,\boldsymbol{\tau}_f) &= \mathbf{F}_f(\boldsymbol{\tau}_f)\,,\\ \mathbf{b}\big(\widehat{\boldsymbol{\sigma}}_f,(\mathbf{v}_f,\boldsymbol{\eta}_f)\big) &= \mathbf{G}_f^{\mathbf{w}}(\mathbf{v}_f,\boldsymbol{\eta}_f)\,, \end{aligned} \tag{3.34}$$

for all $\big(\boldsymbol{\tau}_f,(\mathbf{v}_f,\boldsymbol{\eta}_f)\big)\in\mathrm{H}\times\mathrm{Q}$. In turn, we let $\Theta_s : \mathbf{L}^4(\Omega)\times\mathbf{L}^4(\Omega)\to\mathbf{L}^4(\Omega)$ be the operator given by

$$\Theta_s(\mathbf{w}) \,:=\, \widehat{\mathbf{u}}_s \qquad \forall\,\mathbf{w} := (\mathbf{w}_f,\mathbf{w}_s)\in\mathbf{L}^4(\Omega)\times\mathbf{L}^4(\Omega)\,, \tag{3.35}$$

where $\big(\widehat{\boldsymbol{\sigma}}_s,(\widehat{\mathbf{u}}_s,\widehat{\boldsymbol{\gamma}}_s)\big)\in \mathrm{H}\times\mathrm{Q}$ is the unique solution (to be confirmed below) of the last two equations of (3.18) when the first component $\mathbf{u}_s$ of $\mathbf{d}_s$ and the superscript $\mathbf{u}$ of $\mathbf{F}_s^{\mathbf{u}}$ are replaced by $\mathbf{w}_s$ and $\mathbf{w}$, respectively, that is

$$\begin{aligned} \mathbf{a}_s(\widehat{\boldsymbol{\sigma}}_s,\boldsymbol{\tau}_s)+\mathbf{b}\big(\boldsymbol{\tau}_s,(\widehat{\mathbf{u}}_s,\widehat{\boldsymbol{\gamma}}_s)\big)+\mathbf{c}_s(\widehat{\mathbf{u}}_s,\boldsymbol{\tau}_s)+\mathbf{d}_s(\mathbf{w}_s;\widehat{\mathbf{u}}_s,\boldsymbol{\tau}_s) &= \mathbf{F}_s^{\mathbf{w}}(\boldsymbol{\tau}_s)\,,\\ \mathbf{b}\big(\widehat{\boldsymbol{\sigma}}_s,(\mathbf{v}_s,\boldsymbol{\eta}_s)\big) &= \mathbf{G}_s(\mathbf{v}_s,\boldsymbol{\eta}_s)\,, \end{aligned} \tag{3.36}$$

for all $\big(\boldsymbol{\tau}_s,(\mathbf{v}_s,\boldsymbol{\eta}_s)\big)\in\mathrm{H}\times\mathrm{Q}$. Then, we set the operator $S : \mathbf{L}^4(\Omega)\times\mathbf{L}^4(\Omega)\to\mathbf{L}^4(\Omega)\times\mathbf{L}^4(\Omega)$ as

$$S(\mathbf{w}) := \big(\Theta_f(\mathbf{w}),\,\Theta_s(\mathbf{w})\big) \qquad \forall\,\mathbf{w} := (\mathbf{w}_f,\mathbf{w}_s)\in\mathbf{L}^4(\Omega)\times\mathbf{L}^4(\Omega)\,, \tag{3.37}$$

and readily see that solving (3.18) is equivalent to seeking a fixed-point of S, that is: find $\mathbf{w}\in\mathbf{L}^4(\Omega)\times\mathbf{L}^4(\Omega)$ such that

$$S(\mathbf{w}) \,=\, \mathbf{w}\,. \tag{3.38}$$

Alternatively, one could define an operator $T : \mathbf{L}^4(\Omega)\times\mathbf{L}^4(\Omega)\to\mathbf{L}^4(\Omega)\times\mathbf{L}^4(\Omega)$, either as

$$T(\mathbf{w}) := \big(\Theta_f(\mathbf{w}),\,\Theta_s(\Theta_f(\mathbf{w}),\mathbf{w}_s)\big) \qquad \forall\,\mathbf{w} := (\mathbf{w}_f,\mathbf{w}_s)\in\mathbf{L}^4(\Omega)\times\mathbf{L}^4(\Omega),$$

or

$$T(\mathbf{w}) := \big(\Theta_f(\mathbf{w}_f,\Theta_s(\mathbf{w})),\,\Theta_s(\mathbf{w})\big) \qquad \forall\,\mathbf{w} := (\mathbf{w}_f,\mathbf{w}_s)\in\mathbf{L}^4(\Omega)\times\mathbf{L}^4(\Omega),$$

so that, in both cases, solving (3.18) is equivalent to seeking a fixed-point of T as well, that is: find $\mathbf{w}\in\mathbf{L}^4(\Omega)\times\mathbf{L}^4(\Omega)$ such that

$$T(\mathbf{w}) \,=\, \mathbf{w}.$$

Nevertheless, for sake of clarity of the exposition, in what follows we concentrate only on the operator S. Indeed, while the algebraic manipulations of T are a bit more cumbersome, all the analyses and results that we provide below for S can be extended to T by performing minor modifications.

3.3. Well-definedness of the operators Θ_f and Θ_s

In this section we apply the Banach–Nečas–Babuška Theorem (also know as the generalized Lax–Milgram Lemma), and the classical Babuška–Brezzi theory, both in Banach spaces, to show that the problems (3.34) and (3.36) are well-posed, which means, equivalently, that the operators Θ_f and Θ_s are well-defined. We begin by recalling the aforementioned results (cf. [29, Theorems 2.6 and 2.34]).

Theorem 3.1. *Let* H *and* Q *be Banach spaces such that* Q *is reflexive, and let* $a : \mathrm{H}\times\mathrm{Q}\longrightarrow \mathrm{R}$ *be a bounded bilinear form. Assume that*

(i) *there exists* $\alpha > 0$ *such that*

$$\sup_{\substack{v\in\mathrm{Q}\\ v\neq 0}}\frac{a(w,v)}{\|v\|_{\mathrm{Q}}} \,\ge\, \alpha\,\|w\|_{\mathrm{H}} \qquad \forall\, w\in\mathrm{H}\,, \tag{3.39}$$

(ii) *there holds*

$$\sup_{w\in \mathrm{H}} a(w,v) \,>\, 0 \qquad \forall\, v\in \mathrm{Q},\;\; v\neq 0\,. \tag{3.40}$$

Then, for each $F\in \mathrm{Q}'$ *there exists a unique* $u\in \mathrm{H}$ *such that*

$$a(u,v) \,=\, F(v) \qquad \forall\, v\in \mathrm{Q}\,, \tag{3.41}$$

and the following a priori estimate holds

$$\|u\|_{\mathrm{H}} \,\leq\, \frac{1}{\alpha}\,\|F\|_{\mathrm{Q}'}\,. \tag{3.42}$$

Moreover, (i) *and* (ii) *are also necessary conditions for the well-posedness of* (3.41).

Theorem 3.2. *Let* H *and* Q *be reflexive Banach spaces, and let* $a:\mathrm{H}\times\mathrm{H}\longrightarrow \mathrm{R}$ *and* $b:\mathrm{H}\times\mathrm{Q}\longrightarrow\mathrm{R}$ *be bounded bilinear forms with induced operators* $A\in\mathcal{L}(\mathrm{H},\mathrm{H}')$ *and* $B\in\mathcal{L}(\mathrm{H},\mathrm{Q}')$*, respectively. In addition, let* V *be the null space of* B*, and assume that*

(i) *there exists* $\alpha>0$ *such that*

$$\sup_{\substack{\tau\in \mathrm{V}\\ \tau\neq 0}} \frac{a(\zeta,\tau)}{\|\tau\|_{\mathrm{H}}} \,\geq\, \alpha\,\|\zeta\|_{\mathrm{H}} \qquad \forall\,\zeta\in\mathrm{V}\,, \tag{3.43}$$

(ii) *there holds*

$$\sup_{\tau\in\mathrm{V}} a(\tau,\zeta) > 0 \qquad \forall\,\zeta\in\mathrm{V},\quad \zeta\neq 0\,, \tag{3.44}$$

(iii) *there exists* β *such that*

$$\sup_{\substack{\tau\in H\\ \tau\neq 0}} \frac{b(\tau,v)}{\|\tau\|_{\mathrm{H}}} \,\geq\, \beta\,\|v\|_{\mathrm{Q}} \qquad \forall\, v\in\mathrm{Q}\,. \tag{3.45}$$

Then, for each pair $(F,G)\in\mathrm{H}'\times\mathrm{Q}'$ *there exists a unique* $(\sigma,u)\in\mathrm{H}\times\mathrm{Q}$ *such that*

$$\begin{aligned} a(\sigma,\tau) \,+\, b(\tau,u) \,&=\, F(\tau) \qquad \forall\,\tau\in\mathrm{H}\,,\\ b(\sigma,v) \,&=\, G(v) \qquad \forall\, v\in\mathrm{Q}\,, \end{aligned} \tag{3.46}$$

and the following a priori estimates hold:

$$\begin{aligned} \|\sigma\| &\leq \frac{1}{\alpha}\,\|F\|_{\mathrm{H}'} + \frac{1}{\beta}\Big(1+\frac{\|A\|}{\alpha}\Big)\|G\|_{\mathrm{Q}'}\,,\\ \|u\| &\leq \frac{1}{\beta}\,\Big(1+\frac{\|A\|}{\alpha}\Big)\|F\|_{\mathrm{H}'} \,+\, \frac{\|A\|}{\beta^2}\Big(1+\frac{\|A\|}{\alpha}\Big)\|G\|_{\mathrm{Q}'}\,. \end{aligned} \tag{3.47}$$

Moreover, (i), (ii), *and* (iii) *are also necessary conditions for the well-posedness of* (3.46).

We find it important to stress here that (3.47) is equivalent to a global inf–sup condition for (3.46), which means that there exists a constant $\widetilde{\alpha}>0$, depending only on α, β, and $\|A\|$ (as it follows from the right hand side of (3.47)), such that

$$\sup_{\substack{(\tau,v)\in H\times Q\\ (\tau,v)\neq 0}} \frac{a(\zeta,\tau)+b(\tau,w)+b(\zeta,v)}{\|(\tau,v)\|_{H\times Q}} \,\geq\, \widetilde{\alpha}\,\|(\zeta,w)\|_{H\times Q} \qquad \forall\,(\zeta,w)\in H\times Q\,. \tag{3.48}$$

In order to apply Theorem 3.2 to suitable perturbations of (3.34) and (3.36), which is explained later on, we now let $\mathbb{V}$ be the kernel of the operator induced by $\mathbf{b}$, that is

$$\mathbb{V} := \Big\{\boldsymbol{\tau}\in\mathrm{H}:\quad \mathbf{b}\big(\boldsymbol{\tau},(\mathbf{v},\boldsymbol{\eta})\big)=0\quad\forall\,(\mathbf{v},\boldsymbol{\eta})\in\mathrm{Q}\Big\},$$

which, according to the definitions of $\mathbf{b}$ (cf. (3.4)) and the spaces H and Q (cf. (3.17)), yields

$$\mathbb{V} := \Big\{\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega):\quad \mathbf{div}(\boldsymbol{\tau})=\mathbf{0}\quad\text{and}\quad \boldsymbol{\tau}=\boldsymbol{\tau}^{\mathrm{t}}\quad\text{in}\quad\Omega\Big\}.$$

On the other hand, we recall that a simple modification of the proof of [30, Lemma 2.3] (or [31, Proposition 3.1, Chapter IV]) allows to show (see also [16, Lemma 3.2]) that there exists $c_1>0$, depending only on Ω, such that

$$c_1\,\|\boldsymbol{\tau}\|^2_{0,\Omega} \,\leq\, \|\boldsymbol{\tau}^{\mathrm{d}}\|^2_{0,\Omega} \,+\, \|\mathbf{div}(\boldsymbol{\tau})\|^2_{0,4/3;\Omega} \qquad \forall\,\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,. \tag{3.49}$$

Then, we have the following result establishing the $\mathbb{V}$-ellipticity of $\mathbf{a}_f$.

Lemma 3.3. *There exists a positive constant α_f, depending on c_1 (cf. (3.49)) and μ_f, such that*

$$\mathbf{a}_f(\boldsymbol{\tau},\boldsymbol{\tau}) \,\ge\, \alpha_f\,\|\boldsymbol{\tau}\|^2_{\mathbf{div}_{4/3};\Omega} \qquad \forall\,\boldsymbol{\tau}\in\mathbb{V}\,. \tag{3.50}$$

Proof. According to the definition of $\mathbf{a}_f$ (cf. (3.4)), and employing the inequality (3.49), we find that for each $\boldsymbol{\tau}\in\mathbb{V}$ there holds

$$\mathbf{a}_f(\boldsymbol{\tau},\boldsymbol{\tau}) \,=\, \frac{1}{2\mu_f}\,\|\boldsymbol{\tau}^{\mathrm{d}}\|^2_{0,\Omega} \,\ge\, \frac{c_1}{2\mu_f}\,\|\boldsymbol{\tau}\|^2_{0,\Omega} \,=\, \frac{c_1}{2\mu_f}\,\|\boldsymbol{\tau}\|^2_{\mathbf{div}_{4/3};\Omega}\,,$$

which shows (3.50) with $\alpha_f = \frac{c_1}{2\mu_f}$. □

In turn, the $\mathbb{V}$-ellipticity of the bilinear form $\mathbf{a}_s$ is established as follows.

Lemma 3.4. *There exists a positive constant α_s, depending on c_1 (cf. (3.49)) and μ_2 (cf. (3.26)), such that*

$$\mathbf{a}_s(\boldsymbol{\tau},\boldsymbol{\tau}) \,\ge\, \alpha_s\,\|\boldsymbol{\tau}\|^2_{\mathbf{div}_{4/3};\Omega} \qquad \forall\,\boldsymbol{\tau}\in\mathbb{V}\,. \tag{3.51}$$

Proof. Using now the definition of $\mathbf{a}_s$ (cf. (3.7)), the upper bound of the assumption (3.26), and the inequality (3.49), we find that for each $\boldsymbol{\tau}\in\mathbb{V}$ there holds

$$\mathbf{a}_s(\boldsymbol{\tau},\boldsymbol{\tau}) \,=\, \int_\Omega \frac{1}{2\mu_s(\phi)}\,\|\boldsymbol{\tau}^{\mathrm{d}}\|^2 \,\ge\, \frac{1}{2\mu_2}\,\|\boldsymbol{\tau}^{\mathrm{d}}\|^2_{0,\Omega} \,\ge\, \frac{c_1}{2\mu_2}\,\|\boldsymbol{\tau}\|^2_{0,\Omega} \,=\, \frac{c_1}{2\mu_2}\,\|\boldsymbol{\tau}\|^2_{\mathbf{div}_{4/3};\Omega}\,,$$

which confirms (3.51) with $\alpha_s = \frac{c_1}{2\mu_2}$. □

As a consequence of Lemmas 3.3 and 3.4, we stress here that both $\mathbf{a}_f$ and $\mathbf{a}_s$ satisfy the assumptions (i) and (ii) of Theorem 3.2. Indeed, it is easily seen that

$$\sup_{\substack{\boldsymbol{\tau}\in\mathbb{V}\\ \boldsymbol{\tau}\neq\mathbf{0}}} \frac{\mathbf{a}_f(\boldsymbol{\zeta},\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega}} \,\ge\, \frac{\mathbf{a}_f(\boldsymbol{\zeta},\boldsymbol{\zeta})}{\|\boldsymbol{\zeta}\|_{\mathbf{div}_{4/3};\Omega}} \,\ge\, \alpha_f\,\|\boldsymbol{\zeta}\|_{\mathbf{div}_{4/3};\Omega} \qquad \forall\,\boldsymbol{\zeta}\in\mathbb{V},$$

and

$$\sup_{\boldsymbol{\tau}\in\mathbb{V}} \mathbf{a}_f(\boldsymbol{\tau},\boldsymbol{\zeta}) \,\ge\, \mathbf{a}_f(\boldsymbol{\zeta},\boldsymbol{\zeta}) \,\ge\, \alpha_f\,\|\boldsymbol{\zeta}\|^2_{\mathbf{div}_{4/3};\Omega} > 0 \qquad \forall\,\boldsymbol{\zeta}\in\mathbb{V},\ \boldsymbol{\zeta}\neq\mathbf{0},$$

and analogously for $\mathbf{a}_s$.

Furthermore, the following lemma states that $\mathbf{b}$ satisfies the hypothesis (iii) of Theorem 3.2.

Lemma 3.5. *There exists $\beta > 0$, depending only on Ω, such that*

$$\sup_{\substack{\boldsymbol{\tau}\in\mathrm{H}\\ \boldsymbol{\tau}\neq\mathbf{0}}} \frac{\mathbf{b}\big(\boldsymbol{\tau},(\mathbf{v},\boldsymbol{\eta})\big)}{\|\boldsymbol{\tau}\|_{\mathrm{H}}} \,\ge\, \beta\,\|(\mathbf{v},\boldsymbol{\eta})\|_{\mathrm{Q}} \qquad \forall\,(\mathbf{v},\boldsymbol{\eta})\in\mathrm{Q}\,. \tag{3.52}$$

Proof. Given $(\mathbf{v},\boldsymbol{\eta})\in\mathrm{Q} := \mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\mathtt{skew}}(\Omega)$, we first let $\mathbf{v}_{4/3} := |\mathbf{v}|^2\mathbf{v}$ and observe that $\|\mathbf{v}_{4/3}\|^{4/3}_{0,4/3;\Omega} = \|\mathbf{v}\|^4_{0,4;\Omega}$, which proves that $\mathbf{v}_{4/3}\in\mathbf{L}^{4/3}(\Omega)$ and yields

$$\int_\Omega \mathbf{v}\cdot\mathbf{v}_{4/3} \,=\, \|\mathbf{v}\|^4_{0,4;\Omega} \,=\, \|\mathbf{v}\|_{0,4;\Omega}\,\|\mathbf{v}_{4/3}\|_{0,4/3;\Omega}\,. \tag{3.53}$$

Then, we consider the boundary value problem

$$\mathbf{div}\big(\mathbf{e}(\mathbf{w})\big) \,=\, \mathbf{v}_{4/3} \quad \text{in} \quad \mathcal{D}'(\Omega)\,, \quad \text{and} \quad \mathbf{w} = \mathbf{0} \quad \text{on} \quad \Gamma\,, \tag{3.54}$$

whose weak formulation is: find $\mathbf{w}\in\mathbf{H}^1_0(\Omega)$ such that

$$\int_\Omega \mathbf{e}(\mathbf{w}) : \mathbf{e}(\mathbf{z}) \,=\, -\int_\Omega \mathbf{v}_{4/3}\cdot\mathbf{z} \qquad \forall\,\mathbf{z}\,\in\,\mathbf{H}^1_0(\Omega)\,. \tag{3.55}$$

Note that the right hand side of (3.55) makes sense thanks to the Hölder inequality and the continuous injection $\mathbf{i}_4 : \mathbf{H}^1(\Omega)\to\mathbf{L}^4(\Omega)$ (which is valid in both 2D and 3D). Then, bearing in mind the Poincaré and the first Korn (cf. [32, Theorem 10.1] or [33, Corollaries 9.2.22 and 9.2.25]) inequalities, which establish that

$$\|\mathbf{v}\|^2_{1,\Omega} \,\le\, c_{\mathrm{P}}\,|\mathbf{v}|^2_{1,\Omega} \quad \text{and} \quad |\mathbf{v}|^2_{1,\Omega} \,\le\, 2\,\|\mathbf{e}(\mathbf{v})\|^2_{0,\Omega} \qquad \forall\,\mathbf{v}\in\mathbf{H}^1_0(\Omega),$$

respectively, with a positive constant c_{P} depending only on Ω, and then applying the well-known Lax–Milgram Lemma, we easily deduce that (3.55) has a unique solution $\mathbf{w}\in\mathbf{H}^1_0(\Omega)$, for which there holds

$$\|\mathbf{w}\|_{1,\Omega} \,\le\, 2c_{\mathrm{P}}\|\mathbf{i}_4\|\,\|\mathbf{v}\|_{0,4/3;\Omega}\,.$$

At this point we notice from (3.54) and the previous remarks on $\mathbf{v}_{4/3}$ that $\mathbf{div}\big(\mathbf{e}(\mathbf{w})\big) \in \mathbf{L}^{4/3}(\Omega)$, which, together with the fact that $\mathbf{e}(\mathbf{w}) \in \mathbb{L}^2(\Omega)$, imply that $\mathbf{e}(\mathbf{w})$ belongs to $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$. Hence, we let $\widetilde{\boldsymbol{\tau}}$ be the $\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$-component of $\mathbf{e}(\mathbf{w})$ (cf. (3.1)), and observe that there hold $\mathbf{div}(\widetilde{\boldsymbol{\tau}}) = \mathbf{v}_{4/3}$ and

$$\begin{aligned}\|\widetilde{\boldsymbol{\tau}}\|^2_{\mathbf{div}_{4/3};\Omega} \,&=\, \|\widetilde{\boldsymbol{\tau}}\|^2_{0,\Omega} \,+\, \|\mathbf{div}(\widetilde{\boldsymbol{\tau}})\|^2_{0,4/3;\Omega} \,\le\, \|\mathbf{e}(\mathbf{w})\|^2_{0,\Omega} \,+\, \|\mathbf{v}_{4/3}\|^2_{0,4/3;\Omega} \\ &\le\, \|\mathbf{w}\|^2_{1,\Omega} \,+\, \|\mathbf{v}_{4/3}\|^2_{0,4/3;\Omega} \,\le\, \Big\{1 + 4c_{\mathrm{P}}^2\|\mathbf{i}_4\|^2\Big\}\,\|\mathbf{v}_{4/3}\|^2_{0,4/3;\Omega}\,.\end{aligned} \tag{3.56}$$

In this way, noting that $\widetilde{\boldsymbol{\tau}}$ is symmetric (because $\mathbf{e}(\mathbf{w})$ and the identity matrix $\mathbb{I}$ are), and using (3.53) and (3.56), we find that

$$\sup_{\substack{\boldsymbol{\tau}\in\mathrm{H}\\ \boldsymbol{\tau}\neq\mathbf{0}}} \frac{\mathbf{b}\big(\boldsymbol{\tau},(\mathbf{v},\boldsymbol{\eta})\big)}{\|\boldsymbol{\tau}\|_{\mathrm{H}}} \,\ge\, \frac{\mathbf{b}\big(\widetilde{\boldsymbol{\tau}},(\mathbf{v},\boldsymbol{\eta})\big)}{\|\widetilde{\boldsymbol{\tau}}\|_{\mathrm{H}}} \,=\, \frac{\displaystyle\int_\Omega \mathbf{v}\cdot\mathbf{div}(\widetilde{\boldsymbol{\tau}})}{\|\widetilde{\boldsymbol{\tau}}\|_{\mathbf{div}_{4/3};\Omega}} \,=\, \frac{\displaystyle\int_\Omega \mathbf{v}\cdot\mathbf{v}_{4/3}}{\|\widetilde{\boldsymbol{\tau}}\|_{\mathbf{div}_{4/3};\Omega}} \,\ge\, \beta_1\,\|\mathbf{v}\|_{0,4;\Omega}\,, \tag{3.57}$$

with $\beta_1 \,=\, \Big\{1 + 4c_{\mathrm{P}}^2\|\mathbf{i}_4\|^2\Big\}^{-1/2}$. On the other hand, for the same $(\mathbf{v},\boldsymbol{\eta}) \in \mathrm{Q}$ given at the beginning of the proof, we now consider the boundary value problem

$$\mathbf{div}\big(\mathbf{e}(\mathbf{w})\big) \,=\, -\,\mathbf{div}(\boldsymbol{\eta}) \quad\text{in}\quad \mathcal{D}'(\Omega)\,, \qquad\text{and}\qquad \mathbf{w} \,=\, \mathbf{0} \quad\text{on}\quad \Gamma\,, \tag{3.58}$$

whose weak formulation is: find $\mathbf{w} \in \mathbf{H}^1_0(\Omega)$ such that

$$\int_\Omega \mathbf{e}(\mathbf{w}) : \mathbf{e}(\mathbf{z}) \,=\, -\int_\Omega \boldsymbol{\eta} : \mathbf{e}(\mathbf{z}) \qquad \forall\,\mathbf{z} \,\in\, \mathbf{H}^1_0(\Omega)\,. \tag{3.59}$$

Similarly as for (3.55), and employing again the Poincaré and first Korn inequalities, a straightforward application of the Lax–Milgram Lemma guarantees the existence of a unique solution $\mathbf{w}$ to (3.59), which satisfies

$$\|\mathbf{e}(\mathbf{w})\|_{0,\Omega} \,\le\, \|\boldsymbol{\eta}\|_{0,\Omega}\,. \tag{3.60}$$

In addition, it is clear from (3.58) that $\mathbf{div}\big(\mathbf{e}(\mathbf{w}) + \boldsymbol{\eta}\big) \,=\, \mathbf{0}$, so that $\mathbf{e}(\mathbf{w}) + \boldsymbol{\eta}$ lies in $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$. Thus, defining $\widehat{\boldsymbol{\tau}}$ as the $\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$-component of $\mathbf{e}(\mathbf{w}) + \boldsymbol{\eta}$, we realize that $\widehat{\boldsymbol{\tau}}$ is divergence-free as well, and that $\widehat{\boldsymbol{\tau}} : \boldsymbol{\eta} \,=\, \boldsymbol{\eta} : \boldsymbol{\eta}$, whence, noting that there holds $\|\widehat{\boldsymbol{\tau}}\|_{0,\Omega} \,\le\, \|\mathbf{e}(\mathbf{w})\|_{0,\Omega} + \|\boldsymbol{\eta}\|_{0,\Omega}$, and using (3.60), we deduce that

$$\sup_{\substack{\boldsymbol{\tau}\in\mathrm{H}\\ \boldsymbol{\tau}\neq\mathbf{0}}} \frac{\mathbf{b}\big(\boldsymbol{\tau},(\mathbf{v},\boldsymbol{\eta})\big)}{\|\boldsymbol{\tau}\|_{\mathrm{H}}} \,\ge\, \frac{\mathbf{b}\big(\widehat{\boldsymbol{\tau}},(\mathbf{v},\boldsymbol{\eta})\big)}{\|\widehat{\boldsymbol{\tau}}\|_{\mathrm{H}}} \,=\, \frac{\displaystyle\int_\Omega \boldsymbol{\eta}:\boldsymbol{\eta}}{\|\widehat{\boldsymbol{\tau}}\|_{\mathbf{div}_{4/3};\Omega}} \,=\, \frac{\|\boldsymbol{\eta}\|^2_{0,\Omega}}{\|\widehat{\boldsymbol{\tau}}\|_{0,\Omega}} \,\ge\, \beta_2\,\|\boldsymbol{\eta}\|_{0,\Omega}\,, \tag{3.61}$$

with $\beta_2 = 1/2$. Finally, the required inequality (3.52) follows directly from (3.57) and (3.61) with β depending only on β_1 and β_2. □

We now consider the perturbed formulation arising from (3.34) after eliminating there the terms involving $\mathbf{c}_f$ and $\mathbf{d}_f$. Then, adding the left hand sides of the resulting equations, we obtain the bounded and symmetric bilinear form $A_f : (\mathrm{H}\times\mathrm{Q})\times(\mathrm{H}\times\mathrm{Q}) \to \mathrm{R}$ given by

$$A_f\Big(\big(\boldsymbol{\zeta}_f,(\mathbf{z}_f,\boldsymbol{\xi}_f)\big),\big(\boldsymbol{\tau}_f,(\mathbf{v}_f,\boldsymbol{\eta}_f)\big)\Big) := \mathbf{a}_f(\boldsymbol{\zeta}_f,\boldsymbol{\tau}_f) \,+\, \mathbf{b}\big(\boldsymbol{\tau}_f,(\mathbf{z}_f,\boldsymbol{\xi}_f)\big) \,+\, \mathbf{b}\big(\boldsymbol{\zeta}_f,(\mathbf{v}_f,\boldsymbol{\eta}_f)\big) \tag{3.62}$$

for all $\big(\boldsymbol{\zeta}_f,(\mathbf{z}_f,\boldsymbol{\xi}_f)\big),\ \big(\boldsymbol{\tau}_f,(\mathbf{v}_f,\boldsymbol{\eta}_f)\big) \,\in\, \mathrm{H}\times\mathrm{Q}$. Note that the boundedness of A_f follows directly from (3.19), (3.20), and (3.25). Hence, denoting by $\mathbf{A}_f \in \mathcal{L}\big((H\times Q),(\mathrm{H}\times\mathrm{Q})'\big)$ the operator induced by A_f, and bearing in mind the $\mathbb{V}$-ellipticity of $\mathbf{a}_f$ (cf. Lemma 3.3) and the inf–sup condition for $\mathbf{b}$ (cf. Lemma 3.5), we conclude from a straightforward application of Theorem 3.2 that $\mathbf{A}_f$ is bijective. In addition, it is clear from (3.48) that A_f satisfies a global inf–sup condition, which means that there exists a constant $\bar{\alpha}_f > 0$, depending only on α_f, β, and $\|\mathbf{a}_f\|$ (cf. (3.25)) , such that

$$\sup_{\substack{(\boldsymbol{\tau}_f,(\mathbf{v}_f,\boldsymbol{\eta}_f))\in\mathrm{H}\times\mathrm{Q}\\ (\boldsymbol{\tau}_f(\mathbf{v}_f,\boldsymbol{\eta}_f))\neq\mathbf{0}}} \frac{A_f\Big(\big(\boldsymbol{\zeta}_f,(\mathbf{z}_f,\boldsymbol{\xi}_f)\big),\big(\boldsymbol{\tau}_f,(\mathbf{v}_f,\boldsymbol{\eta}_f)\big)\Big)}{\|\big(\boldsymbol{\tau}_f,(\mathbf{v}_f,\boldsymbol{\eta}_f)\big)\|_{\mathrm{H}\times\mathrm{Q}}} \,\ge\, \bar{\alpha}_f\,\|\big(\boldsymbol{\zeta}_f,(\mathbf{z}_f,\boldsymbol{\xi}_f)\big)\|_{\mathrm{H}\times\mathrm{Q}} \tag{3.63}$$

for all $\big(\boldsymbol{\zeta}_f,(\mathbf{z}_f,\boldsymbol{\xi}_f)\big) \,\in\, \mathrm{H}\times\mathrm{Q}$. Next, in order to apply Theorem 3.1 to (3.34), we introduce the bounded bilinear form $A_{f,\mathbf{w}_f} : (\mathrm{H}\times\mathrm{Q})\times(\mathrm{H}\times\mathrm{Q}) \to \mathrm{R}$ that results after adding the full equations defining that formulation, that is

$$\begin{aligned}A_{f,\mathbf{w}_f}\Big(\big(\boldsymbol{\zeta}_f,(\mathbf{z}_f,\boldsymbol{\xi}_f)\big),\big(\boldsymbol{\tau}_f,(\mathbf{v}_f,\boldsymbol{\eta}_f)\big)\Big) \,&:=\, A_f\Big(\big(\boldsymbol{\zeta}_f,(\mathbf{z}_f,\boldsymbol{\xi}_f)\big),\big(\boldsymbol{\tau}_f,(\mathbf{v}_f,\boldsymbol{\eta}_f)\big)\Big)\\ &\quad+\, \mathbf{c}_f(\mathbf{z}_f,\boldsymbol{\tau}_f) \,+\, \mathbf{d}_f(\mathbf{w}_f;\mathbf{z}_f,\boldsymbol{\tau}_f)\end{aligned} \tag{3.64}$$

for all $\big(\boldsymbol{\zeta}_f,(\mathbf{z}_f,\boldsymbol{\xi}_f)\big),\ \big(\boldsymbol{\tau}_f,(\mathbf{v}_f,\boldsymbol{\eta}_f)\big) \in \mathrm{H}\times\mathrm{Q}$. Knowing that A_f is bounded, the boundedness of $A_{f,\mathbf{w}_f}$ is completed thanks to (3.21), (3.22), and (3.25). In this way, it is clear that (3.34) can be restated as: find $\big(\widehat{\boldsymbol{\sigma}}_f,(\widehat{\mathbf{u}}_f,\widehat{\boldsymbol{\gamma}}_f)\big) \in \mathrm{H}\times\mathrm{Q}$ such that

$$A_{f,\mathbf{w}_f}\Big(\big(\widehat{\boldsymbol{\sigma}}_f,(\widehat{\mathbf{u}}_f,\widehat{\boldsymbol{\gamma}}_f)\big),\big(\boldsymbol{\tau}_f,(\mathbf{v}_f,\boldsymbol{\eta}_f)\big)\Big) \,=\, \mathbf{F}_f(\boldsymbol{\tau}_f) + \mathbf{G}_f^{\mathbf{w}}(\mathbf{v}_f,\boldsymbol{\eta}_f) \tag{3.65}$$

for all $\big(\boldsymbol{\tau}_f, (\mathbf{v}_f, \boldsymbol{\eta}_f)\big) \in \mathrm{H} \times \mathrm{Q}$. Then, it follows straightforwardly from (3.63) and the boundedness estimates for $\mathbf{c}_f$ and $\mathbf{d}_f$ (cf. (3.21), (3.22), (3.25)) that

$$\begin{aligned} \sup_{\substack{(\boldsymbol{\tau}_f,(\mathbf{v}_f,\boldsymbol{\eta}_f))\in \mathrm{H}\times\mathrm{Q}\\ (\boldsymbol{\tau}_f(\mathbf{v}_f,\boldsymbol{\eta}_f))\neq \mathbf{0}}} & \frac{A_{f,\mathbf{w}_f}\Big(\big(\boldsymbol{\zeta}_f,(\mathbf{z}_f,\boldsymbol{\xi}_f)\big),\big(\boldsymbol{\tau}_f,(\mathbf{v}_f,\boldsymbol{\eta}_f)\big)\Big)}{\|\big(\boldsymbol{\tau}_f,(\mathbf{v}_f,\boldsymbol{\eta}_f)\big)\|_{\mathrm{H}\times\mathrm{Q}}} \,\geq\, \bar{\alpha}_f \,\|\big(\boldsymbol{\zeta}_f,(\mathbf{z}_f,\boldsymbol{\xi}_f)\big)\|_{\mathrm{H}\times\mathrm{Q}} \\ & - \frac{1}{\sqrt{n}}\left\|\frac{\nabla\varepsilon}{\varepsilon}\right\|_{0,4;\Omega} \|\mathbf{z}_f\|_{0,4;\Omega} \,-\, \frac{\rho_f}{2\mu_f}\,\|\varepsilon\|_{0,\infty;\Omega}\,\|\mathbf{w}_f\|_{0,4;\Omega}\,\|\mathbf{z}_f\|_{0,4;\Omega} \\ & \geq \left\{\bar{\alpha}_f \,-\, \frac{1}{\sqrt{n}}\left\|\frac{\nabla\varepsilon}{\varepsilon}\right\|_{0,4;\Omega} \,-\, \frac{\rho_f}{2\mu_f}\,\|\mathbf{w}_f\|_{0,4;\Omega}\right\} \|\big(\boldsymbol{\zeta}_f,(\mathbf{z}_f,\boldsymbol{\xi}_f)\big)\|_{\mathrm{H}\times\mathrm{Q}}\,, \end{aligned}$$

where the last inequality uses that $\|\varepsilon\|_{0,\infty;\Omega} \leq 1$. In this way, assuming for instance that

$$\frac{1}{\sqrt{n}}\left\|\frac{\nabla\varepsilon}{\varepsilon}\right\|_{0,4;\Omega} \,\leq\, \frac{\bar{\alpha}_f}{4} \quad \text{and} \quad \|\mathbf{w}_f\|_{0,4;\Omega} \,\leq\, r_f \,:=\, \frac{\bar{\alpha}_f\,\mu_f}{2\rho_f}\,, \tag{3.66}$$

we arrive at

$$\sup_{\substack{(\boldsymbol{\tau}_f,(\mathbf{v}_f,\boldsymbol{\eta}_f))\in \mathrm{H}\times\mathrm{Q}\\ (\boldsymbol{\tau}_f(\mathbf{v}_f,\boldsymbol{\eta}_f))\neq \mathbf{0}}} \frac{A_{f,\mathbf{w}_f}\Big(\big(\boldsymbol{\zeta}_f,(\mathbf{z}_f,\boldsymbol{\xi}_f)\big),\big(\boldsymbol{\tau}_f,(\mathbf{v}_f,\boldsymbol{\eta}_f)\big)\Big)}{\|\big(\boldsymbol{\tau}_f,(\mathbf{v}_f,\boldsymbol{\eta}_f)\big)\|_{\mathrm{H}\times\mathrm{Q}}} \,\geq\, \frac{\bar{\alpha}_f}{2}\,\|\big(\boldsymbol{\zeta}_f,(\mathbf{z}_f,\boldsymbol{\xi}_f)\big)\|_{\mathrm{H}\times\mathrm{Q}} \tag{3.67}$$

for all $\big(\boldsymbol{\zeta}_f, (\mathbf{z}_f, \boldsymbol{\xi}_f)\big) \in \mathrm{H} \times \mathrm{Q}$. Similarly, using the fact that A_f is symmetric, employing the same boundedness estimates for $\mathbf{c}_f$ and $\mathbf{d}_f$, and assuming again (3.66), we are able to prove the companion inf–sup condition to (3.67), in which the supremum is taken with respect to the first component of $A_{f,\mathbf{w}_f}$, that is

$$\sup_{\substack{(\boldsymbol{\zeta}_f,(\mathbf{z}_f,\boldsymbol{\xi}_f))\in \mathrm{H}\times\mathrm{Q}\\ (\boldsymbol{\zeta}_f,(\mathbf{z}_f,\boldsymbol{\xi}_f))\neq \mathbf{0}}} \frac{A_{f,\mathbf{w}_f}\Big(\big(\boldsymbol{\zeta}_f,(\mathbf{z}_f,\boldsymbol{\xi}_f)\big),\big(\boldsymbol{\tau}_f,(\mathbf{v}_f,\boldsymbol{\eta}_f)\big)\Big)}{\|\big(\boldsymbol{\zeta}_f,(\mathbf{z}_f,\boldsymbol{\xi}_f)\big)\|_{\mathrm{H}\times\mathrm{Q}}} \,\geq\, \frac{\bar{\alpha}_f}{2}\,\|\big(\boldsymbol{\tau}_f,(\mathbf{v}_f,\boldsymbol{\eta}_f)\big)\|_{\mathrm{H}\times\mathrm{Q}} \tag{3.68}$$

for all $\big(\boldsymbol{\tau}_f, (\mathbf{v}_f, \boldsymbol{\eta}_f)\big) \in \mathrm{H} \times \mathrm{Q}$.

As a consequence of the previous analysis, we are in position to establish the following result, which confirms that the operator Θ_f (cf. (3.33)) is well-defined.

Theorem 3.6. *Assume that* $\frac{1}{\sqrt{n}}\left\|\frac{\nabla\varepsilon}{\varepsilon}\right\|_{0,4;\Omega} \leq \frac{\bar{\alpha}_f}{4}$. *Then, for each* $\mathbf{w} := (\mathbf{w}_f, \mathbf{w}_s) \in \mathbf{L}^4(\Omega)\times\mathbf{L}^4(\Omega)$ *such that* $\|\mathbf{w}_f\|_{0,4;\Omega} \leq r_f$, *there exists a unique* $\big(\widehat{\boldsymbol{\sigma}}_f, (\widehat{\mathbf{u}}_f, \widehat{\boldsymbol{\gamma}}_f)\big) \in \mathrm{H} \times \mathrm{Q}$ *solution to* (3.65) *(equivalently* (3.34)*). Moreover, there holds*

$$\begin{aligned} \|\Theta_f(\mathbf{w})\|_{0,4;\Omega} \,&=\, \|\widehat{\mathbf{u}}_f\|_{0,4;\Omega} \,\leq\, \|\big(\widehat{\boldsymbol{\sigma}}_f,(\widehat{\mathbf{u}}_f,\widehat{\boldsymbol{\gamma}}_f)\big)\|_{\mathrm{H}\times\mathrm{Q}} \\ &\leq\, \frac{2}{\bar{\alpha}_f}\Big\{\|\mathbf{u}_{D,f}\|_{1/2,\Gamma} \,+\, \|\delta(\phi)\|_{0,\Omega}\,\|\mathbf{w}_f - \mathbf{w}_s\|_{0,4;\Omega} \,+\, |\Omega|^{3/4}\,\rho_f\,\mathbf{g}\,\|\varepsilon\|_{0,\infty;\Omega}\Big\}\,. \end{aligned} \tag{3.69}$$

Proof. It suffices to notice, thanks to (3.67) and (3.68), that $A_{f,\mathbf{w}_f}$ satisfies the hypotheses (i) and (ii) of Theorem 3.1. Therefore, observing that the right hand side of (3.65) defines a functional in $(\mathrm{H} \times \mathrm{Q})'$, a direct application of the aforementioned abstract result implies the existence of a unique solution $\big(\widehat{\boldsymbol{\sigma}}_f, (\widehat{\mathbf{u}}_f, \widehat{\boldsymbol{\gamma}}_f)\big) \in \mathrm{H} \times \mathrm{Q}$ to (3.65), for which there holds

$$\|\big(\widehat{\boldsymbol{\sigma}}_f,(\widehat{\mathbf{u}}_f,\widehat{\boldsymbol{\gamma}}_f)\big)\|_{\mathrm{H}\times\mathrm{Q}} \,\leq\, \frac{2}{\bar{\alpha}_f}\Big\{\|\mathbf{F}_f\| \,+\, \|\mathbf{G}_f^{\mathbf{w}}\|\Big\}.$$

Finally, the foregoing inequality and the upper bounds for $\|\mathbf{F}_f\|$ and $\|\mathbf{G}_f^{\mathbf{w}}\|$ provided in (3.25) yield (3.69) and complete the proof. □

On the other hand, it is not difficult to realize that proving that Θ_s (cf. (3.35)) is well-defined, equivalently that (3.36) is well-posed, proceeds analogously as we already did for Θ_f. Therefore, in what follows we simplify the corresponding presentation and collect only the main aspects of the respective analysis. In fact, we begin by letting $A_s : (\mathrm{H}\times\mathrm{Q})\times(\mathrm{H}\times\mathrm{Q}) \to \mathrm{R}$ be the symmetric bilinear form given by

$$A_s\Big(\big(\boldsymbol{\zeta}_s,(\mathbf{z}_s,\boldsymbol{\xi}_s)\big),\big(\boldsymbol{\tau}_s,(\mathbf{v}_s,\boldsymbol{\eta}_s)\big)\Big) := \mathbf{a}_s(\boldsymbol{\zeta}_s,\boldsymbol{\tau}_s) + \mathbf{b}\big(\boldsymbol{\tau}_s,(\mathbf{z}_s,\boldsymbol{\xi}_s)\big) + \mathbf{b}\big(\boldsymbol{\zeta}_s,(\mathbf{v}_s,\boldsymbol{\eta}_s)\big) \tag{3.70}$$

for all $\big(\boldsymbol{\zeta}_s, (\mathbf{z}_s, \boldsymbol{\xi}_s)\big), \big(\boldsymbol{\tau}_s, (\mathbf{v}_s, \boldsymbol{\eta}_s)\big) \in \mathrm{H} \times \mathrm{Q}$, which, thanks now to (3.27), (3.20), and (3.32), is clearly bounded. Then, in virtue of the $\mathbb{V}$-ellipticity of $\mathbf{a}_s$ (cf. Lemma 3.4) and the inf–sup condition for $\mathbf{b}$ (cf. Lemma 3.5), direct applications of

Theorem 3.1 and the consequent estimate (3.48) imply that there exists a constant $\bar{\alpha}_s > 0$, depending only on α_s, β, and $\|\mathbf{a}_s\|$ (cf. (3.32)), such that

$$\sup_{\substack{(\boldsymbol{\tau}_s,(\mathbf{v}_s,\boldsymbol{\eta}_s))\in \mathrm{H}\times\mathrm{Q}\\ (\boldsymbol{\tau}_s(\mathbf{v}_s,\boldsymbol{\eta}_s))\neq\mathbf{0}}} \frac{A_s\Big((\boldsymbol{\zeta}_s,(\mathbf{z}_s,\boldsymbol{\xi}_s)),(\boldsymbol{\tau}_s,(\mathbf{v}_s,\boldsymbol{\eta}_s))\Big)}{\|(\boldsymbol{\tau}_s,(\mathbf{v}_s,\boldsymbol{\eta}_s))\|_{\mathrm{H}\times\mathrm{Q}}} \geq \bar{\alpha}_s \,\|(\boldsymbol{\zeta}_s,(\mathbf{z}_s,\boldsymbol{\xi}_s))\|_{\mathrm{H}\times\mathrm{Q}} \tag{3.71}$$

for all $(\boldsymbol{\zeta}_s,(\mathbf{z}_s,\boldsymbol{\xi}_s)) \in \mathrm{H}\times\mathrm{Q}$. Then, defining the bilinear form

$$\begin{aligned} A_{s,\mathbf{w}_s}\Big((\boldsymbol{\zeta}_s,(\mathbf{z}_s,\boldsymbol{\xi}_s)),(\boldsymbol{\tau}_s,(\mathbf{v}_s,\boldsymbol{\eta}_s))\Big) \;:=\; & A_s\Big((\boldsymbol{\zeta}_s,(\mathbf{z}_s,\boldsymbol{\xi}_s)),(\boldsymbol{\tau}_s,(\mathbf{v}_s,\boldsymbol{\eta}_s))\Big)\\ & +\; \mathbf{c}_s(\mathbf{z}_s,\boldsymbol{\tau}_s) \;+\; \mathbf{d}_s(\mathbf{w}_s;\mathbf{z}_s,\boldsymbol{\tau}_s) \end{aligned} \tag{3.72}$$

for all $(\boldsymbol{\zeta}_s,(\mathbf{z}_s,\boldsymbol{\xi}_s))$, $(\boldsymbol{\tau}_s,(\mathbf{v}_s,\boldsymbol{\eta}_s)) \in \mathrm{H}\times\mathrm{Q}$, whose boundedness follows now from that of A_s and the estimates (3.28), (3.29), and (3.32), we realize that (3.36) can be restated as: find $(\widehat{\boldsymbol{\sigma}}_s,(\widehat{\mathbf{u}}_s,\widehat{\boldsymbol{\gamma}}_s)) \in \mathrm{H}\times\mathrm{Q}$ such that

$$A_{s,\mathbf{w}_s}\Big((\widehat{\boldsymbol{\sigma}}_s,(\widehat{\mathbf{u}}_s,\widehat{\boldsymbol{\gamma}}_s)),(\boldsymbol{\tau}_s,(\mathbf{v}_s,\boldsymbol{\eta}_s))\Big) \;=\; \mathbf{F}_s^{\mathbf{w}}(\boldsymbol{\tau}_s) + \mathbf{G}_s(\mathbf{v}_s,\boldsymbol{\eta}_s) \tag{3.73}$$

for all $(\boldsymbol{\tau}_s,(\mathbf{v}_s,\boldsymbol{\eta}_s)) \in \mathrm{H}\times\mathrm{Q}$. Then, assuming that

$$\frac{1}{\sqrt{n}}\left\|\frac{\nabla\phi}{\phi}\right\|_{0,4;\Omega} \leq \frac{\bar{\alpha}_s}{4} \quad \text{and} \quad \|\mathbf{w}_s\|_{0,4;\Omega} \leq r_s \;:=\; \frac{\bar{\alpha}_s\mu_1}{2\rho_s}, \tag{3.74}$$

we are able to prove the analogues of the inf–sup conditions (3.67) and (3.68), with $A_{s,\mathbf{w}_s}$ and $\bar{\alpha}_s$ instead of $A_{f,\mathbf{w}_f}$ and $\bar{\alpha}_f$, respectively. In this way, the following theorem confirms that the operator Θ_s (cf. (3.35)) is well-posed

Theorem 3.7. *Assume that* $\frac{1}{\sqrt{n}}\left\|\frac{\nabla\phi}{\phi}\right\|_{0,4;\Omega} \leq \frac{\bar{\alpha}_s}{4}$. *Then, for each* $\mathbf{w} := (\mathbf{w}_f,\mathbf{w}_s) \in \mathbf{L}^4(\Omega)\times\mathbf{L}^4(\Omega)$ *such that* $\|\mathbf{w}_s\|_{0,4;\Omega} \leq r_s$, *there exists a unique* $(\widehat{\boldsymbol{\sigma}}_s,(\widehat{\mathbf{u}}_s,\widehat{\boldsymbol{\gamma}}_s)) \in \mathrm{H}\times\mathrm{Q}$ *solution to* (3.73) *(equivalently* (3.36)*). Moreover, there holds*

$$\begin{aligned} \|\Theta_s(\mathbf{w})\|_{0,4;\Omega} \;&=\; \|\widehat{\mathbf{u}}_s\|_{0,4;\Omega} \;\leq\; \|(\widehat{\boldsymbol{\sigma}}_s,(\widehat{\mathbf{u}}_s,\widehat{\boldsymbol{\gamma}}_s))\|_{\mathrm{H}\times\mathrm{Q}}\\ &\leq\; \frac{2}{\bar{\alpha}_s}\left\{\|\mathbf{u}_{D,s}\|_{1/2,\Gamma} + \frac{\rho_f}{2\mu_1}\|\varepsilon\|_{0,\infty;\Omega}\|\mathbf{w}_f\|_{0,4;\Omega}^2 + |\Omega|^{3/4}\,\mathbf{g}\,\|\varepsilon\rho_f+\phi\rho_s\|_{0,\infty;\Omega}\right\}. \end{aligned} \tag{3.75}$$

Proof. As for the proof of Theorem 3.6, it follows from a straightforward application of Theorem 3.1. We omit further details and just mention that the a priori estimate (3.75) makes use of the upper bounds for $\|\mathbf{F}_s^{\mathbf{w}}\|$ and $\|\mathbf{G}_s\|$ provided in (3.32). □

3.4. Solvability analysis of the fixed-point equation

Knowing from the previous section that the operators Θ_f and Θ_s (cf. (3.33), (3.35)), and consequently S (cf. (3.37)), are well defined, we now focus on the solvability of the corresponding fixed-point equation (3.38). For this purpose, and aiming to apply later on the Banach fixed-point Theorem, we begin by establishing sufficient conditions on the data under which S maps a closed ball into itself. Throughout the rest of the section we assume that ε and ϕ satisfy the hypotheses specified in (3.66) and (3.74), respectively. Hence, denoting from now on

$$r \;:=\; \min\{r_f, r_s\}, \tag{3.76}$$

where r_f and r_s are defined in the aforementioned equations, we have the following result.

Lemma 3.8. *Let* $W := \Big\{\mathbf{w} = (\mathbf{w}_f,\mathbf{w}_s) \in \mathbf{L}^4(\Omega)\times\mathbf{L}^4(\Omega): \quad \|\mathbf{w}\|_{0,4;\Omega} \leq r\Big\}$, *and assume that the data satisfy*

$$\|\mathbf{u}_{D,f}\|_{1/2,\Gamma} + r\,\|\delta(\phi)\|_{0,\Omega} + |\Omega|^{3/4}\,\rho_f\,\mathbf{g}\,\|\varepsilon\|_{0,\infty;\Omega} \;\leq\; \frac{\bar{\alpha}_f}{4}\,r, \tag{3.77}$$

and

$$\|\mathbf{u}_{D,s}\|_{1/2,\Gamma} + \frac{\rho_f}{2\mu_1}r^2\,\|\varepsilon\|_{0,\infty;\Omega} + |\Omega|^{3/4}\,\mathbf{g}\,\|\varepsilon\rho_f+\phi\rho_s\|_{0,\infty;\Omega} \;\leq\; \frac{\bar{\alpha}_s}{4}\,r. \tag{3.78}$$

Then $S(W) \subseteq W$.

Proof. Given $\mathbf{w} = (\mathbf{w}_f,\mathbf{w}_s) \in W$, we first recall from (3.37) that $S(\mathbf{w}) = \big(\Theta_f(\mathbf{w}),\Theta_s(\mathbf{w})\big)$. Then, using that $\|\mathbf{w}_f - \mathbf{w}_s\|_{0,4;\Omega}$ and $\|\mathbf{w}_f\|_{0,4;\Omega}$ are both bounded by $\|\mathbf{w}\|_{0,4;\Omega}$, and hence by r, we easily see that the upper bounds of $\|\Theta_f(\mathbf{w})\|_{0,4;\Omega}$ and $\|\Theta_s(\mathbf{w})\|_{0,4;\Omega}$ provided by (3.69) and (3.75) become the left hand sides of (3.77) and (3.78) multiplied by $\frac{2}{\bar{\alpha}_f}$ and $\frac{2}{\bar{\alpha}_s}$, respectively. In this way, the above assumptions allow to conclude that $\|\Theta_f(\mathbf{w})\|_{0,4;\Omega}$ and $\|\Theta_s(\mathbf{w})\|_{0,4;\Omega}$ are bounded each by $r/2$, which implies that $\|S(\mathbf{w})\|_{0,4;\Omega} \le r$, and hence $S(\mathbf{w}) \in W$. □

It is important to remark at this point that the assumptions (3.77) and (3.78), being linear combinations of data, actually impose that each one of the latter be sufficiently small. In particular, looking for instance at (3.77), it is readily seen that a sufficient condition for its occurrence would be to require that each one of the terms on the left hand side be less than or equal to 1/3 of the right hand side, that is

$$\|\mathbf{u}_{D,f}\|_{1/2,\Gamma} \le \frac{\bar{\alpha}_f}{12}\, r\,, \quad \|\delta(\phi)\|_{0,\Omega} \le \frac{\bar{\alpha}_f}{12} \quad \text{and} \quad \|\varepsilon\|_{0,\infty;\Omega} \le \frac{\bar{\alpha}_f}{12|\Omega|^{3/4}\,\rho_f\,\mathbf{g}}\, r.$$

A similar analysis applies to (3.78), thanks to which one gets individual constraints for the data $\|\mathbf{u}_{D,s}\|_{1/2,\Gamma}$ and $\|\varepsilon\|_{0,\infty;\Omega}$ again, and for $\|\varepsilon\rho_f + \phi\rho_s\|_{0,\infty;\Omega}$. In this way, choosing the smallest bounds for the first two, and keeping as such the ones for $\|\delta(\phi)\|_{0,\Omega}$ and $\|\varepsilon\rho_f + \phi\rho_s\|_{0,\infty;\Omega}$, we obtain a set of four conditions on these data, which guarantee that both (3.77) and (3.78) are satisfied. Nevertheless, the fact that some of the constants involved are not known explicitly, as it is the case for example of $\bar{\alpha}_f$ and $\bar{\alpha}_s$ (because of the unknown constant c_1 from (3.49)), stops us of truly verifying these conditions in practice.

We continue the analysis with the Lipschitz-continuity properties of Θ_f and Θ_s.

Lemma 3.9. *There exists a positive constant L_f, depending on $\bar{\alpha}_f$, ρ_f, and μ_f, such that*

$$\begin{aligned}&\|\Theta_f(\mathbf{w}) - \Theta_f(\mathbf{t})\|_{0,4;\Omega}\\ &\quad\le L_f \left\{ \|\delta(\phi)\|_{0,\Omega} + \|\varepsilon\|_{0,\infty;\Omega}\, \|\Theta_f(\mathbf{t})\|_{0,4;\Omega} \right\} \|\mathbf{w}-\mathbf{t}\|_{0,4;\Omega}\end{aligned} \tag{3.79}$$

for all $\mathbf{w} := (\mathbf{w}_f, \mathbf{w}_s)$, $\mathbf{t} := (\mathbf{t}_f, \mathbf{t}_s) \in \mathbf{L}^4(\Omega) \times \mathbf{L}^4(\Omega)$ such that $\|\mathbf{w}_f\|_{0,4;\Omega}$, $\|\mathbf{t}_f\|_{0,4;\Omega} \le r_f$.

Proof. Given $\mathbf{w} := (\mathbf{w}_f, \mathbf{w}_s)$ and $\mathbf{t} := (\mathbf{t}_f, \mathbf{t}_s)$ as indicated, we set $\Theta_f(\mathbf{w}) := \widehat{\mathbf{u}}_f$ and $\Theta_f(\mathbf{t}) := \widehat{\mathbf{z}}_f$, where $\vec{\boldsymbol{\sigma}}_f := \big(\widehat{\boldsymbol{\sigma}}_f, (\widehat{\mathbf{u}}_f, \widehat{\boldsymbol{\gamma}}_f)\big) \in \mathrm{H}\times\mathrm{Q}$ and $\vec{\boldsymbol{\zeta}}_f := \big(\widehat{\boldsymbol{\zeta}}_f, (\widehat{\mathbf{z}}_f, \widehat{\boldsymbol{\xi}}_f)\big) \in \mathrm{H}\times\mathrm{Q}$ are the unique solutions, guaranteed by Theorem 3.6, of the formulations (cf. (3.34) or (3.65))

$$A_{f,\mathbf{w}_f}\big(\vec{\boldsymbol{\sigma}}_f, \vec{\boldsymbol{\tau}}_f\big) = \mathbf{F}_f(\boldsymbol{\tau}_f) + \mathbf{G}_f^{\mathbf{w}}(\mathbf{v}_f, \boldsymbol{\eta}_f) \tag{3.80}$$

and

$$A_{f,\mathbf{t}_f}\big(\vec{\boldsymbol{\zeta}}_f, \vec{\boldsymbol{\tau}}_f\big) = \mathbf{F}_f(\boldsymbol{\tau}_f) + \mathbf{G}_f^{\mathbf{t}}(\mathbf{v}_f, \boldsymbol{\eta}_f)\,, \tag{3.81}$$

respectively, both for all $\vec{\boldsymbol{\tau}}_f := \big(\boldsymbol{\tau}_f, (\mathbf{v}_f, \boldsymbol{\eta}_f)\big) \in \mathrm{H}\times\mathrm{Q}$. Then, applying the inf–sup condition (3.67) to $\vec{\boldsymbol{\sigma}}_f - \vec{\boldsymbol{\zeta}}_f$, adding and subtracting $A_{f,\mathbf{t}_f}\big(\vec{\boldsymbol{\zeta}}_f, \vec{\boldsymbol{\tau}}_f\big)$, and using (3.80) and (3.81), we obtain

$$\begin{aligned}\frac{\bar{\alpha}_f}{2}\,\|\vec{\boldsymbol{\sigma}}_f - \vec{\boldsymbol{\zeta}}_f\|_{\mathrm{H}\times\mathrm{Q}} &\le \sup_{\substack{\vec{\boldsymbol{\tau}}_f\in\mathrm{H}\times\mathrm{Q}\\ \vec{\boldsymbol{\tau}}_f\neq\mathbf{0}}} \frac{A_{f,\mathbf{w}_f}\big(\vec{\boldsymbol{\sigma}}_f - \vec{\boldsymbol{\zeta}}_f, \vec{\boldsymbol{\tau}}_f\big)}{\|\vec{\boldsymbol{\tau}}_f\|_{\mathrm{H}\times\mathrm{Q}}}\\ &= \sup_{\substack{\vec{\boldsymbol{\tau}}_f\in\mathrm{H}\times\mathrm{Q}\\ \vec{\boldsymbol{\tau}}_f\neq\mathbf{0}}} \frac{(\mathbf{G}_f^{\mathbf{w}} - \mathbf{G}_f^{\mathbf{t}})(\mathbf{v}_f, \boldsymbol{\eta}_f) + \big(A_{f,\mathbf{t}_f} - A_{f,\mathbf{w}_f}\big)\big(\vec{\boldsymbol{\zeta}}_f, \vec{\boldsymbol{\tau}}_f\big)}{\|\vec{\boldsymbol{\tau}}_f\|_{\mathrm{H}\times\mathrm{Q}}}\,.\end{aligned} \tag{3.82}$$

Now, according to the definitions of $\mathbf{G}_f^{\mathbf{w}}$, $\mathbf{G}_f^{\mathbf{t}}$, $F(\mathbf{w})$, and $F(\mathbf{t})$ (cf. (3.15), (2.8)), we readily get (see also the estimate for $\|\mathbf{G}_f^{\mathbf{u}}\|$ in (3.25))

$$\big|(\mathbf{G}_f^{\mathbf{w}} - \mathbf{G}_f^{\mathbf{t}})(\mathbf{v}_f, \boldsymbol{\eta}_f)\big| = \left|\int_\Omega \big(F(\mathbf{w}) - F(\mathbf{t})\big)\cdot\mathbf{v}_f\right| \le \|\delta(\phi)\|_{0,\Omega}\,\|\mathbf{w}-\mathbf{t}\|_{0,4;\Omega}\,\|\mathbf{v}_f\|_{0,4;\Omega}\,. \tag{3.83}$$

In turn, employing (3.64) and the boundedness of $\mathbf{d}_f$ (cf. (3.22), (3.25)), we find that

$$\begin{aligned}\big|\big(A_{f,\mathbf{t}_f} - A_{f,\mathbf{w}_f}\big)\big(\vec{\boldsymbol{\zeta}}_f, \vec{\boldsymbol{\tau}}_f\big)\big| &= \big|\mathbf{d}_f(\mathbf{t}_f - \mathbf{w}_f; \widehat{\mathbf{z}}_f, \boldsymbol{\tau}_f)\big|\\ &\le \frac{\rho_f}{2\mu_f}\,\|\varepsilon\|_{0,\infty;\Omega}\,\|\mathbf{w}_f - \mathbf{t}_f\|_{0,4;\Omega}\,\|\widehat{\mathbf{z}}_f\|_{0,4;\Omega}\,\|\boldsymbol{\tau}_f\|_{0,\Omega}\\ &\le \frac{\rho_f}{2\mu_f}\,\|\varepsilon\|_{0,\infty;\Omega}\,\|\mathbf{w}-\mathbf{t}\|_{0,4;\Omega}\,\|\Theta_f(\mathbf{t})\|_{0,4;\Omega}\,\|\boldsymbol{\tau}_f\|_{0,\Omega}\,.\end{aligned} \tag{3.84}$$

In this way, replacing (3.83) and (3.84) back into (3.82), we deduce that

$$\frac{\bar{\alpha}_f}{2}\,\|\vec{\boldsymbol{\sigma}}_f - \vec{\boldsymbol{\zeta}}_f\|_{\mathrm{H}\times\mathrm{Q}} \le \left\{\|\delta(\phi)\|_{0,\Omega} + \frac{\rho_f}{2\mu_f}\,\|\varepsilon\|_{0,\infty;\Omega}\,\|\Theta_f(\mathbf{t})\|_{0,4;\Omega}\right\}\|\mathbf{w}-\mathbf{t}\|_{0,4;\Omega}\,,$$

which, together with the fact that $\|\Theta_f(\mathbf{w}) - \Theta_f(\mathbf{t})\|_{0,4;\Omega} \le \|\vec{\boldsymbol{\sigma}}_f - \vec{\boldsymbol{\zeta}}_f\|_{\mathrm{H}\times\mathrm{Q}}$, yields (3.79) with $L_f := \dfrac{2}{\bar{\alpha}_f}\max\left\{1, \dfrac{\rho_f}{2\mu_f}\right\}$, thus completing the proof. □

Lemma 3.10. *There exists a positive constant L_s, depending on $\bar{\alpha}_s$, ρ_f, ρ_s, and μ_1, such that*

$$\begin{aligned} &\|\Theta_s(\mathbf{w}) - \Theta_s(\mathbf{t})\|_{0,4;\Omega} \\ &\quad \le L_s \Big\{ \|\varepsilon\|_{0,\infty;\Omega} \|\mathbf{t}_f + \mathbf{w}_f\|_{0,4;\Omega} + \|\phi\|_{0,\infty;\Omega} \|\Theta_s(\mathbf{t})\|_{0,4;\Omega} \Big\} \|\mathbf{w} - \mathbf{t}\|_{0,4;\Omega} \end{aligned} \tag{3.85}$$

for all $\mathbf{w} := (\mathbf{w}_f, \mathbf{w}_s)$, $\mathbf{t} := (\mathbf{t}_f, \mathbf{t}_s) \in \mathbf{L}^4(\Omega) \times \mathbf{L}^4(\Omega)$ such that $\|\mathbf{w}_s\|_{0,4;\Omega}$, $\|\mathbf{t}_s\|_{0,4;\Omega} \le r_s$.

Proof. We proceed similarly to the proof of Lemma 3.9. In this way, given $\mathbf{w} := (\mathbf{w}_f, \mathbf{w}_s)$ and $\mathbf{t} := (\mathbf{t}_f, \mathbf{t}_s)$ as indicated, we set $\Theta_s(\mathbf{w}) := \widehat{\mathbf{u}}_s$ and $\Theta_s(\mathbf{t}) := \widehat{\mathbf{z}}_s$, where $\vec{\boldsymbol{\sigma}}_s := \big(\widehat{\boldsymbol{\sigma}}_s, (\widehat{\mathbf{u}}_s, \widehat{\boldsymbol{\gamma}}_s)\big) \in \mathrm{H} \times \mathrm{Q}$ and $\vec{\boldsymbol{\zeta}}_s := \big(\widehat{\boldsymbol{\zeta}}_s, (\widehat{\mathbf{z}}_s, \widehat{\boldsymbol{\xi}}_s)\big) \in \mathrm{H} \times \mathrm{Q}$ are the unique solutions, guaranteed now by Theorem 3.7, of the formulations (cf. (3.36) or (3.73))

$$A_{s,\mathbf{w}_s}\big(\vec{\boldsymbol{\sigma}}_s, \vec{\boldsymbol{\tau}}_s\big) = \mathbf{F}_s^{\mathbf{w}}(\boldsymbol{\tau}_s) + \mathbf{G}_s(\mathbf{v}_s, \boldsymbol{\eta}_s)$$

and

$$A_{s,\mathbf{t}_s}\big(\vec{\boldsymbol{\zeta}}_s, \vec{\boldsymbol{\tau}}_s\big) = \mathbf{F}_s^{\mathbf{t}}(\boldsymbol{\tau}_s) + \mathbf{G}_s(\mathbf{v}_s, \boldsymbol{\eta}_s),$$

respectively, both for all $\vec{\boldsymbol{\tau}}_s := \big(\boldsymbol{\tau}_s, (\mathbf{v}_s, \boldsymbol{\eta}_s)\big) \in \mathrm{H} \times \mathrm{Q}$. Then, starting from the inf–sup condition for $A_{s,\mathbf{w}_s}$ with constant $\bar{\alpha}_s/2$ (analogue of (3.67)), and employing basically the same kind of arguments that yielded (3.82), we are able to show that

$$\begin{aligned} &\frac{\bar{\alpha}_s}{2} \|\vec{\boldsymbol{\sigma}}_s - \vec{\boldsymbol{\zeta}}_s\|_{\mathrm{H}\times\mathrm{Q}} \\ &\quad \le \sup_{\substack{\vec{\boldsymbol{\tau}}_s \in \mathrm{H}\times\mathrm{Q} \\ \vec{\boldsymbol{\tau}}_s \neq \mathbf{0}}} \frac{\big(\mathbf{F}_s^{\mathbf{w}} - \mathbf{F}_s^{\mathbf{t}}\big)(\boldsymbol{\tau}_s) + \mathbf{d}_s(\mathbf{t}_s - \mathbf{w}_s; \widehat{\mathbf{z}}_s, \boldsymbol{\tau}_s)}{\|\vec{\boldsymbol{\tau}}_s\|_{\mathrm{H}\times\mathrm{Q}}}, \end{aligned} \tag{3.86}$$

where the last term uses, according to (3.72), that $\big(A_{s,\mathbf{t}_s} - A_{s,\mathbf{w}_s}\big)\big(\vec{\boldsymbol{\zeta}}_s, \vec{\boldsymbol{\tau}}_s\big) = \mathbf{d}_s(\mathbf{t}_s - \mathbf{w}_s; \widehat{\mathbf{z}}_s, \boldsymbol{\tau}_s)$. Next, it follows from the definitions of $\mathbf{F}_s^{\mathbf{w}}$ and $\mathbf{F}_s^{\mathbf{t}}$ (cf. (3.8)), and the lower bound of μ_s (cf. (3.26)), that

$$\begin{aligned} \big|\big(\mathbf{F}_s^{\mathbf{w}} - \mathbf{F}_s^{\mathbf{t}}\big)(\boldsymbol{\tau}_s)\big| &= \left| \int_\Omega \frac{\rho_f}{2\mu_s(\phi)} \Big\{ \big((\varepsilon \mathbf{t}_f) \otimes \mathbf{t}_f\big) - \big((\varepsilon \mathbf{w}_f) \otimes \mathbf{w}_f\big) \Big\}^{\mathrm{d}} : \boldsymbol{\tau}_s \right| \\ &\le \frac{\rho_f}{2\mu_1} \|\varepsilon\|_{0,\infty;\Omega} \|\big(\mathbf{t}_f \otimes \mathbf{t}_f\big) - \big(\mathbf{w}_f \otimes \mathbf{w}_f\big)\|_{0,\Omega} \|\boldsymbol{\tau}_s\|_{0,\Omega} \\ &\le \frac{\rho_f}{2\mu_1} \|\varepsilon\|_{0,\infty;\Omega} \|\mathbf{t}_f + \mathbf{w}_f\|_{0,4;\Omega} \|\mathbf{t}_f - \mathbf{w}_f\|_{0,4;\Omega} \|\boldsymbol{\tau}_s\|_{0,\Omega}. \end{aligned} \tag{3.87}$$

In turn, using the boundedness properties of $\mathbf{d}_s$ (cf. (3.29), (3.32)), we find that

$$\big|\mathbf{d}_s(\mathbf{t}_s - \mathbf{w}_s; \widehat{\mathbf{z}}_s, \boldsymbol{\tau}_s)\big| \le \frac{\rho_s}{2\mu_1} \|\phi\|_{0,\infty;\Omega} \|\Theta_s(\mathbf{t})\|_{0,4;\Omega} \|\mathbf{t}_s - \mathbf{w}_s\|_{0,4;\Omega} \|\boldsymbol{\tau}_s\|_{0,\Omega}. \tag{3.88}$$

Therefore, replacing the estimates (3.87) and (3.88) back into (3.86), using that $\|\mathbf{t}_f - \mathbf{w}_f\|_{0,4;\Omega}$ and $\|\mathbf{t}_s - \mathbf{w}_s\|_{0,4;\Omega}$ are bounded by $\|\mathbf{w} - \mathbf{t}\|_{0,4;\Omega}$, and recalling that $\|\Theta_s(\mathbf{w}) - \Theta_s(\mathbf{t})\|_{0,4;\Omega} \le \|\vec{\boldsymbol{\sigma}}_s - \vec{\boldsymbol{\zeta}}_s\|_{\mathrm{H}\times\mathrm{Q}}$, we are led to (3.85) with $L_s := \dfrac{2}{\bar{\alpha}_s} \max\Big\{\dfrac{\rho_f}{2\mu_1}, \dfrac{\rho_s}{2\mu_1}\Big\}$. □

As a straightforward consequence of Lemmas 3.9 and 3.10, we are able to establish now the Lipschitz-continuity of the fixed-point operator S (cf. (3.37)).

Lemma 3.11. *Let W be as in Lemma 3.8 with r given by (3.76), and let L_f and L_s be the constants provided by Lemmas 3.9 and 3.10. Then, there holds*

$$\begin{aligned} \|S(\mathbf{w}) - S(\mathbf{t})\|_{0,4;\Omega} \le &\Big\{ L_f \Big(\|\delta(\phi)\|_{0,\Omega} + \|\varepsilon\|_{0,\infty;\Omega} \|\Theta_f(\mathbf{t})\|_{0,4;\Omega} \Big) \\ &+ L_s \Big(\|\varepsilon\|_{0,\infty;\Omega} \|\mathbf{t}_f + \mathbf{w}_f\|_{0,4;\Omega} + \|\phi\|_{0,\infty;\Omega} \|\Theta_s(\mathbf{t})\|_{0,4;\Omega} \Big) \Big\} \|\mathbf{w} - \mathbf{t}\|_{0,4;\Omega} \end{aligned} \tag{3.89}$$

for all $\mathbf{w} := (\mathbf{w}_f, \mathbf{w}_s)$, $\mathbf{t} := (\mathbf{t}_f, \mathbf{t}_s) \in W$.

Proof. Given $\mathbf{w}$, $\mathbf{t} \in W$, it suffices to observe that

$$\|S(\mathbf{w}) - S(\mathbf{t})\|_{0,4;\Omega} = \|\Theta_f(\mathbf{w}) - \Theta_f(\mathbf{t})\|_{0,4;\Omega} + \|\Theta_s(\mathbf{w}) - \Theta_s(\mathbf{t})\|_{0,4;\Omega},$$

and then apply the estimates (3.79) and (3.85). □

Now, incorporating the upper bounds of $\|\Theta_f(\mathbf{t})\|_{0,4;\Omega}$ and $\|\Theta_s(\mathbf{t})\|_{0,4;\Omega}$ provided by (3.69) and (3.75), respectively, into the right hand side of (3.89), and bounding $\|\mathbf{t}_f - \mathbf{t}_s\|_{0,4;\Omega}$ and $\|\mathbf{t}_f\|_{0,4;\Omega}^2$ by r and r^2, respectively, we arrive at

$$\|S(\mathbf{w}) - S(\mathbf{t})\|_{0,4;\Omega} \le \mathcal{L}\big(\mathtt{data}\big) \|\mathbf{w} - \mathbf{t}\|_{0,4;\Omega}, \tag{3.90}$$

for all $\mathbf{w} := (\mathbf{w}_f, \mathbf{w}_s)$, $\mathbf{t} := (\mathbf{t}_f, \mathbf{t}_s) \in W$, where

$$\begin{aligned}\mathcal{L}(\mathtt{data}) \,:=\, & C_1\,\|\delta(\phi)\|_{0,\Omega} + C_2\,\|\varepsilon\|_{0,\infty;\Omega} + C_3\,\|\varepsilon\|_{0,\infty;\Omega}\,\|\mathbf{u}_{D,f}\|_{1/2,\Gamma}\\ & + C_4\,\|\phi\|_{0,\infty;\Omega}\,\|\mathbf{u}_{D,s}\|_{1/2,\Gamma} + C_5\,\|\varepsilon\|_{0,\infty;\Omega}\,\|\delta(\phi)\|_{0,\Omega} + C_6\,\|\varepsilon\|^2_{0,\infty;\Omega}\\ & + C_7\,\|\phi\|_{0,\infty;\Omega}\,\|\varepsilon\|_{0,\infty;\Omega} + C_8\,\|\phi\|_{0,\infty;\Omega}\,\|\varepsilon\rho_f + \phi\rho_s\|_{0,\infty;\Omega}\,,\end{aligned} \tag{3.91}$$

and C_j, $j \in \{1,\dots,8\}$ are positive constants depending on L_f, L_s, $\bar{\alpha}_f$, $\bar{\alpha}_s$, r, ρ_f, μ_1, $|\Omega|$, and $\mathbf{g}$, as indicated as follows

$$\begin{aligned} & C_1 = L_f\,, \quad C_2 = 2rL_s\,, \quad C_3 = \frac{2L_f}{\bar{\alpha}_f}\,, \quad C_4 = \frac{2L_s}{\bar{\alpha}_s}\,, \quad C_5 = \frac{2rL_f}{\bar{\alpha}_f}\,,\\ & C_6 = \frac{2L_f|\Omega|^{3/4}\rho_f\mathbf{g}}{\bar{\alpha}_f}\,, \quad C_7 = \frac{r^2L_s\rho_f}{\bar{\alpha}_s\mu_1}\,, \quad \text{and} \quad C_8 = \frac{2L_s|\Omega|^{3/4}\mathbf{g}}{\bar{\alpha}_s}\,.\end{aligned} \tag{3.92}$$

We can establish now the main result concerning the solvability of (3.18).

Theorem 3.12. *Let W be as in Lemma 3.8 with r given by (3.76), and assume that the data are sufficiently small so that they satisfy (3.77), (3.78), and*

$$\mathcal{L}(\mathtt{data}) \,<\, 1\,. \tag{3.93}$$

Then, problem (3.18) has a unique solution $(\boldsymbol{\sigma}_f, (\mathbf{u}_f, \boldsymbol{\gamma}_f)) \in \mathrm{H}\times\mathrm{Q}$ and $(\boldsymbol{\sigma}_s, (\mathbf{u}_s, \boldsymbol{\gamma}_s)) \in \mathrm{H}\times\mathrm{Q}$ with $\mathbf{u} := (\mathbf{u}_f, \mathbf{u}_s) \in W$. Moreover, there hold

$$\|(\boldsymbol{\sigma}_f, (\mathbf{u}_f, \boldsymbol{\gamma}_f))\|_{\mathrm{H}\times\mathrm{Q}} \,\le\, \frac{2}{\bar{\alpha}_f}\Big\{\|\mathbf{u}_{D,f}\|_{1/2,\Gamma} \,+\, r\,\|\delta(\phi)\|_{0,\Omega} \,+\, |\Omega|^{3/4}\,\rho_f\,\mathbf{g}\,\|\varepsilon\|_{0,\infty;\Omega}\Big\}\,, \tag{3.94}$$

and

$$\|(\boldsymbol{\sigma}_s, (\mathbf{u}_s, \boldsymbol{\gamma}_s))\|_{\mathrm{H}\times\mathrm{Q}} \,\le\, \frac{2}{\bar{\alpha}_s}\Big\{\|\mathbf{u}_{D,s}\|_{1/2,\Gamma} \,+\, r^2\,\frac{\rho_f}{2\mu_1}\,\|\varepsilon\|_{0,\infty;\Omega} \,+\, |\Omega|^{3/4}\,\mathbf{g}\,\|\varepsilon\rho_f + \phi\rho_s\|_{0,\infty;\Omega}\Big\}\,. \tag{3.95}$$

Proof. According to the equivalence between (3.18) and (3.38), and thanks to Lemma 3.8, the Lipschitz-continuity of S (cf. (3.90)), and the assumption (3.93), the existence of a unique solution of (3.18) with $\mathbf{u} := (\mathbf{u}_f, \mathbf{u}_s) \in W$ follows from a straightforward application of the classical Banach fixed-point Theorem. Then, the a priori estimates (3.69) and (3.75), together with the fact that $\|\mathbf{u}_f\|_{0,4;\Omega}$ and $\|\mathbf{u}_f - \mathbf{u}_s\|_{0,4;\Omega}$ are bounded by r, yield (3.94) and (3.95), which completes the proof. □

Similar remarks to those expressed on the assumptions (3.77) and (3.78) right after the proof of Lemma 3.8, are valid here for (3.93) and the expression $\mathcal{L}(\mathtt{data})$ given by (3.91) and (3.92). We omit further details.

4. The Galerkin scheme

In this section we introduce and analyze a Galerkin scheme for approximating the solution of (3.18). In particular, for the respective solvability analysis we employ basically the same tools and techniques utilized for the continuous case in Section 3, except that now we apply Brouwer and Banach fixed-point Theorems to prove existence and uniqueness of solution, respectively.

4.1. The discrete fixed-point approach

We begin by considering a regular family $\{\mathcal{T}_h\}_{h>0}$ of triangulations of $\bar{\Omega}$, which are made of triangles K (when $\mathrm{n}=2$) or tetrahedra K (when $\mathrm{n}=3$) of diameters h_K, and define the meshsize $h := \max\{h_K : \; K \in \mathcal{T}_h\}$, which also serves as the index of $\mathcal{T}_h$. Then, for each $h>0$ we let H_h^{σ}, $\mathrm{Q}_h^{\mathbf{u}}$, and Q_h^{γ} be arbitrary finite element subspaces of $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$, $\mathbf{L}^4(\Omega)$, and $\mathbb{L}^2_{\mathtt{skew}}(\Omega)$, respectively, and set

$$\mathrm{H}_h \,:=\, \mathrm{H}_h^{\sigma} \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \qquad \text{and} \qquad \mathrm{Q}_h \,:=\, \mathrm{Q}_h^{\mathbf{u}} \times \mathrm{Q}_h^{\gamma}\,. \tag{4.1}$$

Thus, the Galerkin scheme associated with (3.18) reads: find $(\boldsymbol{\sigma}_{fh}, (\mathbf{u}_{fh}, \boldsymbol{\gamma}_{fh})) \in \mathrm{H}_h\times\mathrm{Q}_h$ and $(\boldsymbol{\sigma}_{sh}, (\mathbf{u}_{sh}, \boldsymbol{\gamma}_{sh})) \in \mathrm{H}_h\times\mathrm{Q}_h$ such that, denoting $\mathbf{u}_h := (\mathbf{u}_{fh}, \mathbf{u}_{sh}) \in \mathrm{Q}_h^{\mathbf{u}}\times\mathrm{Q}_h^{\mathbf{u}}$,

$$\begin{aligned}\mathbf{a}_f(\boldsymbol{\sigma}_{fh}, \boldsymbol{\tau}_{fh}) + \mathbf{b}(\boldsymbol{\tau}_{fh}, (\mathbf{u}_{fh}, \boldsymbol{\gamma}_{fh})) + \mathbf{c}_f(\mathbf{u}_{fh}, \boldsymbol{\tau}_{fh}) + \mathbf{d}_f(\mathbf{u}_{fh}; \mathbf{u}_{fh}, \boldsymbol{\tau}_{fh}) &= \mathbf{F}_f(\boldsymbol{\tau}_{fh})\,,\\ \mathbf{b}(\boldsymbol{\sigma}_{fh}, (\mathbf{v}_{fh}, \boldsymbol{\eta}_{fh})) &= \mathbf{G}_f^{\mathbf{u}_h}(\mathbf{v}_{fh}, \boldsymbol{\eta}_{fh})\,,\\ \mathbf{a}_s(\boldsymbol{\sigma}_{sh}, \boldsymbol{\tau}_{sh}) + \mathbf{b}(\boldsymbol{\tau}_{sh}, (\mathbf{u}_{sh}, \boldsymbol{\gamma}_{sh})) + \mathbf{c}_s(\mathbf{u}_{sh}, \boldsymbol{\tau}_{sh}) + \mathbf{d}_s(\mathbf{u}_{sh}; \mathbf{u}_{sh}, \boldsymbol{\tau}_{sh}) &= \mathbf{F}_s^{\mathbf{u}_h}(\boldsymbol{\tau}_{sh})\,,\\ \mathbf{b}(\boldsymbol{\sigma}_{sh}, (\mathbf{v}_{sh}, \boldsymbol{\eta}_{sh})) &= \mathbf{G}_s(\mathbf{v}_{sh}, \boldsymbol{\eta}_{sh})\,,\end{aligned} \tag{4.2}$$

for all $\big(\boldsymbol{\tau}_{fh}, (\mathbf{v}_{fh}, \boldsymbol{\eta}_{fh})\big) \in \mathrm{H}_h \times \mathrm{Q}_h$ and for all $\big(\boldsymbol{\tau}_{sh}, (\mathbf{v}_{sh}, \boldsymbol{\eta}_{sh})\big) \in \mathrm{H}_h \times \mathrm{Q}_h$. Next, we consider the discrete analogue of the fixed-point approach employed in Section 3.2. Indeed, we first introduce the operator $\Theta_{fh} : \mathrm{Q}_h^{\mathbf{u}} \times \mathrm{Q}_h^{\mathbf{u}} \to \mathrm{Q}_h^{\mathbf{u}}$ as

$$\Theta_{fh}(\mathbf{w}_h) := \widehat{\mathbf{u}}_{fh} \qquad \forall\, \mathbf{w}_h := (\mathbf{w}_{fh}, \mathbf{w}_{sh}) \in \mathrm{Q}_h^{\mathbf{u}} \times \mathrm{Q}_h^{\mathbf{u}}\,, \tag{4.3}$$

where $\big(\widehat{\boldsymbol{\sigma}}_{fh}, (\widehat{\mathbf{u}}_{fh}, \widehat{\boldsymbol{\gamma}}_{fh})\big) \in \mathrm{H}_h \times \mathrm{Q}_h$ is the unique solution (to be confirmed below) of the first two equations of (4.2) when the first component $\mathbf{u}_{fh}$ of $\mathbf{d}_f$ and the superscript $\mathbf{u}_h$ of $\mathbf{G}_f^{\mathbf{u}_h}$ are replaced by $\mathbf{w}_{fh}$ and $\mathbf{w}_h$, respectively, that is

$$\begin{aligned} \mathbf{a}_f(\widehat{\boldsymbol{\sigma}}_{fh}, \boldsymbol{\tau}_{fh}) + \mathbf{b}\big(\boldsymbol{\tau}_{fh}, (\widehat{\mathbf{u}}_{fh}, \widehat{\boldsymbol{\gamma}}_{fh})\big) + \mathbf{c}_f(\widehat{\mathbf{u}}_{fh}, \boldsymbol{\tau}_{fh}) + \mathbf{d}_f(\mathbf{w}_{fh}; \widehat{\mathbf{u}}_{fh}, \boldsymbol{\tau}_{fh}) &= \mathbf{F}_f(\boldsymbol{\tau}_{fh})\,, \\ \mathbf{b}\big(\widehat{\boldsymbol{\sigma}}_{fh}, (\mathbf{v}_{fh}, \boldsymbol{\eta}_{fh})\big) &= \mathbf{G}_f^{\mathbf{w}_h}(\mathbf{v}_{fh}, \boldsymbol{\eta}_{fh})\,, \end{aligned} \tag{4.4}$$

for all $\big(\boldsymbol{\tau}_{fh}, (\mathbf{v}_{fh}, \boldsymbol{\eta}_{fh})\big) \in \mathrm{H}_h \times \mathrm{Q}_h$. In addition, we let $\Theta_{sh} : \mathrm{Q}_h^{\mathbf{u}} \times \mathrm{Q}_h^{\mathbf{u}} \to \mathrm{Q}_h^{\mathbf{u}}$ be the operator given by

$$\Theta_{sh}(\mathbf{w}_h) := \widehat{\mathbf{u}}_{sh} \qquad \forall\, \mathbf{w}_h := (\mathbf{w}_{fh}, \mathbf{w}_{sh}) \in \mathrm{Q}_h^{\mathbf{u}} \times \mathrm{Q}_h^{\mathbf{u}}\,, \tag{4.5}$$

where $\big(\widehat{\boldsymbol{\sigma}}_{sh}, (\widehat{\mathbf{u}}_{sh}, \widehat{\boldsymbol{\gamma}}_{sh})\big) \in \mathrm{H}_h \times \mathrm{Q}_h$ is the unique solution (to be confirmed below) of the last two equations of (4.2) when the first component $\mathbf{u}_{sh}$ of $\mathbf{d}_s$ and the superscript $\mathbf{u}_h$ of $\mathbf{F}_s^{\mathbf{u}_h}$ are replaced by $\mathbf{w}_{sh}$ and $\mathbf{w}_h$, respectively, that is

$$\begin{aligned} \mathbf{a}_s(\widehat{\boldsymbol{\sigma}}_{sh}, \boldsymbol{\tau}_{sh}) + \mathbf{b}\big(\boldsymbol{\tau}_{sh}, (\widehat{\mathbf{u}}_{sh}, \widehat{\boldsymbol{\gamma}}_{sh})\big) + \mathbf{c}_s(\widehat{\mathbf{u}}_{sh}, \boldsymbol{\tau}_{sh}) + \mathbf{d}_s(\mathbf{w}_{sh}; \widehat{\mathbf{u}}_{sh}, \boldsymbol{\tau}_{sh}) &= \mathbf{F}_s^{\mathbf{w}_h}(\boldsymbol{\tau}_{sh})\,, \\ \mathbf{b}\big(\widehat{\boldsymbol{\sigma}}_{sh}, (\mathbf{v}_{sh}, \boldsymbol{\eta}_{sh})\big) &= \mathbf{G}_s(\mathbf{v}_{sh}, \boldsymbol{\eta}_{sh})\,, \end{aligned} \tag{4.6}$$

for all $\big(\boldsymbol{\tau}_{sh}, (\mathbf{v}_{sh}, \boldsymbol{\eta}_{sh})\big) \in \mathrm{H}_h \times \mathrm{Q}_h$. Finally, we define the operator $S_h : \mathrm{Q}_h^{\mathbf{u}} \times \mathrm{Q}_h^{\mathbf{u}} \to \mathrm{Q}_h^{\mathbf{u}} \times \mathrm{Q}_h^{\mathbf{u}}$ as

$$S_h(\mathbf{w}_h) := \big(\Theta_{fh}(\mathbf{w}_h), \Theta_{sh}(\mathbf{w}_h)\big) \qquad \forall\, \mathbf{w}_h := (\mathbf{w}_{fh}, \mathbf{w}_{sh}) \in \mathrm{Q}_h^{\mathbf{u}} \times \mathrm{Q}_h^{\mathbf{u}}\,, \tag{4.7}$$

and notice that solving (4.2) is equivalent to seeking a fixed-point of S_h, that is: find $\mathbf{w}_h \in \mathrm{Q}_h^{\mathbf{u}} \times \mathrm{Q}_h^{\mathbf{u}}$ such that

$$S_h(\mathbf{w}_h) = \mathbf{w}_h\,. \tag{4.8}$$

4.2. Well-definedness of the operators Θ_{fh} and Θ_{sh}

In this section we apply the discrete versions of Theorems 3.1 and 3.2 to prove that problems (4.4) and (4.6) are well-posed, thus confirming that the operators Θ_{fh} and Θ_{sh} are well-defined. Regarding the aforementioned versions of those theorems, which certainly involve finite dimensional subspaces, we stress that in this case each assumption (i) (cf. (3.39) and (3.43)) is equivalent to its corresponding assumption (ii) (cf. (3.40) and (3.44)), so that in what follows we choose to stay with the (i) ones. Moreover, for the stability of the associated discrete schemes, we require the respective constants α to be independent of the meshsize h.

In order to proceed as announced, we need to incorporate some hypotheses on the arbitrary discrete spaces $\mathrm{H}_h^{\boldsymbol{\sigma}}$, $\mathrm{Q}_h^{\mathbf{u}}$, and $\mathrm{Q}_h^{\boldsymbol{\gamma}}$. Specific finite element subspaces verifying these conditions will be introduced later on. More precisely, from now on we assume the following:

(**H.1**) $\mathrm{H}_h^{\boldsymbol{\sigma}}$ contains the multiples of the identity tensor $\mathbb{I}$.

(**H.2**) $\mathbf{div}\big(\mathrm{H}_h^{\boldsymbol{\sigma}}\big) \subseteq \mathrm{Q}_h^{\mathbf{u}}$.

(**H.3**) There exists a positive constant β_{d}, independent of h, such that

$$\sup_{\substack{\boldsymbol{\tau}_h \in \mathrm{H}_h \\ \boldsymbol{\tau}_h \neq \mathbf{0}}} \frac{\mathbf{b}\big(\boldsymbol{\tau}_h, (\mathbf{v}_h, \boldsymbol{\eta}_h)\big)}{\|\boldsymbol{\tau}_h\|_{\mathrm{H}}} \geq \beta_{\mathrm{d}}\, \|(\mathbf{v}_h, \boldsymbol{\eta}_h)\|_{\mathrm{Q}} \qquad \forall\, (\mathbf{v}_h, \boldsymbol{\eta}_h) \in \mathrm{Q}_h\,. \tag{4.9}$$

Hence, thanks to (**H.1**) and the decomposition (3.1), the subspace H_h (cf. (4.1)) can be redefined, at least from a theoretical point of view, as:

$$\mathrm{H}_h := \left\{ \boldsymbol{\tau}_h - \left(\frac{1}{n|\Omega|} \int_\Omega \mathrm{tr}(\boldsymbol{\tau}_h) \right) \mathbb{I} : \quad \boldsymbol{\tau}_h \in \mathrm{H}_h^{\boldsymbol{\sigma}} \right\}.$$

However, for the computational implementation of the Galerkin scheme (4.2), which is addressed below in Section 6, the null mean value condition for the traces of the unknown tensors living in H_h will be imposed via real Lagrange multipliers.

On the other hand, the kernel of the operator induced by the bilinear form $\mathbf{b}$ restricted to $\mathrm{H}_h \times \mathrm{Q}_h$, is given by

$$\mathbb{V}_h := \Big\{ \boldsymbol{\tau}_h \in \mathrm{H}_h : \quad \mathbf{b}\big(\boldsymbol{\tau}_h, (\mathbf{v}_h, \boldsymbol{\eta}_h)\big) = 0 \quad \forall\, (\mathbf{v}_h, \boldsymbol{\eta}_h) \in \mathrm{Q}_h \Big\},$$

from which, bearing in mind the definitions of $\mathbf{b}$ (cf. (3.4)) and Q_h, and the assumption (**H.2**), we find that

$$\mathbb{V}_h := \Big\{ \boldsymbol{\tau}_h \in \mathrm{H}_h : \quad \mathbf{div}(\boldsymbol{\tau}_h) = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \int_\Omega \boldsymbol{\eta}_h : \boldsymbol{\tau}_h = 0 \quad \forall\, \boldsymbol{\eta}_h \in \mathrm{Q}_h^{\boldsymbol{\gamma}} \Big\}. \tag{4.10}$$

In this way, noticing from Lemmas 3.3 and 3.4 that the $\mathbb{V}$-ellipticity of the bilinear forms $\mathbf{a}_f$ and $\mathbf{a}_s$ only makes use of the divergence-free property of the tensors of $\mathbb{V}$, we conclude from (4.10) that $\mathbf{a}_f$ and $\mathbf{a}_s$ are $\mathbb{V}_h$-elliptic as well, with the same positive constants α_f and α_s provided by those lemmas, that is there hold

$$\mathbf{a}_f(\boldsymbol{\tau}_h, \boldsymbol{\tau}_h) \geq \alpha_f \, \|\boldsymbol{\tau}_h\|^2_{\mathbf{div}_{4/3};\Omega} \qquad \forall\, \boldsymbol{\tau}_h \in \mathbb{V}_h\,, \tag{4.11}$$

and

$$\mathbf{a}_s(\boldsymbol{\tau}_h, \boldsymbol{\tau}_h) \geq \alpha_s \, \|\boldsymbol{\tau}_h\|^2_{\mathbf{div}_{4/3};\Omega} \qquad \forall\, \boldsymbol{\tau}_h \in \mathbb{V}_h\,. \tag{4.12}$$

Therefore, in virtue of (**H.3**), (4.11), and (4.12), straightforward applications of the discrete version of Theorem 3.2, and particularly of the corresponding estimate (3.48), imply that A_f (cf. (3.62)) and A_s (cf. (3.70)) satisfy global discrete inf–sup conditions on $\mathrm{H}_h \times \mathrm{Q}_h$, that is the discrete analogues of (3.63) and (3.71), with constants $\bar{\alpha}_{f,\mathrm{d}} > 0$, depending only on α_f, β_{d}, and $\|\mathbf{a}_f\|$ (cf. (3.25)), and $\bar{\alpha}_{s,\mathrm{d}} > 0$, depending only on α_s, β_{d}, and $\|\mathbf{a}_s\|$ (cf. (3.32)), respectively. Moreover, given $\mathbf{w}_h := (\mathbf{w}_{fh}, \mathbf{w}_{sh}) \in \mathrm{Q}_h^{\mathbf{u}} \times \mathrm{Q}_h^{\mathbf{u}}$, and proceeding analogously as we did in Section 3.3, we are able to show that, under the following pairs of conditions

$$\frac{1}{\sqrt{n}} \left\| \frac{\nabla \varepsilon}{\varepsilon} \right\|_{0,4;\Omega} \leq \frac{\bar{\alpha}_{f,\mathrm{d}}}{4} \quad \text{and} \quad \|\mathbf{w}_{fh}\|_{0,4;\Omega} \leq r_{f,\mathrm{d}} := \frac{\bar{\alpha}_{f,\mathrm{d}}\,\mu_f}{2\rho_f}\,, \tag{4.13}$$

and

$$\frac{1}{\sqrt{n}} \left\| \frac{\nabla \phi}{\phi} \right\|_{0,4;\Omega} \leq \frac{\bar{\alpha}_{s,\mathrm{d}}}{4} \quad \text{and} \quad \|\mathbf{w}_{sh}\|_{0,4;\Omega} \leq r_{s,\mathrm{d}} := \frac{\bar{\alpha}_{s,\mathrm{d}}\,\mu_1}{2\rho_s}\,, \tag{4.14}$$

the bilinear forms $A_{f,\mathbf{w}_{fh}}$ (cf. (3.64)) and $A_{s,\mathbf{w}_{sh}}$ (cf. (3.72)) satisfy global discrete inf–sup conditions on $\mathrm{H}_h \times \mathrm{Q}_h$ with constants $\bar{\alpha}_{f,\mathrm{d}}/2$ and $\bar{\alpha}_{s,\mathrm{d}}/2$, respectively. Consequently, rewriting (4.4) and (4.6) as the discrete analogues of (3.65) and (3.73), respectively, and applying now the discrete version of Theorem 3.1, we obtain the following results confirming that the discrete operators Θ_{fh} (cf. (4.3)) and Θ_{sh} (cf. (4.5)) are well-defined. The respective proofs, being almost verbatim to those of Theorems 3.6 and 3.7, are omitted.

Theorem 4.1. *Assume that* $\frac{1}{\sqrt{n}} \left\| \frac{\nabla \varepsilon}{\varepsilon} \right\|_{0,4;\Omega} \leq \frac{\bar{\alpha}_{f,\mathrm{d}}}{4}$. *Then, for each* $\mathbf{w}_h := (\mathbf{w}_{fh}, \mathbf{w}_{sh}) \in \mathrm{Q}_h^{\mathbf{u}} \times \mathrm{Q}_h^{\mathbf{u}}$ *such that* $\|\mathbf{w}_{fh}\|_{0,4;\Omega} \leq r_{f,\mathrm{d}}$, *there exists a unique* $\big(\widehat{\boldsymbol{\sigma}}_{fh}, (\widehat{\mathbf{u}}_{fh}, \widehat{\boldsymbol{\gamma}}_{fh})\big) \in \mathrm{H}_h \times \mathrm{Q}_h$ *solution to* (4.4)*. Moreover, there holds*

$$\begin{aligned} \|\Theta_{fh}(\mathbf{w}_h)\|_{0,4;\Omega} &= \|\widehat{\mathbf{u}}_{fh}\|_{0,4;\Omega} \leq \|\big(\widehat{\boldsymbol{\sigma}}_{fh}, (\widehat{\mathbf{u}}_{fh}, \widehat{\boldsymbol{\gamma}}_{fh})\big)\|_{\mathrm{H}\times\mathrm{Q}} \\ &\leq \frac{2}{\bar{\alpha}_{f,\mathrm{d}}} \Big\{ \|\mathbf{u}_{D,f}\|_{1/2,\Gamma} + \|\delta(\phi)\|_{0,\Omega}\, \|\mathbf{w}_{fh} - \mathbf{w}_{sh}\|_{0,4;\Omega} + |\Omega|^{3/4}\, \rho_f\, \mathbf{g}\, \|\varepsilon\|_{0,\infty;\Omega} \Big\}\,. \end{aligned} \tag{4.15}$$

Theorem 4.2. *Assume that* $\frac{1}{\sqrt{n}} \left\| \frac{\nabla \phi}{\phi} \right\|_{0,4;\Omega} \leq \frac{\bar{\alpha}_{s,\mathrm{d}}}{4}$. *Then, for each* $\mathbf{w}_h := (\mathbf{w}_{fh}, \mathbf{w}_{sh}) \in \mathrm{Q}_h^{\mathbf{u}} \times \mathrm{Q}_h^{\mathbf{u}}$ *such that* $\|\mathbf{w}_{sh}\|_{0,4;\Omega} \leq r_{s,\mathrm{d}}$, *there exists a unique* $\big(\widehat{\boldsymbol{\sigma}}_{sh}, (\widehat{\mathbf{u}}_{sh}, \widehat{\boldsymbol{\gamma}}_{sh})\big) \in \mathrm{H}_h \times \mathrm{Q}_h$ *solution to* (4.6)*. Moreover, there holds*

$$\begin{aligned} \|\Theta_{sh}(\mathbf{w}_h)\|_{0,4;\Omega} &= \|\widehat{\mathbf{u}}_{sh}\|_{0,4;\Omega} \leq \|\big(\widehat{\boldsymbol{\sigma}}_{sh}, (\widehat{\mathbf{u}}_{sh}, \widehat{\boldsymbol{\gamma}}_{sh})\big)\|_{\mathrm{H}\times\mathrm{Q}} \\ &\leq \frac{2}{\bar{\alpha}_{s,\mathrm{d}}} \Big\{ \|\mathbf{u}_{D,s}\|_{1/2,\Gamma} + \frac{\rho_f}{2\mu_1}\, \|\varepsilon\|_{0,\infty;\Omega}\, \|\mathbf{w}_{fh}\|^2_{0,4;\Omega} + |\Omega|^{3/4}\, \mathbf{g}\, \|\varepsilon\rho_f + \phi\rho_s\|_{0,\infty;\Omega} \Big\}\,. \end{aligned} \tag{4.16}$$

Regarding the assumptions on $\frac{\nabla \varepsilon}{\varepsilon}$ and $\frac{\nabla \phi}{\phi}$ specified in Theorems 4.1 and 4.2, whose continuous analogues are required by Theorems 3.6 and 3.7, respectively, and observing that these expressions require both ε and ϕ to be bounded from below, we find it important to state here some remarks.

First of all, and while in many industrial situations there will certainly be regions in which the particle concentration will be zero (zones depleted of particles) or, equivalently, the void fraction will be one (pure fluid regions), in our present formulation it is only the concentration ϕ that has to be bounded from below, so that it avoids the zero value. The void fraction ε can never be zero as the upper bound of the model is $\phi \approx \phi_p = 0.65$ (and not $\phi = 1$), since the particle pressure and particle viscosity functions (cf. (2.6), (2.7)) would be singular at ϕ_p as ϕ increases from a typical ϕ_0 to ϕ_p. In this model, bubbles, or large amplitude concentration instabilities, are obtained as very low concentration regions, but never reaching true zero values inside them. Indeed, a quick search in the literature shows that the lowest concentrations obtained in structures resembling (and being analyzed as) bubbles were $\phi = 0.14$ in [4], $\phi = 0.11$ in [34], and $\phi = 0.01$ in [9] (in which a slightly different model and a very low ϕ_0 were used). The reason why simulations cannot reach very low values of ϕ is due to the requirement that the mass of particles is conserved, which forces that the mean volume fraction should be ϕ_0 in the *small-ish* numerical domains. In a real scale fluidized bed, the ratio of the size of the fluidization domain to the size of the bubbles is much larger than 10, allowing for local rearrangements of the particles flowing out of the structures that will become bubbles, whereas in numerical simulations bubbles occupy an important part of the domain of the simulation. The minimum values of ϕ obtained with this model depend not only on the flow properties, but also on the mean particle concentration ϕ_0 and on the size of the numerical domain that is used. Therefore, one would not expect to find true zero values of ϕ in numerical simulation of bubbles using averaged models in normal scale simulations.

Nevertheless, and despite the above comments, in Section 6 we consider two examples to test the validity of the model in the $\phi \approx 0$ limit. The first one represents a fluidized bed that is expanded homogeneously up to a certain height and, from there, after a sharp transition, the concentration decreases very rapidly to zero. The second test represents a bubble placed at the center of the domain, and whose particle concentration distribution is such that it is actually zero at the center of the bubble.

4.3. Solvability of the discrete fixed-point equation

We now address the solvability of the fixed-point equation (4.8), which is equivalent to analyzing the existence and uniqueness of solution of the Galerkin scheme (4.2). To this end, we proceed very similarly to the continuous case and establish first the discrete versions of the preliminary lemmas from Section 3.4. Bearing this in mind, we assume in what follows that ε and ϕ satisfy the conditions indicated in (4.13) and (4.14), respectively, and we set

$$r_{\mathrm{d}} \;:=\; \min\big\{r_{f,\mathrm{d}}, r_{s,\mathrm{d}}\big\}\,. \tag{4.17}$$

Then, we begin with the result that provides sufficient conditions on the data for S_h mapping a closed ball into itself.

Lemma 4.3. *Let $W_h := \Big\{\mathbf{w}_h = (\mathbf{w}_{fh}, \mathbf{w}_{sh}) \in \mathrm{Q}_h^{\mathbf{u}} \times \mathrm{Q}_h^{\mathbf{u}} : \quad \|\mathbf{w}_h\|_{0,4;\Omega} \le r_{\mathrm{d}}\Big\}$, and assume that the data satisfy*

$$\|\mathbf{u}_{D,f}\|_{1/2,\Gamma} \;+\; r_{\mathrm{d}}\,\|\delta(\phi)\|_{0,\Omega} \;+\; |\Omega|^{3/4}\,\rho_f\,\mathbf{g}\,\|\varepsilon\|_{0,\infty;\Omega} \;\le\; \frac{\bar{\alpha}_{f,\mathrm{d}}}{4}\,r_{\mathrm{d}}\,, \tag{4.18}$$

and

$$\|\mathbf{u}_{D,s}\|_{1/2,\Gamma} \;+\; \frac{\rho_f}{2\mu_1}\,r_{\mathrm{d}}^2\,\|\varepsilon\|_{0,\infty;\Omega} \;+\; |\Omega|^{3/4}\,\mathbf{g}\,\|\varepsilon\rho_f + \phi\rho_s\|_{0,\infty;\Omega} \;\le\; \frac{\bar{\alpha}_{s,\mathrm{d}}}{4}\,r_{\mathrm{d}}\,. \tag{4.19}$$

Then $S_h(W_h) \subseteq W_h$.

Proof. It proceeds analogously to the proof of Lemma 3.8, but now using the well-posedness and associated a priori estimates of Θ_{fh} and Θ_{sh} provided by Theorems 4.1 and 4.2. We omit further details. □

Next, we establish the Lipschitz-continuity properties of Θ_{fh} and Θ_{sh}.

Lemma 4.4. *There exists a positive constant $L_{f,\mathrm{d}}$, depending on $\bar{\alpha}_{f,\mathrm{d}}$, ρ_f, and μ_f, such that*

$$\begin{aligned}&\|\Theta_{fh}(\mathbf{w}_h) \;-\; \Theta_{fh}(\mathbf{t}_h)\|_{0,4;\Omega}\\ &\quad\le\; L_{f,\mathrm{d}}\,\Big\{\|\delta(\phi)\|_{0,\Omega} \;+\; \|\varepsilon\|_{0,\infty;\Omega}\,\|\Theta_{fh}(\mathbf{t}_h)\|_{0,4;\Omega}\Big\}\,\|\mathbf{w}_h - \mathbf{t}_h\|_{0,4;\Omega}\end{aligned}$$

for all $\mathbf{w}_h := (\mathbf{w}_{fh}, \mathbf{w}_{sh})$, $\mathbf{t}_h := (\mathbf{t}_{fh}, \mathbf{t}_{sh}) \in \mathrm{Q}_h^{\mathbf{u}} \times \mathrm{Q}_h^{\mathbf{u}}$ such that $\|\mathbf{w}_{fh}\|_{0,4;\Omega}\,,\ \|\mathbf{t}_{fh}\|_{0,4;\Omega} \le r_{f,\mathrm{d}}$.

Proof. Given $\mathbf{w}_h := (\mathbf{w}_{fh}, \mathbf{w}_{sh})$ and $\mathbf{t}_h := (\mathbf{t}_{fh}, \mathbf{t}_{sh})$ as indicated, we set $\Theta_{fh}(\mathbf{w}_h) := \widehat{\mathbf{u}}_{fh}$ and $\Theta_{fh}(\mathbf{t}_h) := \widehat{\mathbf{z}}_{fh}$, where $\vec{\boldsymbol{\sigma}}_{fh} := \big(\widehat{\boldsymbol{\sigma}}_{fh}, (\widehat{\mathbf{u}}_{fh}, \widehat{\boldsymbol{\gamma}}_{fh})\big) \in \mathrm{H}_h \times \mathrm{Q}_h$ and $\vec{\boldsymbol{\zeta}}_{fh} := \big(\widehat{\boldsymbol{\zeta}}_{fh}, (\widehat{\mathbf{z}}_{fh}, \widehat{\boldsymbol{\xi}}_{fh})\big) \in \mathrm{H}_h \times \mathrm{Q}_h$ are the unique solutions, guaranteed by Theorem 4.1, of the formulations

$$A_{f,\mathbf{w}_{fh}}\big(\vec{\boldsymbol{\sigma}}_{fh}, \vec{\boldsymbol{\tau}}_{fh}\big) \;=\; \mathbf{F}_f(\boldsymbol{\tau}_{fh}) + \mathbf{G}_f^{\mathbf{w}_h}(\mathbf{v}_{fh}, \boldsymbol{\eta}_{fh})$$

and

$$A_{f,\mathbf{t}_{fh}}\big(\vec{\boldsymbol{\zeta}}_{fh}, \vec{\boldsymbol{\tau}}_{fh}\big) \;=\; \mathbf{F}_f(\boldsymbol{\tau}_{fh}) + \mathbf{G}_f^{\mathbf{t}_h}(\mathbf{v}_{fh}, \boldsymbol{\eta}_{fh}),$$

respectively, both for all $\vec{\boldsymbol{\tau}}_{fh} := \big(\boldsymbol{\tau}_{fh}, (\mathbf{v}_{fh}, \boldsymbol{\eta}_{fh})\big) \in \mathrm{H}_h \times \mathrm{Q}_h$. We refer to (3.64) for the definitions of $A_{f,\mathbf{w}_{fh}}$ and $A_{f,\mathbf{t}_{fh}}$. The rest of the proof follows similarly to the one of Lemma 3.9, using now the discrete inf–sup condition satisfied by $A_{f,\mathbf{w}_{fh}}$ with constant $\bar{\alpha}_{f,\mathrm{d}}/2$, adding and subtracting suitable expressions, and employing the boundedness properties of the linear forms involved. Further details are omitted. □

Lemma 4.5. *There exists a positive constant $L_{s,\mathrm{d}}$, depending on $\bar{\alpha}_{s,\mathrm{d}}$, ρ_f, ρ_s, and μ_1, such that*

$$\begin{aligned}&\|\Theta_{sh}(\mathbf{w}_h) \;-\; \Theta_{sh}(\mathbf{t}_h)\|_{0,4;\Omega}\\ &\quad\le\; L_{s,\mathrm{d}}\,\Big\{\|\varepsilon\|_{0,\infty;\Omega}\,\|\mathbf{t}_{fh} + \mathbf{w}_{fh}\|_{0,4;\Omega} \;+\; \|\phi\|_{0,\infty;\Omega}\,\|\Theta_{sh}(\mathbf{t}_h)\|_{0,4;\Omega}\Big\}\,\|\mathbf{w}_h - \mathbf{t}_h\|_{0,4;\Omega}\end{aligned}$$

for all $\mathbf{w}_h := (\mathbf{w}_{fh}, \mathbf{w}_{sh})$, $\mathbf{t}_h := (\mathbf{t}_{fh}, \mathbf{t}_{sh}) \in \mathrm{Q}_h^{\mathbf{u}} \times \mathrm{Q}_h^{\mathbf{u}}$ such that $\|\mathbf{w}_{sh}\|_{0,4;\Omega}\,,\ \|\mathbf{t}_{sh}\|_{0,4;\Omega} \le r_{s,\mathrm{d}}$.

Proof. It begins analogously to the proof of Lemma 4.4, and then it continues similarly to the one of Lemma 3.10, employing now the discrete inf–sup condition satisfied by $A_{s,\mathbf{w}_{sh}}$ (cf. (3.72)) with constant $\bar{\alpha}_{s,\mathrm{d}}/2$. □

We are now in position to state the Lipschitz-continuity of the discrete fixed-point operator S_h. More precisely, as a direct consequence of Lemmas 4.4 and 4.5, we have the following result, which constitutes the discrete analogue of Lemma 3.11.

Lemma 4.6. *Let W_h be as in Lemma 4.3 with r_d given by (4.17), and let $L_{f,\mathrm{d}}$ and $L_{s,\mathrm{d}}$ be the constants provided by Lemmas 4.4 and 4.5. Then, there holds*

$$\begin{aligned}\|S_h(\mathbf{w}_h) - S_h(\mathbf{t}_h)\|_{0,4;\Omega} \le \Big\{ & L_{f,\mathrm{d}}\Big(\|\delta(\phi)\|_{0,\Omega} + \|\varepsilon\|_{0,\infty;\Omega}\,\|\Theta_{fh}(\mathbf{t}_h)\|_{0,4;\Omega}\Big) \\ & + L_{s,\mathrm{d}}\Big(\|\varepsilon\|_{0,\infty;\Omega}\,\|\mathbf{t}_{fh} + \mathbf{w}_{fh}\|_{0,4;\Omega} + \|\phi\|_{0,\infty;\Omega}\,\|\Theta_{sh}(\mathbf{t}_h)\|_{0,4;\Omega}\Big)\Big\}\,\|\mathbf{w}_h - \mathbf{t}_h\|_{0,4;\Omega}\end{aligned} \tag{4.20}$$

for all $\mathbf{w}_h := (\mathbf{w}_{fh}, \mathbf{w}_{sh})$, $\mathbf{t}_h := (\mathbf{t}_{fh}, \mathbf{t}_{sh}) \in W_h$.

Next, we proceed as in the last part of Section 3.4 to continue bounding the right hand side of (4.20). Indeed, employing the upper bounds of $\|\Theta_{fh}(\mathbf{t}_h)\|_{0,4;\Omega}$ and $\|\Theta_{sh}(\mathbf{t}_h)\|_{0,4;\Omega}$ provided by (4.15) and (4.16), respectively, and bounding $\|\mathbf{t}_{fh} - \mathbf{t}_{sh}\|_{0,4;\Omega}$ and $\|\mathbf{t}_{fh}\|^2_{0,4;\Omega}$ by r_d and r_d^2, respectively, we arrive at

$$\|S_h(\mathbf{w}_h) - S_h(\mathbf{t}_h)\|_{0,4;\Omega} \le \mathcal{L}_\mathrm{d}\big(\mathtt{data}\big)\,\|\mathbf{w}_h - \mathbf{t}_h\|_{0,4;\Omega}\,, \tag{4.21}$$

for all $\mathbf{w}_h := (\mathbf{w}_{fh}, \mathbf{w}_{sh})$, $\mathbf{t}_h := (\mathbf{t}_{fh}, \mathbf{t}_{sh}) \in W_h$, where $\mathcal{L}_\mathrm{d}\big(\mathtt{data}\big)$ is defined exactly as in (3.91), except that the constants from (3.92) are computed now employing $L_{f,\mathrm{d}}$, $L_{s,\mathrm{d}}$, $\bar{\alpha}_{f,\mathrm{d}}$, $\bar{\alpha}_{s,\mathrm{d}}$, and r_d, instead of L_f, L_s, $\bar{\alpha}_f$, $\bar{\alpha}_s$, and r, respectively.

Consequently, the main result concerning the solvability of (4.8) (equivalently (4.2)) is stated as follows thanks to the Brouwer and Banach fixed-point Theorems.

Theorem 4.7. *Let W_h be as in Lemma 4.3 with r_d given by (4.17), and assume that the data are sufficiently small so that they satisfy* (4.18) and (4.19)*. Then, problem (4.2) has at least one solution $\big(\boldsymbol{\sigma}_{fh}, (\mathbf{u}_{fh}, \boldsymbol{\gamma}_{fh})\big) \in \mathrm{H}_h \times \mathrm{Q}_h$ and $\big(\boldsymbol{\sigma}_{sh}, (\mathbf{u}_{sh}, \boldsymbol{\gamma}_{sh})\big) \in \mathrm{H}_h \times \mathrm{Q}_h$ with $\mathbf{u}_h := (\mathbf{u}_{fh}, \mathbf{u}_{sh}) \in W_h$. Moreover, under the further assumption*

$$\mathcal{L}_\mathrm{d}\big(\mathtt{data}\big) < 1\,, \tag{4.22}$$

this solution is unique. In addition, in both cases there hold

$$\|\big(\boldsymbol{\sigma}_{fh}, (\mathbf{u}_{fh}, \boldsymbol{\gamma}_{fh})\big)\|_{\mathrm{H}\times\mathrm{Q}} \le \frac{2}{\bar{\alpha}_{f,\mathrm{d}}}\Big\{\|\mathbf{u}_{D,f}\|_{1/2,\Gamma} + r_\mathrm{d}\,\|\delta(\phi)\|_{0,\Omega} + |\Omega|^{3/4}\,\rho_f\,\mathbf{g}\,\|\varepsilon\|_{0,\infty;\Omega}\Big\}\,, \tag{4.23}$$

and

$$\begin{aligned}&\|\big(\boldsymbol{\sigma}_{sh}, (\mathbf{u}_{sh}, \boldsymbol{\gamma}_{sh})\big)\|_{\mathrm{H}\times\mathrm{Q}} \\ &\quad\le \frac{2}{\bar{\alpha}_{s,\mathrm{d}}}\Big\{\|\mathbf{u}_{D,s}\|_{1/2,\Gamma} + r_\mathrm{d}^2\,\frac{\rho_f}{2\mu_1}\,\|\varepsilon\|_{0,\infty;\Omega} + |\Omega|^{3/4}\,\mathbf{g}\,\|\varepsilon\rho_f + \phi\rho_s\|_{0,\infty;\Omega}\Big\}.\end{aligned} \tag{4.24}$$

Proof. The fact that W_h is certainly a compact and convex subset of $\mathrm{Q}_h^{\mathbf{u}} \times \mathrm{Q}_h^{\mathbf{u}}$, together with Lemma 4.3 and the continuity of S_h (cf. (4.20) or (4.21)), allow to apply the Brouwer Theorem (cf. [35, Theorem 9.9-2]) to conclude the existence of at least a solution to (4.8), and hence to (4.2). Next, the assumption (4.22) and the Banach fixed-point Theorem imply the uniqueness. Finally, (4.23) and (4.24) follow from the a priori estimates (4.15) and (4.16), taking also into account that $\|\mathbf{u}_{fh}\|_{0,4;\Omega}$ and $\|\mathbf{u}_{fh} - \mathbf{u}_{sh}\|_{0,4;\Omega}$ are bounded by r_d. □

4.4. Specific finite element subspaces

In this section we describe a way of choosing finite element subspaces $\mathrm{H}_h^{\boldsymbol{\sigma}}$, $\mathrm{Q}_h^{\mathbf{u}}$, and $\mathrm{Q}_h^{\boldsymbol{\gamma}}$ of $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$, $\mathbf{L}^4(\Omega)$, and $\mathbb{L}^2_{\mathtt{skew}}(\Omega)$, respectively, that satisfy the hypotheses (**H.1**), (**H.2**), and (**H.3**) stated in Section 4.2, and then we provide two specific examples of them. More precisely, given a stable triplet of finite element subspaces for the usual Hilbertian framework of mixed linear elasticity, such that it verifies (**H.1**) and (**H.2**) (which is actually a common feature to most of such triplets), we add a couple of additional feasible assumptions that allow to conclude that (**H.3**) is also satisfied. Before dealing with the respective analysis in Section 4.4.2, we collect in what follows some definitions and results that are needed later on.

4.4.1. Preliminaries

Hereafter, we make use of the notations from Section 4.1. In particular, given an integer $\ell \ge 0$ and $K \in \mathcal{T}_h$, we let $\mathrm{P}_\ell(K)$ be the space of polynomials of degree $\le \ell$ defined on K with vector and tensorial versions denoted by $\mathbf{P}_\ell(K) := [\mathrm{P}_\ell(K)]^n$ and $\mathbb{P}_\ell(K) := [\mathrm{P}_\ell(K)]^{n\times n}$, respectively. In addition, we let $\mathbf{RT}_\ell(K) := \mathbf{P}_\ell(K) \oplus \mathrm{P}_\ell(K)\,\mathbf{x}$ be the local Raviart–Thomas space of order ℓ defined on K, where $\mathbf{x}$ stands for a generic vector in R^n. Furthermore, denoting by b_K the bubble function on K, which is given by the product of its $n+1$ barycentric coordinates, we set the local bubble space of order ℓ as

$$\mathbf{B}_\ell(K) := \operatorname{curl}\big(b_K\,\mathrm{P}_\ell(K)\big) \quad\text{if}\quad n=2\,, \quad\text{and}\quad \mathbf{B}_\ell(K) := \operatorname{curl}\big(b_K\,\mathbf{P}_\ell(K)\big) \quad\text{if}\quad n=3,$$

where $\operatorname{curl}(v) := \big(\frac{\partial v}{\partial x_2}, -\frac{\partial v}{\partial x_1}\big)$ if $n=2$ and $v: K\to \mathrm{R}$, and $\operatorname{curl}(\boldsymbol{v}) := \nabla\times\boldsymbol{v}$ if $n=3$ and $\boldsymbol{v}: K\to\mathrm{R}^3$. Then, having defined the above local spaces, we now introduce corresponding global subspaces of $\mathbf{L}^2(\Omega)$, $\mathbb{L}^2(\Omega)$, and $\mathbb{H}(\mathbf{div};\Omega)$ (cf. (2.1)), as follows

$$\mathbf{P}_\ell(\Omega) := \Big\{\mathbf{v}_h \in \mathbf{L}^2(\Omega):\quad \mathbf{v}_h|_K\in\mathbf{P}_\ell(K)\quad\forall\,K\in\mathcal{T}_h\Big\},$$

$$\mathbb{P}_\ell(\Omega) := \Big\{\boldsymbol{\eta}_h \in \mathbb{L}^2(\Omega):\quad \boldsymbol{\eta}_h|_K\in\mathbb{P}_\ell(K)\quad\forall\,K\in\mathcal{T}_h\Big\},$$

$$\mathbb{RT}_\ell(\Omega) := \Big\{\boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div};\Omega):\quad \boldsymbol{\tau}_{h,i}|_K\in\mathbf{RT}_\ell(K)\quad\forall\,i\in\{1,\dots,n\}\,,\quad\forall\,K\in\mathcal{T}_h\Big\},$$

and

$$\mathbb{B}_\ell(\Omega) := \Big\{\boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div};\Omega):\quad \boldsymbol{\tau}_{h,i}|_K\in\mathbf{B}_\ell(K)\quad\forall\,i\in\{1,\dots,n\}\,,\quad\forall\,K\in\mathcal{T}_h\Big\},$$

where $\boldsymbol{\tau}_{h,i}$ denotes the ith-row of $\boldsymbol{\tau}_h$. We remark here that $\mathbf{P}_\ell(\Omega)$ and $\mathbb{P}_\ell(\Omega)$ are also subspaces of $\mathbf{L}^4(\Omega)$ and $\mathbb{L}^4(\Omega)$, respectively. In addition, the fact that $\mathbf{L}^2(\Omega)$ is clearly contained in $\mathbf{L}^{4/3}(\Omega)$ with bounded injection, implies that $\mathbb{H}(\mathbf{div};\Omega)$ is in turn continuously embedded in $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$ and there holds

$$\|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega} \le c(\Omega)\,\|\boldsymbol{\tau}\|_{\mathbf{div};\Omega} \qquad \forall\,\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div};\Omega)\,, \tag{4.25}$$

where $c(\Omega)$ is a positive constant depending only on $|\Omega|$. It follows then that $\mathbb{RT}_\ell(\Omega)$ and $\mathbb{B}_\ell(\Omega)$ are subspaces of $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$ as well. Moreover, denoting $\mathbb{RT}_{\ell,0}(\Omega) := \mathbb{RT}_\ell(\Omega)\cap\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$ (cf. (3.2)), we recall from [17, Lemma 5.5] that, for each integer $\ell\ge 0$, there exists a positive constant β_0, independent of h, such that

$$\sup_{\substack{\boldsymbol{\tau}_h\in\mathbb{RT}_{\ell,0}(\Omega)\\ \boldsymbol{\tau}_h\neq\mathbf{0}}} \frac{\displaystyle\int_\Omega \mathbf{v}_h\cdot\mathbf{div}(\boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega}} \ge \beta_0\,\|\mathbf{v}_h\|_{0,4;\Omega} \qquad \forall\,\mathbf{v}_h\in\mathbf{P}_\ell(\Omega)\,. \tag{4.26}$$

4.4.2. Stable triplets for mixed linear elasticity and (H.3)

We now let H_h^σ, $\mathrm{Q}_h^{\mathbf{u}}$, and Q_h^γ be finite element subspaces of $\mathbb{H}(\mathbf{div};\Omega)$, $\mathbf{L}^2(\Omega)$, and $\mathbb{L}^2_{\mathtt{skew}}(\Omega)$, respectively, which satisfy (**H.1**) and (**H.2**), and conform a stable triplet for mixed linear elasticity. In particular, denoting $\mathrm{H}_h := \mathrm{H}_h^\sigma\cap\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, the above means that there exists a positive constant β_1, independent of h, such that

$$\sup_{\substack{\boldsymbol{\tau}_h\in\mathrm{H}_h\\ \boldsymbol{\tau}_h\neq\mathbf{0}}} \frac{\mathbf{b}\big(\boldsymbol{\tau}_h,(\mathbf{v}_h,\boldsymbol{\eta}_h)\big)}{\|\boldsymbol{\tau}_h\|_{\mathbf{div};\Omega}} \ge \beta_1\Big\{\|\mathbf{v}_h\|_{0,\Omega}+\|\boldsymbol{\eta}_h\|_{0,\Omega}\Big\} \qquad \forall\,(\mathbf{v}_h,\boldsymbol{\eta}_h)\in\mathrm{Q}_h^{\mathbf{u}}\times\mathrm{Q}_h^\gamma\,. \tag{4.27}$$

Then, employing (4.25) and (4.27), we deduce that

$$\sup_{\substack{\boldsymbol{\tau}_h\in\mathrm{H}_h\\ \boldsymbol{\tau}_h\neq\mathbf{0}}} \frac{\mathbf{b}\big(\boldsymbol{\tau}_h,(\mathbf{v}_h,\boldsymbol{\eta}_h)\big)}{\|\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega}} \ge \frac{1}{c(\Omega)}\sup_{\substack{\boldsymbol{\tau}_h\in\mathrm{H}_h\\ \boldsymbol{\tau}_h\neq\mathbf{0}}} \frac{\mathbf{b}\big(\boldsymbol{\tau}_h,(\mathbf{v}_h,\boldsymbol{\eta}_h)\big)}{\|\boldsymbol{\tau}_h\|_{\mathbf{div};\Omega}} \ge \frac{\beta_1}{c(\Omega)}\Big\{\|\mathbf{v}_h\|_{0,\Omega}+\|\boldsymbol{\eta}_h\|_{0,\Omega}\Big\},$$

and hence

$$\sup_{\substack{\boldsymbol{\tau}_h\in\mathrm{H}_h\\ \boldsymbol{\tau}_h\neq\mathbf{0}}} \frac{\mathbf{b}\big(\boldsymbol{\tau}_h,(\mathbf{v}_h,\boldsymbol{\eta}_h)\big)}{\|\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega}} \ge \frac{\beta_1}{c(\Omega)}\|\boldsymbol{\eta}_h\|_{0,\Omega} \quad \forall\,(\mathbf{v}_h,\boldsymbol{\eta}_h)\in\mathrm{Q}_h^{\mathbf{u}}\times\mathrm{Q}_h^\gamma\,. \tag{4.28}$$

In turn, assuming that there exists an integer $\ell\ge 0$ such that $\mathbb{RT}_\ell(\Omega)\subseteq\mathrm{H}_h^\sigma$, which certainly yields $\mathbb{RT}_{\ell,0}(\Omega)\subseteq\mathrm{H}_h$, we find that

$$\sup_{\substack{\boldsymbol{\tau}_h\in\mathrm{H}_h\\ \boldsymbol{\tau}_h\neq\mathbf{0}}} \frac{\mathbf{b}\big(\boldsymbol{\tau}_h,(\mathbf{v}_h,\boldsymbol{\eta}_h)\big)}{\|\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega}} \ge \sup_{\substack{\boldsymbol{\tau}_h\in\mathbb{RT}_{\ell,0}(\Omega)\\ \boldsymbol{\tau}_h\neq\mathbf{0}}} \frac{\mathbf{b}\big(\boldsymbol{\tau}_h,(\mathbf{v}_h,\boldsymbol{\eta}_h)\big)}{\|\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega}} \ge \sup_{\substack{\boldsymbol{\tau}_h\in\mathbb{RT}_{\ell,0}(\Omega)\\ \boldsymbol{\tau}_h\neq\mathbf{0}}} \frac{\displaystyle\int_\Omega \mathbf{v}_h\cdot\mathbf{div}(\boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega}} - \|\boldsymbol{\eta}_h\|_{0,\Omega}\,,$$

from which, assuming additionally that $\mathrm{Q}_h^{\mathbf{u}}\subseteq\mathbf{P}_\ell(\Omega)$, and using (4.26), we conclude that

$$\sup_{\substack{\boldsymbol{\tau}_h\in\mathrm{H}_h\\ \boldsymbol{\tau}_h\neq\mathbf{0}}} \frac{\mathbf{b}\big(\boldsymbol{\tau}_h,(\mathbf{v}_h,\boldsymbol{\eta}_h)\big)}{\|\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega}} \ge \beta_0\,\|\mathbf{v}_h\|_{0,4;\Omega} - \|\boldsymbol{\eta}_h\|_{0,\Omega} \qquad \forall\,(\mathbf{v}_h,\boldsymbol{\eta}_h)\in\mathrm{Q}_h^{\mathbf{u}}\times\mathrm{Q}_h^\gamma\,. \tag{4.29}$$

In this way, a suitable linear combination of (4.28) and (4.29) imply that H_h and $\mathrm{Q}_h := \mathrm{Q}_h^{\mathbf{u}} \times \mathrm{Q}_h^{\gamma}$ satisfy (**H.3**) (cf. (4.9)) with a positive constant β_{d} depending only on β_0, β_1, and $c(\Omega)$.

We have thus proved the following result.

Lemma 4.8. *Let H_h^{σ}, $\mathrm{Q}_h^{\mathbf{u}}$, and Q_h^{γ} be finite element subspaces of $\mathbb{H}(\mathbf{div};\Omega)$, $\mathbf{L}^2(\Omega)$, and $\mathbb{L}^2_{\mathtt{skew}}(\Omega)$, respectively, such that they conform a stable triplet for linear elasticity. In addition, assume that there exists an integer $\ell \geq 0$ such that $\mathbb{RT}_\ell(\Omega) \subseteq \mathrm{H}_h^{\sigma}$ and $\mathrm{Q}_h^{\mathbf{u}} \subseteq \mathbf{P}_\ell(\Omega)$. Then, $\mathrm{H}_h := \mathrm{H}_h^{\sigma} \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$ and $\mathrm{Q}_h := \mathrm{Q}_h^{\mathbf{u}} \times \mathrm{Q}_h^{\gamma}$ verify (**H.3**) with a positive constant β_{d} independent of h.*

4.4.3. Two specific examples

In order to define specific finite element subspaces yielding the well-posedness of the Galerkin scheme introduced and analyzed in Section 4, we now identify stable triplets for linear elasticity that satisfy (**H.1**), (**H.2**), and the hypotheses of Lemma 4.8.

Our first example is PEERS$_\ell$, the plane elasticity element with reduced symmetry of order $\ell \geq 0$, whose stability was originally proved in [36] for $\ell = 0$ and $n = 2$, and later on established for $\ell \geq 0$ and $n \in \{2,3\}$ (see. e.g. [37]). Letting $\mathbb{C}(\bar{\Omega}) := [C(\bar{\Omega})]^{n\times n}$, the corresponding subspaces are given as follows:

$$\mathrm{H}_h^{\sigma} := \mathbb{RT}_\ell(\Omega) \oplus \mathbb{B}_\ell(\Omega), \quad \mathrm{Q}_h^{\mathbf{u}} := \mathrm{P}_\ell(\Omega), \quad \text{and} \quad \mathrm{Q}_h^{\gamma} := \mathbb{C}(\bar{\Omega}) \cap \mathbb{L}^2_{\mathtt{skew}}(\Omega) \cap \mathbb{P}_{\ell+1}(\Omega), \tag{4.30}$$

which are easily seen to satisfy the aforementioned requirements. In particular, (**H.2**) follows from the divergence-free property of $\mathbb{B}_\ell(\Omega)$ and the inclusion $\mathbf{div}\big(\mathbb{RT}_\ell(\Omega)\big) \subseteq \mathbf{P}_\ell(\Omega)$, whereas the hypotheses of Lemma 4.8 are trivially met.

Our second example is AFW$_\ell$, the Arnold–Falk–Winther element of order $\ell \geq 0$, which, introduced and proved to be stable in [38], is defined as:

$$\mathrm{H}_h^{\sigma} := \mathbb{P}_{\ell+1}(\Omega) \cap \mathbb{H}(\mathbf{div};\Omega), \quad \mathrm{Q}_h^{\mathbf{u}} := \mathrm{P}_\ell(\Omega), \quad \text{and} \quad \mathrm{Q}_h^{\gamma} := \mathbb{L}^2_{\mathtt{skew}}(\Omega) \cap \mathbb{P}_\ell(\Omega). \tag{4.31}$$

Again, (**H.1**) and (**H.2**) are straightforward, whereas the fact that $\mathbf{RT}_\ell(K) \subseteq \mathbf{P}_{\ell+1}(K)$ for each $K \in \mathcal{T}_h$, completes the hypotheses of Lemma 4.8.

The approximation properties of the finite element subspaces defining PEERS$_\ell$ (cf. (4.30)) and AFW$_\ell$ (cf. (4.31)), which basically follow from the analogue properties of the Raviart–Thomas and AFW interpolation operators, and of the orthogonal projectors $\boldsymbol{\mathcal{P}}_h^\ell : \mathbf{L}^p(\Omega) \to \mathbf{P}_\ell(\Omega)$ and $\mathbb{P}_h^\ell : \mathbb{L}^p(\Omega) \to \mathbb{P}_\ell(\Omega)$ (cf. [29, Proposition 1.135]), and which make use of the commuting diagram properties and of the interpolation estimates of Sobolev spaces as well, are given as follows (see also [31,39], [17, eqs. (5.37) and (5.40)], [30]):

($\mathbf{AP}_h^{\sigma}$): there exists $C > 0$, independent of h, such that for each $r \in [0, \ell+1]$, and for each $\boldsymbol{\tau} \in \mathbb{H}^r(\Omega) \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$ with $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{W}^{r,4/3}(\Omega)$, there holds

$$\mathrm{dist}(\boldsymbol{\tau}, \mathrm{H}_h) := \inf_{\boldsymbol{\tau}_h \in \mathrm{H}_h} \|\boldsymbol{\tau} - \boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega} \leq C\,h^r \left\{ \|\boldsymbol{\tau}\|_{r,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{r,4/3;\Omega} \right\}. \tag{4.32}$$

($\mathbf{AP}_h^{\mathbf{u}}$): there exists $C > 0$, independent of h, such that for each $r \in [0, \ell+1]$, and for each $\mathbf{v} \in \mathbf{W}^{r,4}(\Omega)$ there holds

$$\mathrm{dist}(\mathbf{v}, \mathrm{Q}_h^{\mathbf{u}}) := \inf_{\mathbf{v}_h \in \mathrm{Q}_h^{\mathbf{u}}} \|\mathbf{v} - \mathbf{v}_h\|_{0,4;\Omega} \leq C\,h^r\,\|\mathbf{v}\|_{r,4;\Omega}\,. \tag{4.33}$$

($\mathbf{AP}_h^{\gamma}$): there exists $C > 0$, independent of h, such that for each $r \in [0, \ell+1]$, and for each $\boldsymbol{\eta} \in \mathbb{H}^r(\Omega) \cap \mathbb{L}^2_{\mathtt{skew}}(\Omega)$ there holds

$$\mathrm{dist}(\boldsymbol{\eta}, \mathrm{Q}_h^{\gamma}) := \inf_{\boldsymbol{\eta}_h \in \mathrm{Q}_h^{\gamma}} \|\boldsymbol{\eta} - \boldsymbol{\eta}_h\|_{0,\Omega} \leq C\,h^r\,\|\boldsymbol{\eta}\|_{r,\Omega}\,. \tag{4.34}$$

The associated rates of convergence of our Galerkin scheme (4.2), implemented in each case with $\mathrm{H}_h := \mathrm{H}_h^{\sigma} \cap \mathbb{H}(\mathbf{div}_{4/3};\Omega)$ and $\mathrm{Q}_h := \mathrm{Q}_h^{\mathbf{u}} \times \mathrm{Q}_h^{\gamma}$, are provided below in Section 5 after performing the respective a priori error analysis.

5. A priori error analysis

In this section we derive the a priori error analysis for the Galerkin scheme (4.2) considering arbitrary finite element subspaces satisfying hypotheses (**H.1**), (**H.2**) and (**H.3**) (cf. Section 4.2). In addition, we define postprocessed approximations of further variables of interest and establish its corresponding rates of convergence, which coincide with those of the original unknowns. This fact constitutes a clear advantage of the present approach with respect to the usual primal method since, in order for the latter to be able to provide approximations of those additional variables, numerical differentiation procedures would need to be applied with the consequent loss of accuracy that they imply.

5.1. The main estimates

We begin by introducing the following Strang-type estimate for saddle point problems. Its proof follows after slight modifications of that of [30, Theorem 2.6].

Lemma 5.1. *Let* H *and* Q *be reflexive Banach spaces, and let* $a : \mathrm{H}\times\mathrm{H} \to \mathrm{R}$ *and* $b : \mathrm{H}\times\mathrm{Q}\to\mathrm{R}$ *be bounded bilinear forms with bounding constants* $\|a\|$ *and* $\|b\|$*, respectively. Furthermore, let* $\{\mathrm{H}_h\}_{h>0}$ *and* $\{\mathrm{Q}_h\}_{h>0}$ *be sequences of finite dimensional subspaces of* H *and* Q*, respectively, and assume that* a *and* b *satisfy the hypotheses of Theorem 3.2 on* $\mathrm{H}\times\mathrm{Q}$ *and* $\mathrm{H}_h\times\mathrm{Q}_h$*. In turn, given* $F\in\mathrm{H}'$*,* $G\in\mathrm{Q}'$*, and the sequences of functionals* $\{F_h\}_{h>0}$ *with* $F_h\in\mathrm{H}_h'$ *for each* $h>0$ *and* $\{G_h\}_{h>0}$ *with* $G_h\in\mathrm{Q}_h'$ *for each* $h>0$*, we let* $(\sigma,u)\in\mathrm{H}\times\mathrm{Q}$ *and* $(\sigma_h,u_h)\in\mathrm{H}_h\times\mathrm{Q}_h$ *be the unique solutions, respectively, to the problems*

$$\begin{aligned} a(\sigma,\tau) + b(\tau,u) &= F(\tau) \qquad \forall\,\tau\in\mathrm{H},\\ b(\sigma,v) &= G(v) \qquad \forall\,v\in\mathrm{Q}, \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} a(\sigma_h,\tau_h) + b(\tau_h,u_h) &= F_h(\tau_h) \qquad \forall\,\tau_h\in\mathrm{H}_h,\\ b(\sigma_h,v_h) &= G_h(v_h) \qquad \forall\,v_h\in\mathrm{Q}_h. \end{aligned} \tag{5.2}$$

Then, there holds

$$\begin{aligned} \|\sigma-\sigma_h\|_{\mathrm{H}} + \|u-u_h\|_{\mathrm{Q}} \le\ & C_{S,1}\,\mathrm{dist}(\sigma,\mathrm{H}_h) + C_{S,2}\,\mathrm{dist}(u,\mathrm{Q}_h)\\ & + C_{S,3}\,\|F-F_h\|_{\mathrm{H}_h'} + C_{S,4}\,\|G-G_h\|_{\mathrm{Q}_h'}, \end{aligned} \tag{5.3}$$

with

$$\begin{aligned} C_{S,1} &:= \left(1+\frac{\|a\|}{\tilde\alpha}\right)\left(1+\frac{\|a\|}{\tilde\beta}\right)\left(1+\frac{\|b\|}{\tilde\beta}\right),\\ C_{S,2} &:= 1+\frac{\|b\|}{\tilde\alpha}+\frac{\|b\|}{\tilde\beta}+\frac{\|a\|\,\|b\|}{\tilde\alpha\tilde\beta},\\ C_{S,3} &:= \frac{1}{\tilde\alpha}+\frac{1}{\tilde\beta}\left(1+\frac{\|a\|}{\tilde\alpha}\right),\\ C_{S,4} &:= \frac{1}{\tilde\beta}\left(1+\frac{\|a\|}{\tilde\alpha}\right)\left(1+\frac{\|a\|}{\tilde\beta}\right), \end{aligned} \tag{5.4}$$

where $\tilde\alpha$ *and* $\tilde\beta$ *are the positive constants satisfying (3.43) and (3.45), respectively, on* $\mathrm{H}_h\times\mathrm{Q}_h$*.*

Now, for the subsequent analysis we let $\big(\boldsymbol{\sigma}_f,(\mathbf{u}_f,\boldsymbol{\gamma}_f)\big)\in\mathrm{H}\times\mathrm{Q}$, $\big(\boldsymbol{\sigma}_s,(\mathbf{u}_s,\boldsymbol{\gamma}_s)\big)\in\mathrm{H}\times\mathrm{Q}$ and $\big(\boldsymbol{\sigma}_{fh},(\mathbf{u}_{fh},\boldsymbol{\gamma}_{fh})\big)\in\mathrm{H}_h\times\mathrm{Q}_h$, $\big(\boldsymbol{\sigma}_{sh},(\mathbf{u}_{sh},\boldsymbol{\gamma}_{sh})\big)\in\mathrm{H}_h\times\mathrm{Q}_h$ be the solutions of (3.18) and (4.2), respectively, and for the sake of convenience, we rewrite both problems as follows:

$$\begin{aligned} \mathbf{a}_f(\boldsymbol{\sigma}_f,\boldsymbol{\tau}_f)+\mathbf{b}\big(\boldsymbol{\tau}_f,(\mathbf{u}_f,\boldsymbol{\gamma}_f)\big) &= \widehat{\mathbf{F}}_f^{\mathbf{u}}(\boldsymbol{\tau}_f), && \forall\,\boldsymbol{\tau}_f\in\mathrm{H},\\ \mathbf{b}\big(\boldsymbol{\sigma}_f,(\mathbf{v}_f,\boldsymbol{\eta}_f)\big) &= \mathbf{G}_f^{\mathbf{u}}(\mathbf{v}_f,\boldsymbol{\eta}_f), && \forall\,(\mathbf{v}_f,\boldsymbol{\eta}_f)\in\mathrm{Q},\\ \mathbf{a}_s(\boldsymbol{\sigma}_s,\boldsymbol{\tau}_s)+\mathbf{b}\big(\boldsymbol{\tau}_s,(\mathbf{u}_s,\boldsymbol{\gamma}_s)\big) &= \widehat{\mathbf{F}}_s^{\mathbf{u}}(\boldsymbol{\tau}_s), && \forall\,\boldsymbol{\tau}_s\in\mathrm{H},\\ \mathbf{b}\big(\boldsymbol{\sigma}_s,(\mathbf{v}_s,\boldsymbol{\eta}_s)\big) &= \mathbf{G}_s(\mathbf{v}_s,\boldsymbol{\eta}_s), && \forall\,(\mathbf{v}_s,\boldsymbol{\eta}_s)\in\mathrm{Q}, \end{aligned} \tag{5.5}$$

and

$$\begin{aligned} \mathbf{a}_f(\boldsymbol{\sigma}_{fh},\boldsymbol{\tau}_{fh})+\mathbf{b}\big(\boldsymbol{\tau}_{fh},(\mathbf{u}_{fh},\boldsymbol{\gamma}_{fh})\big) &= \widehat{\mathbf{F}}_f^{\mathbf{u}_h}(\boldsymbol{\tau}_{fh}), && \forall\,\boldsymbol{\tau}_{fh}\in\mathrm{H}_h\\ \mathbf{b}\big(\boldsymbol{\sigma}_{fh},(\mathbf{v}_{fh},\boldsymbol{\eta}_{fh})\big) &= \mathbf{G}_f^{\mathbf{u}_h}(\mathbf{v}_{fh},\boldsymbol{\eta}_{fh}), && \forall\,(\mathbf{v}_{fh},\boldsymbol{\eta}_{fh})\in\mathrm{Q}_h\\ \mathbf{a}_s(\boldsymbol{\sigma}_{sh},\boldsymbol{\tau}_{sh})+\mathbf{b}\big(\boldsymbol{\tau}_{sh},(\mathbf{u}_{sh},\boldsymbol{\gamma}_{sh})\big) &= \widehat{\mathbf{F}}_s^{\mathbf{u}_h}(\boldsymbol{\tau}_{sh}), && \forall\,\boldsymbol{\tau}_{sh}\in\mathrm{H}_h\\ \mathbf{b}\big(\boldsymbol{\sigma}_{sh},(\mathbf{v}_{sh},\boldsymbol{\eta}_{sh})\big) &= \mathbf{G}_s(\mathbf{v}_{sh},\boldsymbol{\eta}_{sh}), && \forall\,(\mathbf{v}_{sh},\boldsymbol{\eta}_{sh})\in\mathrm{Q}_h, \end{aligned} \tag{5.6}$$

with

$$\begin{aligned} \widehat{\mathbf{F}}_f^{\mathbf{u}}(\boldsymbol{\tau}_f) &:= \mathbf{F}_f(\boldsymbol{\tau}_f) - \mathbf{c}_f(\mathbf{u}_f,\boldsymbol{\tau}_f) - \mathbf{d}_f(\mathbf{u}_f;\mathbf{u}_f,\boldsymbol{\tau}_f), && \forall\,\boldsymbol{\tau}_f\in\mathrm{H},\\ \widehat{\mathbf{F}}_s^{\mathbf{u}}(\boldsymbol{\tau}_s) &:= \mathbf{F}_s^{\mathbf{u}}(\boldsymbol{\tau}_s) - \mathbf{c}_s(\mathbf{u}_s,\boldsymbol{\tau}_s) - \mathbf{d}_s(\mathbf{u}_s;\mathbf{u}_s,\boldsymbol{\tau}_s), && \forall\,\boldsymbol{\tau}_s\in\mathrm{H},\\ \widehat{\mathbf{F}}_f^{\mathbf{u}_h}(\boldsymbol{\tau}_{fh}) &:= \mathbf{F}_f(\boldsymbol{\tau}_{fh}) - \mathbf{c}_f(\mathbf{u}_{fh},\boldsymbol{\tau}_{fh}) - \mathbf{d}_f(\mathbf{u}_{fh};\mathbf{u}_{fh},\boldsymbol{\tau}_{fh}), && \forall\,\boldsymbol{\tau}_{fh}\in\mathrm{H}_h,\\ \widehat{\mathbf{F}}_s^{\mathbf{u}_h}(\boldsymbol{\tau}_{sh}) &:= \mathbf{F}_s^{\mathbf{u}_h}(\boldsymbol{\tau}_{sh}) - \mathbf{c}_s(\mathbf{u}_{sh},\boldsymbol{\tau}_{sh}) - \mathbf{d}_s(\mathbf{u}_{sh};\mathbf{u}_{sh},\boldsymbol{\tau}_{sh}), && \forall\,\boldsymbol{\tau}_{sh}\in\mathrm{H}_h. \end{aligned} \tag{5.7}$$

Then, since (5.5) and (5.6) have the same structure of the abstract problems (5.1) and (5.2), respectively, in what follows we proceed similarly to [17] and apply Lemma 5.1 to derive the a priori error estimate for the Galerkin scheme (4.2). Let us first establish the following upper bounds for the differences between the functionals introduced above.

Lemma 5.2. *There holds,*

$$\left\|\widehat{\mathbf{F}}_f^{\mathbf{u}} - \widehat{\mathbf{F}}_f^{\mathbf{u}_h}\right\|_{\mathrm{H}_h'} \le \left(\frac{1}{\sqrt{n}}\left\|\frac{\nabla\varepsilon}{\varepsilon}\right\|_{0,4;\Omega} + \frac{\rho_f}{2\mu_f}\|\varepsilon\|_{0,\infty;\Omega}\left(\|\mathbf{u}_f\|_{0,4;\Omega} + \|\mathbf{u}_{fh}\|_{0,4;\Omega}\right)\right)\|\mathbf{u}_f - \mathbf{u}_{fh}\|_{0,4;\Omega}, \tag{5.8}$$

$$\begin{aligned}\left\|\widehat{\mathbf{F}}_s^{\mathbf{u}} - \widehat{\mathbf{F}}_s^{\mathbf{u}_h}\right\|_{\mathrm{H}_h'} \le\ & \left(\frac{1}{\sqrt{n}}\left\|\frac{\nabla\phi}{\phi}\right\|_{0,4;\Omega} + \frac{\rho_s}{2\mu_1}\|\phi\|_{0,\infty;\Omega}(\|\mathbf{u}_s\|_{0,4;\Omega} + \|\mathbf{u}_{sh}\|_{0,4;\Omega})\right)\|\mathbf{u}_s - \mathbf{u}_{sh}\|_{0,4;\Omega} \\ & + \frac{\rho_f}{2\mu_1}\|\varepsilon\|_{0,\infty;\Omega}(\|\mathbf{u}_f\|_{0,4;\Omega} + \|\mathbf{u}_{fh}\|_{0,4;\Omega})\|\mathbf{u}_f - \mathbf{u}_{fh}\|_{0,4;\Omega}.\end{aligned} \tag{5.9}$$

Proof. Recalling that $\mathbf{c}_f$ and $\mathbf{d}_f$ (cf. (3.4)) are bilinear and trilinear forms, respectively, and summing and subtracting $\mathbf{u}_{fh}$ in the second component of $\mathbf{d}_f$, we deduce from (5.7) that for each $\boldsymbol{\tau}_{fh} \in \mathrm{H}_h$ there holds

$$\left(\widehat{\mathbf{F}}_f^{\mathbf{u}} - \widehat{\mathbf{F}}_f^{\mathbf{u}_h}\right)(\boldsymbol{\tau}_{fh}) = -\mathbf{c}_f(\mathbf{u}_f - \mathbf{u}_{fh}, \boldsymbol{\tau}_{fh}) - \left(\mathbf{d}_f(\mathbf{u}_f; \mathbf{u}_f - \mathbf{u}_{fh}, \boldsymbol{\tau}_{fh}) - \mathbf{d}_f(\mathbf{u}_f - \mathbf{u}_{fh}; \mathbf{u}_{fh}, \boldsymbol{\tau}_{fh})\right),$$

which together with (3.21) and (3.22), yields

$$\left|\left(\widehat{\mathbf{F}}_f^{\mathbf{u}} - \widehat{\mathbf{F}}_f^{\mathbf{u}_h}\right)(\boldsymbol{\tau}_{fh})\right| \le \left(\|\mathbf{c}_f\| + \|\mathbf{d}_f\|(\|\mathbf{u}_f\|_{0,4;\Omega} + \|\mathbf{u}_{fh}\|_{0,4;\Omega})\right)\|\mathbf{u}_f - \mathbf{u}_{fh}\|_{0,4;\Omega}\|\boldsymbol{\tau}_{fh}\|_{0,\Omega}.$$

Then, using the definitions of $\|\mathbf{c}_f\|$ and $\|\mathbf{d}_f\|$ (cf. (3.25)), the foregoing inequality implies (5.8). Analogously, according to the definitions of $\widehat{\mathbf{F}}_s^{\mathbf{u}}$ and $\widehat{\mathbf{F}}_s^{\mathbf{u}_h}$ (cf. (5.7)), it is easy to see that for each $\boldsymbol{\tau}_{sh} \in \mathrm{H}_h$ there holds

$$\begin{aligned}\left(\widehat{\mathbf{F}}_s^{\mathbf{u}} - \widehat{\mathbf{F}}_s^{\mathbf{u}_h}\right)(\boldsymbol{\tau}_{sh}) &= \mathbf{F}_s^{\mathbf{u}}(\boldsymbol{\tau}_{sh}) - \mathbf{F}_s^{\mathbf{u}_h}(\boldsymbol{\tau}_{sh}) - \mathbf{c}_s(\mathbf{u}_s - \mathbf{u}_{sh}, \boldsymbol{\tau}_{sh}) \\ &\quad - \left(\mathbf{d}_s(\mathbf{u}_s; \mathbf{u}_s - \mathbf{u}_{sh}, \boldsymbol{\tau}_{sh}) - \mathbf{d}_s(\mathbf{u}_s - \mathbf{u}_{sh}; \mathbf{u}_{sh}, \boldsymbol{\tau}_{sh})\right),\end{aligned}$$

and then, from (3.28), (3.29), (3.87), and the definitions of $\|\mathbf{c}_s\|$ and $\|\mathbf{d}_s\|$ (cf. (3.32)), we conclude (5.9), which ends the proof. □

Now we proceed to establish preliminary estimates for $(\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{fh}, (\mathbf{u}_f - \mathbf{u}_{fh}, \boldsymbol{\gamma}_f - \boldsymbol{\gamma}_{fh}))$ and $(\boldsymbol{\sigma}_s - \boldsymbol{\sigma}_{sh}, (\mathbf{u}_s - \mathbf{u}_{sh}, \boldsymbol{\gamma}_s - \boldsymbol{\gamma}_{sh}))$.

Lemma 5.3. *There exist positive constants $C_{f,i}$, $i \in \{1, 2, 3, 4\}$, depending on μ_f and other constants independent of the discretization and physical parameters, such that*

$$\begin{aligned}&\|\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{fh}\|_{\mathrm{H}} + \|\mathbf{u}_f - \mathbf{u}_{fh}\|_{0,4;\Omega} + \|\boldsymbol{\gamma}_f - \boldsymbol{\gamma}_{fh}\|_{0,\Omega} \le C_{f,1}\,\mathrm{dist}(\boldsymbol{\sigma}_f, \mathrm{H}_h) \\ &+ C_{f,2}\,\mathrm{dist}((\mathbf{u}_f, \boldsymbol{\gamma}_f), \mathrm{Q}_h) + \mathcal{J}_f(\mathtt{data})\left\|\mathbf{u}_f - \mathbf{u}_{fh}\right\|_{0,4;\Omega} + \mathcal{K}_f(\mathtt{data})\|\mathbf{u}_s - \mathbf{u}_{sh}\|_{0,4;\Omega},\end{aligned} \tag{5.10}$$

with $\mathcal{J}_f(\mathtt{data})$ and $\mathcal{K}_f(\mathtt{data})$ given by

$$\begin{aligned}\mathcal{J}_f(\mathtt{data}) :=\ & C_{f,3}\frac{1}{\sqrt{n}}\left\|\frac{\nabla\varepsilon}{\varepsilon}\right\|_{0,4;\Omega} + C_{f,3}\frac{\rho_f}{\mu_f}\min\{\bar{\alpha}_f, \bar{\alpha}_{f,\mathrm{d}}\}\|\varepsilon\|_{0,\infty;\Omega}\Big(2\|\mathbf{u}_{D,f}\|_{1/2,\Gamma} \\ & + (r + r_{\mathrm{d}})\|\delta(\phi)\|_{0,\Omega} + 2|\Omega|^{3/4}\rho_f\mathbf{g}\|\varepsilon\|_{0,\infty;\Omega}\Big) + C_{f,4}\|\delta(\phi)\|_{0,\Omega}, \\ \mathcal{K}_f(\mathtt{data}) :=\ & C_{f,4}\|\delta(\phi)\|_{0,\Omega}\,.\end{aligned} \tag{5.11}$$

Proof. By applying Lemma 5.1 to the first and second equations of (5.5) and (5.6), we find that

$$\begin{aligned}&\|\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{fh}\|_{\mathrm{H}} + \|\mathbf{u}_f - \mathbf{u}_{fh}\|_{0,4;\Omega} + \|\boldsymbol{\gamma}_f - \boldsymbol{\gamma}_{fh}\|_{0,\Omega} \le C_{f,1}\,\mathrm{dist}(\boldsymbol{\sigma}_f, \mathrm{H}_h) \\ &+ C_{f,2}\,\mathrm{dist}((\mathbf{u}_f, \boldsymbol{\gamma}_f), \mathrm{Q}_h) + C_{f,3}\left\|\widehat{\mathbf{F}}_f^{\mathbf{u}} - \widehat{\mathbf{F}}_f^{\mathbf{u}_h}\right\|_{\mathrm{H}_h'} + C_{f,4}\|\mathbf{G}_f^{\mathbf{u}} - \mathbf{G}_f^{\mathbf{u}_h}\|_{\mathrm{Q}_h'},\end{aligned} \tag{5.12}$$

where the constants $C_{f,i}$, $i \in \{1, \ldots, 4\}$, are given by (5.4) with $\|a\| = \|\mathbf{a}_f\| = \frac{1}{2\mu_f}$, $\|b\| = \|\mathbf{b}\| = 1$, $\tilde{\alpha} = \alpha_f = \frac{c_1}{2\mu_f}$ and $\tilde{\beta} = \beta_{\mathrm{d}} > 0$ (cf. (3.25), (3.50), (4.9)). In turn, from (3.83) we observe that

$$\left|\left(\mathbf{G}_f^{\mathbf{u}} - \mathbf{G}_f^{\mathbf{u}_h}\right)(\mathbf{v}_{fh}, \boldsymbol{\eta}_{fh})\right| \le \|\delta(\phi)\|_{0,\Omega}\left\{\|\mathbf{u}_f - \mathbf{u}_{fh}\|_{0,4;\Omega} + \|\mathbf{u}_s - \mathbf{u}_{sh}\|_{0,4;\Omega}\right\}\|\mathbf{v}_{fh}\|_{0,4;\Omega},$$

for all $(\mathbf{v}_{fh}, \boldsymbol{\eta}_{fh}) \in \mathrm{Q}_h$, which implies

$$\left\|\mathbf{G}_f^{\mathbf{u}} - \mathbf{G}_f^{\mathbf{u}_h}\right\|_{\mathrm{Q}_h'} \le \|\delta(\phi)\|_{0,\Omega}\left\{\|\mathbf{u}_f - \mathbf{u}_{fh}\|_{0,4;\Omega} + \|\mathbf{u}_s - \mathbf{u}_{sh}\|_{0,4;\Omega}\right\}. \tag{5.13}$$

In this way, from estimates (5.8) (5.12) and (5.13), and the fact that (see (3.94) and (4.23))

$$\begin{aligned}&\|\mathbf{u}_f\|_{0,4;\Omega} + \|\mathbf{u}_{fh}\|_{0,4;\Omega} \\ &\quad \le 2\min\{\bar{\alpha}_f, \bar{\alpha}_{f,\mathrm{d}}\}\Big\{2\|\mathbf{u}_{D,f}\|_{1/2,\Gamma} + (r + r_{\mathrm{d}})\|\delta(\phi)\|_{0,\Omega} + 2|\Omega|^{3/4}\rho_f\mathbf{g}\|\varepsilon\|_{0,\infty;\Omega}\Big\},\end{aligned}$$

we readily obtain (5.10). □

Lemma 5.4. *There exist positive constants $C_{s,i}$, $i \in \{1,2,3\}$, depending on μ_1, μ_2, and other constants independent of the discretization and physical parameters, such that*

$$\begin{aligned} \|\boldsymbol{\sigma}_s - \boldsymbol{\sigma}_{sh}\|_{\mathrm{H}} + \|\mathbf{u}_s - \mathbf{u}_{sh}\|_{0,4;\Omega} + \|\boldsymbol{\gamma}_s - \boldsymbol{\gamma}_{sh}\|_{0,\Omega} \le C_{s,1}\,\mathrm{dist}(\boldsymbol{\sigma}_s, \mathrm{H}_h) \\ + C_{s,2}\,\mathrm{dist}((\mathbf{u}_s, \boldsymbol{\gamma}_s), \mathrm{Q}_h) + \mathcal{J}_s(\mathtt{data})\,\|\mathbf{u}_f - \mathbf{u}_{fh}\|_{0,4;\Omega} + \mathcal{K}_s(\mathtt{data})\,\|\mathbf{u}_s - \mathbf{u}_{sh}\|_{0,4;\Omega}\,, \end{aligned} \tag{5.14}$$

with $\mathcal{J}_s(\mathtt{data})$ and $\mathcal{K}_s(\mathtt{data})$ given by

$$\begin{aligned} \mathcal{J}_s(\mathtt{data}) :=\ & C_{s,3}\frac{\rho_f}{\mu_1}\min\{\bar{\alpha}_f, \bar{\alpha}_{f,\mathrm{d}}\}\|\varepsilon\|_{0,\infty;\Omega}\Big(2\|\mathbf{u}_{D,f}\|_{1/2,\Gamma} + (r + r_{\mathrm{d}})\|\delta(\phi)\|_{0,\Omega} \\ & + 2|\Omega|^{3/4}\rho_f\mathbf{g}\|\varepsilon\|_{0,\infty;\Omega}\Big), \\ \mathcal{K}_s(\mathtt{data}) :=\ & C_{s,3}\frac{1}{\sqrt{n}}\left\|\frac{\nabla\phi}{\phi}\right\|_{0,4;\Omega} + C_{s,3}\frac{\rho_s}{\mu_1}\min\{\bar{\alpha}_s, \bar{\alpha}_{s,\mathrm{d}}\}\|\phi\|_{0,\infty;\Omega}\Big(2\|\mathbf{u}_{D,s}\|_{1/2,\Gamma} \\ & + (r^2 + r_{\mathrm{d}}^2)\frac{\rho_f}{2\mu_1}\|\varepsilon\|_{0,\infty;\Omega} + 2|\Omega|^{3/4}\mathbf{g}\|\varepsilon\rho_f + \phi\rho_s\|_{0,\infty;\Omega}\Big)\,. \end{aligned} \tag{5.15}$$

Proof. Analogously to the proof of Lemma 5.3, we apply Lemma 5.1 to the third and fourth equations of (5.5) and (5.6), to obtain

$$\begin{aligned} & \|\boldsymbol{\sigma}_s - \boldsymbol{\sigma}_{sh}\|_{\mathrm{H}} + \|\mathbf{u}_s - \mathbf{u}_{sh}\|_{0,4;\Omega} + \|\boldsymbol{\gamma}_s - \boldsymbol{\gamma}_{sh}\|_{0,\Omega} \\ & \quad \le C_{s,1}\,\mathrm{dist}(\boldsymbol{\sigma}_s, \mathrm{H}_h) + C_{s,2}\,\mathrm{dist}((\mathbf{u}_s, \boldsymbol{\gamma}_s), \mathrm{Q}_h) + C_{s,3}\,\left\|\widehat{\mathbf{F}}_s^{\mathbf{u}} - \widehat{\mathbf{F}}_s^{\mathbf{u}_h}\right\|_{\mathrm{H}_h'}\,, \end{aligned} \tag{5.16}$$

where the constants $C_{s,i}$, $i \in \{1,\dots,3\}$, are given by (5.4) with $\|a\| = \|\mathbf{a}_s\| = \frac{1}{2\mu_1}$, $\|b\| = \|\mathbf{b}\| = 1$, $\tilde{\alpha} = \alpha_s = \frac{c_1}{2\mu_2}$ and $\tilde{\beta} = \beta_{\mathrm{d}} > 0$ (cf. (3.32), (3.51), (4.9)). Then, the result follows from (5.9), (5.16), (3.94), (3.95), (4.23) and (4.24). □

The a priori error estimate for the Galerkin scheme (4.2) is provided next.

Theorem 5.5. *Assume that the hypotheses of Theorems 3.12 and 4.7 hold, and let $\big(\boldsymbol{\sigma}_f, (\mathbf{u}_f, \boldsymbol{\gamma}_f)\big) \in \mathrm{H}\times\mathrm{Q}$, $\big(\boldsymbol{\sigma}_s, (\mathbf{u}_s, \boldsymbol{\gamma}_s)\big) \in \mathrm{H}\times\mathrm{Q}$ and $\big(\boldsymbol{\sigma}_{fh}, (\mathbf{u}_{fh}, \boldsymbol{\gamma}_{fh})\big) \in \mathrm{H}_h \times \mathrm{Q}_h$, $\big(\boldsymbol{\sigma}_{sh}, (\mathbf{u}_{sh}, \boldsymbol{\gamma}_{sh})\big) \in \mathrm{H}_h \times \mathrm{Q}_h$ be the unique solutions of (3.18) and (4.2), respectively. Assume further that*

$$\mathcal{J}_f(\mathtt{data}) + \mathcal{J}_s(\mathtt{data}) \le \frac{1}{2} \quad \text{and} \quad \mathcal{K}_f(\mathtt{data}) + \mathcal{K}_s(\mathtt{data}) \le \frac{1}{2}, \tag{5.17}$$

with $\mathcal{J}_f$, $\mathcal{K}_f$ and $\mathcal{J}_s$, $\mathcal{K}_s$ given by (5.11) and (5.15), respectively. Then, there holds

$$\begin{aligned} & \sum_{j\in\{f,s\}}\Big\{\|\boldsymbol{\sigma}_j - \boldsymbol{\sigma}_{jh}\|_{\mathrm{H}} + \|\mathbf{u}_j - \mathbf{u}_{jh}\|_{0,4;\Omega} + \|\boldsymbol{\gamma}_j - \boldsymbol{\gamma}_{jh}\|_{0,\Omega}\Big\} \\ & \quad \le \sum_{j\in\{f,s\}}\Big\{C_{j,1}\,\mathrm{dist}(\boldsymbol{\sigma}_j, \mathrm{H}_h) + C_{j,2}\,\mathrm{dist}((\mathbf{u}_i, \boldsymbol{\gamma}_j), \mathrm{Q}_h)\Big\}\,, \end{aligned} \tag{5.18}$$

with $C_{f,i}$ and $C_{s,i}$, $i = 1,2$, specified in Lemmas 5.3 and 5.4, respectively.

Proof. Employing assumption (5.17), the result is a direct consequence of Lemmas 5.3 and 5.4. We omit further details. □

Similarly as we did for the assumptions (3.77), (3.78), and (3.93), we stress here that the feasibility of the hypotheses in (5.17) depends finally on the data defining $\mathcal{J}_f$, $\mathcal{K}_f$, $\mathcal{J}_s$, and $\mathcal{K}_s$ (cf. (5.11) and (5.15)). In particular, it is easy to see that a sufficient condition for (5.17) is given by the set of assumptions

$$\mathcal{J}_f(\mathtt{data}) \le \frac{1}{4}\,, \quad \mathcal{K}_f(\mathtt{data}) \le \frac{1}{4}\,, \quad \mathcal{J}_s(\mathtt{data}) \le \frac{1}{4}\,, \quad \text{and} \quad \mathcal{K}_s(\mathtt{data}) \le \frac{1}{4},$$

which, in turn, are satisfied if, proceeding as for (3.77) and (3.78), individual constraints on each one of the terms defining them are imposed. However, as already noticed in the case of the aforementioned hypotheses, the fact that some of the constants involved are not known explicitly stops us of checking in practice the verification of those conditions. Furthermore, we believe that only unrealistic data, with very sharp and unusual gradients, might fail (5.17). Indeed, some of the numerical essays reported in Section 6 consider even delicate cases for which our proposed algorithm still performs very well. In other words, and summarizing our point of view, (5.17) seems to be more a technical issue of the analysis rather than a real limitation of the applicability of the method.

We end this section with the theoretical rate of convergence for the Galerkin scheme (4.2) discretized by the finite element spaces introduced in Section 4.4.3.

Theorem 5.6. *Assume that (3.93) holds and let $\big(\boldsymbol{\sigma}_f, (\mathbf{u}_f, \boldsymbol{\gamma}_f)\big) \in \mathrm{H}\times\mathrm{Q}$, $\big(\boldsymbol{\sigma}_s, (\mathbf{u}_s, \boldsymbol{\gamma}_s)\big) \in \mathrm{H}\times\mathrm{Q}$ be the unique solution of (3.18). In addition, given $\ell \ge 0$, we let $\mathrm{H}_h \times \mathrm{Q}_h$ be the pair defined by the PEERS_ℓ or AFW_ℓ elements introduced in (4.30) and (4.31), respectively, and under assumption (4.22), we let $\big(\boldsymbol{\sigma}_{fh}, (\mathbf{u}_{fh}, \boldsymbol{\gamma}_{fh})\big) \in \mathrm{H}_h \times \mathrm{Q}_h$, $\big(\boldsymbol{\sigma}_{sh}, (\mathbf{u}_{sh}, \boldsymbol{\gamma}_{sh})\big) \in \mathrm{H}_h \times \mathrm{Q}_h$ be the* *unique solution of (4.2). Assume further that (5.17) holds and that, given $r \in [0, \ell+1]$, $\boldsymbol{\sigma}_j \in \mathbb{H}^r(\Omega) \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$ with $\mathbf{div}(\boldsymbol{\sigma}_j) \in \mathbf{W}^{r,4/3}(\Omega)$, $\mathbf{u}_j \in \mathbf{W}^{r,4}(\Omega)$ and $\boldsymbol{\gamma}_j \in \mathbb{H}^r(\Omega) \cap \mathbb{L}^2_{\mathtt{skew}}(\Omega)$ for $j \in \{f,s\}$. Then there exists $C > 0$, independent of h, such that*

$$\sum_{j\in\{f,s\}} \Big\{ \|\boldsymbol{\sigma}_j - \boldsymbol{\sigma}_{jh}\|_{\mathrm{H}} + \|\mathbf{u}_j - \mathbf{u}_{jh}\|_{0,4;\Omega} + \|\boldsymbol{\gamma}_j - \boldsymbol{\gamma}_{jh}\|_{0,\Omega} \Big\}$$
$$\leq C\,h^r \sum_{j\in\{f,s\}} \Big\{ \|\boldsymbol{\sigma}_j\|_{r,\Omega} + \|\mathbf{div}(\boldsymbol{\sigma}_j)\|_{r,4/3;\Omega} + \|\mathbf{u}_j\|_{r,4;\Omega} + \|\boldsymbol{\gamma}_j\|_{r,\Omega} \Big\}.$$

Proof. The result follows straightforwardly from Theorem 5.5 and the approximation properties (4.32), (4.33) and (4.34). □

5.2. Computing further variables of interest

Here we introduce suitable approximations of further variables of interest, such as the stresses T_f and T_s, the fluid pressure p_f and the gradient of the fluid and particle velocities $\nabla\mathbf{u}_f$ and $\nabla\mathbf{u}_s$, respectively, all of them written in terms of the solution of the discrete problem (4.2). To that end, we first notice from (2.10) and (2.11) that T_f and T_s satisfy the identities

$$T_f = \boldsymbol{\sigma}_f + \rho_f(\varepsilon\mathbf{u}_f)\otimes\mathbf{u}_f \quad \text{and} \quad T_s = \boldsymbol{\sigma}_s + \rho_s(\phi\mathbf{u}_s)\otimes\mathbf{u}_s + \rho_f(\varepsilon\mathbf{u}_f)\otimes\mathbf{u}_f\,. \tag{5.19}$$

Next, we observe that p_f, $\nabla\mathbf{u}_f$ and $\nabla\mathbf{u}_s$ can be written in terms of T_f and T_s as follows:

$$\begin{aligned} p_f = -\frac{1}{n}\,\mathrm{tr}\big(T_f\big), \quad \nabla\mathbf{u}_f = \frac{1}{2\mu_f}T_f^{\mathrm{d}} + \boldsymbol{\gamma}_f - \frac{1}{n}\left(\frac{\nabla\varepsilon}{\varepsilon}\cdot\mathbf{u}_f\right)\mathbb{I}, \\ \nabla\mathbf{u}_s = \frac{1}{2\mu_s(\phi)}T_s^{\mathrm{d}} + \boldsymbol{\gamma}_s - \frac{1}{n}\left(\frac{\nabla\phi}{\phi}\cdot\mathbf{u}_s\right)\mathbb{I}\,. \end{aligned} \tag{5.20}$$

Therefore, given the discrete solution $\big(\boldsymbol{\sigma}_{fh}, (\mathbf{u}_{fh}, \boldsymbol{\gamma}_{fh})\big) \in \mathrm{H}_h \times \mathrm{Q}_h$, $\big(\boldsymbol{\sigma}_{sh}, (\mathbf{u}_{sh}, \boldsymbol{\gamma}_{sh})\big) \in \mathrm{H}_h \times \mathrm{Q}_h$, of (4.2), we propose the post processed approximations of T_s, T_f, p_f, $\nabla\mathbf{u}_f$ and $\nabla\mathbf{u}_s$ defined by the following expressions

$$\begin{aligned} T_{fh} = \boldsymbol{\sigma}_{fh} + \rho_f(\varepsilon\mathbf{u}_{fh})\otimes\mathbf{u}_{fh}, \quad T_{sh} = \boldsymbol{\sigma}_{sh} + \rho_s(\phi\mathbf{u}_{sh})\otimes\mathbf{u}_{sh} + \rho_f(\varepsilon\mathbf{u}_{fh})\otimes\mathbf{u}_{fh}, \\ p_{fh} = -\frac{1}{n}\,\mathrm{tr}\big(T_{fh}\big), \quad (\nabla\mathbf{u}_f)_h = \frac{1}{2\mu_f}T_{fh}^{\mathrm{d}} + \boldsymbol{\gamma}_{fh} - \frac{1}{n}\left(\frac{\nabla\varepsilon}{\varepsilon}\cdot\mathbf{u}_{fh}\right)\mathbb{I}, \\ (\nabla\mathbf{u}_s)_h = \frac{1}{2\mu_s(\phi)}T_{sh}^{\mathrm{d}} + \boldsymbol{\gamma}_{sh} - \frac{1}{n}\left(\frac{\nabla\phi}{\phi}\cdot\mathbf{u}_{sh}\right)\mathbb{I}\,. \end{aligned} \tag{5.21}$$

Notice that all the variables in (5.21) can be obtained in terms of the solution of (4.2) without applying any numerical differentiation procedure, thus avoiding further sources of error, which constitutes a significant advantage of the present mixed finite element method as compared with the usual primal formulation. In addition, it is easy to see that they optimally converge to their exact counterparts. The latter is established in the following corollary whose proof is omitted since it follows directly from (5.19), (5.20), (5.21), and Theorem 5.6.

Corollary 5.7. *Given $\ell \geq 0$, let $\mathrm{H}_h \times \mathrm{Q}_h$ be the pair defined by the PEERS_ℓ or AFW_ℓ elements introduced in (4.30) and (4.31), respectively, and let $\big(\boldsymbol{\sigma}_f, (\mathbf{u}_f, \boldsymbol{\gamma}_f)\big) \in \mathrm{H}\times\mathrm{Q}$, $\big(\boldsymbol{\sigma}_s, (\mathbf{u}_s, \boldsymbol{\gamma}_s)\big) \in \mathrm{H}\times\mathrm{Q}$ and $\big(\boldsymbol{\sigma}_{fh}, (\mathbf{u}_{fh}, \boldsymbol{\gamma}_{fh})\big) \in \mathrm{H}_h \times \mathrm{Q}_h$, $\big(\boldsymbol{\sigma}_{sh}, (\mathbf{u}_{sh}, \boldsymbol{\gamma}_{sh})\big) \in \mathrm{H}_h \times \mathrm{Q}_h$ be the unique solutions of (3.18) and (4.2), respectively. Assume that the regularity hypotheses of Theorem 5.6 hold with $r \in [0, \ell+1]$. Then, there exists $\tilde{C} > 0$, independent of h, such that*

$$\begin{aligned} \|p_f - p_{fh}\|_{0,\Omega} + \sum_{j\in\{f,s\}} \Big\{ \|T_j - T_{jh}\|_{0,\Omega} + \|\nabla\mathbf{u}_j - (\nabla\mathbf{u}_j)_h\|_{0,\Omega} \Big\} \\ \leq \tilde{C}h^r \sum_{j\in\{f,s\}} \Big\{ \|\boldsymbol{\sigma}_j\|_{r,\Omega} + \|\mathbf{div}(\boldsymbol{\sigma}_j)\|_{r,4/3;\Omega} + \|\mathbf{u}_j\|_{r,4;\Omega} + \|\boldsymbol{\gamma}_j\|_{r,\Omega} \Big\}. \end{aligned} \tag{5.22}$$

6. Numerical results

The realization of the numerical methods described in Section 5 has been carried out using the open-source finite element library FEniCS [40]. A Newton–Raphson algorithm with null initial guesses and exact Jacobian is used to solve the nonlinear set of equations. The condition of zero-averaged fluid pressure (translated in terms of tensor traces) is imposed through a real Lagrange multiplier, which amounts to augmenting the algebraic systems by one row and one column; and the solution of all linear systems appearing at each Newton–Raphson iteration is conducted with the multifrontal massively parallel sparse direct solver MUMPS.

Table 6.1
Example 1. Convergence history for the mixed finite element approximations of the coupled nonlinear problem in 2D, for different variants of the scheme. DoF stands for the number of degrees of freedom associated with each method on each mesh refinement.

DoF	$e(\boldsymbol{\sigma}_f)$	$r(\boldsymbol{\sigma}_f)$	$e(\mathbf{u}_f)$	$r(\mathbf{u}_f)$	$e(\boldsymbol{\gamma}_f)$	$r(\boldsymbol{\gamma}_f)$	$e(\boldsymbol{\sigma}_s)$	$r(\boldsymbol{\sigma}_s)$	$e(\mathbf{u}_s)$	$r(\mathbf{u}_s)$	$e(\boldsymbol{\gamma}_s)$	$r(\boldsymbol{\gamma}_s)$
AFW_ℓ-based formulation with $\ell = 0$												
54	2.19e+0	–	6.42e−1	–	1.08e+0	–	1.07e+1	–	1.07e+0	–	1.04e+0	–
178	1.23e+0	0.832	3.27e−1	0.973	5.80e−1	0.900	5.32e+00	1.009	5.53e−1	0.955	4.86e−1	1.104
642	6.20e−1	0.987	1.66e−1	0.983	3.93e−1	0.559	2.59e+00	1.040	2.79e−1	0.990	2.51e−1	0.951
2434	3.11e−1	0.998	8.29e−2	0.997	2.07e−1	0.930	1.28e+00	1.018	1.40e−1	0.998	1.27e−1	0.980
9474	1.55e−1	1.000	4.15e−2	1.000	1.04e−1	0.984	6.36e−1	1.007	6.98e−2	0.999	6.39e−2	0.995
37 378	7.76e−2	1.000	2.07e−2	1.000	5.24e−2	0.996	3.17e−1	1.002	3.49e−2	1.000	3.20e−2	0.999
AFW_ℓ-based formulation with $\ell = 1$												
122	7.45e−1	–	1.30e−1	–	2.19e−1	–	2.96e+0	–	2.20e−1	–	2.46e−1	–
434	1.89e−1	1.975	3.84e−2	1.765	1.67e−1	0.389	6.39e−1	2.214	5.60e−2	1.973	8.61e−2	1.513
1634	4.70e−2	2.012	9.37e−3	2.034	4.70e−2	1.832	1.60e−1	1.994	1.41e−2	1.994	2.47e−2	1.799
6338	1.17e−2	2.005	2.32e−3	2.015	1.22e−2	1.949	4.03e−2	1.991	3.52e−3	1.999	6.45e−3	1.939
24 962	2.92e−3	2.002	5.78e−4	2.004	3.10e−3	1.974	1.01e−2	2.000	8.79e−4	2.000	1.64e−3	1.979
99 074	7.29e−4	2.001	1.44e−4	2.001	7.83e−4	1.986	2.52e−3	2.002	2.20e−4	2.000	4.12e−4	1.992
PEERS_ℓ-based formulation with $\ell = 0$												
46	2.49e+0	–	8.65e−1	–	4.47e+0	–	1.07e+1	–	1.08e+0	–	1.57e+0	–
148	1.59e+0	0.647	6.45e−1	0.423	3.75e+0	0.252	5.73e+0	0.898	5.58e−1	0.955	5.56e−1	1.500
532	7.26e−1	1.131	1.91e−1	1.756	9.99e−1	1.909	2.92e+0	0.972	2.79e−1	0.999	1.97e−1	1.500
2020	3.60e−1	1.011	8.78e−2	1.120	4.26e−1	1.230	1.47e+0	0.993	1.40e−1	0.999	8.81e−2	1.159
7876	1.76e−1	1.034	4.22e−2	1.057	1.71e−1	1.312	7.30e−1	1.009	6.98e−2	1.000	3.88e−2	1.184
31 108	8.70e−2	1.016	2.08e−2	1.020	6.38e−2	1.428	3.64e−1	1.005	3.49e−2	1.000	1.55e−2	1.327
PEERS_ℓ-based formulation with $\ell = 1$												
124	7.63e−1	–	1.36e−1	–	4.34e−1	–	2.71e+0	–	2.24e−1	–	5.33e−1	–
436	2.10e−1	1.863	4.02e−2	1.761	1.77e−1	1.490	7.01e−1	1.950	5.63e−2	1.992	1.50e−1	1.830
1636	5.39e−2	1.962	1.00e−2	2.006	7.22e−2	1.596	1.90e−1	1.882	1.41e−2	1.996	4.46e−2	1.848
6340	1.36e−2	1.983	2.39e−3	2.063	2.44e−2	1.668	4.95e−2	1.943	3.52e−3	2.001	1.31e−2	1.868
24 964	3.43e−3	1.988	5.85e−4	2.034	7.07e−3	1.785	1.26e−2	1.977	8.80e−4	2.001	3.58e−3	1.873
99 076	8.62e−4	1.994	1.45e−4	2.012	1.90e−3	1.895	3.17e−3	1.990	2.20e−4	2.001	9.30e−4	1.945

6.1. Test 1: accuracy verification using smooth manufactured solutions

We assess the convergence of the mixed finite element discretizations by manufacturing an exact solution of the coupled system (2.10) defined over the domain $\Omega := (0,1)^2$

$$\phi(\mathbf{x}) = \frac{1}{2} - \frac{1}{4}\sin(x_1)\cos(x_2),$$
$$\mathbf{u}_s(\mathbf{x}) = \left(\frac{\begin{matrix} 4\cos(x_1)\sin(x_2) \\ \frac{1}{4}(\sin^2(x_1) - \cos^2(x_1))\cos(2x_2) - 2\sin(x_1)\cos(x_2)\end{matrix}}{\phi(\mathbf{x})}\right),$$
$$\varepsilon(\mathbf{x}) = 1 - \phi(\mathbf{x}), \quad \mathbf{u}_f(\mathbf{x}) = \frac{\phi(\mathbf{x})}{\varepsilon(\mathbf{x})}\mathbf{u}_s(\mathbf{x}), \quad p(\mathbf{x}) = x_1^4 - x_2^4.$$

The exact velocities and the smooth particle distribution are such they satisfy the mass conservation equations. Using these closed-form solutions we require additional right-hand side load terms in (2.13), and the boundary Dirichlet velocities are also adjusted in terms of these manufactured solutions. The model constants for the convergence test assume the values

$$\rho_f = 1, \quad \rho_s = 2.2, \quad \mu_f = 0.1, \quad d_s = 0.1, \quad \phi_p = 0.65, \quad \mathbf{g} = (0,-1)^{\mathrm{t}},$$
$$P = 1.266, \quad r = 0.3, \quad M = 0.571, \quad m = 3.65, \quad \phi_0 = 0.61, \quad v_t = 14.3.$$

Errors between exact and approximate solutions are denoted as

$$e(\boldsymbol{\sigma}_f) := \|\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{fh}\|_{\mathbf{div}_{4/3};\Omega}, \quad e(\mathbf{u}_f) := \|\mathbf{u}_f - \mathbf{u}_h\|_{0,4;\Omega}, \quad e(\boldsymbol{\gamma}_f) := \|\boldsymbol{\gamma}_f - \boldsymbol{\gamma}_{fh}\|_{0,\Omega},$$
$$e(\boldsymbol{\sigma}_s) := \|\boldsymbol{\sigma}_s - \boldsymbol{\sigma}_{sh}\|_{\mathbf{div}_{4/3};\Omega}, \quad e(\mathbf{u}_s) := \|\mathbf{u}_s - \mathbf{u}_{sh}\|_{0,4;\Omega}, \quad e(\boldsymbol{\gamma}_s) := \|\boldsymbol{\gamma}_s - \boldsymbol{\gamma}_{sh}\|_{0,\Omega},$$

and by $r(\star)$ we denote their corresponding rates of convergence, that is

$$r(\star) := \frac{\log(e(\star)/e'(\star))}{\log(h/h')} \qquad \forall \star \in \left\{\boldsymbol{\sigma}_f, \mathbf{u}_f, \boldsymbol{\gamma}_f, \boldsymbol{\sigma}_s, \mathbf{u}_s, \boldsymbol{\gamma}_s\right\},$$

where h and h' denote two consecutive mesh sizes with errors $e(\star)$ and $e'(\star)$, respectively. Errors and corresponding convergence rates are summarized in Table 6.1, focusing on approximations using AFW_ℓ and PEERS_ℓ elements for the two

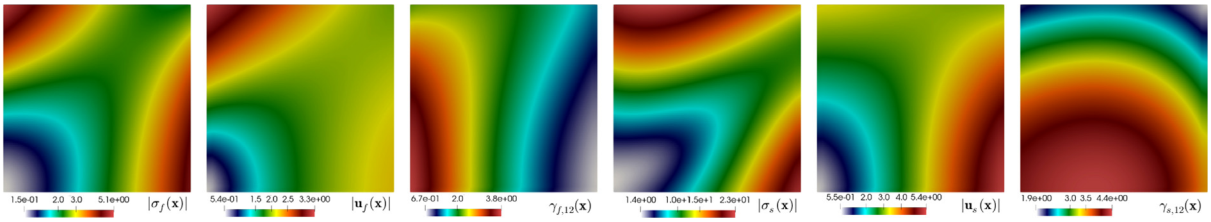

Fig. 6.1. Example 1. Approximate solutions computed with PEERS$_\ell$ method with $\ell = 1$. Magnitude of fluid pseudostress, fluid velocity magnitude, component (1,2) of the vorticity fluid tensor, magnitude of particle pseudostress, solid velocity magnitude, and component (1,2) of the vorticity solid tensor.

lowest-order polynomial degrees $\ell = 0, 1$. In all cases we see the optimal convergence rates predicted by Theorem 5.6 for all individual unknowns. Also, we mention that in every run the number of Newton–Raphson iterations needed to reach a residual-based convergence criterion with tolerance of 1e-6 was less than 4. Samples of approximate solutions are shown in Fig. 6.1.

6.2. Test 2: velocity fields generated by synthetic particle distributions

For our second test we consider a two-dimensional fluidized bed of size 15×30 cm^2, where the problem configuration follows a simplification of the applicative cases discussed in [4,13]. The inlet boundary Γ^{in} is defined as a nozzle of 1 cm width which is located at the center of the lower horizontal boundary, and through which fluid is injected with a uniform profile. In addition, we generate a synthetic particle distribution

$$\phi(\mathbf{x}) = \frac{\phi_0}{2} - \frac{\phi_0}{4}\sin\left(\frac{1}{5}x_1\right)\cos\left(\frac{1}{5}x_2\right), \tag{6.1}$$

where ϕ_0 is the mean concentration of the particles in the fluidized bed occupying Ω. Note that, because of (2.16) the formulation requires ϕ to be smooth and non-zero. The boundary conditions are now slightly different than in Example 1. The fluid velocity is still prescribed on the whole boundary, but it is split as follows

$$\mathbf{u}_{f,D}(\mathbf{x}) = \begin{cases} (0, U)^{\mathtt{t}} & \text{on } \Gamma^{\text{in}}, \\ \left(0, \dfrac{6U}{15^3}x_1(15 - x_1)\right)^{\mathtt{t}} & \text{on } \Gamma^{\text{out}}, \\ \mathbf{0} & \text{on } \Gamma^{\text{wall}} = \Gamma \setminus \{\Gamma^{\text{in}} \cup \Gamma^{\text{out}}\}, \end{cases}$$

whereas the particle velocities are allowed to slip on all boundaries

$$\mathbf{u}_s \cdot \mathbf{n} = 0 \qquad \text{on } \Gamma, \tag{6.2}$$

which implies that, at the discrete level, the first pairing appearing in the definition of $\mathbf{F}_s^{\mathbf{u}_h}(\boldsymbol{\tau}_{sh})$ (see the specification for the continuous variational form in (3.8)) is replaced by

$$\left\langle \tilde{\mathbf{t}} \cdot \boldsymbol{\tau}_{sh}\mathbf{n}, \mathbf{u}_{sh} \cdot \tilde{\mathbf{t}} \right\rangle, \tag{6.3}$$

in 2D, where $\tilde{\mathbf{t}}$ is the tangent vector on the boundary. Such a boundary setup expresses, respectively, that we apply exactly the fluidization uniform velocity at the entrance of the fluidized bed and that this is the same profile with which the fluid leaves the bed, that there is no-slip boundary conditions for the fluid at rigid walls, and that there is slip of solid particles at rigid walls but no particles should leave the fluidized bed. The remaining parameters characterizing (2.6), (2.7) and (2.9), (6.1), are

$$\rho_f = 1.205, \quad \rho_s = 2.7, \quad \mu_f = 1.8 \cdot 10^{-2}, \quad d_s = 4 \cdot 10^{-1}, \quad \phi_p = 0.65,$$
$$P = 10.78, \quad r = 0.3, \quad M = 0.571, \quad m = 4.25, \quad U \in \{1, 2.2\}, \quad \phi_0 = 0.61,$$

taken as in [4,13] using CGS units, and representing the interaction between a liquid and solids in a fluidized bed. We employ a uniform mesh and run the simulation of the interaction between the mass and momentum conservations in the steady case. The outcome is shown in Fig. 6.2, where we see how the particle distribution generates velocity patterns going from the nozzle to the outlet boundary, and how the particles slip on the boundaries. We also compare two cases for different inlet velocities. The plots in Fig. 6.3 show distinct velocity patterns generated using the same particle distribution, but where the intensity of the nozzle varies from $U = 1$ to $U = 2.2$.

We also conduct a modification of the previous tests by considering different specifications for the particle distribution. Using the same rectangular channel as before we set

$$\phi_C(\mathbf{x}) = \frac{\phi_0}{2}\left(1 - \tanh\left(\frac{x_2 - 15}{5}\right)\right), \qquad \phi_D(\mathbf{x}) = \phi_0\left(1 - \exp\left[-0.5(x_1 - 7.5)^2 - (x_2 - 15)^2\right]\right),$$

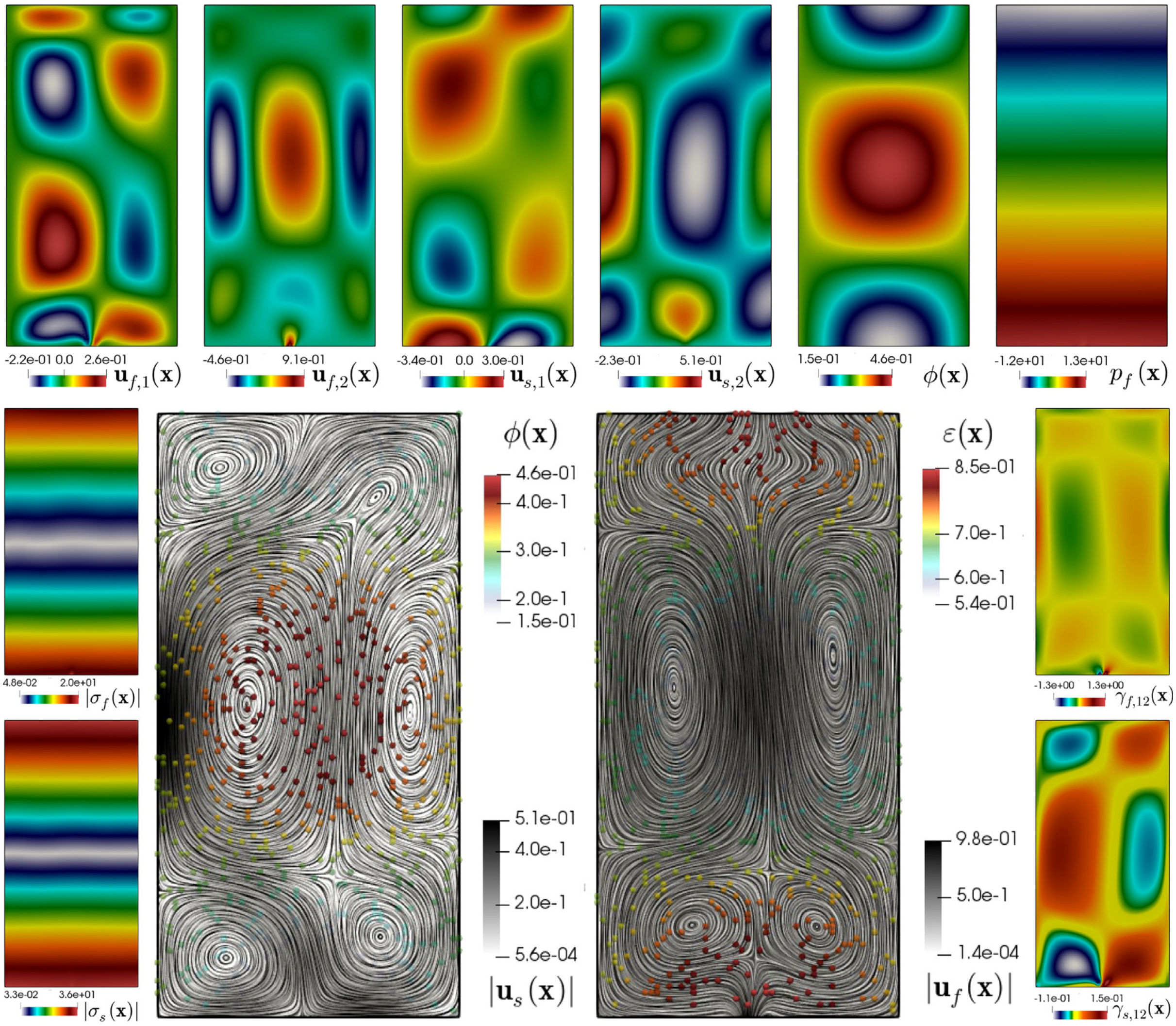

Fig. 6.2. Example 2A. Approximate solutions obtained with a lowest-order AFW_ℓ method. Velocity components and postprocessed pressure (top), magnitude of fluid and particle pseudostress (bottom left), synthetic particle distribution with particle velocity line integral contours and sample of distribution of ε and fluid velocity (bottom center panels), and entry (1,2) of the fluid and particle vorticity fields.

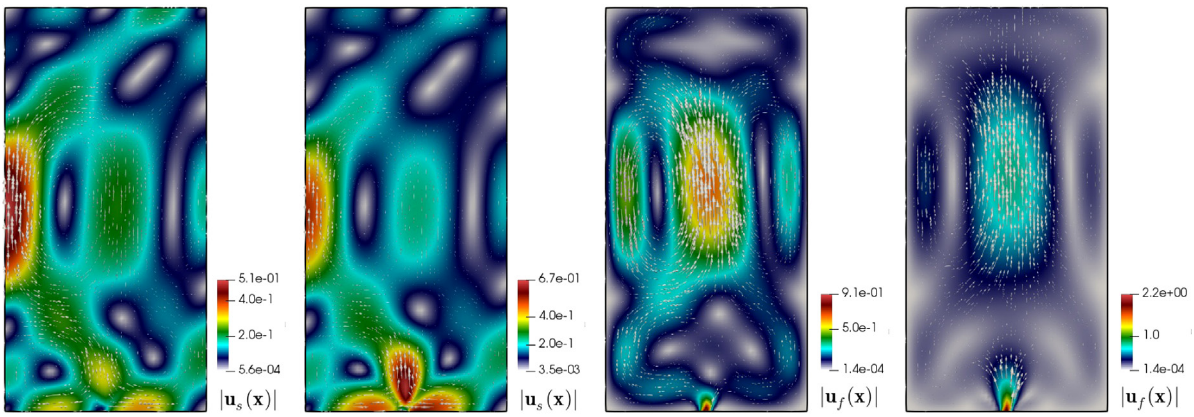

Fig. 6.3. Example 2B. From left to right: Magnitudes of the particle and fluid velocities for two different magnitudes of the inlet fluid velocity.

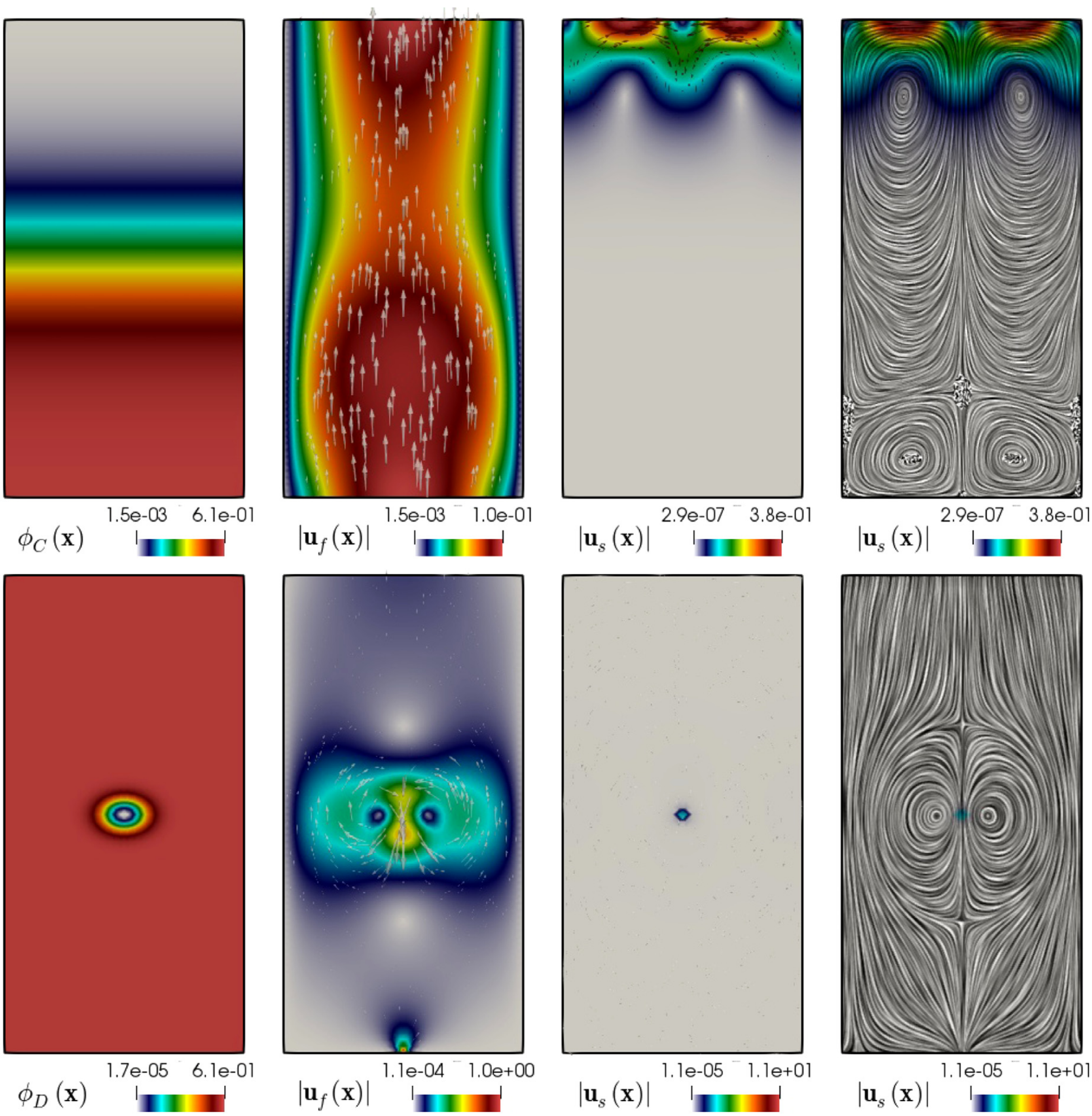

Fig. 6.4. Examples 2C (top row)-2D (bottom row). Particle distribution, magnitudes of the fluid and particle velocities, and particle velocity line integral contours, for two different particle distributions.

where ϕ_0 is as in examples 2A-B. In test 2C we also modify the boundary conditions setting the same parabolic profile for the fluid velocity on the inlet and the outlet, in order to generate a Poiseuille fluid flow. For test 2D the boundary setup is as in examples 2A–2B. As announced at the end of Section 4.1, the aim of these additional tests is to assess how the method behaves in the low concentration limit $\phi \approx 0$. Test 2C mimics a fluidized bed that is homogeneously expanded to a height of approximately 15 and, from there, after a fast transition of the concentration field from approximately ϕ_0 to approximately 0, the flow would only be of almost pure fluid up until the outlet of the reservoir. For test 2D, we mimic a (stationary) bubble at the center of the domain. In the center of the bubble, the particle concentration is actually zero. The results indicate that the formulation performs relatively well in these scenarios, suggesting that the restrictions of maintaining ϕ away from zero dictated by (2.16) and the corresponding hypotheses in Theorems 3.6, 3.7, 4.1 and 4.2, may be waived at the implementation level. The numerical solutions for tests 2C and 2D are portrayed in Fig. 6.4.

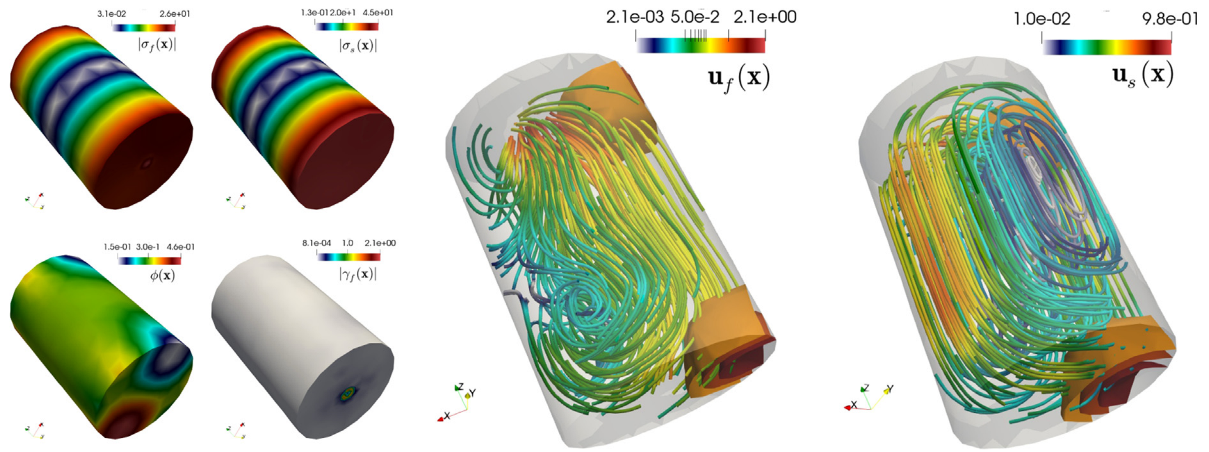

Fig. 6.5. Example 3. Magnitude of fluid and particle pseudostress (left top), particle distribution and magnitude fluid vorticity (left bottom), and fluid (center) and particle (right) velocity streamlines showing also contour plots of the particle distribution. The simulations were performed with the lowest-order PEERS$_\ell$ method.

6.3. Test 3: 3D version of Test 2B

As a proof of concept of the need of multidimensional models for fluidized beds we present a simple extension of the previous tests to the 3D case. We consider a cylinder of height 30 cm and radius 10 cm. Again we start from a given smooth particle distribution that we choose as

$$\phi(\mathbf{x}) = \frac{\phi_0}{2} - \frac{\phi_0}{900}\sin\left(\frac{1}{5}x_1\right)\cos\left(\frac{1}{5}x_2\right)(x_3 - 15)^2.$$

The boundary conditions are set similarly as above, on the bottom disk of the cylinder we define as $\Gamma^{\rm in}$ a smaller region of radius 1 cm on which we impose a uniform fluid velocity $(0, 0, 1)^{\rm t}$, on $x_3 = 30$ we define another disk region $\Gamma^{\rm out}$ centered at $(0, 0, 30)$ and with radius 3 cm, where we set the parabolic outlet fluid velocity profile $(0, 0, \frac{1}{12}x_1(15-x_1)x_2(15-x_2))^{\rm t}$; and the remainder of the boundary conforms $\Gamma^{\rm wall}$. Again, we impose slip-velocity conditions for the solid particles according to (6.2), and instead of (6.3) we now set $\langle \boldsymbol{\tau}_{sh}\mathbf{n} \times \mathbf{n}, \mathbf{u}_{sh} \times \mathbf{n}\rangle$. The remaining parameters assume the same values as in the 2D case. The computations were performed with a coarse unstructured tetrahedral mesh for which the lowest-order PEERS$_\ell$ elements use around 110k DoFs. The outcomes are collected in Fig. 6.5. The larger plots on the center and right panels show the streamlines of fluid and particle velocities, where we also show also contours of ϕ that go over the threshold 0.35. The fluid velocity streamlines indicate the direction of the flow and the generation of non-axisymmetric recirculation patterns. The remaining panels show the magnitude of pseudostress and vorticity.

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