

## Finite element analysis for a Herrmann pressure formulation of the elasto-acoustic eigenvalue problem with variable coefficients

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[Received on 3 November 2025; revised on 14 August 2026]

In two and three dimensions, this study is focused on the numerical analysis of an eigenproblem associated with a fluid-structure model for sloshing and elasto-acoustic vibration. We use a displacement-Herrmann pressure formulation for the solid, while for the fluid, a pure displacement formulation is considered. Under this approach we propose a conforming locking-free method based on classic finite elements to approximate the natural frequencies (of the eigenmodes) of the coupled system. Employing the theory for non-compact operators we prove convergence and error estimates. Also we propose an a posteriori error estimator for this coupled problem which is shown to be efficient and reliable. We present a set of numerical tests in 2D and 3D.

**Keywords:** Fluid structure problems; Herrmann pressure; eigenvalue problems; finite elements; a priori error estimates; a posteriori error bounds

### 1. Introduction

The dynamics of fluid-structure interaction systems are of primary interest in a number of scientific and industrial applications. For example, vibrations occurring in pipes, parts of aerospace vessels, tanks, and many others. These applications are related to the design and development of different components, structures, and devices that are needed in different contexts. Several specific frameworks of interest are described in, e.g., [42]. Although this problem has been studied for many years, new applications, formulations, methods, and challenges are still emerging in the literature, proving that research on this problem is in ongoing progress.

In the present paper, our contribution is related to the development of numerical methods for an elasto-acoustic problem where we model the motion of a fluid in a solid container. Regarding the mathematical formulation of this classical problem (see, e.g., [17]), it is possible to use a displacement

formulation for both subdomains and employ, at the discrete level, simple Lagrangian finite elements (FE) for the solid and Raviart–Thomas elements for the fluid. This type of formulation has been used for elasto-acoustic vibration models as well as hydroelastic and sloshing models. Some important references, such as [8, 9, 12], are focused on the study of formulations where the main unknowns are the displacements of the solid and the fluid. Hence, this choice of formulation leads to a numerical analysis that involves classical finite elements, such as piecewise linear functions to approximate the displacement of the solid and Raviart–Thomas elements for the displacement of the fluid. This choice of finite elements is shown to be spurious free and capable of approximating the spectrum of the elasto-acoustic problem accurately. However, the method depending on these families of finite elements leads to a non-conforming approximation scheme, which has its inherent difficulties. To avoid this non-conformity, [40] proposes an alternative mixed formulation based on the pressure of the fluid and the stress of the solid, which, when numerically analyzed by approximating the stress with Brezzi–Douglas–Marini elements and the pressure with piecewise linear functions, is shown to be locking free, spurious free, and conforming. This formulation is consistent with the classical results in the literature. An alternative method, such as the one introduced in [18], where only piecewise linear polynomials are considered for both media, also turns out to be theoretically and computationally accurate. Time-dependent elasto-acoustic problems can also be studied, such as in [16], where a semi-implicit time discretization with distributed Lagrange multipliers is analyzed, or [4], where a spatial discretization based on the finite element method is first considered and error estimates are also provided, followed by a fully discrete approximation based on a family of implicit finite difference schemes in time. Let us remark that an important physical property is the absence of viscosity in the model, particularly in the fluid. This assumption leads to a linear eigenvalue problem, which is not the case when internal dissipation is considered, where naturally the eigenvalue problem is quadratic, as in [10]. Hence, to begin with a simple formulation, the presence of viscosity is not considered in this paper.

All the aforementioned references (and the references therein) are focused on a priori error estimates. However, the a posteriori analysis can also be performed. Indeed, we can mention [5, 6, 34, 38] as particularly interesting and well-developed contributions on this subject, which certainly provide inspiration for our work. We extend these works by addressing both a priori and a posteriori analysis, and employ a formulation for the elastic structure that accommodates the nearly and perfectly incompressible regimes, and that is also valid for heterogeneous material parameters. This brings the advantage that the error estimates are robust with respect to the model coefficients.

In the present paper, our contribution is focused on the mathematical and numerical analysis of an extension of the elasto-acoustic model studied in, for instance, [8], where sloshing effects are also considered, as in [40]. More precisely, the classical elasto-acoustic model of [8], which considers the typical formulation for the elastic domain, is here extended with the incorporation of the so-called Herrmann pressure [32]. This new variable creates a connection between the classical Stokes problem and the elasticity problem. More precisely, for eigenvalue problems, it is possible to prove a relation between the spectra of the Stokes and elasticity eigenproblems via the Herrmann pressure. This fact has been proved and numerically analyzed in [36] with a finite element method involving inf-sup stable families of finite elements for Stokes. More precisely, the approach of [36] has the advantage of providing a framework in which one can analyze both a fully incompressible solid (Stokes’ limit) and the typical elastic structure by means of a locking-free numerical method. With this formulation we are capable of considering a Poisson ratio equal to  $1/2$ , which classical formulations of the elasto-acoustic problem are not able to handle. Indeed, as in [36] for the elasticity eigenproblem, for the elasto-acoustic eigenvalue problem the locking-free property remains valid, as we present in our numerical findings. On the other hand, we allow the physical coefficients to vary in space, accounting for possible variations

in viscosity, stiffness, or incompressibility across the geometry of the structure. This introduces an additional layer of complexity in the analysis. Specifically, these variable coefficients demand the use of a weighted norm (see eq. (2.5) below) with spatially dependent parameters in order to ensure robustness of the different estimates that we derive. For the fluid part, we consider a similar model to that of [18]. However, there are two main differences with respect to the aforementioned study. First, the numerical scheme in the present paper considers an inf-sup stable family for the solid domain, such as Taylor–Hood or MINI elements, while the fluid domain is discretized with Brezzi–Douglas–Marini elements. These approximations provide conformity along the interface at the expense of additional degrees of freedom. Second, we propose an a posteriori error analysis that relies on the weighted norm and a Helmholtz decomposition to provide a reliable and efficient estimator.

Despite these relevant features of the Herrmann formulation and the framework of our analysis, the approximation of the solutions must be studied not only on the domains in which the fluid or the solid are located, but also on the interface. This drawback was rigorously studied in [8], where a corrected interpolant was constructed in order to obtain a proper approximation on the contact interface while accounting for the non-conformity of the method in that paper. For the mixed formulation in [40], this corrected interpolant operator was also needed for the same reason, but its analysis is based on the conformity of the method on the interface and hence, its construction was analyzed with a suitable extension operator. For the analysis of our model and its numerical method, we also need this corrected operator, and we construct it inspired by the aforementioned references.

Continuing with the contributions of the paper, we introduce a novel residual-based a posteriori error estimator tailored to the proposed model. The formulation of the estimator for the solid domain draws inspiration from the work of [36]. Leveraging the Helmholtz decomposition, we devise an innovative estimator for the fluid domain. Furthermore, we establish the reliability and efficiency of the proposed estimator, which are proved under the use of weighted norms and using the standard techniques for a posteriori error analysis such as [1, 45].

The paper is structured in the following manner: In Section 2 we describe the governing equations and derive a weak formulation of the eigenvalue problem, noting that the solution operator associated with the corresponding source problem is non-compact. Next, Section 3 describes the numerical discretization, introducing the main assumptions for the discrete spaces used herein (Taylor–Hood, MINI element, and Brezzi–Douglas–Marini), and stating approximation properties of the needed modification of classical interpolants. We discuss both the source and the discrete eigenvalue problem. The spectral approximation is postponed to Section 4 according to [25], where we show the convergence (with optimal order) of the nonconforming method in a suitably defined mesh-dependent norm. In Section 5 we propose a residual-based a posteriori error estimator and show its robustness. We continue in Section 6 with a set of numerical experiments that report on the accuracy of the proposed schemes and that also illustrate the properties of the a posteriori error indicators. Finally, we give a summary of our results and discuss possible extensions in Section 7.

## 2. Model problem

We briefly describe the model problem according to [40], which represents the small-amplitude motion of a fluid in a container with a free surface for the fluid. Let  $\Omega_f$  and  $\Omega_s$  two polygonal/polyhedral bounded domains of  $\mathbb{R}^d$ , where  $d \in \{2, 3\}$ , with Lipschitz boundary. We assume that  $\Omega_f$  is the domain where the fluid is contained whereas  $\Omega_s$  is the domain occupied by the structure. We split in two parts the boundary  $\partial\Omega_f$ : the first part corresponds to the interface contact with the structure, which we denote by  $\Sigma$ , and the second part is an open boundary that we denote by  $\Gamma_0$ . On the other hand, the boundary

$\partial\Omega_s$  is such that  $\partial\Omega_s := \Sigma \cup \Gamma_D \cup \Gamma_N$ , where  $\Gamma_D$  is the part of the solid that we consider as clamped. We assume that  $\Sigma$  is oriented by the normal vector  $\mathbf{n}$  outward to the boundary of  $\Omega_f$ . The unit normal outward to  $\partial\Omega_s$  is also denoted by  $\mathbf{n}$  (see Figure 2.1).

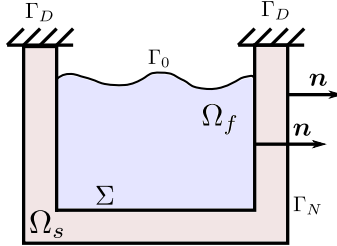


FIG. 2.1. Sketch of a fluid-structure interaction domain with sub-boundaries. Here,  $\Omega_s$  and  $\Omega_f$  denote the solid and fluid subdomains, respectively.

On the solid subdomain, let us denote by  $\boldsymbol{\sigma}$  the Cauchy stress tensor, defined as

$$\boldsymbol{\sigma} := 2\mu \boldsymbol{\varepsilon}(\mathbf{u}) + \lambda \operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{u})) \mathbf{I},$$

where  $\mathbf{u}$  denotes the displacement of the solid,  $\mathbf{I} \in \mathbb{R}^{d \times d}$  with  $d \in \{2, 3\}$  is the identity matrix,  $\boldsymbol{\varepsilon}(\mathbf{u}) := \frac{1}{2}[\nabla \mathbf{u} + (\nabla \mathbf{u})^\top]$  is the infinitesimal strain tensor, and the Lamé parameters are given by

$$\lambda := \frac{E(\mathbf{x})\nu}{(1+\nu)(1-2\nu)} \quad \text{and} \quad \mu := \frac{E(\mathbf{x})}{2(1+\nu)},$$

with  $E$  and  $\nu$  representing the Young modulus and Poisson ratio, respectively, both assumed heterogeneous but uniformly bounded away from zero. Let us define the solid pressure variable  $p := -\lambda \operatorname{div} \mathbf{u}$ . Under the assumption of small oscillations and inspired in [40], if  $\omega > 0$  represents the natural frequencies of the eigenmodes, the system of interest consists in finding the values of the frequencies  $\omega$  for which there is a tuple of solid and fluid displacements, together with Herrmann and fluid pressures  $((\mathbf{u}, \mathbf{w}), (p, p_F))$  that satisfy the following system

$$\begin{aligned} -\operatorname{div}(2\mu \boldsymbol{\varepsilon}(\mathbf{u})) + \nabla p &= \omega^2 \rho_s \mathbf{u} && \text{in } \Omega_s, \\ \operatorname{div} \mathbf{u} + \frac{1}{\lambda} p &= 0 && \text{in } \Omega_s, \\ \nabla p_F &= \omega^2 \rho_f \mathbf{w} && \text{in } \Omega_f, \\ p_F + \rho_f c^2 \operatorname{div} \mathbf{w} &= 0 && \text{in } \Omega_f, \\ (2\mu \boldsymbol{\varepsilon}(\mathbf{u}) - p \mathbf{I}) \mathbf{n} - (\rho_f c^2 \operatorname{div} \mathbf{w}) \mathbf{n} &= \mathbf{0} && \text{on } \Sigma, \\ \mathbf{w} \cdot \mathbf{n} - \mathbf{u} \cdot \mathbf{n} &= 0 && \text{on } \Sigma, \\ \rho_f g(\mathbf{w} \cdot \mathbf{n}) + \rho_f c^2 \operatorname{div} \mathbf{w} &= 0 && \text{on } \Gamma_0, \\ \mathbf{u} &= \mathbf{0} && \text{on } \Gamma_D, \\ (2\mu \boldsymbol{\varepsilon}(\mathbf{u}) - p \mathbf{I}) \mathbf{n} &= \mathbf{0} && \text{on } \Gamma_N, \end{aligned} \tag{2.1}$$



where  $g$  is the gravity magnitude,  $c$  is the sound speed,  $\rho_s$  and  $\rho_f$  denote the density of the solid and the fluid, respectively, and  $\mathbf{w}$  is the fluid displacement. We observe that the pressure  $p_F$  of the fluid can be eliminated replacing the fourth equation on (2.1) on the third, leading to an expression for the fluid depending only on the displacement  $\mathbf{w}$ .

Let us consider the following spaces

$$\mathbf{V} := \left\{ \mathbf{v} \in \mathbf{H}^1(\Omega_s)^d : \mathbf{v} := \mathbf{0} \text{ on } \Gamma_D \right\}, \quad \mathbf{Q} := L^2(\Omega_s), \quad \mathbf{W} := \left\{ \boldsymbol{\tau} \in \mathbf{H}(\text{div}; \Omega_f) : \boldsymbol{\tau} \cdot \mathbf{n} \in L^2(\Gamma_0) \right\},$$

and proceed to test system (2.1) by suitable test functions over  $\mathbf{V}$ ,  $\mathbf{Q}$  and  $\mathbf{W}$ , and imposing the boundary and interface conditions, we obtain the following variational problem: find  $\omega \in \mathbb{R}^+$  and  $(\mathbf{0}, \mathbf{0}, 0) \neq (\mathbf{u}, \mathbf{w}, p) \in \mathbf{V} \times \mathbf{W} \times \mathbf{Q}$  such that

$$\begin{aligned} \int_{\Omega_s} 2\mu(\mathbf{x}) \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) - \int_{\Omega_s} p \text{div } \mathbf{v} + \int_{\Sigma} (\rho_f c^2 \text{div } \mathbf{w})(\mathbf{v} \cdot \mathbf{n}) &= \omega^2 \int_{\Omega_s} \rho_s \mathbf{u} \cdot \mathbf{v}, \quad \forall \mathbf{v} \in \mathbf{V}, \\ - \int_{\Omega_s} \text{div } \mathbf{u} q - \int_{\Omega_s} \frac{1}{\lambda(\mathbf{x})} p q &= 0, \quad \forall q \in \mathbf{Q}, \quad (2.2) \\ \int_{\Omega_f} c^2 \rho_f \text{div } \mathbf{w} \text{div } \boldsymbol{\tau} - \int_{\Sigma} (\rho_f c^2 \text{div } \mathbf{w})(\boldsymbol{\tau} \cdot \mathbf{n}) + \int_{\Gamma_0} (g \rho_f \mathbf{w} \cdot \mathbf{n})(\boldsymbol{\tau} \cdot \mathbf{n}) &= \omega^2 \int_{\Omega_f} \rho_f \mathbf{w} \cdot \boldsymbol{\tau}, \quad \forall \boldsymbol{\tau} \in \mathbf{W}. \end{aligned}$$

Note that the formulation is not well-defined for a Darcy flux merely in  $\mathbf{H}(\text{div}; \Omega_f)$ , since in that case we only have  $\mathbf{w} \cdot \mathbf{n}, \boldsymbol{\tau} \cdot \mathbf{n} \in H^{-1/2}(\partial\Omega_f)$  and therefore the last term on the left-hand side of the third equation in (2.2) might not be bounded. This explains the need for the space  $\mathbf{W}$  defined above, which in turn makes sense as long as  $\Gamma_0$  is sufficiently regular (see also, e.g., [31]). As commonly done in other formulations for fluid-structure interaction problems, we observe from the sixth equation in system (2.1) that, in order to enforce the continuity of the normal displacement across the interface, we can define a functional space of kinematically admissible displacements as follows:

$$\mathbf{Y} := \{(\mathbf{v}, \boldsymbol{\tau}) \in \mathbf{V} \times \mathbf{W} : \mathbf{v} \cdot \mathbf{n} - \boldsymbol{\tau} \cdot \mathbf{n} = 0 \text{ on } \Sigma\},$$

where the transmission condition is understood (at least) in the  $H^{-1/2}(\Sigma)$  sense. This allows us to define a strongly coupled system. Hence, using  $(\mathbf{v}, \boldsymbol{\tau}) \in \mathbf{Y}$  in (2.2), we arrive at the problem: find  $\omega \in \mathbb{R}^+$  and  $((\mathbf{0}, \mathbf{0}), 0) \neq ((\mathbf{u}, \mathbf{w}), p) \in \mathbf{Y} \times \mathbf{Q}$  such that

$$\begin{aligned} \int_{\Omega_s} 2\mu(\mathbf{x}) \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) - \int_{\Omega_s} p \text{div } \mathbf{v} + \int_{\Omega_f} c^2 \rho_f \text{div } \mathbf{w} \text{div } \boldsymbol{\tau} + \int_{\Gamma_0} g \rho_f (\mathbf{w} \cdot \mathbf{n})(\boldsymbol{\tau} \cdot \mathbf{n}) \\ = \omega^2 \left( \int_{\Omega_s} \rho_s \mathbf{u} \cdot \mathbf{v} + \int_{\Omega_f} \rho_f \mathbf{w} \cdot \boldsymbol{\tau} \right) \quad \forall (\mathbf{v}, \mathbf{w}) \in \mathbf{Y}, \quad (2.3) \\ - \int_{\Omega_s} q \text{div } \mathbf{u} - \int_{\Omega_s} \frac{1}{\lambda(\mathbf{x})} p q = 0 \quad \forall q \in \mathbf{Q}. \end{aligned}$$

Let us define  $\kappa := \omega^2$  and  $\mathbf{H} := \mathbf{Y} \times \mathbf{Q}$  in order to rewrite (2.3) as follows: find  $\kappa \in \mathbb{R}^+$  and  $((\mathbf{0}, \mathbf{0}), 0) \neq ((\mathbf{u}, \mathbf{w}), p) \in \mathbf{H}$  such that

$$A(((\mathbf{u}, \mathbf{w}), p), ((\mathbf{v}, \boldsymbol{\tau}), q)) = (\kappa + 1)B(((\mathbf{u}, \mathbf{w}), p), ((\mathbf{v}, \boldsymbol{\tau}), q)), \quad \forall ((\mathbf{v}, \boldsymbol{\tau}), q) \in \mathbf{H}, \quad (2.4)$$

where the continuous bilinear forms  $A : \mathbf{H} \times \mathbf{H} \rightarrow \mathbb{R}$  and  $B : \mathbf{H} \times \mathbf{H} \rightarrow \mathbb{R}$  are defined by

$$A(((\mathbf{u}, \mathbf{w}), p), ((\mathbf{v}, \boldsymbol{\tau}), q)) := \int_{\Omega_s} 2\mu(\mathbf{x}) \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) - \int_{\Omega_s} p \text{div } \mathbf{v} + \int_{\Omega_f} c^2 \rho_f \text{div } \mathbf{w} \text{div } \boldsymbol{\tau}$$

$$\begin{aligned}
& + \int_{\Gamma_0} g \rho_f (\mathbf{w} \cdot \mathbf{n}) (\boldsymbol{\tau} \cdot \mathbf{n}) - \int_{\Omega_s} q \operatorname{div} \mathbf{u} - \int_{\Omega_s} \frac{1}{\lambda(\mathbf{x})} p q + B(((\mathbf{u}, \mathbf{w}), p), ((\mathbf{v}, \boldsymbol{\tau}), q)), \\
B(((\mathbf{u}, \mathbf{w}), p), ((\mathbf{v}, \boldsymbol{\tau}), q)) &:= \int_{\Omega_s} \rho_s \mathbf{u} \cdot \mathbf{v} + \int_{\Omega_f} \rho_f \mathbf{w} \cdot \boldsymbol{\tau},
\end{aligned}$$

for all  $((\mathbf{u}, \mathbf{w}), p), ((\mathbf{v}, \boldsymbol{\tau}), q) \in \mathbf{H}$ . We observe that the formulation associated with the fluid corresponds to the acoustic equations written in terms of the displacement of the fluid. Following the ideas of [8], it is necessary to decompose  $\mathbf{H}(\operatorname{div}; \Omega_f)$  in a suitable way. Also, we observe that the eigenspace associated with  $\kappa = 0$  is  $\mathcal{K} := \{((\mathbf{0}, \operatorname{curl} \boldsymbol{\xi}), 0) : \boldsymbol{\xi} \in \mathbf{H}_0^1(\Omega_f)\} \subset \mathbf{H}$ , and its orthogonal complement (in  $L^2$ ) denoted as  $\mathcal{G}$  and defined by (see, e.g., [8])

$$\mathcal{G} := \{((\mathbf{u}, \nabla \varphi), p) : \mathbf{u} \in \mathbf{V}, \varphi \in H^1(\Omega_f), p \in Q\}.$$

We also define the subspace  $\mathcal{G}_{\mathbf{H}} := \mathcal{G} \cap \mathbf{H}$ . Moreover, it holds that  $\mathbf{H} = \mathcal{K} \oplus \mathcal{G}_{\mathbf{H}}$  (similarly as obtained in [8, Lemma 2.3]).

To perform the analysis, we define the following parameter-weighted norm for all  $((\mathbf{v}, \boldsymbol{\tau}), q) \in \mathbf{H}$

$$\begin{aligned}
\|((\mathbf{v}, \boldsymbol{\tau}), q)\|_{\mathbf{H}}^2 &:= \|\mu(\mathbf{x})^{1/2} \nabla \mathbf{v}\|_{0, \Omega_s}^2 + \|\mu(\mathbf{x})^{-1/2} q\|_{0, \Omega_s}^2 + \|\lambda(\mathbf{x})^{-1/2} q\|_{0, \Omega_s}^2 + \|\rho_s^{1/2} \mathbf{v}\|_{0, \Omega_s}^2 \\
&\quad + \|(c^2 \rho_f)^{1/2} \operatorname{div} \boldsymbol{\tau}\|_{0, \Omega_f}^2 + \|\rho_f^{1/2} \boldsymbol{\tau}\|_{0, \Omega_f}^2 + \|(g \rho_f)^{1/2} \boldsymbol{\tau} \cdot \mathbf{n}\|_{0, \Gamma_0}^2. \quad (2.5)
\end{aligned}$$

Let us now define the solution operator

$$\mathbf{T} : \mathbf{H} \rightarrow \mathbf{H}, \quad ((\mathbf{f}, \mathbf{g}), 0) \mapsto \mathbf{T}((\mathbf{f}, \mathbf{g}), 0) := ((\bar{\mathbf{u}}, \bar{\mathbf{w}}), \bar{p}),$$

where the triplet  $((\bar{\mathbf{u}}, \bar{\mathbf{w}}), \bar{p}) \in \mathbf{H}$  is the solution of the following source problem:

$$A(((\bar{\mathbf{u}}, \bar{\mathbf{w}}), \bar{p}), ((\mathbf{v}, \boldsymbol{\tau}), q)) = B(((\mathbf{f}, \mathbf{g}), 0), ((\mathbf{v}, \boldsymbol{\tau}), q)), \quad \forall ((\mathbf{v}, \boldsymbol{\tau}), q) \in \mathbf{H}. \quad (2.6)$$

**Lemma 2.1** *If  $((\mathbf{f}, \mathbf{g}), 0) \in \mathcal{G}$ , then  $\mathbf{T}((\mathbf{f}, \mathbf{g}), 0) := ((\bar{\mathbf{u}}, \bar{\mathbf{w}}), \bar{p}) \in \mathcal{G}_{\mathbf{H}}$  is the unique solution of (2.6). Moreover, there exist  $\alpha \in (1/2, 1]$ ,  $\beta \in (0, 1]$  and  $C > 0$ , independent of  $\lambda$ , such that the following estimate holds true*

$$\|\bar{\mathbf{u}}\|_{1+\beta, \Omega_s} + \|\bar{\mathbf{w}}\|_{\alpha, \Omega_f} + \|\operatorname{div} \bar{\mathbf{w}}\|_{1, \Omega_f} + \|\bar{p}\|_{\beta, \Omega_s} \leq C \|((\mathbf{f}, \mathbf{g}), 0)\|_{L^2(\Omega_s)^d \times L^2(\Omega_f)^d \times L^2(\Omega_s)}.$$

*Proof* We proceed similarly as in [8, Theorem 2.5], adapting the solid sub-problem to the Herrmann formulation to ensure the regularity bounds are uniform with respect to the Lamé parameter  $\lambda$ . We include the details for sake of completeness. This also provides an explicit proof that was not included in [36, Lemma 2.1].

The existence and uniqueness of the solution  $((\bar{\mathbf{u}}, \bar{\mathbf{w}}), \bar{p}) \in \mathbf{H}$  to the source problem (2.6) follow from the standard inf-sup theory for saddle-point problems [15, Section 4.1] applied to the coupled Herrmann-fluid formulation. Next, we note that the solution operator  $\mathbf{T}$  is continuous from  $L^2(\Omega_s)^d \times L^2(\Omega_f)^d \times L^2(\Omega_s)$  to  $\mathbf{H}$ , yielding the basic stability estimate:

$$\|\bar{\mathbf{u}}\|_{1, \Omega_s} + \|\bar{\mathbf{w}}\|_{\mathbf{H}(\operatorname{div}, \Omega_f)} + \|\bar{p}\|_{0, \Omega_s} \leq C \|((\mathbf{f}, \mathbf{g}), 0)\|_{L^2(\Omega_s)^d \times L^2(\Omega_f)^d \times L^2(\Omega_s)}. \quad (2.7)$$

Furthermore, since the source  $((\mathbf{f}, \mathbf{g}), 0)$  belongs to the irrotational subspace  $\mathcal{G}$ , the fluid displacement operator preserves this structure, meaning that  $((\bar{\mathbf{u}}, \bar{\mathbf{w}}), \bar{p}) \in \mathcal{G}_{\mathbf{H}}$  and  $\bar{\mathbf{w}}$  is a gradient field.

For the divergence estimate in the fluid domain, by choosing test functions  $((\mathbf{0}, \boldsymbol{\tau}), 0)$  with  $\boldsymbol{\tau} \in \mathcal{D}(\Omega_f)^d$  in (2.6), we recover the strong form of the momentum equation in the fluid domain:

$$-\nabla(\operatorname{div} \bar{\mathbf{w}}) + \bar{\mathbf{w}} = \mathbf{g} \quad \text{in } \Omega_f, \quad (2.8)$$

which implies  $\nabla(\operatorname{div} \bar{\mathbf{w}}) = \mathbf{g} - \bar{\mathbf{w}}$  in  $\Omega_f$ . Since  $\mathbf{g} \in L^2(\Omega_f)^d$  and  $\bar{\mathbf{w}} \in L^2(\Omega_f)^d$  (from (2.7)), it follows that  $\nabla(\operatorname{div} \bar{\mathbf{w}}) \in L^2(\Omega_f)^d$ . Thus,  $\operatorname{div} \bar{\mathbf{w}} \in H^1(\Omega_f)$  and we obtain the bound:

$$\|\operatorname{div} \bar{\mathbf{w}}\|_{1, \Omega_f} \leq C \left( \|\mathbf{g}\|_{0, \Omega_f} + \|\bar{\mathbf{w}}\|_{0, \Omega_f} \right) \leq C \|((\mathbf{f}, \mathbf{g}), 0)\|_{L^2(\Omega_s)^d \times L^2(\Omega_f)^d \times L^2(\Omega_s)}. \quad (2.9)$$

For the regularity of the fluid velocity  $\bar{\mathbf{w}}$ , first we remark that since  $((\mathbf{f}, \mathbf{g}), 0) \in \mathcal{G}$ , then the fluid velocity is irrotational. Thus, there exists a potential field  $\bar{q} \in H^1(\Omega_f)$  such that  $\bar{\mathbf{w}} = \nabla \bar{q}$ . Integrating by parts in (2.6),  $\bar{q}$  is shown to be the weak solution of the following compatible Neumann problem:

$$\begin{cases} \Delta \bar{q} = \operatorname{div} \bar{\mathbf{w}} & \text{in } \Omega_f, \\ \nabla \bar{q} \cdot \mathbf{n} = \bar{\mathbf{u}} \cdot \mathbf{n} & \text{on } \Sigma, \\ \nabla \bar{q} \cdot \mathbf{n} = -\rho_f c^2 \operatorname{div} \bar{\mathbf{w}} & \text{on } \Gamma_0, \end{cases}$$

where we have used problem (2.6) and (2.8). Moreover,  $\mathbf{n}$  is the outward unit normal to  $\Omega_f$ . From the trace theorem, trace of the divergence, (2.7) and (2.9), the boundary data satisfies:

$$\|\bar{\mathbf{u}} \cdot \mathbf{n}\|_{1/2, \Sigma} + \|\operatorname{div} \bar{\mathbf{w}}\|_{1/2, \Gamma_0} \leq C(\|\bar{\mathbf{u}}\|_{1, \Omega_s} + \|\operatorname{div} \bar{\mathbf{w}}\|_{1, \Omega_f}) \leq C \|((\mathbf{f}, \mathbf{g}), 0)\|_{L^2(\Omega_s)^d \times L^2(\Omega_f)^d \times L^2(\Omega_s)}.$$

Applying classical a priori estimates for the Neumann problem on the polygonal domain  $\Omega_f$ , there exists a regularity exponent  $\alpha \in (1/2, 1]$  such that  $\bar{q} \in H^{1+\alpha}(\Omega_f)$  and:

$$\begin{aligned} \|\bar{\mathbf{w}}\|_{\alpha, \Omega_f} &= \|\nabla \bar{q}\|_{\alpha, \Omega_f} \leq C \left( \|\operatorname{div} \bar{\mathbf{w}}\|_{0, \Omega_f} + \|\bar{\mathbf{u}} \cdot \mathbf{n}\|_{1/2, \Sigma} + \|\operatorname{div} \bar{\mathbf{w}}\|_{1/2, \Gamma_0} \right) \\ &\leq C \|((\mathbf{f}, \mathbf{g}), 0)\|_{L^2(\Omega_s)^d \times L^2(\Omega_f)^d \times L^2(\Omega_s)}. \end{aligned}$$

Combining this with (2.9) yields the fluid domain control:

$$\|\bar{\mathbf{w}}\|_{\alpha, \Omega_f} + \|\operatorname{div} \bar{\mathbf{w}}\|_{1, \Omega_f} \leq C \|((\mathbf{f}, \mathbf{g}), 0)\|_{L^2(\Omega_s)^d \times L^2(\Omega_f)^d \times L^2(\Omega_s)}. \quad (2.10)$$

Regarding the regularity in the solid domain, by testing (2.6) with functions supported in  $\Omega_s$ , we find that  $(\bar{\mathbf{u}}, \bar{p})$  satisfies the strong form of the elasticity problem in the Herrmann formulation:

$$\begin{cases} -\operatorname{div} \boldsymbol{\sigma}(\bar{\mathbf{u}}, \bar{p}) + \bar{\mathbf{u}} = \mathbf{f} & \text{in } \Omega_s, \\ \operatorname{div} \bar{\mathbf{u}} + \frac{1}{\lambda} \bar{p} = 0 & \text{in } \Omega_s, \end{cases}$$

where  $\boldsymbol{\sigma}(\bar{\mathbf{u}}, \bar{p}) = 2\mu \boldsymbol{\varepsilon}(\bar{\mathbf{u}}) + \bar{p} \mathbf{I}$ . The boundary conditions on  $\partial\Omega_s$  are the following

$$\begin{cases} \boldsymbol{\sigma}(\bar{\mathbf{u}}, \bar{p}) \mathbf{n} = (\rho_f c^2 \operatorname{div} \bar{\mathbf{w}}) \mathbf{n} & \text{on } \Sigma, \\ \boldsymbol{\sigma}(\bar{\mathbf{u}}, \bar{p}) \mathbf{n} = \mathbf{0} & \text{on } \Gamma_N, \\ \bar{\mathbf{u}} = \mathbf{0} & \text{on } \Gamma_D. \end{cases}$$

For the Herrmann formulation, the inf-sup condition and corresponding regularity estimates are uniform with respect to  $\lambda$ . Proceeding as in the proof of [8, Theorem 2.5] we have that there exists a geometric

regularity exponent  $\beta \in (0, 1]$ , depending on the boundary conditions and the Lamé coefficients, and a constant  $C > 0$  independent of  $\lambda$ , such that:

$$\|\bar{\mathbf{u}}\|_{1+\beta, \Omega_s} + \|\bar{p}\|_{\beta, \Omega_s} \leq C (\|\mathbf{f}\|_{0, \Omega_s} + \|\sigma(\bar{\mathbf{u}}, \bar{p})\mathbf{n}\|_{1/2, \Sigma}).$$

Since the traction boundary condition on  $\Sigma$  is bounded by the trace of the fluid divergence, we have  $\|\sigma(\bar{\mathbf{u}}, \bar{p})\mathbf{n}\|_{1/2, \Sigma} \leq C \|\operatorname{div} \bar{\mathbf{w}}\|_{1/2, \Sigma}$ . Then, by using the trace theorem for divergence and (2.9), we obtain

$$\|\bar{\mathbf{u}}\|_{1+\beta, \Omega_s} + \|\bar{p}\|_{\beta, \Omega_s} \leq C \|((\mathbf{f}, \mathbf{g}), 0)\|_{L^2(\Omega_s)^d \times L^2(\Omega_f)^d \times L^2(\Omega_s)}. \quad (2.11)$$

Summing the uniform fluid estimate (2.10) and the uniform solid estimate (2.11) readily gives:

$$\|\bar{\mathbf{u}}\|_{1+\beta, \Omega_s} + \|\bar{\mathbf{w}}\|_{\alpha, \Omega_f} + \|\operatorname{div} \bar{\mathbf{w}}\|_{1, \Omega_f} + \|\bar{p}\|_{\beta, \Omega_s} \leq C \|((\mathbf{f}, \mathbf{g}), 0)\|_{L^2(\Omega_s)^d \times L^2(\Omega_f)^d \times L^2(\Omega_s)},$$

where  $\alpha \in (1/2, 1]$ ,  $\beta \in (0, 1]$ , and  $C > 0$  is independent of  $\lambda$ . This completes the proof.  $\square$

**Remark 2.2** *Let us notice that the regularity parameters  $\alpha$  and  $\beta$  depend on the configuration of the problem. More precisely, as is indicated on the proof of [8, Theorem 2.5], if  $\Omega_f$  is convex,  $\alpha = 1$  whereas if  $\alpha = \pi/\theta$ , with  $\theta$  the biggest reentrant corner, otherwise. On the other hand,  $\beta$  depends on the reentrant corners of  $\partial\Omega_s$ , the angles between  $\Gamma_D$  and  $\Gamma_N$ , and on the Lamé coefficients.*

We observe that Lemma 2.1 establishes that source problem (2.6) is well posed and, as a consequence of this, we have that  $\mathbf{T}$  is well defined. We also notice that operator  $\mathbf{T}$  is non-compact, since  $H(\operatorname{div}; \Omega_f)$  is not compactly embedded in  $L^2(\Omega_f)^d$ . Moreover, we observe that  $(\kappa, ((\mathbf{u}, \mathbf{w}), p)) \in \mathbb{R}^+ \times \mathbf{H}$  solves (2.4) if and only if  $(\vartheta, ((\mathbf{u}, \mathbf{w}), p))$  is an eigenpair of  $\mathbf{T}$ , i.e., if  $((\mathbf{u}, \mathbf{w}), p) \neq ((\mathbf{0}, \mathbf{0}), 0)$  and  $\mathbf{T}((\mathbf{u}, \mathbf{w}), p) = (\vartheta, ((\mathbf{u}, \mathbf{w}), p))$ , with  $\vartheta := (\kappa + 1)^{-1} \neq 0$ .

Moreover, as a consequence of Lemma 2.1, we have the following additional regularity for the eigenfunctions, which holds when  $((\mathbf{f}, \mathbf{g}), 0) = (((\kappa + 1)\mathbf{u}, (\kappa + 1)\mathbf{w}), 0)$ .

**Corollary 2.3** *There exist  $\alpha_1 \in (1/2, 1]$ ,  $\beta_1 \in (0, 1]$  and  $C > 0$  independent of  $\lambda$  but depending on the eigenvalue  $\kappa$ , such that the following estimate holds true*

$$\|\mathbf{u}\|_{1+\beta_1, \Omega_s} + \|\mathbf{w}\|_{\alpha_1, \Omega_f} + \|\operatorname{div} \mathbf{w}\|_{1, \Omega_f} + \|p\|_{\beta_1, \Omega_s} \leq C \|((\mathbf{u}, \mathbf{w}), p)\|_{L^2(\Omega_s)^d \times L^2(\Omega_f)^d \times L^2(\Omega_s)}.$$

Next, the following result provides a spectral characterization of  $\mathbf{T}$ . It follows from a slight modification of the proof of [8, Theorem 2.7] together with the regularity provided by Corollary 2.3.

**Theorem 2.4** *The spectrum of  $\mathbf{T}$  decomposes as follows:  $\operatorname{sp}(\mathbf{T}) = \{0, 1\} \cup \{\vartheta_k\}_{k \in \mathbb{N}}$ , where:*

- i)  $\vartheta = 1$  is an infinite-multiplicity eigenvalue of  $\mathbf{T}$  and its associated eigenspace is  $\mathcal{H}$ .
- ii)  $\{\vartheta_k\}_{k \in \mathbb{N}} \subset (0, 1)$  is a sequence of finite-multiplicity eigenvalues of  $\mathbf{T}$  which converge to 0 and the corresponding eigenspaces lie in  $\{((\mathbf{u}, \mathbf{w}), p) \in H^{1+\beta_1}(\Omega_s)^d \times H^{\alpha_1}(\Omega_f)^d \times H^{\beta_1}(\Omega_s) : \operatorname{div} \mathbf{w} \in H^1(\Omega_f)\}$ .

### 3. Numerical discretization

In this section our aim is to describe a FE discretization of problem (2.4). Let  $\mathcal{T}_h(\Omega_s)$  and  $\mathcal{T}_h(\Omega_f)$  be a conforming partition of the polyhedral domains  $\bar{\Omega}_s$  and  $\Omega_f$ , respectively, into triangles (tetrahedrons)

$T$  with size  $h_T = \text{diam}(T)$ . Define  $h := \max\{h_T : T \in \mathcal{T}_h(\Omega_s) \cup \Omega_f\}$ . Both the interface and the meshes are assumed conformal, that is, meshes are constructed such that the vertices of  $\mathcal{T}_h(\Omega_s)$  and  $\mathcal{T}_h(\Omega_f)$  coincide on  $\Sigma_h = \Sigma$ . On the other hand, the numerical method that we propose is nonconforming. To make matters precise, on the solid part of the problem, the discretization will be considered under the approach of inf-sup stable families of FEs for Stokes, whereas for the fluid, Brezzi–Douglas–Marini elements will be considered. This is the key point on the conforming or non conforming nature of the methods.

Let  $k \geq 1$  and  $S \subseteq \mathbb{R}^d$ . We denote by  $\mathbb{P}_k(S)$  the space of polynomial functions defined on  $S$  of total degree  $\leq k$ . In particular, given two families of inf-sup stable FEs  $\mathbf{V}_h$  and  $\mathbf{Q}_h$  to approximate the solid displacement  $\mathbf{u}_s$  and the pressure  $p$  we can take, for example,

- (a) the MINI element [28, Section 4.2.4]:

$$\begin{aligned}\mathbf{V}_h &= \{\mathbf{v}_h \in \mathbf{C}(\overline{\Omega}) : \mathbf{v}_h|_T \in [\mathbb{P}_1(T) \oplus \mathbb{B}(T)]^d \forall T \in \mathcal{T}_h(\Omega_s)\} \cap \mathbf{V}, \\ \mathbf{Q}_h &= \{q_h \in C(\overline{\Omega}) : q_h|_T \in \mathbb{P}_1(T) \forall T \in \mathcal{T}_h(\Omega_s)\},\end{aligned}$$

where  $\mathbb{B}(T)$  denotes the space spanned by local bubble functions; or

- (b) the Taylor–Hood element [28, Section 4.2.5]:

$$\begin{aligned}\mathbf{V}_h &= \{\mathbf{v}_h \in \mathbf{C}(\overline{\Omega}) : \mathbf{v}_h|_T \in \mathbb{P}_{k+1}(T)^d \forall T \in \mathcal{T}_h(\Omega_s)\} \cap \mathbf{V}, \\ \mathbf{Q}_h &= \{q_h \in C(\overline{\Omega}) : q_h|_T \in \mathbb{P}_k(T) \forall T \in \mathcal{T}_h(\Omega_s)\}.\end{aligned}$$

Let us introduce the space to approximate the fluid displacement. We use the well-known Brezzi–Douglas–Marini finite element space  $\mathbf{BDM}_k := \mathbb{P}_k(\mathcal{T}_h((\Omega_f)))^d$  with  $k \geq 1$ . We set  $\mathbf{W}_h := \mathbf{BDM}_k \cap \mathbf{W}$  to be the corresponding global space. Then, we need to impose the condition  $(\mathbf{w} - \mathbf{u}) \cdot \mathbf{n} = 0$  on  $\Sigma$  at discrete level. Thus, we introduce the following space

$$\mathbf{Y}_h := \{(\mathbf{v}_h, \boldsymbol{\tau}_h) \in \mathbf{V}_h \times \mathbf{W}_h : \mathbf{v}_h \cdot \mathbf{n} - \boldsymbol{\tau}_h \cdot \mathbf{n} = 0 \text{ on } \Sigma\},$$

where the MINI element is coupled with  $\mathbf{BDM}_1$  and Taylor–Hood element for  $k \geq 1$  is coupled with  $\mathbf{BDM}_{k+1}$  (see [39, Table 1] for further details). Let us remark that the choice of BDM elements to approximate what concerns to the fluid leads to a conforming discretization since  $\mathbf{Y}_h \subset \mathbf{Y}$ .

Now we recall some well-known approximation properties for this FE family. First, for the fluid, given  $s \in (0, 1]$ , let  $\boldsymbol{\Pi}_h : \mathbf{H}^s(\Omega_f)^d \cap \mathbf{W} \rightarrow \mathbf{W}_h$  be the classic global lowest order BDM interpolant operator which satisfies

$$\int_{\ell} (\boldsymbol{\Pi}_h \boldsymbol{\tau} \cdot \mathbf{n}_{\ell}) \zeta = \int_{\ell} (\boldsymbol{\tau} \cdot \mathbf{n}_{\ell}) \zeta, \quad \forall \zeta \in \mathbb{P}_1(\ell)^d,$$

where  $\ell$  is an edge of any  $T \in \mathcal{T}_h(\Omega_f)$ , and the following commutative diagram property holds true

$$\text{div}(\boldsymbol{\Pi}_h \boldsymbol{\tau}) = \mathcal{P}_h(\text{div} \boldsymbol{\tau}) \quad \forall \boldsymbol{\tau} \in \mathbf{H}^s(\Omega_f)^d \cap \mathbf{H}(\text{div}; \Omega_f),$$

where  $\mathcal{P}_h : \mathbf{L}^2(\Omega_f) \rightarrow \mathcal{U}_h$  is the  $\mathbf{L}^2(\Omega_f)$ -orthogonal projection onto

$$\mathcal{U}_h := \{q_h \in \mathbf{L}^2(\Omega_f) : q_h|_T \in \mathbb{P}_{k-1}(T) \forall T \in \mathcal{T}_h(\Omega_f)\}.$$

Moreover, for  $s \in (0, 1]$ , the operators  $\boldsymbol{\Pi}_h$  and  $\mathcal{P}_h$  satisfy the following approximation properties (see [41, eq. (4.2), (4.3) and (4.4)])

$$\|\boldsymbol{\tau} - \boldsymbol{\Pi}_h \boldsymbol{\tau}\|_{0, \Omega_f} \leq Ch^s \left( \|\boldsymbol{\tau}\|_{s, \Omega_f} + \|\text{div} \boldsymbol{\tau}\|_{0, \Omega_f} \right) \quad \forall \boldsymbol{\tau} \in \mathbf{H}(\text{div}; \Omega_f) \cap \mathbf{H}^s(\Omega_f)^d, \quad (3.1a)$$

$$\|\boldsymbol{\tau} - \mathbf{\Pi}_h \boldsymbol{\tau}\|_{\text{div}, \Omega_f} \leq Ch^s \|\boldsymbol{\tau}\|_{\mathbf{H}^s(\text{div}; \Omega_f)} \quad \forall \boldsymbol{\tau} \in \mathbf{H}^s(\text{div}; \Omega_f) \cap \mathbf{W}, \quad (3.1b)$$

$$\|\mathbf{v} - \mathcal{P}_h \mathbf{v}\|_{0, \Omega_f} \leq Ch^s \|\mathbf{v}\|_{s, \Omega_f} \quad \forall \mathbf{v} \in \mathbf{H}^s(\Omega_f)^d. \quad (3.1c)$$

In the solid domain, we also introduce the orthogonal projection  $\Lambda_h : \mathbf{H}^1(\Omega_s)^d \rightarrow \mathcal{M}_h$ , where

$$\mathcal{M}_h := \{\mathbf{v}_h \in \mathbf{V}_h : \mathbf{v}_h|_T \in \mathbb{P}_{k+1}(T) \ \forall T \in \mathcal{T}_h(\Omega_s)\},$$

with  $k = 0$  for MINI element, and  $k \geq 1$  for Taylor–Hood elements. The projection satisfies

$$\|\mathbf{v} - \Lambda_h \mathbf{v}\|_{1, \Omega_s} \leq Ch^s \|\mathbf{v}\|_{1+s, \Omega_s} \quad \forall \mathbf{v} \in \mathbf{H}^{1+s}(\Omega_s)^d. \quad (3.2)$$

We will also consider  $\mathcal{P}_h^c : \mathbf{L}^2(\Omega_s) \rightarrow \mathbf{Q}_h$  as the  $\mathbf{L}^2(\Omega_s)$ -orthogonal projection onto  $\mathbf{Q}_h$ .

$$\|q - \mathcal{P}_h^c q\|_{0, \Omega_s} \leq Ch^s \|q\|_{s, \Omega_s} \quad \forall q \in \mathbf{H}^s(\Omega_s). \quad (3.3)$$

### 3.1. The interface approximation

In standard finite element approximations of fluid–structure interaction problems, directly interpolating the fluid and solid variables independently often leads to a mismatch of the normal traces on the interface  $\Sigma$ . To ensure that our discrete functions belong to the conforming space  $\mathbf{Y}_h$ , our goal in this section is to construct a *corrected* Brezzi–Douglas–Marini (BDM) interpolation operator that properly approximates the continuous solutions while strictly preserving the interface transmission conditions. To achieve this, we follow the theoretical framework established in [40, Section 5] by introducing a specialized divergence-free extension map. Given  $\mathbf{v} \in \mathbf{H}^1(\Omega_s)^d$ , let  $\boldsymbol{\varphi} \in \mathbf{H}^1(\Omega_f)$  be the unique zero-mean solution of the following Neumann problem:

$$-\Delta \boldsymbol{\varphi} = \frac{1}{|\Omega_f|} \int_{\Sigma} \mathbf{v} \cdot \mathbf{n} \quad \text{in } \Omega_f, \quad \frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{n}} = 0 \quad \text{in } \Gamma_0, \quad \frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{n}} = \mathbf{v} \cdot \mathbf{n} \quad \text{on } \Sigma. \quad (3.4)$$

Applying classic regularity results for elliptic problems (see, e.g., [23, 33] for the case of Lipschitz domains that are not necessarily convex), we know that there exists  $\hat{v} > 0$  such that  $\boldsymbol{\varphi} \in \mathbf{W}^{2, \hat{v}}(\Omega_f)$  with  $\frac{3}{2} - \varepsilon < \hat{v} < 3 + \varepsilon$  for a small  $\varepsilon > 0$  depending on the domain. Therefore  $\nabla \boldsymbol{\varphi} \in \mathbf{H}^v(\Omega_f)^d$  for every  $v < 2 - \frac{2}{\hat{v}}$ , giving a guaranteed lower bound of  $v < \frac{2}{3}$ , for more details see also [24, Theorem 4.57]. As a consequence of this regularity and the trace inequality applied to  $\mathbf{v} \cdot \mathbf{n}$  on  $\Sigma$ , the following stability estimate holds true:

$$\|\nabla \boldsymbol{\varphi}\|_{v, \Omega_f} \leq C \|\mathbf{v}\|_{1, \Omega_s}. \quad (3.5)$$

On the other hand, we define  $\hat{\boldsymbol{\psi}} := -\nabla \boldsymbol{\varphi}$  such that  $\text{div } \hat{\boldsymbol{\psi}} \in \mathbb{R}$ , since the right-hand side of the first equation in (3.4) is explicitly a constant. Let us define the following linear operator

$$\mathbf{E} : \mathbf{H}^1(\Omega_s)^d \rightarrow \mathbf{W}, \quad \mathbf{v} \mapsto \mathbf{E} \mathbf{v} := -\hat{\boldsymbol{\psi}} = \nabla \boldsymbol{\varphi}.$$

Observe that  $\mathbf{E}$  is bounded and provides an extension from  $\Omega_s$  to  $\Omega_f$  that lifts the normal trace of  $\mathbf{v}$  from the interface to the fluid domain. According to (3.5), it satisfies  $\mathbf{E} \mathbf{v} \in \mathbf{H}^v(\Omega_f)^d$  with  $\|\mathbf{E} \mathbf{v}\|_{v, \Omega_f} \leq C \|\mathbf{v}\|_{1, \Omega_s}$  for all  $\mathbf{v} \in \mathbf{H}_{\Gamma_D}^1(\Omega_s)^d$ . Furthermore, by construction of the Neumann problem,  $\mathbf{E} \mathbf{v} \cdot \mathbf{n} = \mathbf{v} \cdot \mathbf{n}$  on  $\Sigma$  and  $\mathbf{E} \mathbf{v} \cdot \mathbf{n} = 0$  on  $\Gamma_0$ . The following properties of  $\mathbf{E}$  hold true.

**Lemma 3.1** *There exists a constant  $C > 0$  independent of  $h$  such that*

$$\begin{aligned} \|\Pi_h \mathbf{E} \mathbf{v}\|_{0, \Omega_f} &\leq C \|\mathbf{v}\|_{1, \Omega_s} \quad \forall \mathbf{v} \in \mathbf{H}_{\Gamma_D}^1(\Omega_s)^d, \\ \|\mathbf{E} \mathbf{v} - \Pi_h \mathbf{E} \mathbf{v}\|_{0, \Omega_f} &\leq Ch^v \|\mathbf{v}\|_{1, \Omega_s} \quad \forall \mathbf{v} \in \mathbf{H}_{\Gamma_D}^1(\Omega_s)^d. \end{aligned}$$

*Proof* Although the detailed technical proof is available in [40, Lemma 5.1], the rationale is as follows: since  $\operatorname{div}(\mathbf{E} \mathbf{v})$  is constant over  $\Omega_f$ , the BDM interpolant  $\Pi_h$  preserves it exactly due to the commuting diagram property. The first bound then follows directly from the uniform stability of the BDM interpolant in  $L^2(\Omega_f)^d$  for functions with constant divergence, combined with the bound  $\|\mathbf{E} \mathbf{v}\|_{v, \Omega_f} \leq C \|\mathbf{v}\|_{1, \Omega_s}$ . The second estimate is a standard interpolation error bound for  $\Pi_h$  acting on the  $H^v(\Omega_f)^d$ -regular field  $\mathbf{E} \mathbf{v}$ .  $\square$

Next, let  $\mathbf{E}_h$  be the discrete counterpart of  $\mathbf{E}$ , defined by  $\mathbf{E}_h \mathbf{v} := \Pi_h \mathbf{E}(\Lambda_h \mathbf{v}) \in \mathbf{W}_h$ , for any  $\mathbf{v} \in \mathbf{H}_{\Gamma_D}^1(\Omega_s)^d$ . The following result provides an approximation property between operators  $\mathbf{E}$  and  $\mathbf{E}_h$ .

**Lemma 3.2** *There exists a constant  $C > 0$ , independent of  $h$ , such that*

$$\|\mathbf{E} \mathbf{v} - \mathbf{E}_h \mathbf{v}\|_{\operatorname{div}, \Omega_f} \leq C (h^v \|\mathbf{v}\|_{1, \Omega_s} + \|\mathbf{v} - \Lambda_h \mathbf{v}\|_{1, \Omega_s}) \quad \forall \mathbf{v} \in \mathbf{H}_{\Gamma_D}^1(\Omega_s)^d.$$

*Proof* Following the arguments in [40, Lemma 5.2], this bound is achieved by adding and subtracting  $\Pi_h \mathbf{E} \mathbf{v}$  and employing the triangle inequality. The term  $\|\mathbf{E} \mathbf{v} - \Pi_h \mathbf{E} \mathbf{v}\|_{\operatorname{div}, \Omega_f}$  is bounded using Lemma 3.1, while the remaining term  $\|\Pi_h \mathbf{E}(\mathbf{v} - \Lambda_h \mathbf{v})\|_{\operatorname{div}, \Omega_f}$  is controlled by the continuity of  $\mathbf{E}$  and the stability of the projection operators, leading directly to the term  $\|\mathbf{v} - \Lambda_h \mathbf{v}\|_{1, \Omega_s}$ .  $\square$

Our next task is to prove that any smooth enough  $(\mathbf{u}, \mathbf{w}) \in \mathbf{Y}$  is well approximated by functions of  $\mathbf{Y}_h$ . With this aim, we need to correct the standard BDM interpolant on the fluid to safely ensure the continuity of the normal traces across the interface  $\Sigma$  against the continuous piecewise linear solid displacements. To do this task, we define the corrected interpolant operator  $\widehat{\Pi}_h : \mathbf{Y} \rightarrow \mathbf{Y}_h$  as follows

$$\widehat{\Pi}_h(\mathbf{u}, \mathbf{w}) := (\Lambda_h \mathbf{u}, \Pi_h \mathbf{w} + (\mathbf{E}_h \mathbf{u} - \Pi_h \mathbf{E} \mathbf{u})) \quad \forall (\mathbf{u}, \mathbf{w}) \in \mathbf{Y}.$$

By adding the correction term  $(\mathbf{E}_h \mathbf{u} - \Pi_h \mathbf{E} \mathbf{u})$ , we adjust the normal component of the fluid variable on  $\Sigma$  to exactly match  $\Lambda_h \mathbf{u} \cdot \mathbf{n}$ , without perturbing the normal trace on the rigid boundary  $\Gamma_0$  (since both  $\mathbf{E}_h \mathbf{u} \cdot \mathbf{n} = 0$  and  $\Pi_h \mathbf{E} \mathbf{u} \cdot \mathbf{n} = 0$  on  $\Gamma_0$ ). The following lemma guarantees that  $\widehat{\Pi}_h(\mathbf{u}, \mathbf{w})$  indeed lies in  $\mathbf{Y}_h$  and establishes its error estimate.

**Lemma 3.3** *Let  $(\mathbf{u}, \mathbf{w}) \in \mathbf{Y}$  with  $\mathbf{w} \in H^v(\Omega_f)^d$  and let  $(\mathbf{u}_h, \mathbf{w}_h) := \widehat{\Pi}_h(\mathbf{u}, \mathbf{w})$ . Then,  $(\mathbf{u}_h, \mathbf{w}_h) \in \mathbf{Y}_h$  and*

$$\|(\mathbf{u}, \mathbf{w}) - (\mathbf{u}_h, \mathbf{w}_h)\|_{\mathbf{W} \times \mathbf{V}} \leq C \left( \|\mathbf{w} - \Pi_h \mathbf{w}\|_{\operatorname{div}, \Omega_f} + \|\mathbf{u} - \Lambda_h \mathbf{u}\|_{1, \Omega_s} \right).$$

*Proof* The conformity  $(\mathbf{u}_h, \mathbf{w}_h) \in \mathbf{Y}_h$  follows immediately from the trace preservation properties discussed above. The error bound is obtained by applying the triangle inequality to separate the pure interpolation errors from the correction term, and then bounding the correction term  $\|\mathbf{E}_h \mathbf{u} - \Pi_h \mathbf{E} \mathbf{u}\|_{\operatorname{div}, \Omega_f}$  using Lemma 3.2. For further details, we refer the reader to [40, Lemma 5.3].  $\square$



The first step towards analyzing the global scheme is to construct an adequate interpolant that acts on the full product space, covering both the domain occupied by the fluid, the domain of the solid, and the interface. With this aim, we introduce the *global corrected* operator  $\mathbf{M}_h : \mathbf{H} \rightarrow \mathbf{Y}_h \times \mathbf{Q}_h$ , defined as follows:

$$\mathbf{M}_h((\mathbf{v}, \boldsymbol{\tau}), q) := \begin{cases} ((\mathbf{0}, \mathbf{0}), \mathcal{P}_h^c q) & \text{in } \Omega_s, \\ (\widehat{\Pi}_h(\mathbf{v}, \boldsymbol{\tau}), 0), & \text{otherwise.} \end{cases}$$

Moreover, the operator  $\mathbf{M}_h$  satisfies the following approximation property.

**Lemma 3.4** *There exists a positive constant  $C$  such that, for all  $((\mathbf{v}, \boldsymbol{\tau}), q) \in \mathbf{H}^{1+\beta}(\Omega_s)^d \times \mathbf{H}^\alpha(\Omega_f)^d \times \mathbf{H}^\beta(\Omega_s)$  with  $\operatorname{div} \boldsymbol{\tau} \in \mathbf{H}^1(\Omega_f)$  there holds*

$$\|((\mathbf{v}, \boldsymbol{\tau}), q) - \mathbf{M}_h((\mathbf{v}, \boldsymbol{\tau}), q)\|_{\mathbf{H}} \leq Ch^r (\|\boldsymbol{\tau}\|_{\alpha, \Omega_f} + \|\operatorname{div} \boldsymbol{\tau}\|_{1, \Omega_f} + \|\mathbf{v}\|_{1+\beta, \Omega_s} + \|q\|_{\beta, \Omega_s}),$$

where  $\|\cdot\|_{\mathbf{H}}$  is the norm defined in (2.5),  $r = \min\{\alpha - 1/2, \beta, v, k\}$  where the parameters  $\alpha \in (1/2, 1]$  and  $\beta \in (0, 1]$  are as in Lemma 2.1,  $v$  is as in Lemma 3.1, and  $k \geq 1$  is the respective polynomial degree.

*Proof* Let  $((\mathbf{v}, \boldsymbol{\tau}), q) \in \mathbf{H}^{1+\beta}(\Omega_s)^d \times \mathbf{H}^\alpha(\Omega_f)^d \times \mathbf{H}^\beta(\Omega_s)$  with  $\operatorname{div} \boldsymbol{\tau} \in \mathbf{H}^1(\Omega_f)$ . From the definition of  $\mathbf{M}_h$ , direct computations expanding the  $\mathbf{H}$ -norm reveal that

$$\begin{aligned} \|((\mathbf{v}, \boldsymbol{\tau}), q) - \mathbf{M}_h((\mathbf{v}, \boldsymbol{\tau}), q)\|_{\mathbf{H}}^2 &\leq \|\mu(\mathbf{x})^{1/2} \nabla(\mathbf{v} - \Lambda_h \mathbf{v})\|_{0, \Omega_s}^2 + \|\mu(\mathbf{x})^{-1/2} (q - \mathcal{P}_h^c q)\|_{0, \Omega_s}^2 \\ &\quad + \|\lambda(\mathbf{x})^{-1/2} (q - \mathcal{P}_h^c q)\|_{0, \Omega_s}^2 + \|\rho_s^{1/2} (\mathbf{v} - \Lambda_h \mathbf{v})\|_{0, \Omega_s}^2 \\ &\quad + \|(c^2 \rho_f)^{1/2} \operatorname{div}(\boldsymbol{\tau} - (\Pi_h \boldsymbol{\tau} + \mathbf{E}_h \mathbf{v} - \Pi_h \mathbf{E} \mathbf{v}))\|_{0, \Omega_f}^2 \\ &\quad + \|\rho_f^{1/2} (\boldsymbol{\tau} - (\Pi_h \boldsymbol{\tau} + \mathbf{E}_h \mathbf{v} - \Pi_h \mathbf{E} \mathbf{v}))\|_{0, \Omega_f}^2 \\ &\quad + \|(g \rho_f)^{1/2} (\boldsymbol{\tau} - (\Pi_h \boldsymbol{\tau} + \mathbf{E}_h \mathbf{v} - \Pi_h \mathbf{E} \mathbf{v})) \cdot \mathbf{n}\|_{0, \Gamma_0}^2 \\ &\leq \mathbf{I}_1 + \mathbf{I}_2, \end{aligned}$$

where we have separated the standard interpolation errors ( $\mathbf{I}_1$ ) from the extension correction terms ( $\mathbf{I}_2$ ):

$$\begin{aligned} \mathbf{I}_1 &:= C_1 \left( \|\nabla(\mathbf{v} - \Lambda_h \mathbf{v})\|_{0, \Omega_s}^2 + \|q - \mathcal{P}_h^c q\|_{0, \Omega_s}^2 + \|\mathbf{v} - \Lambda_h \mathbf{v}\|_{0, \Omega_s}^2 + \|\operatorname{div}(\boldsymbol{\tau} - \Pi_h \boldsymbol{\tau})\|_{0, \Omega_f}^2 \right. \\ &\quad \left. + \|\boldsymbol{\tau} - \Pi_h \boldsymbol{\tau}\|_{0, \Omega_f}^2 + \|(g \rho_f)^{1/2} (\boldsymbol{\tau} - \Pi_h \boldsymbol{\tau}) \cdot \mathbf{n}\|_{0, \Gamma_0}^2 \right), \\ \mathbf{I}_2 &:= C_2 \left( \|\operatorname{div}(\Pi_h \mathbf{E}(\mathbf{v} - \Lambda_h \mathbf{v}))\|_{0, \Omega_f}^2 + \|\Pi_h \mathbf{E}(\mathbf{v} - \Lambda_h \mathbf{v})\|_{0, \Omega_f}^2 + \|(\Pi_h \mathbf{E}(\mathbf{v} - \Lambda_h \mathbf{v}) \cdot \mathbf{n})\|_{0, \Gamma_0}^2 \right), \end{aligned}$$

with constants  $C_1, C_2 > 0$  depending on  $\mu_{\max}, \mu_{\min}, \lambda_{\min}, \rho_s, \rho_f, c$  and  $g$ .

From standard boundary approximation properties and the following estimate (valid for the BDM interpolation applied to any  $\boldsymbol{\tau} \in \mathbf{H}^\alpha(\Omega_f)^d$ ),  $\|\Pi_h \boldsymbol{\tau} \cdot \mathbf{n} - \boldsymbol{\tau} \cdot \mathbf{n}\|_{0, \Gamma_0} \leq Ch^\zeta |\boldsymbol{\tau}|_{\alpha, \Omega_f}$ , with  $\zeta := \min\{\alpha - 1/2, k\}$ , we have by the triangle inequality:

$$\begin{aligned} \|\Pi_h \mathbf{E}(\mathbf{v} - \Lambda_h \mathbf{v}) \cdot \mathbf{n}\|_{0, \Gamma_0} &\leq \|\Pi_h \mathbf{E}(\mathbf{v} - \Lambda_h \mathbf{v}) \cdot \mathbf{n} - \mathbf{E}(\mathbf{v} - \Lambda_h \mathbf{v}) \cdot \mathbf{n}\|_{0, \Gamma_0} + \|\mathbf{E}(\mathbf{v} - \Lambda_h \mathbf{v}) \cdot \mathbf{n}\|_{0, \Gamma_0} \\ &\leq C \left( h^\zeta |\mathbf{E}(\mathbf{v} - \Lambda_h \mathbf{v})|_{1/2+\zeta, \Omega_f} + \|\mathbf{E}(\mathbf{v} - \Lambda_h \mathbf{v})\|_{1/2+\zeta, \Omega_f} \right) \end{aligned}$$

$$\leq C \left( h^\zeta \|\mathbf{v} - \Lambda_h \mathbf{v}\|_{1, \Omega_s} + \|\mathbf{v} - \Lambda_h \mathbf{v}\|_{1, \Omega_s} \right).$$

Hence, the desired estimate follows by replacing these intermediate bounds back into  $\mathbf{I}_1$  and  $\mathbf{I}_2$ , and directly applying the classical convergence rates (3.1a)-(3.2), Lemma 3.1 and Lemma 3.2.  $\square$

**Remark 3.5** *It might seem contradictory to introduce the specialized extension operator  $\mathbf{E}$  to handle the interface  $\Sigma$ , given that the  $\mathbf{H}$ -norm evaluated in Lemma 3.4 consists purely of bulk terms and does not explicitly contain any interface integrals. The necessity for this correction lies in the conformity required by the discrete space  $\mathbf{Y}_h$ . The global continuous space  $\mathbf{Y}$  inherently encodes the kinematic transmission condition across the interface  $\mathbf{w} \cdot \mathbf{n} = \mathbf{u} \cdot \mathbf{n}$  on  $\Sigma$ .*

*With a standard, uncorrected interpolant pair  $(\Lambda_h \mathbf{u}, \Pi_h \mathbf{w})$ , the mismatch in their normal traces on  $\Sigma$  would imply that  $(\Lambda_h \mathbf{u}, \Pi_h \mathbf{w}) \notin \mathbf{Y}_h$ . Consequently, this uncorrected pair is no longer a candidate in the infimum of the discrete best-approximation estimates (such as property P2). Furthermore, evaluating the discrete variational formulation with a non-conforming test/trial function would invoke Strang's Second Lemma, introducing an interface consistency error term. This non-vanishing boundary integral would fail to achieve the optimal  $\mathcal{O}(h^r)$  rate. Therefore, the correction via  $\mathbf{E}_h$  is required to enforce strict conformity in  $\mathbf{Y}_h$ , which in turn allows the global error to be bounded cleanly via the bulk terms in the  $\mathbf{H}$ -norm.*

### 3.2. The discrete problem

In this section we introduce the discrete version of (2.4) and the corresponding discrete solution operator. The aim is to prove that the discrete source problem is well posed (implying that the discrete solution operator is well defined) and that its solution is capable to approximate the continuous source solution of (2.6). To achieve this goal, the properties of operator  $\mathbf{M}_h$  proved in the previous section are crucial.

Let us begin by introducing the finite element discretization of (2.4) which reads as follows: find  $((\mathbf{u}_h, \mathbf{w}_h), p_h) \in \mathbf{H}_h := \mathbf{Y}_h \times Q_h$  such that

$$A(((\mathbf{u}_h, \mathbf{w}_h), p_h), ((\mathbf{v}_h, \boldsymbol{\tau}_h), q_h)) = (\kappa_h + 1)B(((\mathbf{u}_h, \mathbf{w}_h), p_h), ((\mathbf{v}_h, \boldsymbol{\tau}_h), q_h)) \quad \forall ((\mathbf{v}_h, \boldsymbol{\tau}_h), q_h) \in \mathbf{H}_h. \quad (3.6)$$

Let us define the corresponding discrete solution operator

$$\mathbf{T}_h : \mathbf{H} \rightarrow \mathbf{H}_h, \quad ((\mathbf{f}, \mathbf{g}), 0) \mapsto \mathbf{T}_h((\mathbf{f}, \mathbf{g}), 0) := ((\bar{\mathbf{u}}_h, \bar{\mathbf{w}}_h), \bar{p}_h),$$

where  $((\bar{\mathbf{u}}_h, \bar{\mathbf{w}}_h), \bar{p}_h)$  is the solution of the following discrete source problem:

$$A(((\bar{\mathbf{u}}_h, \bar{\mathbf{w}}_h), \bar{p}_h), ((\mathbf{v}_h, \boldsymbol{\tau}_h), q_h)) = B(((\mathbf{f}, \mathbf{g}), 0), ((\mathbf{v}_h, \boldsymbol{\tau}_h), q_h)), \quad \forall ((\mathbf{v}_h, \boldsymbol{\tau}_h), q_h) \in \mathbf{H}_h. \quad (3.7)$$

It is straightforward to prove that problem (3.7) is well posed and hence, the operator  $\mathbf{T}_h$  is well defined. On the other hand, we observe that  $(\kappa_h, ((\mathbf{u}_h, \mathbf{w}_h), p_h)) \in \mathbb{R}^+ \times \mathbf{H}_h$  solves (3.6) if and only if  $(\vartheta_h, ((\mathbf{u}_h, \mathbf{w}_h), p_h))$  is an eigenpair of  $\mathbf{T}_h$ , i.e., if  $((\mathbf{u}_h, \mathbf{w}_h), p_h) \neq ((\mathbf{0}, \mathbf{0}), 0)$  and  $\mathbf{T}_h((\mathbf{u}_h, \mathbf{w}_h), p_h) = (\vartheta_h((\mathbf{u}_h, \mathbf{w}_h), p_h), \text{with } \vartheta_h := (\kappa_h + 1)^{-1} \neq 0.$

We observe that  $\kappa_h = 0$  is an eigenvalue of the problem (3.6) and its associated eigenspace is  $\mathcal{K}_h := \mathcal{K} \cap \mathbf{H}_h$ . To continue with the analysis, we need the following decomposition  $\mathbf{H}_h = \mathcal{G}_h \oplus \mathcal{K}_h$ , where  $\mathcal{G}_h$  is the discrete versions of  $\mathcal{G}$ . Additionally, we introduce the following suitable Helmholtz decomposition. This result has been stated in [10, Lemma 4.1], and with minor manipulations can be adapted to our purposes. We skip details for brevity.

**Lemma 3.6** For  $((\mathbf{u}_h, \mathbf{w}_h), p_h) \in \mathcal{G}_h$  we have the following decomposition

$$((\mathbf{u}_h, \mathbf{w}_h), p_h) = ((\mathbf{u}_h, \nabla \xi), p_h) + ((\mathbf{0}, \boldsymbol{\chi}), 0) \quad \text{with} \quad \xi \in H^{1+\tilde{\nu}}(\Omega_f) \quad \text{and} \quad ((\mathbf{0}, \boldsymbol{\chi}), 0) \in \mathcal{K},$$

which satisfies

$$\begin{aligned} \|\nabla \xi\|_{\tilde{\nu}, \Omega_f} &\leq C(\|\operatorname{div} \mathbf{u}_h\|_{0, \Omega_f} + \|\mathbf{w}_h\|_{1, \Omega_s} + \|p_h\|_{0, \Omega_s}), \quad \text{and} \\ \|\boldsymbol{\chi}\|_{0, \Omega_f} &\leq Ch^{\tilde{\nu}}(\|\operatorname{div} \mathbf{u}_h\|_{0, \Omega_f} + \|\mathbf{w}_h\|_{1, \Omega_s} + \|p_h\|_{0, \Omega_s}). \end{aligned}$$

where  $\tilde{\nu} := \min\{\frac{1}{2}, \nu\}$ .

Next, since  $\mathbf{T}$  is non-compact, we need to consider for the analysis a convergence in a weak sense. We have that  $\mathbf{H}_h \subset \mathbf{H}$ , thus for any discrete source  $\mathbf{0} \neq ((\mathbf{f}_h, \mathbf{g}_h), 0) \in \mathbf{H}_h$ , operators  $\mathbf{T}$  and  $\mathbf{T}_h$  are well defined. In what follows, we will consider discrete source  $\mathbf{0} \neq ((\mathbf{f}_h, \mathbf{g}_h), 0) \in \mathbf{H}_h$  for the error analysis between problems (2.6) and (3.7). More precisely, since  $\mathcal{G}_h \not\subseteq \mathcal{G}$  we need to apply the second Strang lemma in order to derive an approximation error for the solutions of the source problems. For problems (2.6) and (3.7) the Strang estimate reads as follows:

$$\begin{aligned} \|((\bar{\mathbf{u}}, \bar{\mathbf{w}}), \bar{p}) - ((\bar{\mathbf{u}}_h, \bar{\mathbf{w}}_h), \bar{p}_h)\|_{\mathbf{H}} &\leq \left\{ C \inf_{((\mathbf{v}_h, \boldsymbol{\tau}_h), q_h) \in \mathcal{G}_h} \|((\bar{\mathbf{u}}, \bar{\mathbf{w}}), \bar{p}) - ((\mathbf{v}_h, \boldsymbol{\tau}_h), q_h)\|_{\mathbf{H}} \right. \\ &\quad \left. + \sup_{((\mathbf{0}, \mathbf{0}), 0) \neq ((\mathbf{v}_h, \boldsymbol{\tau}_h), q_h) \in \mathcal{G}_h} \frac{A(((\bar{\mathbf{u}}, \bar{\mathbf{w}}), \bar{p}) - ((\bar{\mathbf{u}}_h, \bar{\mathbf{w}}_h), \bar{p}_h), ((\mathbf{v}_h, \boldsymbol{\tau}_h), q_h))}{\|((\mathbf{v}_h, \boldsymbol{\tau}_h), q_h)\|_{\mathbf{H}}} \right\}. \end{aligned} \quad (3.8)$$

Since  $\mathbf{M}_h((\bar{\mathbf{u}}, \bar{\mathbf{w}}), \bar{p}) \in \mathbf{H}_h$  and  $\mathbf{H}_h = \mathcal{G}_h \oplus \mathcal{K}_h$ , there exist  $((\tilde{\mathbf{u}}_h, \tilde{\mathbf{w}}_h), \tilde{p}_h) \in \mathcal{G}_h$  and  $((\tilde{\mathbf{u}}_{\mathcal{K}_h}, \tilde{\mathbf{w}}_{\mathcal{K}_h}), \tilde{p}_{\mathcal{K}_h}) \in \mathcal{K}_h$  such that  $\mathbf{M}_h((\bar{\mathbf{u}}, \bar{\mathbf{w}}), \bar{p}) = ((\tilde{\mathbf{u}}_h, \tilde{\mathbf{w}}_h), \tilde{p}_h) + ((\tilde{\mathbf{u}}_{\mathcal{K}_h}, \tilde{\mathbf{w}}_{\mathcal{K}_h}), \tilde{p}_{\mathcal{K}_h})$ . Then, since  $((\bar{\mathbf{u}}, \bar{\mathbf{w}}), \bar{p}) \in \mathcal{G}$  is orthogonal to  $((\bar{\mathbf{u}}_{\mathcal{K}_h}, \bar{\mathbf{w}}_{\mathcal{K}_h}), \bar{p}_{\mathcal{K}_h}) \in \mathcal{K}_h$ , we observe that

$$\begin{aligned} \|((\bar{\mathbf{u}}, \bar{\mathbf{w}}), \bar{p}) - ((\tilde{\mathbf{u}}_h, \tilde{\mathbf{w}}_h), \tilde{p}_h)\|_{\mathbf{H}}^2 &\leq \|((\bar{\mathbf{u}}, \bar{\mathbf{w}}), \bar{p}) - ((\tilde{\mathbf{u}}_h, \tilde{\mathbf{w}}_h), \tilde{p}_h)\|_{\mathbf{H}}^2 + \|((\tilde{\mathbf{u}}_{\mathcal{K}_h}, \tilde{\mathbf{w}}_{\mathcal{K}_h}), \tilde{p}_{\mathcal{K}_h})\|_{\mathbf{H}}^2 \\ &= \|((\bar{\mathbf{u}}, \bar{\mathbf{w}}), \bar{p}) - ((\tilde{\mathbf{u}}_h, \tilde{\mathbf{w}}_h), \tilde{p}_h) + ((\tilde{\mathbf{u}}_{\mathcal{K}_h}, \tilde{\mathbf{w}}_{\mathcal{K}_h}), \tilde{p}_{\mathcal{K}_h})\|_{\mathbf{H}}^2 \\ &= \|((\bar{\mathbf{u}}, \bar{\mathbf{w}}), \bar{p}) - \mathbf{M}_h((\bar{\mathbf{u}}, \bar{\mathbf{w}}), \bar{p})\|_{\mathbf{H}}^2 \\ &\leq Ch^r(\|\bar{\mathbf{w}}\|_{\alpha, \Omega_f} + \|\operatorname{div} \bar{\mathbf{w}}\|_{1, \Omega_f} + \|\bar{\mathbf{u}}\|_{1+\beta, \Omega_s} + \|\bar{p}\|_{1, \Omega_s}), \end{aligned} \quad (3.9)$$

where we have used the approximation property of  $\mathbf{M}_h$  given by Lemma 3.4, and the additional regularity of the solution  $((\bar{\mathbf{u}}, \bar{\mathbf{w}}), \bar{p})$  given in Lemma 2.1.

Now for the consistency term, we proceed as follows: Let  $((\mathbf{v}_h, \boldsymbol{\tau}_h), q_h) \in \mathbf{H}_h$ . Invoking Lemma 3.6, let us consider the following decomposition  $((\mathbf{v}_h, \boldsymbol{\tau}_h), q_h) = ((\mathbf{v}_h, \nabla \xi), q_h) + ((\mathbf{0}, \boldsymbol{\chi}), 0)$ . Hence

$$\begin{aligned} A(((\bar{\mathbf{u}}, \bar{\mathbf{w}}), \bar{p}), ((\mathbf{v}_h, \boldsymbol{\tau}_h), q_h)) &= 2 \int_{\Omega_s} \mu(\mathbf{x}) \boldsymbol{\varepsilon}(\bar{\mathbf{u}}) : \boldsymbol{\varepsilon}(\mathbf{v}_h) - \int_{\Omega_s} \bar{p} \operatorname{div} \mathbf{v}_h + \int_{\Omega_f} c^2 \rho_f \operatorname{div} \bar{\mathbf{w}} \operatorname{div} \boldsymbol{\tau}_h \\ &\quad + \langle g \rho_f \bar{\mathbf{w}} \cdot \mathbf{n}, \boldsymbol{\tau}_h \cdot \mathbf{n} \rangle_{\Gamma_0} - \int_{\Omega_s} \operatorname{div} \bar{\mathbf{u}} q_h - \int_{\Omega_s} \frac{1}{\lambda} \bar{p} q_h + \int_{\Omega_s} \rho_s \bar{\mathbf{u}} \cdot \mathbf{v}_h + \int_{\Omega_f} \rho_f \bar{\mathbf{w}} \cdot \boldsymbol{\tau}_h \\ &= 2 \int_{\Omega_s} \mu(\mathbf{x}) \boldsymbol{\varepsilon}(\bar{\mathbf{u}}) : \boldsymbol{\varepsilon}(\mathbf{v}_h) - \int_{\Omega_s} \bar{p} \operatorname{div} \mathbf{v}_h + \int_{\Omega_f} c^2 \rho_f \operatorname{div} \bar{\mathbf{w}} (\nabla \xi + \boldsymbol{\chi}) \end{aligned}$$

$$\begin{aligned}
& + \langle g \rho_f \bar{\mathbf{w}} \cdot \mathbf{n}, (\nabla \xi + \boldsymbol{\chi}) \cdot \mathbf{n} \rangle_{\Gamma_0} - \int_{\Omega_s} \operatorname{div} \bar{\mathbf{u}} q_h - \int_{\Omega_s} \frac{1}{\lambda} \bar{p} q_h + \int_{\Omega_s} \rho_s \bar{\mathbf{u}} \cdot \mathbf{v}_h \\
& + \int_{\Omega_f} \rho_f \bar{\mathbf{w}} \cdot (\nabla \xi + \boldsymbol{\chi}) \\
& = 2 \int_{\Omega_s} \mu(\mathbf{x}) \boldsymbol{\varepsilon}(\bar{\mathbf{u}}) : \boldsymbol{\varepsilon}(\mathbf{v}_h) - \int_{\Omega_s} \bar{p} \operatorname{div} \mathbf{v}_h - \int_{\Omega_s} \frac{1}{\lambda} \bar{p} q_h + \int_{\Omega_f} c^2 \rho_f \operatorname{div} \bar{\mathbf{w}} \operatorname{div} \nabla \xi \\
& + \langle g \rho_f \bar{\mathbf{w}} \cdot \mathbf{n}, \nabla \xi \cdot \mathbf{n} \rangle_{\Gamma_0} - \int_{\Omega_s} \operatorname{div} \bar{\mathbf{u}} q_h + \int_{\Omega_s} \rho_s \bar{\mathbf{u}} \cdot \mathbf{v}_h + \int_{\Omega_f} \rho_f \bar{\mathbf{w}} \cdot \nabla \xi \\
& = \int_{\Omega_s} \rho_s \mathbf{f}_h \cdot \mathbf{v}_h + \int_{\Omega_f} \rho_f \mathbf{g}_h \cdot \nabla \xi, \tag{3.10}
\end{aligned}$$

where we have used that  $\mathcal{G} \oplus \mathcal{H}$ , where  $\operatorname{div} \boldsymbol{\chi} = 0$  in  $\Omega_f$  and  $\boldsymbol{\chi} \cdot \mathbf{n} = 0$  on  $\Gamma_0$  and that  $((\bar{\mathbf{u}}, \bar{\mathbf{w}}), \bar{p})$  is the solution of the source problem (2.6).

On the other hand, for the discrete source problem there holds

$$\begin{aligned}
A((\bar{\mathbf{u}}_h, \bar{\mathbf{w}}_h), \bar{p}_h), ((\mathbf{v}_h, \boldsymbol{\tau}_h), q_h)) & = 2 \int_{\Omega_s} \mu(\mathbf{x}) \boldsymbol{\varepsilon}(\bar{\mathbf{u}}_h) : \boldsymbol{\varepsilon}(\mathbf{v}_h) - \int_{\Omega_s} \bar{p}_h \operatorname{div} \mathbf{v}_h + \int_{\Omega_f} c^2 \rho_f \operatorname{div} \bar{\mathbf{w}}_h \operatorname{div} \boldsymbol{\tau}_h \\
& + \int_{\Gamma_0} (g \rho_f \bar{\mathbf{w}}_h \cdot \mathbf{n})(\boldsymbol{\tau}_h \cdot \mathbf{n}) - \int_{\Omega_s} \operatorname{div} \bar{\mathbf{u}}_h q_h - \int_{\Omega_s} \frac{1}{\lambda} \bar{p}_h q_h + \int_{\Omega_s} \rho_s \bar{\mathbf{u}}_h \cdot \mathbf{v}_h + \int_{\Omega_f} \rho_f \bar{\mathbf{w}}_h \cdot \boldsymbol{\tau}_h \\
& = \int_{\Omega_s} \rho_s \mathbf{f}_h \cdot \mathbf{v}_h + \int_{\Omega_f} \rho_f \mathbf{g}_h \cdot (\nabla \xi + \boldsymbol{\chi}). \tag{3.11}
\end{aligned}$$

Now, subtracting (3.11) from (3.10), applying the Cauchy–Schwarz inequality and invoking Lemma 3.6 we have

$$\begin{aligned}
A((\bar{\mathbf{u}}, \bar{\mathbf{w}}), \bar{p}) - ((\bar{\mathbf{u}}_h, \bar{\mathbf{w}}_h), \bar{p}_h), ((\mathbf{v}_h, \boldsymbol{\tau}_h), q_h)) & = \int_{\Omega_f} \rho_f \mathbf{g}_h \cdot \boldsymbol{\chi} \\
& \leq C_{\rho_f} \|\mathbf{g}_h\|_{0, \Omega_f} \|\boldsymbol{\chi}\|_{0, \Omega_f} \leq Ch^{\tilde{\nu}} \|\mathbf{g}_h\|_{0, \Omega_f} (\|\operatorname{div} \boldsymbol{\tau}_h\|_{0, \Omega_f} + \|\mathbf{v}_h\|_{1, \Omega_s} + \|q_h\|_{0, \Omega_s}).
\end{aligned}$$

Finally, taking supremum, we readily obtain the bound

$$\sup_{((\mathbf{0}, \mathbf{0}), 0) \neq ((\mathbf{v}_h, \boldsymbol{\tau}_h), q_h) \in \mathcal{G}_h} \frac{A((\bar{\mathbf{u}}, \bar{\mathbf{w}}), \bar{p}) - ((\bar{\mathbf{u}}_h, \bar{\mathbf{w}}_h), \bar{p}_h), ((\mathbf{v}_h, \boldsymbol{\tau}_h), q_h))}{\|((\mathbf{v}_h, \boldsymbol{\tau}_h), q_h)\|_{\mathbf{H}}} \leq Ch^r \|\mathbf{g}_h\|_{0, \Omega_f}. \tag{3.12}$$

Then, replacing the estimates (3.9) and (3.12) in (3.8), we obtain an error estimate for the source problem, which will be used later in the next section. In fact, we have that

$$\|((\bar{\mathbf{u}}, \bar{\mathbf{w}}), \bar{p}) - ((\bar{\mathbf{u}}_h, \bar{\mathbf{w}}_h), \bar{p}_h)\|_{\mathbf{H}} \leq Ch^r \|((\mathbf{f}_h, \mathbf{g}_h), 0)\|_{L^2(\Omega_s)^d \times L^2(\Omega_f)^d \times L^2(\Omega_s)}. \tag{3.13}$$

#### 4. Spectral approximation

We begin this section by recalling some standard definitions from classical spectral theory (we refer the reader to, e.g., [35, Chapter III] or [19, Chapter 6] for a comprehensive background). Let  $\mathcal{X}$  be a

generic Hilbert space and let  $\mathbf{S}$  be a linear bounded operator defined by  $\mathbf{S} : \mathcal{X} \rightarrow \mathcal{X}$ . If  $\mathbf{I}$  represents the identity operator, the spectrum of  $\mathbf{S}$  is defined by  $\text{sp}(\mathbf{S}) := \{z \in \mathbb{C} : (z\mathbf{I} - \mathbf{S}) \text{ is not invertible}\}$  and the resolvent is its complement  $\text{res}(\mathbf{S}) := \mathbb{C} \setminus \text{sp}(\mathbf{S})$ . For any  $z \in \text{res}(\mathbf{S})$ , we define the resolvent operator of  $\mathbf{S}$  corresponding to  $z$  by  $R_z(\mathbf{S}) := (z\mathbf{I} - \mathbf{S})^{-1} : \mathcal{X} \rightarrow \mathcal{X}$ .

Also, if  $\mathcal{X}$  and  $\mathcal{Y}$  are vectorial fields, we denote by  $\mathcal{L}(\mathcal{X}, \mathcal{Y})$  the space of all the linear and bounded operators acting from  $\mathcal{X}$  to  $\mathcal{Y}$ . We define  $\mathcal{X} := \mathcal{W} \times \mathcal{Q}$  and  $\mathcal{X}_h := \mathcal{W}_h \times \mathcal{Q}_h$ .

Let  $\mathcal{S}$  be a linear operator defined by  $\mathcal{S} : \mathcal{X} \rightarrow \mathcal{X}$ . We define the norm  $\|\cdot\|_h$ , associated with  $\mathcal{S}$  as follows

$$\|\mathcal{S}\|_h := \sup_{\mathbf{x}_h \in \mathcal{X}_h} \frac{\|\mathcal{S}\mathbf{x}_h\|_{\mathbf{H}}}{\|\mathbf{x}_h\|_{\mathbf{H}}}. \quad (4.1)$$

Let  $\mathbf{x} \in \mathcal{X}$  and  $\mathcal{H}$  and  $\mathcal{J}$  be two closed subspaces of  $\mathcal{X}$ . We define the following distances

$$\delta(\mathbf{x}, \mathcal{H}) := \inf_{\mathbf{y} \in \mathcal{H}} \|\mathbf{x} - \mathbf{y}\|, \quad \delta(\mathcal{H}, \mathcal{J}) := \sup_{\mathbf{y} \in \mathcal{H}, \|\mathbf{y}\|=1} \delta(\mathbf{y}, \mathcal{J}),$$

and hence, the so-called gap between two subspaces

$$\widehat{\delta}(\mathcal{H}, \mathcal{J}) := \max \{ \delta(\mathcal{H}, \mathcal{J}), \delta(\mathcal{J}, \mathcal{H}) \}. \quad (4.2)$$

We now address the core of this section. As noted in Section 2, the solution operator  $\mathbf{T}$  is non-compact. Consequently, the classical spectral approximation framework for compact operators, as described in [13, Part 2, Section 8], cannot be applied since uniform convergence  $\|\mathbf{T} - \mathbf{T}_h\| \rightarrow 0$  in the natural norm is not achieved. To circumvent this limitation, we rely on the framework established in [25, Section 2] for non-compact operators. The main idea of this theory is to replace uniform convergence in the natural norm with uniform convergence in the weaker, mesh-dependent norm (4.1), which evaluates the operators strictly on discrete source terms. Under this approach, the absence of spurious modes and the convergence of the discrete eigenspaces are guaranteed once the following two properties are established:

- P1.  $\|\mathbf{T} - \mathbf{T}_h\|_h \rightarrow 0$ , as  $h \rightarrow 0$ .
- P2.  $\forall \mathbf{x} \in \mathcal{X}, \lim_{h \rightarrow 0} \delta(\mathbf{x}, \mathcal{X}_h) = 0$ .

We observe that property P2 holds due to the smoothness of the eigenfunctions given by Corollary 2.3 and the approximation properties (3.1a), (3.1b), (3.1c), (3.2), and Lemma 3.4. The task now is to prove P1 which establishes the convergence of  $\mathbf{T}_h$  to  $\mathbf{T}$  as  $h \rightarrow 0$  in the norm (4.1). This is addressed in the following result.

**Lemma 4.1** *Property P1 holds true in the following sense: there exists a constant  $C > 0$  independent of  $h$ , such that*

$$\|\mathbf{T} - \mathbf{T}_h\|_h \leq Ch^r,$$

where the parameter  $r$  is given by Lemma 3.4.

*Proof* Since  $\mathbf{H}_h \subset \mathbf{H}$ , for any discrete source  $\mathbf{0} \neq ((\mathbf{f}_h, \mathbf{g}_h), 0) \in \mathbf{H}_h$ , operators  $\mathbf{T}$  and  $\mathbf{T}_h$  are well defined. Then, utilizing the definition of the discrete norm (4.1), the stability bound for the continuous operator

$\mathbf{T}$  (i.e., continuous dependence on the data, cf. Corollary 2.3), and the estimate (3.13), we obtain

$$\begin{aligned} \|\mathbf{T} - \mathbf{T}_h\|_h &= \sup_{\mathbf{0} \neq ((\mathbf{f}_h, \mathbf{g}_h), 0) \in \mathbf{H}_h} \frac{\|(\mathbf{T} - \mathbf{T}_h)((\mathbf{f}_h, \mathbf{g}_h), g_h)\|_{\mathbf{H}}}{\|((\mathbf{f}_h, \mathbf{g}_h), 0)\|_{\mathbf{H}}} \\ &= \sup_{\mathbf{0} \neq ((\mathbf{f}_h, \mathbf{g}_h), 0) \in \mathbf{H}_h} \frac{\|((\bar{\mathbf{u}}, \bar{\mathbf{w}}), \bar{p}) - ((\bar{\mathbf{u}}_h, \bar{\mathbf{w}}_h), \bar{p}_h)\|_{\mathbf{H}}}{\|((\mathbf{f}_h, \mathbf{g}_h), 0)\|_{\mathbf{H}}} \\ &\leq Ch^r. \end{aligned}$$

This concludes the proof.  $\square$

A fundamental consequence of the convergence in the discrete norm established in Lemma 4.1 is the preclusion of spectral pollution. Specifically, the uniform convergence of  $\mathbf{T}_h$  to  $\mathbf{T}$  in  $\|\cdot\|_h$  ensures that our numerical method does not produce spurious eigenvalues. This property is formalized in the following theorem (for a detailed abstract proof, we refer the reader to [13, Theorem 9.1] or the foundational work [35, Chapter IV, Section 1]).

**Theorem 4.2** *Let  $V \subset \mathbb{C}$  be an open set containing  $\text{sp}(\mathbf{T})$ . Then, there exists  $h_0 > 0$  such that  $\text{sp}(\mathbf{T}_h) \subset V$  for all  $h < h_0$ .*

Let us recall the definition of spectral projectors. Let  $\mu$  be a nonzero isolated eigenvalue of  $\mathbf{T}$  with algebraic multiplicity  $m$  and let  $\Gamma$  be a disk of the complex plane centered in  $\vartheta$ , such that  $\vartheta$  is the only eigenvalue of  $\mathbf{T}$  lying in  $\Gamma$  and  $\partial\Gamma \cap \text{sp}(\mathbf{T}) = \emptyset$ . With these considerations at hand, we define the spectral projections of  $\mathbf{E}$  associated with  $\mathbf{T}$  as follows:

$$\mathbf{E} := \frac{1}{2\pi i} \int_{\partial\Gamma} (z\mathbf{I} - \mathbf{T})^{-1} dz.$$

Note that  $\mathbf{E}$  is the projection onto the generalized eigenvector space  $R(\mathbf{E})$ . A consequence of Lemma 4.1 is that there exist  $m$  eigenvalues, which lie in  $\Gamma$ , namely  $\vartheta_h^{(1)}, \dots, \vartheta_h^{(m)}$ , repeated according their respective multiplicities, that converge to  $\vartheta$  as  $h$  goes to zero. With this result at hand, we introduce a spectral projection

$$\mathbf{E}_h := \frac{1}{2\pi i} \int_{\partial\Gamma} (z\mathbf{I} - \mathbf{T}_h)^{-1} dz,$$

which is a projection onto the discrete invariant subspace  $R(\mathbf{E}_h)$  of  $\mathbf{T}$ , spanned by the generalized eigenvectors of  $\mathbf{T}_h$  corresponding to  $\vartheta_h^{(1)}, \dots, \vartheta_h^{(m)}$ .

For the eigenfunctions and eigenvalues, the following estimates hold true. This is a direct implication of properties P1-P2 together with the abstract framework of [26, Theorem 1]. This framework guarantees that once discrete norm convergence (P1) and eigenspace approximability (P2) are satisfied, the distance between the eigenspaces is bounded by the best approximation error of the continuous generalized eigenspace  $R(\mathbf{E})$ , provided it is sufficiently regular.

**Theorem 4.3** *There exist constants  $h_0 > 0$  and  $C > 0$  such that, for all  $h \leq h_0$ ,*

$$\widehat{\delta}(R(\mathbf{E}_h), R(\mathbf{E})) \leq Ch^{\widehat{r}}, \quad \text{and} \quad |\vartheta - \vartheta_h^{(i)}| \leq Ch^{\widehat{r}}, \quad i = 1, \dots, m,$$

where

$$\widehat{r} := \min\{\alpha_1, \beta_1, \nu, k\} > 0 \quad (4.3)$$

denotes the combined regularity exponent of the eigenfunctions on the generalized eigenspace, derived from the individual bounds in Corollary 2.3,  $\nu$  is as in Lemma 3.1, and  $k \geq 1$  is the respective polynomial degree, and  $\widehat{\delta}$  is the gap between closed subspaces defined in (4.2).

**Remark 4.4** *Let us remark that the error estimate for eigenvalues provided by Theorem 4.3 of the solution operator  $\mathbf{T}$  leads to a similar approximation for the eigenvalue  $\kappa = \frac{1}{\vartheta} - 1$  involved in problem (2.4) by the eigenvalues  $\kappa_h^{(i)} = \frac{1}{\vartheta_h^{(i)}} - 1$  of problem (3.6).*

Let us observe that is possible to improve the convergence rate of approximation for the eigenvalues given in Theorem 4.3. This is proved in the following result. To establish this double order of convergence, we rely on a classical Babuška–Osborn type argument tailored to our mixed variational formulation.

**Lemma 4.5** *Let  $\kappa \in \mathbb{R}^+$  be a non-zero eigenvalue of (2.4) and let  $\kappa_h^{(i)} \in \mathbb{R}^+$  with  $1 \leq i \leq m$  be a discrete eigenvalue given by problem (3.6) that approximates  $\kappa$ . Then, there exists a positive constant  $C$ , independent of  $\lambda$ , such that*

$$|\kappa - \kappa_h^{(i)}| \leq Ch^{2\widehat{r}}, \quad i = 1, \dots, m,$$

where  $\widehat{r} > 0$  is as in (4.3).

*Proof* Let us define  $\mathbf{U} := ((\mathbf{u}, \mathbf{w}), p)$  and  $\mathbf{U}_h := ((\mathbf{u}_h, \mathbf{w}_h), p_h)$ . By evaluating the discrete and continuous eigenvalue problems against test functions spanning the associated spaces, standard algebraic manipulation yields the following identity:

$$A(\mathbf{U} - \mathbf{U}_h, \mathbf{U} - \mathbf{U}_h) - \kappa B(\mathbf{U} - \mathbf{U}_h, \mathbf{U} - \mathbf{U}_h) = (\kappa_h^{(i)} - \kappa) B(\mathbf{U}_h, \mathbf{U}_h). \quad (4.4)$$

Without loss of regularity, through this proof, we will assume that  $\|\mathbf{U}_h\|_{\mathbf{H}} = 1$ . Taking modulus on the above identity and using the continuity of the bilinear forms along with Young's inequality, we have

$$\begin{aligned} & |A(\mathbf{U} - \mathbf{U}_h, \mathbf{U} - \mathbf{U}_h)| \\ & \leq 2\|\mu^{1/2} \boldsymbol{\varepsilon}(\mathbf{u} - \mathbf{u}_h)\|_{0, \Omega_s}^2 + 2\|p - p_h\|_{0, \Omega_s} \|\operatorname{div}(\mathbf{u} - \mathbf{u}_h)\|_{0, \Omega_s} + \|(c\rho_f)^{1/2} \operatorname{div}(\mathbf{w} - \mathbf{w}_h)\|_{0, \Omega_f}^2 \\ & \quad + \|(g\rho_f)^{1/2} (\mathbf{w} - \mathbf{w}_h) \cdot \mathbf{n}\|_{0, \Gamma_0}^2 + \|\lambda^{-1/2} (p - p_h)\|_{0, \Omega_s}^2 \\ & \quad + \|\rho_f^{1/2} (\mathbf{u} - \mathbf{u}_h)\|_{0, \Omega_f}^2 + \|\rho_s^{1/2} (\mathbf{w} - \mathbf{w}_h)\|_{0, \Omega_f}^2 \\ & \leq 2C\|\boldsymbol{\nabla}(\mathbf{u} - \mathbf{u}_h)\|_{0, \Omega_s}^2 + (C_{\lambda_{\min}} + 1)\|\lambda^{-1/2} (p - p_h)\|_{0, \Omega_s}^2 + \|\rho_s^{1/2} (\mathbf{u} - \mathbf{u}_h)\|_{0, \Omega_s}^2 \end{aligned}$$

$$\begin{aligned}
& + \|\operatorname{div}(\mathbf{u} - \mathbf{u}_h)\|_{0,\Omega_s}^2 + (C_{\rho_f,c} + 1) \|(\rho_f c^2)^{1/2} \operatorname{div}(\mathbf{w} - \mathbf{w}_h)\|_{0,\Omega_f}^2 \\
& + \|(g\rho_f)^{1/2}(\mathbf{w} - \mathbf{w}_h) \cdot \mathbf{n}\|_{0,\Gamma_0}^2 + \|\rho_s^{1/2}(\mathbf{w} - \mathbf{w}_h)\|_{0,\Omega_f}^{1/2} \\
& \leq C(\widehat{\delta}(R(\mathbf{E}_h), R(\mathbf{E})))^2 \leq Ch^{2\tilde{r}},
\end{aligned} \tag{4.5}$$

where we have used the definition of the continuous and discrete spectral projections and the gap (cf. Theorem 4.3). Also

$$|\kappa| |B(\mathbf{U} - \mathbf{U}_h, \mathbf{U} - \mathbf{U}_h)| \leq C(\|\rho_s^{1/2}(\mathbf{w} - \mathbf{w}_h)\|_{0,\Omega_f}^{1/2} + \|\rho_s^{1/2}(\mathbf{u} - \mathbf{u}_h)\|_{0,\Omega_s}^2) \leq C(\widehat{\delta}(R(\mathbf{E}_h), R(\mathbf{E})))^2 \leq Ch^{2\tilde{r}}. \tag{4.6}$$

On the other hand, it is straightforward to prove that  $B(\mathbf{U}_h, \mathbf{U}_h) > 0$ . Indeed, let us consider the eigenvalue problem

$$A(\mathbf{U}_h, \mathbf{U}_h) = (\kappa_h + 1)B(\mathbf{U}_h, \mathbf{U}_h). \tag{4.7}$$

From (4.7) and the definition of  $\|\cdot\|_{\mathbf{H}}$  we have

$$B(\mathbf{U}_h, \mathbf{U}_h) = \frac{1}{\kappa_h + 1} A(\mathbf{U}_h, \mathbf{U}_h) = \frac{1}{\kappa_h + 1}.$$

Since  $\kappa_h \rightarrow \kappa$  as  $h \rightarrow 0$  and  $\kappa \in \mathbb{R}^+$  (cf. Theorem 4.3 and Remark 4.4), and  $\|\mathbf{U}_h\|_{\mathbf{H}} = 1$ , we have

$$B(\mathbf{U}_h, \mathbf{U}_h) = \frac{1}{\kappa + 1} A(\mathbf{U}_h, \mathbf{U}_h) = \frac{\|\mathbf{U}_h\|_{\mathbf{H}}}{\kappa + 1} = \frac{1}{\kappa + 1} > C_\star > 0,$$

where the constant  $C_\star > 0$  is independent of  $h$ . Finally, replacing (4.5), (4.6) in (4.4), we conclude the proof.  $\square$

## 5. A posteriori error analysis

In this section, we introduce a residual-based a posteriori error estimator and establish reliability and efficiency bounds relating the estimator to the approximation error. Throughout this section, we restrict the analysis to the two-dimensional case and simple eigenvalues. Nevertheless, the estimator can also be naturally defined in three dimensions, and its performance in a three-dimensional setting is numerically assessed in Section 6.4.

We first introduce some notation. For each  $T \in \mathcal{T}_h$ , let  $\mathcal{E}_T$  denote the set of its edges, and define

$$\mathcal{E}_h := \bigcup_{T \in \mathcal{T}_h} \mathcal{E}_T.$$

We decompose  $\mathcal{E}_h$  as  $\mathcal{E}_h = \mathcal{E}_h^{\text{int}} \cup \mathcal{E}_h^\partial$ , where  $\mathcal{E}_h^\partial$  is the set of edges contained in  $\partial\Omega$ , and  $\mathcal{E}_h^{\text{int}}$  is the set of interior edges. Let  $\ell \in \mathcal{E}_h^{\text{int}}$  be shared by two elements  $T$  and  $T'$ , and let  $\mathbf{n}_T$  and  $\mathbf{n}_{T'}$  denote the corresponding outward unit normal vectors. For any sufficiently smooth vector field  $\mathbf{v}$ , we define its normal-derivative jump across  $\ell$  by

$$\left[ \left[ \frac{\partial \mathbf{v}}{\partial \mathbf{n}} \right] \right]_\ell := \nabla(\mathbf{v}|_T) \mathbf{n}_T + \nabla(\mathbf{v}|_{T'}) \mathbf{n}_{T'}.$$



### 5.1. Residual-based a posteriori error estimator

For the elastic structure, we define the local element indicator  $\eta_{T,S}$  and the edge jump indicator  $\eta_{\ell,S}$  by

$$\eta_{T,S}^2 := h_T^2 \|\rho_1^S \mathbf{R}_{1,T}^S\|_{0,T}^2 + \|(\rho_2^S)^{1/2} R_{2,T}^S\|_{0,T}^2, \quad \eta_{\ell,S}^2 := h_\ell \|\rho_\ell^S J_\ell^S\|_{0,\ell}^2.$$

where, for each  $T \subset \Omega_s$ ,

$$\mathbf{R}_{1,T}^S := \operatorname{div}(2\mu_h \boldsymbol{\varepsilon}(\mathbf{u}_h)) - \nabla p_h + \omega_h^2 \rho_s \mathbf{u}_h, \quad R_{2,T}^S := \operatorname{div} \mathbf{u}_h + \frac{1}{\lambda(\mathbf{x})} p_h.$$

Let  $\mathcal{E}_{h,s}^{\text{int}}$  denote the set of interior edges contained in  $\overline{\Omega}_s$ . For the edge residuals, we define

$$J_\ell^S := \begin{cases} \frac{1}{2} [[(2\mu_h \boldsymbol{\varepsilon}(\mathbf{u}_h) - p_h \mathbf{I}) \mathbf{n}]]_\ell, & \ell \in \mathcal{E}_{h,s}^{\text{int}}, \\ (2\mu_h \boldsymbol{\varepsilon}(\mathbf{u}_h) - p_h \mathbf{I}) \mathbf{n}, & \ell \subset \Gamma_N, \\ \mathbf{0}, & \ell \subset \Gamma_D. \end{cases}$$

The solid weighting parameters  $\rho_1^S$ ,  $\rho_2^S$ , and  $\rho_\ell^S$  are defined by

$$\rho_1^S := (2\mu_h)^{-1/2}, \quad \rho_2^S := [(2\mu_h)^{-1} + \lambda(\mathbf{x})^{-1}]^{-1}, \quad \rho_\ell^S := \frac{(2\mu_h)^{-1/2}}{\sqrt{2}}.$$

Here,  $\mu_h$  denotes the elementwise  $L^2$ -orthogonal polynomial projection of  $\mu$ . The local data oscillation is defined by  $\Theta_{T,S}^2 := \|\rho_1^S (\mu - \mu_h) \boldsymbol{\varepsilon}(\mathbf{u}_h)\|_{0,T}^2$ . The global solid estimator and the corresponding data oscillation are then defined by

$$\eta_S^2 := \sum_{T \in \mathcal{T}_h(\Omega_s)} \eta_{T,S}^2 + \sum_{\ell \in \mathcal{E}_{h,s}^{\text{int}}} \eta_{\ell,S}^2 + \sum_{\ell \subset \Gamma_N} \eta_{\ell,S}^2, \quad \Theta_S^2 := \sum_{T \in \mathcal{T}_h(\Omega_s)} \Theta_{T,S}^2,$$

where  $\mathcal{T}_h(\Omega_s) := \{T \in \mathcal{T}_h : T \subset \Omega_s\}$ .

For the fluid part, we define the local element indicator  $\eta_{T,F}$  and the edge jump indicator  $\eta_{\ell,F}$  by

$$\eta_{T,F}^2 := h_T^2 \|\rho^F \mathbf{R}_{1,T}^F\|_{0,T}^2 + h_T^2 \|\rho^F R_{2,T}^F\|_{0,T}^2, \quad \eta_{\ell,F}^2 := h_\ell \|\rho_\ell^F J_{1,\ell}^F\|_{0,\ell}^2 + h_\ell \|\rho_\ell^F J_{2,\ell}^F\|_{0,\ell}^2.$$

where, for each  $T \subset \Omega_f$  we define

$$\mathbf{R}_{1,T}^F := c^2 \rho_f \nabla(\operatorname{div} \mathbf{w}_h) + \omega_h^2 \rho_f \mathbf{w}_h, \quad R_{2,T}^F := \operatorname{rot}(\omega_h^2 \rho_f \mathbf{w}_h).$$

Let  $\mathcal{E}_{h,f}^{\text{int}}$  denote the set of interior edges contained in  $\overline{\Omega}_f$ . For the edge residuals, we define

$$J_{1,\ell}^F := \begin{cases} \frac{1}{2} [[c^2 \rho_f (\operatorname{div} \mathbf{w}_h) \mathbf{n}]]_\ell, & \ell \in \mathcal{E}_{h,f}^{\text{int}}, \\ (c^2 \rho_f \operatorname{div} \mathbf{w}_h + g \rho_f \mathbf{w}_h \cdot \mathbf{n}) \mathbf{n}, & \ell \subset \Gamma_0, \end{cases}, \quad J_{2,\ell}^F := \begin{cases} \frac{1}{2} [[\omega_h^2 \rho_f \mathbf{w}_h \times \mathbf{n}]]_\ell, & \ell \in \mathcal{E}_{h,f}^{\text{int}}, \\ \omega_h^2 \rho_f \mathbf{w}_h \times \mathbf{n}, & \ell \subset \Gamma_0. \end{cases}$$

On the other hand, the fluid weighting parameters  $\rho^F$  and  $\rho_\ell^F$  are defined by

$$\rho^F := (c^2 \rho_f)^{-1/2}, \quad \rho_\ell^F := \min \left\{ \frac{(c^2 \rho_f)^{-1/2}}{\sqrt{2}}, \frac{(\omega_h^2 \rho_f)^{-1/2}}{\sqrt{2}} \right\}.$$

Finally, setting  $\rho^\ell := \min\{\rho_\ell^F, \rho_\ell^S\}$ , for each  $\ell \in \Sigma$  we define the interface indicator by

$$\eta_{\ell,I}^2 := h_\ell \left\| \rho^\ell \left[ (2\mu_h \boldsymbol{\varepsilon}(\mathbf{u}_h) - p_h \mathbf{I}) \mathbf{n} - c^2 \rho_f (\operatorname{div} \mathbf{w}_h) \mathbf{n} \right] \right\|_{0,\ell}^2 + h_\ell \left\| \rho_\ell^F \omega_h^2 \rho_f (\mathbf{w}_h \times \mathbf{n}) \right\|_{0,\ell}^2.$$

The global fluid estimator, the global interface estimator, and the fluid data oscillation are defined by

$$\eta_F^2 := \sum_{T \in \mathcal{T}_h(\Omega_f)} \eta_{T,F}^2 + \sum_{\ell \in \mathcal{E}_{h,f}^{\text{int}}} \eta_{\ell,F}^2 + \sum_{\ell \in \Gamma_0} \eta_{\ell,F}^2, \quad \eta_I^2 := \sum_{\ell \in \mathcal{E}_h^\Sigma} \eta_{\ell,I}^2, \quad \Theta_F^2 := \sum_{T \in \mathcal{T}_h(\Omega_f)} \Theta_{T,F}^2.$$

### 5.2. Reliability

We begin with the reliability analysis of the a posteriori error estimator. The main goal is to bound the error in the energy norm in terms of the residual estimator and the corresponding data oscillation terms.

The following stability result will be instrumental.

**Lemma 5.1** *For every  $((\mathbf{u}, \mathbf{w}), p) \in \mathbf{H}$ , there exists  $((\mathbf{v}, \boldsymbol{\tau}), q) \in \mathbf{H}$  with  $\|((\mathbf{v}, \boldsymbol{\tau}), q)\|_{\mathbf{H}} \leq \bar{C}_1 \|((\mathbf{u}, \mathbf{w}), p)\|_{\mathbf{H}}$  such that*

$$\bar{C}_2 \|((\mathbf{u}, \mathbf{w}), p)\|_{\mathbf{H}}^2 \leq A(((\mathbf{u}, \mathbf{w}), p), ((\mathbf{v}, \boldsymbol{\tau}), q)),$$

where  $\bar{C}_1$  and  $\bar{C}_2$  are positive constants independent of  $\lambda$ .

*Proof* From the definition of bilinear form  $A(\cdot, \cdot)$ , it follows that

$$\begin{aligned} A(((\mathbf{u}, \mathbf{w}), p), ((\mathbf{u}, \mathbf{w}), -p)) &= \|(2\mu(\mathbf{x}))^{1/2} \boldsymbol{\varepsilon}(\mathbf{u})\|_{0,\Omega_s}^2 + \|\lambda(\mathbf{x})^{-1/2} p\|_{0,\Omega_s}^2 + \|\rho_s^{1/2} \mathbf{u}\|_{0,\Omega_s}^2 \\ &\quad + \|(c^2 \rho_f)^{1/2} \operatorname{div} \mathbf{w}\|_{0,\Omega_f}^2 + \|\rho_f^{1/2} \mathbf{w}\|_{0,\Omega_f}^2 + \|(g \rho_f)^{1/2} \mathbf{w} \cdot \mathbf{n}\|_{0,\Gamma_0}^2. \end{aligned}$$

Applying the standard inf-sup condition [36, Eq. 2.6] shows that for every  $p \in Q$ , there exists an element  $\tilde{\mathbf{v}} \in \mathbf{V}$  with  $\tilde{\mathbf{v}} = 0$  on  $\Sigma$  such that

$$-(p, \operatorname{div} \tilde{\mathbf{v}}) \geq C_2 \|\mu(\mathbf{x})^{-1/2} p\|_{0,\Omega_s}^2, \quad \|\mu(\mathbf{x})^{1/2} \boldsymbol{\varepsilon}(\tilde{\mathbf{v}})\|_{0,\Omega_s} \leq C_1 \|\mu(\mathbf{x})^{-1/2} p\|_{0,\Omega_s}. \quad (5.1)$$

The constants  $C_1$  and  $C_2$  depend only on the ratio  $E_{\max}/E_{\min}$  and on the inf-sup constant, but are independent of  $\lambda$ . Using  $-(p, \operatorname{div} \tilde{\mathbf{v}}) \geq C_2 \|\mu(\mathbf{x})^{-1/2} p\|_{0,\Omega_s}^2$  and Cauchy–Schwarz’s inequality, gives

$$\begin{aligned} A(((\mathbf{u}, \mathbf{w}), p), ((\tilde{\mathbf{v}}, \mathbf{0}), 0)) &= 2 \int_{\Omega_s} \mu(\mathbf{x}) \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\tilde{\mathbf{v}}) - \int_{\Omega_s} p \operatorname{div} \tilde{\mathbf{v}} + \int_{\Omega_s} \rho_s \mathbf{u} \cdot \tilde{\mathbf{v}} \\ &\geq C_2 \|\mu(\mathbf{x})^{-1/2} p\|_{0,\Omega_s}^2 - C_1 \|\mu(\mathbf{x})^{1/2} \boldsymbol{\varepsilon}(\mathbf{u})\|_{0,\Omega_s} \|\mu(\mathbf{x})^{-1/2} p\|_{0,\Omega_s} - \|\rho_s^{1/2} \mathbf{u}\|_{0,\Omega_s} \|\rho_s^{1/2} \tilde{\mathbf{v}}\|_{0,\Omega_s}. \end{aligned}$$

By the Korn and Poincaré inequalities, together with the second estimate of (5.1), there exists a constant  $C_3 > 0$ , independent of  $\lambda$ , such that

$$\|\rho_s^{1/2} \tilde{\mathbf{v}}\|_{0,\Omega_s} \leq C_3 \|\mu(\mathbf{x})^{-1/2} p\|_{0,\Omega_s}.$$

Using the preceding estimate and Young's inequality, we infer that, for any  $\delta_1, \delta_2 > 0$ ,

$$\begin{aligned} A(((\mathbf{u}, \mathbf{w}), p), ((\tilde{\mathbf{v}}, \mathbf{0}), 0)) &\geq \left( C_2 - \frac{C_1^2}{\delta_1} - \frac{C_3^2}{4\delta_2} \right) \|\mu(\mathbf{x})^{-1/2} p\|_{0,\Omega_s}^2 \\ &\quad - \delta_1 \|\mu(\mathbf{x})^{1/2} \boldsymbol{\varepsilon}(\mathbf{u})\|_{0,\Omega_s}^2 - \delta_2 \|\rho_s^{1/2} \mathbf{u}\|_{0,\Omega_s}^2. \end{aligned}$$

We now set  $\mathbf{v} := \mathbf{u} + \gamma \tilde{\mathbf{v}}$ ,  $\boldsymbol{\tau} := \mathbf{w}$ ,  $q := -p$ . Since  $\tilde{\mathbf{v}} = \mathbf{0}$  on  $\Sigma$ , the transmission condition is preserved, and hence  $((\mathbf{v}, \boldsymbol{\tau}), q) \in \mathbf{Y} \times \mathbf{Q}$ . Then, combining the preceding estimates, we obtain

$$A(((\mathbf{u}, \mathbf{w}), p), ((\mathbf{v}, \boldsymbol{\tau}), q)) = A(((\mathbf{u}, \mathbf{w}), p), ((\mathbf{u}, \mathbf{w}), -p)) + \gamma A(((\mathbf{u}, \mathbf{w}), p), ((\tilde{\mathbf{v}}, \mathbf{0}), 0)).$$

Choosing  $\delta_1, \delta_2 > 0$  sufficiently large so that

$$C_2 - \frac{C_1^2}{\delta_1} - \frac{C_3^2}{4\delta_2} > 0,$$

and then choosing  $\gamma > 0$  sufficiently small, independently of  $\lambda$ , we obtain

$$A(((\mathbf{u}, \mathbf{w}), p), ((\mathbf{v}, \boldsymbol{\tau}), q)) \geq \bar{C}_2 \|((\mathbf{u}, \mathbf{w}), p)\|_{\mathbf{H}}^2.$$

Here, we have also used the Korn inequality to control  $\|\mu(\mathbf{x})^{1/2} \nabla \mathbf{u}\|_{0,\Omega_s}$  by  $\|\mu(\mathbf{x})^{1/2} \boldsymbol{\varepsilon}(\mathbf{u})\|_{0,\Omega_s}$ .

Moreover, it also holds that

$$\|((\mathbf{v}, \boldsymbol{\tau}), q)\|_{\mathbf{H}} = \|((\mathbf{u} + \gamma \tilde{\mathbf{v}}, \mathbf{w}), -p)\|_{\mathbf{H}} \leq \bar{C}_1 \|((\mathbf{u}, \mathbf{w}), p)\|_{\mathbf{H}},$$

which completes the proof.  $\square$

To derive an upper bound of the the fluid contribution, we employ a classical Helmholtz decomposition. For sufficiently smooth scalar  $\psi$ , and vector  $\mathbf{v} := (v_1, v_2)^t$ , we let

$$\mathbf{curl}(\psi) := \left( \frac{\partial \psi}{\partial x_2}, -\frac{\partial \psi}{\partial x_1} \right)^t, \quad \text{rot}(\mathbf{v}) := \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}.$$

By applying [21, Lemma 4.4], for each  $\boldsymbol{\tau} \in \mathbf{W}$ , we can find  $z \in H^2(\Omega_f)$  and  $\phi \in H_\Sigma^1(\Omega_f)$ , where  $H_\Sigma^1(\Omega_f) := \{\phi \in H^1(\Omega_f) : \phi = 0 \text{ on } \Sigma\}$ , such that

$$\boldsymbol{\tau} = \nabla z + \mathbf{curl} \phi \text{ in } \Omega_f, \quad \text{and} \quad \|z\|_{2,\Omega_f} + \|\phi\|_{1,\Omega_f} \leq C_4 \|\boldsymbol{\tau}\|_{\mathbf{H}(\text{div}; \Omega_f)},$$

where  $C_4$  is positive constant. Finally, we can define a discrete function  $\boldsymbol{\tau}_h \in \mathbf{W}_h$  such that

$$\boldsymbol{\tau}_h = \boldsymbol{\Pi}_h(\nabla z) + \mathbf{curl} \phi_h.$$

where  $\boldsymbol{\Pi}_h$  is the BDM interpolation and  $\phi_h$  is a piecewise linear approximation. Then

$$\text{div}(\boldsymbol{\tau} - \boldsymbol{\tau}_h) = \text{div}(\nabla z - \boldsymbol{\Pi}_h(\nabla z)) = (\mathbf{I} - \mathcal{P}_h)(\text{div} \boldsymbol{\tau}),$$

is  $L^2(\Omega)$ -orthogonal.

**Theorem 5.2** *Let  $(\kappa, ((\mathbf{u}, \mathbf{w}), p)) \in \mathbb{R}^+ \times \mathbf{H}$  be the solution of the spectral problem ((2.4)) and let  $(\kappa_h, ((\mathbf{u}_h, \mathbf{w}_h), p_h)) \in \mathbb{R}^+ \times \mathbf{H}_h$  be the corresponding discrete solution of the spectral problem (3.6). Then, for every  $h_0 \geq h$  there holds:*

$$\begin{aligned} \|((\mathbf{u} - \mathbf{u}_h, \mathbf{w} - \mathbf{w}_h), p - p_h)\|_{\mathbf{H}} &\leq C_5(\eta + \Theta + \|(\rho_s(\kappa \mathbf{u} - \kappa_h \mathbf{u}_h))\|_{0, \Omega_s} + \|(\rho_s(\mathbf{u} - \mathbf{u}_h))\|_{0, \Omega_s} \\ &\quad + \|(\rho_f(\kappa \mathbf{w} - \kappa_h \mathbf{w}_h))\|_{0, \Omega_f} + \|(\rho_f(\mathbf{w} - \mathbf{w}_h))\|_{0, \Omega_f}), \end{aligned}$$

where  $C_5 > 0$  is independent of the mesh size,  $\lambda$ , and the discrete solution. Moreover, the eigenvalue also satisfies the following reliability bound:

$$\begin{aligned} |\kappa - \kappa_h| &\leq C_6(\eta^2 + \Theta^2 + \|(\rho_s(\kappa \mathbf{u} - \kappa_h \mathbf{u}_h))\|_{0, \Omega_s}^2 + \|(\rho_s(\mathbf{u} - \mathbf{u}_h))\|_{0, \Omega_s}^2 \\ &\quad + \|(\rho_f(\kappa \mathbf{w} - \kappa_h \mathbf{w}_h))\|_{0, \Omega_f}^2 + \|(\rho_f(\mathbf{w} - \mathbf{w}_h))\|_{0, \Omega_f}^2), \end{aligned}$$

where  $C_6$  is the positive constant independent of the mesh size,  $\lambda$  and the discrete solutions.

*Proof* By applying an application of the stability result of Lemma 5.1, for  $((\mathbf{u} - \mathbf{u}_h, \mathbf{w} - \mathbf{w}_h), p - p_h) \in \mathbf{H}$ , there exists  $((\mathbf{v}, \boldsymbol{\tau}), q) \in \mathbf{H}$  with  $\|((\mathbf{v}, \boldsymbol{\tau}), q)\|_{\mathbf{H}} \leq \bar{C}_1 \|((\mathbf{u} - \mathbf{u}_h, \mathbf{w} - \mathbf{w}_h), p - p_h)\|_{\mathbf{H}}$  such that

$$\bar{C}_2 \|((\mathbf{u} - \mathbf{u}_h, \mathbf{w} - \mathbf{w}_h), p - p_h)\|_{\mathbf{H}}^2 \leq A(((\mathbf{u} - \mathbf{u}_h, \mathbf{w} - \mathbf{w}_h), p - p_h), ((\mathbf{v}, \boldsymbol{\tau}), q)).$$

Using the definition of the weak formulation (2.4), it follows that

$$\begin{aligned} \bar{C}_2 \|((\mathbf{u} - \mathbf{u}_h, \mathbf{w} - \mathbf{w}_h), p - p_h)\|_{\mathbf{H}}^2 &\leq (\kappa + 1)B(((\mathbf{u}, \mathbf{w}), p), ((\mathbf{v}, \boldsymbol{\tau}), q)) - A(((\mathbf{u}_h, \mathbf{w}_h), p_h), ((\mathbf{v}, \boldsymbol{\tau}), q)), \\ &\leq (\kappa + 1)B(((\mathbf{u}, \mathbf{w}), p), ((\mathbf{v}, \boldsymbol{\tau}), q)) - (\kappa_h + 1)B(((\mathbf{u}_h, \mathbf{w}_h), p_h), ((\mathbf{v}, \boldsymbol{\tau}), q)) \\ &\quad + (\kappa_h + 1)B(((\mathbf{u}_h, \mathbf{w}_h), p_h), ((\mathbf{v}, \boldsymbol{\tau}), q)) - A(((\mathbf{u}_h, \mathbf{w}_h), p_h), ((\mathbf{v}, \boldsymbol{\tau}), q)). \end{aligned}$$

By the definition of the bilinear form  $B(\cdot, \cdot)$  and using Cauchy–Schwarz’s inequality, we obtain

$$\begin{aligned} |(\kappa + 1)B(((\mathbf{u}, \mathbf{w}), p), ((\mathbf{v}, \boldsymbol{\tau}), q)) - (\kappa_h + 1)B(((\mathbf{u}_h, \mathbf{w}_h), p_h), ((\mathbf{v}, \boldsymbol{\tau}), q))| \\ = |((\kappa + 1)\rho_s \mathbf{u} - (\kappa_h + 1)\rho_s \mathbf{u}_h, \mathbf{v}) + ((\kappa + 1)\rho_f \mathbf{w} - (\kappa_h + 1)\rho_f \mathbf{w}_h, \boldsymbol{\tau})| \\ \leq C_7(\|\rho_s(\kappa \mathbf{u} - \kappa_h \mathbf{u}_h)\|_{0, \Omega_s} + \|\rho_s(\mathbf{u} - \mathbf{u}_h)\|_{0, \Omega_s})\|\mathbf{v}\|_{0, \Omega_s} \\ + (\|\rho_f(\kappa \mathbf{w} - \kappa_h \mathbf{w}_h)\|_{0, \Omega_f} + \|\rho_f(\mathbf{w} - \mathbf{w}_h)\|_{0, \Omega_f})\|\boldsymbol{\tau}\|_{0, \Omega_f}, \end{aligned}$$

where  $C_7$  is the positive constant independent of the mesh size,  $\lambda$  and the discrete solutions. In order to estimate

$$(\kappa_h + 1)B(((\mathbf{u}_h, \mathbf{w}_h), p_h), ((\mathbf{v}, \boldsymbol{\tau}), q)) - A(((\mathbf{u}_h, \mathbf{w}_h), p_h), ((\mathbf{v}, \boldsymbol{\tau}), q)),$$

we first add

$$A(((\mathbf{u}_h, \mathbf{w}_h), p_h), ((\mathbf{v}_h, \boldsymbol{\tau}_h), q_h)) - (\kappa_h + 1)B(((\mathbf{u}_h, \mathbf{w}_h), p_h), ((\mathbf{v}_h, \boldsymbol{\tau}_h), q_h)) = 0,$$

and then apply integration by parts element-wise with the Helmholtz decomposition for  $\boldsymbol{\tau} - \boldsymbol{\tau}_h$ . Then we obtain

$$\begin{aligned} & (\kappa_h + 1)B(((\mathbf{u}_h, \mathbf{w}_h), p_h), ((\mathbf{v} - \mathbf{v}_h, \boldsymbol{\tau} - \boldsymbol{\tau}_h), q - q_h)) - A(((\mathbf{u}_h, \mathbf{w}_h), p_h), ((\mathbf{v} - \mathbf{v}_h, \boldsymbol{\tau} - \boldsymbol{\tau}_h), q - q_h)) \\ & \quad =: I_S + I_F + I_I, \end{aligned}$$

where

$$\begin{aligned} I_S &:= \sum_{T \in \mathcal{T}_h \cap \Omega_s} \int_T (\operatorname{div}(2\mu(\mathbf{x})\boldsymbol{\varepsilon}(\mathbf{u}_h)) - \nabla p_h + \kappa_h \rho_s \mathbf{u}_h) \cdot (\mathbf{v} - \mathbf{v}_h) + \int_{\Omega_s} \left( \operatorname{div} \mathbf{u}_h + \frac{1}{\lambda} p_h \right) (q - q_h) \\ &\quad - \sum_{\ell \in \mathcal{E}(\mathcal{T}_h) \cap \Omega_s} \int_{\ell} ((2\mu_h \boldsymbol{\varepsilon}(\mathbf{u}_h) - p_h \mathbf{I}) \mathbf{n}) \cdot (\mathbf{v} - \mathbf{v}_h), \\ I_F &:= \int_{\Omega_f} (\nabla(c^2 \rho_f \operatorname{div} \mathbf{w}_h) + \kappa_h \rho_f \mathbf{w}_h) \cdot (\nabla \mathbf{z} - \boldsymbol{\Pi}_h(\nabla \mathbf{z})) + \int_{\Omega_f} (\kappa_h \rho_f \mathbf{w}_h) \cdot (\operatorname{curl} \phi - \operatorname{curl} \phi_h) \\ &\quad - \sum_{\ell \in \mathcal{E}(\mathcal{T}_h) \cap \Omega_f} \int_{\ell} ((c^2 \rho_f \operatorname{div} \mathbf{w}_h) \mathbf{n}) \cdot (\nabla \mathbf{z} - \boldsymbol{\Pi}_h(\nabla \mathbf{z})) - \int_{\Gamma_0} (g \rho_f \mathbf{w}_h \cdot \mathbf{n}) ((\nabla \mathbf{z} - \boldsymbol{\Pi}_h(\nabla \mathbf{z})) \cdot \mathbf{n}), \\ I_I &:= \sum_{\ell \in \mathcal{E}(\mathcal{T}_h) \cap \Sigma} \int_{\ell} ((2\mu_h \boldsymbol{\varepsilon}(\mathbf{u}_h) - p_h \mathbf{I}) \mathbf{n}) \cdot (\mathbf{v} - \mathbf{v}_h) - \sum_{\ell \in \mathcal{E}(\mathcal{T}_h) \cap \Sigma} \int_{\ell} ((c^2 \rho_f \operatorname{div} \mathbf{w}_h) \mathbf{n}) \cdot (\nabla \mathbf{z} - \boldsymbol{\Pi}_h(\nabla \mathbf{z})). \end{aligned}$$

Applying integration by parts, along with the Cauchy–Schwarz inequality and the approximation results of the Cl  ment interpolant (see [22, Theorem 1] for instance), yields:

$$I_S \leq C_8(\eta_S + \Theta_S) \|((\mathbf{v}, \boldsymbol{\tau}), q)\|_{\mathbf{H}},$$

where  $C_8$  is the positive constant independent of the mesh size,  $\lambda$  and the discrete solutions. Using integration by parts and the Cauchy–Schwarz inequality implies that

$$|I_F| \leq C_9(\eta_F + \Theta_F) \|\boldsymbol{\tau}\|_{\mathbf{H}(\operatorname{div}, \Omega_f)}, \quad |I_I| \leq C_{10} \eta_I \|((\mathbf{v}, \boldsymbol{\tau}), q)\|_{\mathbf{H}},$$

where  $C_9$  and  $C_{10}$  are positive constants independent of the mesh size,  $\lambda$  and the discrete solutions. This completes the proof.  $\square$

### 5.3. Efficiency

We now establish the efficiency of the a posteriori error estimator. The proof relies on standard localization arguments based on element and edge bubble functions. We first recall the properties of the interior bubble functions; see [1, Theorem 2.2].

**Lemma 5.3** (Interior bubble functions) *For any  $T \in \mathcal{T}_h$ , let  $\psi_T$  be the corresponding interior bubble function. Then, the following estimates hold:*

$$\begin{aligned} C_{11} \|q\|_{0,T}^2 &\leq \int_T \psi_T q^2 \leq \|q\|_{0,T}^2 \quad \forall q \in \mathbb{P}_k(T), \\ C_{12} \|q\|_{0,T} &\leq \|\psi_T q\|_{0,T} + h_T \|\nabla(\psi_T q)\|_{0,T} \leq C_{13} \|q\|_{0,T} \quad \forall q \in \mathbb{P}_k(T), \end{aligned}$$

where  $C_{11}$ ,  $C_{12}$  and  $C_{13}$  are positive constants independent of  $h_T$ .

For the edge bubble functions, we have the following properties [1, Theorem 2.4].

**Lemma 5.4** (Edge bubble functions) *For any  $T \in \mathcal{T}_h$  and  $\ell \in \mathcal{E}_T$ , let  $\psi_\ell$  be the corresponding edge bubble function. Then, there holds*

$$C_{14} \|q\|_{0,\ell}^2 \leq \int_\ell \psi_\ell q^2 \leq \|q\|_{0,\ell}^2 \quad \forall q \in \mathbb{P}_k(\ell).$$

Moreover, for every  $q \in \mathbb{P}_k(\ell)$ , there exists an extension of  $q$  to  $\mathbb{P}_k(T)$  (again denoted by  $q$ ) such that

$$h_T^{-1/2} \|\psi_\ell q\|_{0,T} + h_T^{1/2} \|\nabla(\psi_\ell q)\|_{0,T} \leq C_{15} \|q\|_{0,\ell},$$

where  $C_{14}$  and  $C_{15}$  are positive constants independent of  $h_T$ .

The efficiency of the solid estimator  $\eta_S$  directly follows from [36, Lemma 4.3]. On the other hand, the efficiency of the fluid estimator  $\eta_F$  follows by collecting the bounds in the following result.

**Lemma 5.5** *There holds:*

$$\begin{aligned} h_T^2 \|\rho^F \mathbf{R}_{1,T}^F\|_{0,T}^2 &\leq \bar{C}_3 \left( \|c \rho_f^{1/2} \operatorname{div}(\mathbf{w} - \mathbf{w}_h)\|_{0,T}^2 + \|(\rho^F)^{-1/2} (\omega_h^2 \rho_f \mathbf{w}_h - \omega^2 \rho_f \mathbf{w})\|_{0,T}^2 \right), \\ h_T^2 \|\rho^F \mathbf{R}_{2,T}^S\|_{0,T}^2 &\leq \bar{C}_4 \|(\rho^F)^{-1/2} (\omega_h^2 \rho_f \mathbf{w}_h - \omega^2 \rho_f \mathbf{w})\|_{0,T}^2, \\ h_\ell \|\rho_\ell^F J_{1,\ell}^F\|_{0,\ell}^2 &\leq \bar{C}_5 \left( \|c \rho_f^{1/2} \operatorname{div}(\mathbf{w} - \mathbf{w}_h)\|_{0,\omega_\ell}^2 + \|(\rho^F)^{-1/2} (\omega_h^2 \rho_f \mathbf{w}_h - \omega^2 \rho_f \mathbf{w})\|_{0,\omega_\ell}^2 \right), \\ h_\ell \|\rho_\ell^F J_{2,\ell}^F\|_{0,\ell}^2 &\leq \bar{C}_6 \|(\rho^F)^{-1/2} (\omega_h^2 \rho_f \mathbf{w}_h - \omega^2 \rho_f \mathbf{w})\|_{0,\omega_\ell}^2, \end{aligned}$$

where  $\bar{C}_3$ ,  $\bar{C}_4$ ,  $\bar{C}_5$  and  $\bar{C}_6$  are positive constants independent of  $h_T$  and  $\lambda$ . Here,  $\omega_\ell := T \cup T'$  denotes the element patch associated with the interior edge  $\ell$ , with  $T$  and  $T'$  sharing the edge  $\ell$ .

*Proof* Firstly, we define  $\mathbf{v}_T := \chi_T h_T^2 (\rho^F)^2 \mathbf{R}_{1,T}^F$ , where  $\chi_T$  is the element bubble function defined on  $T$ . Then

$$\begin{aligned} C_{11} h_T^2 \|\rho^F \mathbf{R}_{1,T}^F\|_{0,T}^2 &\leq (\mathbf{R}_{1,T}^F, \mathbf{v}_T) = \int_T (c^2 \rho_f \nabla(\operatorname{div} \mathbf{w}_h) + \omega_h^2 \rho_f \mathbf{w}_h) \mathbf{v}_T \\ &= \int_T (c^2 \rho_f \nabla(\operatorname{div} \mathbf{w}_h) + \omega_h^2 \rho_f \mathbf{w}_h - c^2 \rho_f \nabla(\operatorname{div} \mathbf{w}) + \omega^2 \rho_f \mathbf{w}) \mathbf{v}_T \\ &= \int_T (-c^2 \rho_f \nabla(\operatorname{div}(\mathbf{w} - \mathbf{w}_h))) \mathbf{v}_T + \int_T (\omega_h^2 \rho_f \mathbf{w}_h - \omega^2 \rho_f \mathbf{w}) \mathbf{v}_T =: D_1 + D_2. \end{aligned}$$

An application of integration of parts on  $D_1$  gives that

$$D_1 = \int_T (-c^2 \rho_f \nabla(\operatorname{div}(\mathbf{w} - \mathbf{w}_h))) \mathbf{v}_T \leq \|c \rho_f^{1/2} \operatorname{div}(\mathbf{w} - \mathbf{w}_h)\|_{0,T} h_T \|\rho^F \mathbf{R}_{1,T}^F\|_{0,T}.$$

Finally, in order to estimate  $D_2$  we simply apply the Cauchy–Schwarz inequality to obtain

$$D_2 = \int_T (\omega_h^2 \rho_f \mathbf{w}_h - \omega^2 \rho_f \mathbf{w}) \mathbf{v}_T \leq \|(\rho^F)^{-1/2} (\omega_h^2 \rho_f \mathbf{w}_h - \omega^2 \rho_f \mathbf{w})\|_{0,T} h_T \|\rho^F \mathbf{R}_{1,T}^F\|_{0,T}.$$

By applying the inverse inequality, we derive the second stated result. Next, we define

$$\mathbf{v}_\ell := \psi_\ell h_\ell (\rho_\ell^F)^2 \llbracket c^2 \rho_f (\operatorname{div} \mathbf{w}_h) \mathbf{n} \rrbracket,$$

where  $\psi_\ell$  is the bubble function that satisfies the properties outlined in Lemma 5.4. We then proceed to estimate the term  $h_\ell \|\rho_\ell^F J_{1,\ell}\|_{0,\ell}^2$ , leading to

$$C_{14} h_\ell \|\rho_\ell^F J_{1,\ell}\|_{0,\ell}^2 \leq (\llbracket c^2 \rho_f (\operatorname{div} \mathbf{w}_h) \mathbf{n} \rrbracket, \mathbf{v}_\ell)_\ell. \quad (5.2)$$

Applying integration by parts on  $\omega_\ell$  yields

$$\begin{aligned} (\llbracket c^2 \rho_f (\operatorname{div} \mathbf{w}_h) \mathbf{n} \rrbracket, \mathbf{v}_\ell)_\ell &= \sum_{T \in \omega_\ell} \left( (c^2 \rho_f (\operatorname{div} \mathbf{w}_h) - c^2 \rho_f (\operatorname{div} \mathbf{w}), \operatorname{div}(\mathbf{v}_\ell))_T \right. \\ &\quad \left. + (\mathbf{R}_{1,T}^F, \mathbf{v}_\ell)_T + (\omega_h^2 \rho_f \mathbf{w}_h - \omega^2 \rho_f \mathbf{w}, \mathbf{v}_\ell)_T \right). \end{aligned}$$

Using the Cauchy–Schwarz inequality along with Lemma 5.4 and combining it with (5.2), we obtain that

$$\begin{aligned} C_{14} h_\ell \|\rho_\ell^F J_{1,\ell}\|_{0,\ell}^2 &\leq C_{16} h_\ell^{1/2} (\|c \rho_f^{1/2} \operatorname{div}(\mathbf{w} - \mathbf{w}_h)\|_{0,\omega_\ell} \\ &\quad + \|(\rho^F)^{-1/2} (\omega_h^2 \rho_f \mathbf{w}_h - \omega^2 \rho_f \mathbf{w})\|_{0,\omega_\ell}) \|\rho_\ell^F J_{1,\ell}\|_{0,\ell}. \end{aligned}$$

The last estimate is a rescaled version of the interior edge bound proven in [27, Theorem 2].  $\square$

Lastly, the efficiency of the first term of the interface estimator  $\eta_I$  is established in the following lemma, which follows directly from [36, Lemma 4.3] in conjunction with Lemma 5.5. The efficiency of the second term of  $\eta_I$  has already been established in Lemma 5.5.

**Lemma 5.6** *There holds:*

$$\begin{aligned} &h_\ell \|\rho^I [(2\mu_h \boldsymbol{\varepsilon}(\mathbf{u}_h) - p_h \mathbf{I}) \mathbf{n} - c^2 \rho_f (\operatorname{div} \mathbf{w}_h) \mathbf{n}]\|_{0,\ell}^2 \\ &\leq \bar{C}_7 \left( \|(2\mu)^{1/2} \boldsymbol{\varepsilon}(\mathbf{u} - \mathbf{u}_h)\|_{0,\omega_\ell \cap \Omega_s}^2 + \|(2\mu)^{-1/2} (p - p_h)\|_{0,\omega_\ell \cap \Omega_s}^2 + \|\rho_s^{-1/2} (\omega_h^2 \rho_s \mathbf{u}_h - \omega^2 \rho_s \mathbf{u})\|_{0,\omega_\ell \cap \Omega_s}^2 \right. \\ &\quad \left. + \|c \rho_f^{1/2} \operatorname{div}(\mathbf{w} - \mathbf{w}_h)\|_{0,\omega_\ell \cap \Omega_f}^2 + \|(\rho^F)^{-1/2} (\omega_h^2 \rho_f \mathbf{w}_h - \omega^2 \rho_f \mathbf{w})\|_{0,\omega_\ell \cap \Omega_f}^2 \right) \end{aligned}$$

where  $\bar{C}_7$  is positive constant independent of  $h_T$  and  $\lambda$ . Here,  $\omega_\ell := T \cup T'$  denotes the element patch associated with the interior edge  $\ell$ , with  $T$  and  $T'$  sharing the edge  $\ell$ .

*Proof* First, we define

$$\mathbf{v}_\ell := \psi_\ell h_\ell (\rho^I)^2 ((2\mu \boldsymbol{\varepsilon}(\mathbf{u}_h) - p \mathbf{I}) \mathbf{n} - (\rho_f c^2 \operatorname{div} \mathbf{w}_h) \mathbf{n}),$$

where  $\psi_\ell$  is the bubble function that satisfies the properties outlined in Lemma 5.4. We then proceed to estimate the first term of the interface estimator  $\eta_I$ , leading to

$$C_{14} h_\ell \|\rho^I ((2\mu \boldsymbol{\varepsilon}(\mathbf{u}_h) - p \mathbf{I}) \mathbf{n} - (\rho_f c^2 \operatorname{div} \mathbf{w}_h) \mathbf{n})\|_{0,\ell}^2 \leq (((2\mu \boldsymbol{\varepsilon}(\mathbf{u}_h) - p \mathbf{I}) \mathbf{n} - (\rho_f c^2 \operatorname{div} \mathbf{w}_h) \mathbf{n}), \mathbf{v}_\ell)_\ell. \quad (5.3)$$

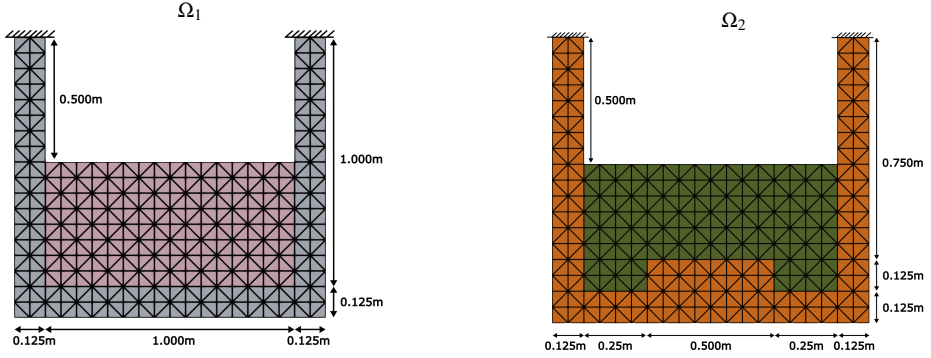


FIG. 6.1. Test 6.1. Schematic of 2D computational domains indicating dimensions, boundaries, subdomains, and using a coarse mesh with  $N = 2$ .

An application of integration by parts gives

$$\begin{aligned}
 (((2\mu\boldsymbol{\varepsilon}(\mathbf{u}_h) - p\mathbf{I})\mathbf{n} - (\rho_f c^2 \operatorname{div} \mathbf{w}_h)\mathbf{n}), \mathbf{v}_\ell)_\ell &= \sum_{T \in \omega_\ell \cap \Omega_s} \left( (2\mu\boldsymbol{\varepsilon}(\mathbf{u}) - 2\mu\boldsymbol{\varepsilon}(\mathbf{u}_h), \boldsymbol{\varepsilon}(\mathbf{v}_\ell))_T + (p - p_h, \operatorname{div} \mathbf{v}_\ell)_T \right. \\
 &\quad \left. + (\mathbf{R}_{1,T}^S, \mathbf{v}_\ell)_T + (\omega_h^2 \rho_s \mathbf{u}_h - \omega^2 \rho_s \mathbf{u}, \mathbf{v}_\ell)_T \right) \\
 &+ \sum_{T \in \omega_\ell \cap \Omega_f} \left( (c^2 \rho_f (\operatorname{div} \mathbf{w}_h) - c^2 \rho_f (\operatorname{div} \mathbf{w}), \operatorname{div}(\mathbf{v}_\ell))_T \right. \\
 &\quad \left. + (\mathbf{R}_{1,T}^F, \mathbf{v}_\ell)_T + (\omega_h^2 \rho_f \mathbf{w}_h - \omega^2 \rho_f \mathbf{w}, \mathbf{v}_\ell)_T \right).
 \end{aligned}$$

Using the Cauchy–Schwarz inequality along with Lemma 5.4 and combining it with (5.3), we obtain that

$$\begin{aligned}
 &C_{14} h_\ell \|\rho^I ((2\mu\boldsymbol{\varepsilon}(\mathbf{u}_h) - p\mathbf{I})\mathbf{n} - (\rho_f c^2 \operatorname{div} \mathbf{w}_h)\mathbf{n})\|_{0,\ell}^2 \\
 &\leq \left( \| (2\mu)^{1/2} \boldsymbol{\varepsilon}(\mathbf{u} - \mathbf{u}_h) \|_{0,\omega_\ell \cap \Omega_s} + \| (2\mu)^{-1/2} (p - p_h) \|_{0,\omega_\ell \cap \Omega_s} + \| (\rho_s)^{-1/2} (\omega_h^2 \rho_s \mathbf{u}_h - \omega^2 \rho_s \mathbf{u}) \|_{0,\omega_\ell \cap \Omega_s} \right) \\
 &\quad + \left( \| c \rho_f^{1/2} \operatorname{div}(\mathbf{w} - \mathbf{w}_h) \|_{0,\omega_\ell \cap \Omega_f} + \| (\rho^F)^{-1/2} (\omega_h^2 \rho_f \mathbf{w}_h - \omega^2 \rho_f \mathbf{w}) \|_{0,\omega_\ell \cap \Omega_f} \right) \\
 &C_{17} h_\ell^{1/2} \|\rho^I ((2\mu\boldsymbol{\varepsilon}(\mathbf{u}_h) - p\mathbf{I})\mathbf{n} - (\rho_f c^2 \operatorname{div} \mathbf{w}_h)\mathbf{n})\|_{0,\ell},
 \end{aligned}$$

which completes the proof.  $\square$

## 6. Numerical experiments

This section presents a series of computational tests using the open-source FE library FEniCS [2] together with the special modules FEniCSii [37] and multiphenics [7] for the treatment of bulk-surface coupling mechanisms. The meshes have been constructed using the mesh generation/manipulation library GMSH [30].

As we typically do not know the closed-form solutions for the eigenvalues, their convergence rates have been obtained with a standard least-squares fitting and highly refined meshes. We denote by  $N$



the mesh refinement level (number of edges along the shortest edge) and  $\text{dof}$  denotes the number of degrees of freedom. We denote by  $\omega_{h,i}$  the  $i$ -th discrete eigenfrequency and denote the error on the  $i$ -th eigenvalue by  $\text{err}(\omega_i^2)$  with

$$\text{err}(\omega_i^2) := |\omega_{h,i}^2 - \omega_i^2|.$$

### 6.1. Accuracy test

In this section we perform a convergence analysis with respect to mesh refinement using the first three eigenvalues of the coupled problem. With the aim of comparing the numerically obtained results against the simulations reported in [40], the first series of tests employ the following parameter values  $\rho_s = 7700\text{kg/m}^3$ ,  $E = 1.44 \times 10^{11}\text{Pa}$ ,  $\nu = 0.35$ ,  $\rho_f = 1000\text{kg/m}^3$ ,  $c = 1430\text{m/s}$ ,  $g = 9.8\text{m/s}^2$ , and we focus on the elasto-acoustic modes. We consider two configurations for the geometry. First, we consider  $\Omega_1$  such that the fluid domain is a rectangle and the solid container is a polygon as depicted in Figure 6.1(left). The second geometry, denoted as  $\Omega_2$ , is such that we have re-entrant corners in the solid and fluid subdomains 6.1(right). The domains are discretized such that there is conformity between  $\Omega_s$  and  $\Omega_f$ .

The error history is outlined in the two blocks of Table 6.1 for the elasto-acoustic modes in  $\Omega_1$  and  $\Omega_2$ , respectively. We observe an asymptotic linear convergence for the case of Taylor–Hood elements, and suboptimal rate for MINI-elements. However, results on Taylor–Hood on the first part of Table 6.1 are closer to the reference values from [40]. Suboptimal convergence results are expected in  $\Omega_i$  because of the strong interaction between the solid and the fluid, where the solid contains at least two angle singularities and four points where boundary conditions change from Dirichlet to Neumann.

### 6.2. Fluid-structure interaction in a half-filled barrel

In this experiment we study the behavior of the scheme when considering a non-polygonal domain in three dimensions. The solid and fluid sub-domains (see a sketch in Figure 6.3) are defined as follows

$$\begin{aligned}\Omega_{s_1} &:= \{(x, y, z) \in \mathbb{R}^3 : y^2 + z^2 = 1, x \in [-2, 1]\}, \\ \Omega_{s_2} &:= \{(x, y, z) \in \mathbb{R}^3 : y^2 + z^2 = 0.75^2, x \in [-1.75, 0.75]\}, \\ \Omega_s &:= \Omega_{s_1} \setminus \Omega_{s_2}, \quad \Omega_f := \{(x, y, z) \in \mathbb{R}^3 : \sqrt{y^2 + z^2} = \sqrt{0.75}, x \in [-1.75, 0.75]\}.\end{aligned}$$

The barrel is clamped on the external circular faces and free of stress in the rest. In order to compute the references eigenfrequencies, we have used a highly refined mesh and the higher order family  $\mathbb{P}_3 + \mathbb{P}_2 + \mathbf{BDM}_3$  for the solid displacement, solid pressure and fluid displacement, respectively. The mesh level is defined as  $h \approx 1/N$ . The error history is reported in Table 6.2. Approximate solutions are shown in Figure 6.4. A linear rate of convergence is observed, justified by the fluid-solid interaction.

### 6.3. A posteriori error estimation

With the aim of assessing the performance of the a posteriori error estimator, we consider domains with singularities in two and three dimensions. On each adaptive iteration, we use the blue-green marking strategy to refine each  $T' \in \mathcal{T}_h$  whose indicator  $\eta_{T'}$  satisfies  $\eta_{T'} \geq 0.5 \max\{\eta_T : T \in \mathcal{T}_h\}$ . The effectivity indexes with respect to  $\eta$  and the eigenvalue  $\omega_i$  are defined by

$$\text{eff}(\omega_i) := \text{err}(\omega_i) / \eta^2.$$

For simplicity, the adaptive algorithm uses the MINI-element family for the solid displacement and pressure.

TABLE 6.1 *Test 6.1. Lowest computed elasto-acoustic eigenvalues on two different geometry configurations  $\Omega_i$ .*

|                          |                                       |           |           |           |       |                 |                                 |
|--------------------------|---------------------------------------|-----------|-----------|-----------|-------|-----------------|---------------------------------|
| Configuration $\Omega_1$ | $N = 8$                               | $N = 10$  | $N = 12$  | $N = 14$  | Order | $\omega_{extr}$ | Meddahi <i>et al.</i><br>(2014) |
|                          | MINI-element + <b>BDM<sub>1</sub></b> |           |           |           |       |                 |                                 |
|                          | 452.4554                              | 449.4600  | 447.7540  | 446.6792  | 1.83  | 443.3200        | 442.71                          |
|                          | 1495.3899                             | 1487.1441 | 1482.4956 | 1479.5959 | 1.88  | 1470.8135       | 1469.4                          |
|                          | 2634.2701                             | 2618.4326 | 2609.1272 | 2603.1127 | 1.68  | 2582.4208       | 2578.33                         |
|                          | 2813.9435                             | 2796.5619 | 2786.7383 | 2780.5963 | 1.87  | 2761.8755       | 2758.94                         |
|                          | Taylor–Hood + <b>BDM<sub>2</sub></b>  |           |           |           |       |                 |                                 |
|                          | 443.7421                              | 443.5416  | 443.4114  | 443.3204  | 1.18  | 442.8549        | 442.71                          |
|                          | 1471.5727                             | 1471.1864 | 1470.9341 | 1470.7567 | 1.15  | 1469.8245       | 1469.4                          |
|                          | 2586.2547                             | 2584.7255 | 2583.7213 | 2583.0128 | 1.12  | 2579.1717       | 2578.33                         |
|                          | 2763.4399                             | 2762.6174 | 2762.0797 | 2761.7013 | 1.14  | 2759.6912       | 2758.94                         |

|                          |                                       |           |           |           |       |                 |
|--------------------------|---------------------------------------|-----------|-----------|-----------|-------|-----------------|
| Configuration $\Omega_2$ | $N = 10$                              | $N = 12$  | $N = 14$  | $N = 16$  | Order | $\omega_{extr}$ |
|                          | MINI-element + <b>BDM<sub>1</sub></b> |           |           |           |       |                 |
|                          | 405.0943                              | 403.5271  | 402.5353  | 401.8617  | 1.76  | 399.2754        |
|                          | 1597.7890                             | 1592.0717 | 1588.3495 | 1585.7601 | 1.59  | 1574.6229       |
|                          | 2634.5808                             | 2625.2576 | 2619.2071 | 2615.0098 | 1.61  | 2597.2107       |
|                          | 2654.9989                             | 2645.1907 | 2638.9963 | 2634.7990 | 1.78  | 2618.9133       |
|                          | Taylor–Hood + <b>BDM<sub>2</sub></b>  |           |           |           |       |                 |
|                          | 399.6299                              | 399.4973  | 399.4046  | 399.3362  | 1.48  | 399.0376        |
|                          | 1576.9642                             | 1576.2474 | 1575.7438 | 1575.3713 | 1.14  | 1573.0690       |
|                          | 2600.9286                             | 2599.7732 | 2598.9654 | 2598.3703 | 1.17  | 2594.7947       |
|                          | 2620.8404                             | 2620.0780 | 2619.5463 | 2619.1554 | 1.18  | 2616.8270       |

TABLE 6.2 *Example 6.2. Lowest computed eigenvalues using different combinations of FE families in the half filled barrel domain.*

|                                       |           |           |           |       |                 |
|---------------------------------------|-----------|-----------|-----------|-------|-----------------|
| $N = 8$                               | $N = 10$  | $N = 12$  | $N = 14$  | Order | $\omega_{extr}$ |
| MINI-element + <b>BDM<sub>1</sub></b> |           |           |           |       |                 |
| 752.0366                              | 750.1402  | 735.6936  | 731.3101  | 1.11  | 712.2499        |
| 837.3585                              | 833.0175  | 807.9182  | 799.7536  | 1.17  | 768.1502        |
| 1072.1009                             | 1066.3581 | 1022.3635 | 1009.2311 | 1.30  | 963.4901        |
| 1137.4735                             | 1133.2683 | 1101.1474 | 1091.0507 | 1.30  | 1057.3622       |
| Taylor–Hood + <b>BDM<sub>2</sub></b>  |           |           |           |       |                 |
| 715.9959                              | 715.2956  | 714.2166  | 713.8267  | 1.26  | 712.2499        |
| 772.4715                              | 771.5097  | 770.2985  | 769.8680  | 1.33  | 768.1502        |
| 968.7934                              | 967.6739  | 966.1642  | 965.6427  | 1.31  | 963.4901        |
| 1061.7571                             | 1060.9426 | 1059.6572 | 1059.1596 | 1.29  | 1057.3622       |

### 6.3.1. 2D example

From Section 6.1 we observed that elasto-acoustic modes converges with suboptimal rate because of the point singularities in the reentrant corner. In this example we study the convergence on the adaptive process when this configuration is considered. The domain under consideration is the same as in Figure 6.1(right). We will analyze the convergence of the first eigenvalue for several values of  $\nu$ . This is observed in the intermediate meshes portrayed in Figure 6.5. Also, we note that the refinements are

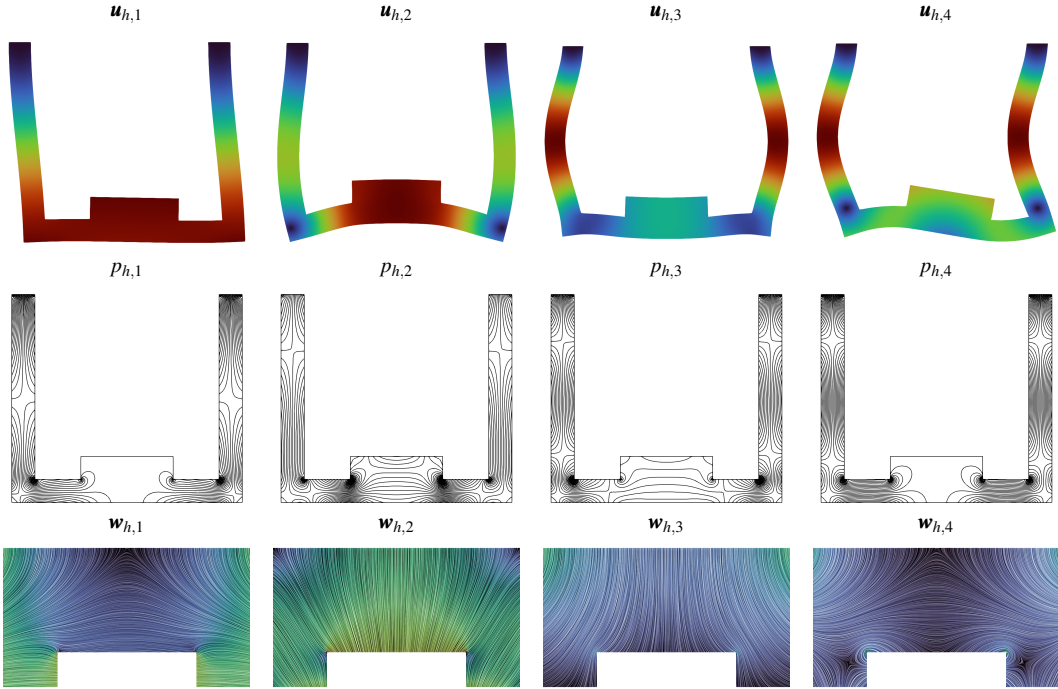


FIG. 6.2. Example 6.1. Comparison between the first fourth lowest order computed elasto-acoustic modes on  $\Omega_2$ . The solid domain have been warped by a sufficiently large factor in order to observe the deformation.

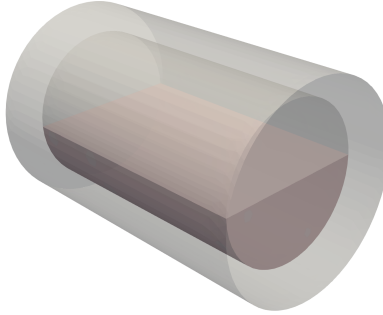


FIG. 6.3. Example 6.2. The computational domain of the half filled barrel.

concentrated where the solid pressure  $p_{h,1}$  becomes singular, namely, near the re-entrant corner and where the boundary conditions change from Dirichlet to Neumann type.

In order to further examine the locking-free property of our scheme, we proceed to change the Poisson ratio but keeping the rest of the physical parameters unmodified. Adaptive meshes for  $\nu = 0.49$  and  $\nu = 0.5$  are also depicted in Figure 6.5. Here, we observe that less elements are used the closer we get to  $\nu = 0.5$ . The error decay is presented in Figure 6.6 for different values of  $\nu$ . We observe that optimal rates are attained for some  $h_0$  as predicted in Section 5. We end the experiment by investigating

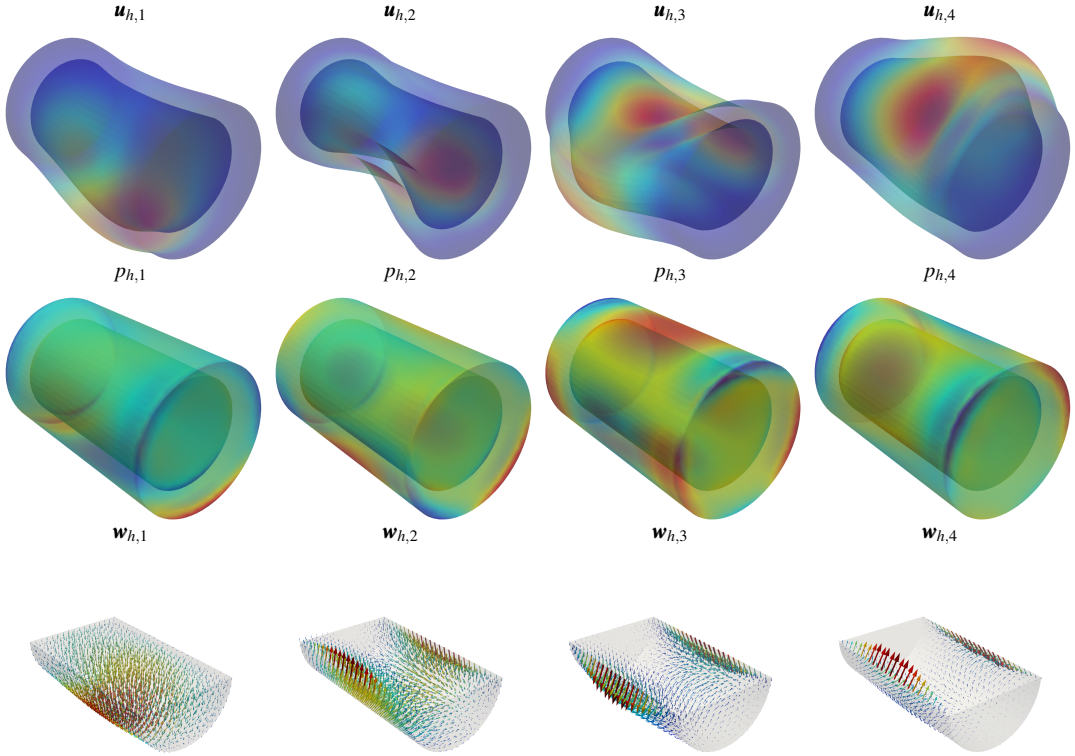


FIG. 6.4. Example 6.2. Sample numerical solutions of the first four elasto-acoustic modes on the barrel domain. The solid subdomain has been warped by a sufficiently large factor in order to observe the deformation.

the effectivity curves, reported in Figure 6.7. Properly bounded effectivity values are observed for the studied eigenvalues with the selected values for  $\nu$ .

#### 6.4. 3D example

In this experiment, we test the estimator on a three-dimensional geometry exhibiting both dihedral and trihedral singularities. Although the a posteriori analysis presented in Section 5 is restricted to two dimensions, the estimator admits a natural three-dimensional counterpart, with a similar definition, that can be evaluated within the present discrete framework. Hence, the goal of this experiment is to provide numerical evidence of the above. We consider the domain defined by  $\Omega_{CF} := \Omega_s \cup \Omega_f$ , where

$$\Omega_s := (-1, 1)^3,$$

$$\Omega_f := (-0.75, -0.3) \times (-0.75, -0.2) \times (-0.75, -0.1) \setminus ((-0.5, -0.3) \times (-0.5, -0.2) \times (-0.5, -0.1)).$$

Here,  $\Omega_f$  is a Fichera-like domain. The physical parameters are the same as those of Section 6.1. A sample of this domain is depicted in Figure 6.8. The refinement level is such that  $1/N \approx h$ . We assume that  $\Omega_{CF}$  is clamped in the  $xz$ -plane with  $y = -1$  and consider  $\Gamma_0 = \emptyset$ .

We start by presenting the lowest computed elasto-acoustic frequencies in Table 6.3. We observe a linear rate of convergence in all the cases, denoting a strong coupling between solid and fluid. This

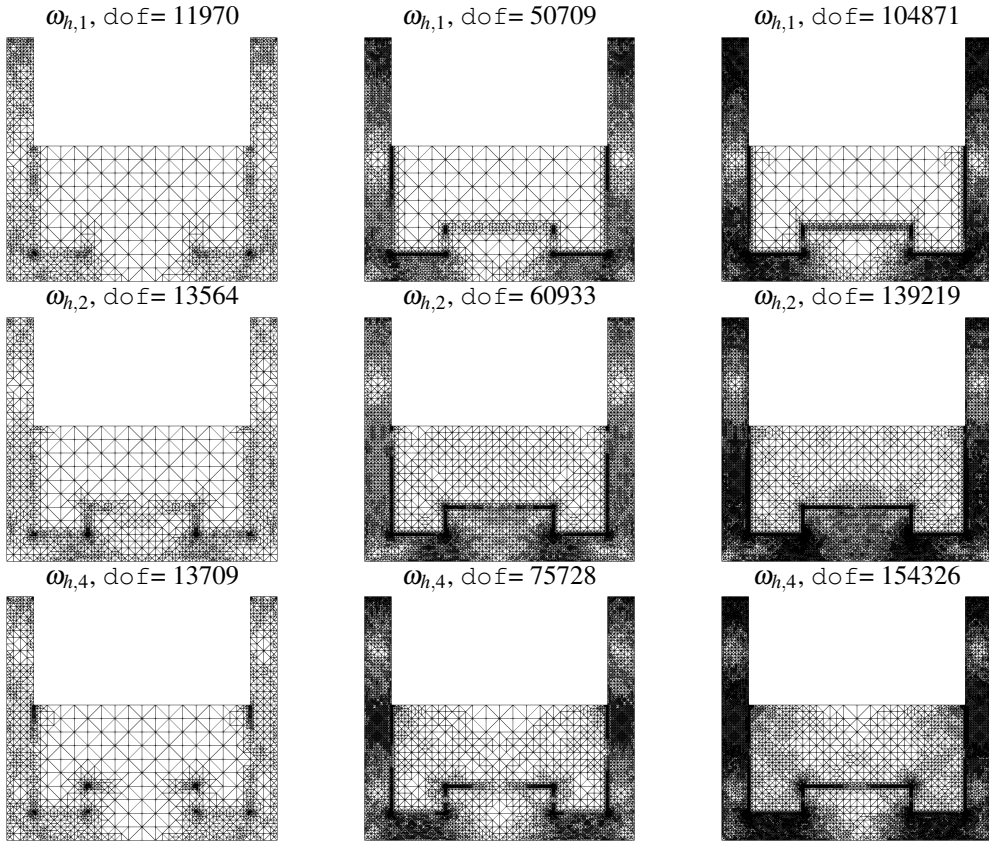


FIG. 6.5. Example 6.3. Comparison between intermediate meshes on the adaptive process for the first, second and fourth eigenfrequencies in  $\Omega_2$  with  $\nu = 0.35$ .

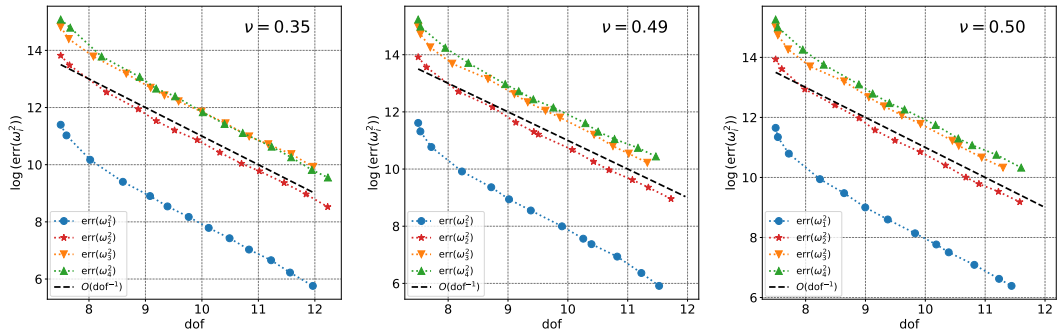


FIG. 6.6. Example 6.3. Error history for the first four lowest computed frequencies in the adaptive algorithm on  $\Omega_2$  with different values of  $\nu$ .

was also observed in [11] for the two dimensional case. Samples of the eigenmodes are portrayed in Figure 6.4. Here we note the effects of having a bottom clamped cube and a singular fluid domain. The

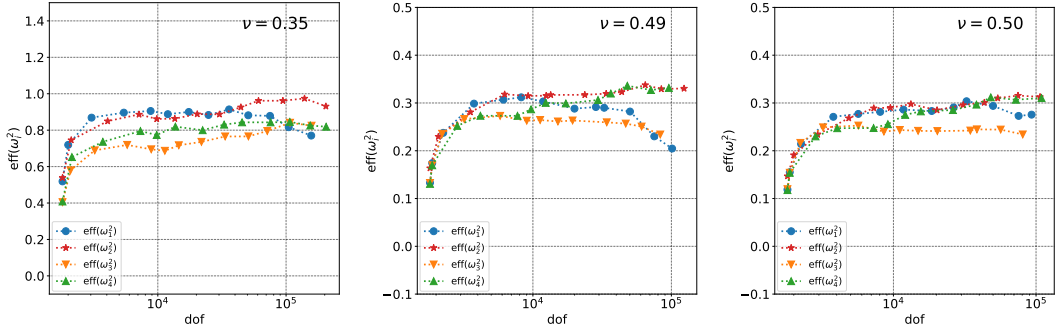


FIG. 6.7. Example 6.3. Effectivity indexes for the first four lowest computed frequencies in the adaptive algorithm on  $\Omega_2$  with different values of  $\nu$ .

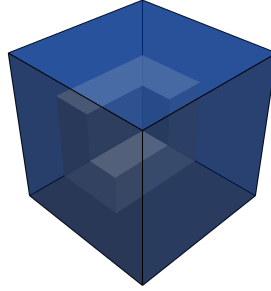


FIG. 6.8. Example 6.4. Sketch of the computational domain  $\Omega_{CF}$ .

pressure modes show high gradients near  $z = 0$  and on the dihedral singularities from the solid domain side. On the fluid part, we observe that the third and fourth modes have high gradients across the Fichera singularities.

Considering the extrapolated values from Table 6.3, we now perform a total of 12 adaptive refinement for different values of  $\nu$ . The results on the error and efficiency are depicted in figures 6.11 and 6.12. We observe that the fourth eigenmode is the one who takes more iterations to converge optimally, and it is the one with less marked elements. This behavior increases when we approach to the incompressible limit. In all the cases, optimal rates of  $\mathcal{O}(h^2) \approx \mathcal{O}(\text{dof}^{-1})$  is observed for some  $h_0$ . Also, the estimator remains reliable and efficient.

A sample of the adaptive meshes for  $\nu = 0.35$  is displayed in Figure 6.8. Here we present a bottom view of the solid domain in order to observe the refinements on the edges where boundary conditions change. It is clear that the algorithm detects and refine near the singularities according to the computed eigenmode. The pressure modes in Figure 6.4 gives a clue of the zones to be refined. It notes that a dihedral singularity for the solid is not necessary a singularity for the fluid.

## 7. Concluding remarks

We have presented a numerical analysis for a coupled elasto-acoustic eigenvalue problem with variable physical coefficients, utilizing a Herrmann pressure formulation in the solid and a pure displacement formulation in the fluid. Standard conforming/non-conforming finite element discretizations were



TABLE 6.3 *Example 6.4. Lowest computed eigenvalues using different combinations of FE families in the cube-fichera domain.*

| $N = 8$                                | $N = 16$  | $N = 24$  | $N = 32$  | Order | $\omega_{extr}$ |
|----------------------------------------|-----------|-----------|-----------|-------|-----------------|
| $\nu = 0.35$                           |           |           |           |       |                 |
| MINI-element + <b>BDM</b> <sub>1</sub> |           |           |           |       |                 |
| 2611.7912                              | 2533.9313 | 2506.0877 | 2494.1838 | 1.03  | 2468.8218       |
| 2704.2740                              | 2646.4223 | 2626.2310 | 2617.1568 | 1.06  | 2599.8235       |
| 3921.0288                              | 3804.9109 | 3772.5662 | 3759.9489 | 1.37  | 3741.8883       |
| 6100.5965                              | 5981.2481 | 5938.0631 | 5918.0167 | 1.03  | 5879.1927       |
| Taylor–Hood + <b>BDM</b> <sub>2</sub>  |           |           |           |       |                 |
| 2488.4968                              | 2477.4893 | 2473.7092 | 2472.1574 | 1.09  | 2468.9218       |
| 2611.8620                              | 2604.6753 | 2602.3378 | 2601.3331 | 1.13  | 2599.4235       |
| 3747.8091                              | 3742.5131 | 3740.6487 | 3739.9089 | 1.07  | 3738.2751       |
| 5903.2999                              | 5888.5302 | 5883.9274 | 5881.8062 | 1.16  | 5878.1927       |
| $\nu = 0.49$                           |           |           |           |       |                 |
| MINI-element + <b>BDM</b> <sub>1</sub> |           |           |           |       |                 |
| 2648.6718                              | 2556.4203 | 2522.2810 | 2507.4712 | 1.01  | 2476.1459       |
| 2724.4364                              | 2653.7656 | 2626.9209 | 2614.6971 | 1.04  | 2591.6592       |
| 3755.3424                              | 3641.8972 | 3609.7036 | 3597.0934 | 1.24  | 3574.6994       |
| 6175.0254                              | 6050.0035 | 5995.0476 | 5970.3920 | 0.96  | 5920.2144       |
| Taylor–Hood + <b>BDM</b> <sub>2</sub>  |           |           |           |       |                 |
| 2497.1129                              | 2484.6219 | 2480.2789 | 2478.4733 | 1.12  | 2474.8066       |
| 2605.4853                              | 2596.3463 | 2593.3193 | 2591.9961 | 1.16  | 2589.5652       |
| 3584.0291                              | 3579.0381 | 3577.2039 | 3576.4756 | 1.07  | 3574.8222       |
| 5941.0869                              | 5929.5876 | 5925.5804 | 5923.7570 | 1.08  | 5920.1135       |
| $\nu = 0.5$                            |           |           |           |       |                 |
| MINI-element + <b>BDM</b> <sub>1</sub> |           |           |           |       |                 |
| 2657.6245                              | 2561.6934 | 2526.2274 | 2510.8320 | 1.00  | 2476.0557       |
| 2731.1055                              | 2657.5435 | 2629.3030 | 2616.5388 | 0.93  | 2585.4029       |
| 3745.9572                              | 3631.9913 | 3599.5220 | 3586.8093 | 1.34  | 3567.8426       |
| 6181.2730                              | 6060.4663 | 6003.3623 | 5977.8556 | 1.01  | 5932.4767       |
| Taylor–Hood + <b>BDM</b> <sub>2</sub>  |           |           |           |       |                 |
| 2499.4290                              | 2486.7652 | 2482.3614 | 2480.5172 | 1.12  | 2476.7811       |
| 2606.4451                              | 2597.1011 | 2594.0051 | 2592.6502 | 1.16  | 2590.1541       |
| 3573.4814                              | 3568.4880 | 3566.6480 | 3565.9128 | 1.07  | 3564.2578       |
| 5946.5744                              | 5935.1881 | 5931.1799 | 5929.3456 | 1.07  | 5925.6497       |

proved to be locking-free and spurious-free, complemented by an efficient and reliable residual-based a posteriori error estimator.

We have several directions for future work. A natural physical extension consists in adding internal dissipation or viscous damping terms in either the fluid or structural subdomains. Incorporating such damping mechanisms transforms the linear spectral system into a quadratic eigenvalue problem of the form  $(\lambda^2 \mathbf{M} + \lambda \mathbf{C} + \mathbf{K})\mathbf{u} = \mathbf{0}$ , requiring quadratic spectral analysis tools. Additionally, exploring alternative mixed formulations that inherently provide exact momentum conservativity across subdomains remains a direction of interest.

Extending our residual-based a posteriori error estimator to 3D geometries presents technical challenges due to the vector nature of 3D curl operators and potential topology issues, as the present analysis relied on a 2D scalar Helmholtz decomposition for  $H(\text{div})$  spaces. We believe this limitation can be overcome by exploiting vector potentials for non-smooth domains following [3], utilizing 3D

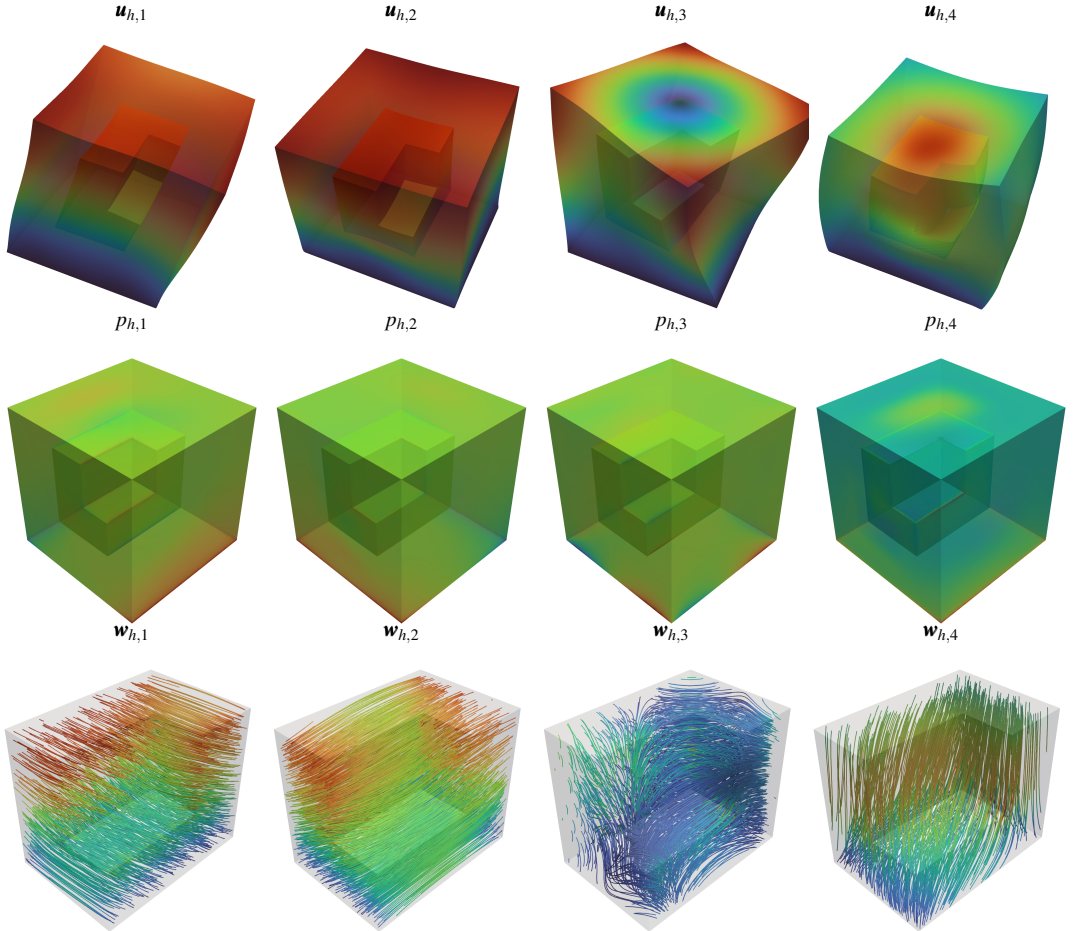


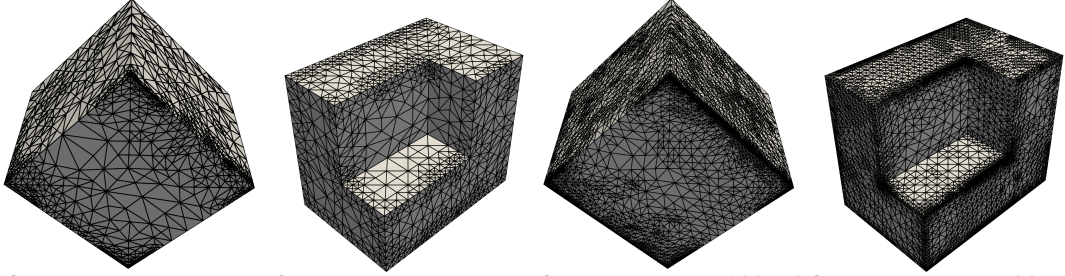
FIG. 6.9. Example 6.4. Comparison between the first four lowest order computed elasto-acoustic modes on  $\Omega_{CF}$ . The solid domain have been warped by a sufficiently large factor in order to observe the deformation.

Helmholtz-type decompositions developed by [29] for mixed systems, or applying  $H(\text{div})$ -conforming potential techniques proposed in [44].

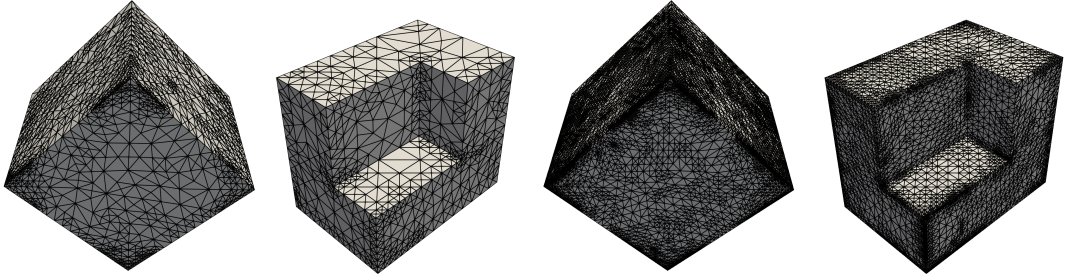
Although the present work is based on conforming mixed finite element discretizations, extending the proposed analysis to polytopal methods, particularly the Hybrid High-Order (HHO) framework, is a promising direction for future research. HHO methods naturally support general polygonal and polyhedral meshes while maintaining high-order accuracy, local conservation, and computational efficiency through static condensation. Recent advances in HHO methods for coupled elasto-acoustic wave propagation on general meshes [43] further demonstrate the robustness and flexibility of this approach for complex coupled wave phenomena. Motivated by these developments, it would be of considerable interest to develop and analyze HHO discretizations for primal fluid–structure interaction eigenvalue problems, including a rigorous study of spectral correctness, convergence, and optimal error estimates on general polytopal meshes. We would be also interested in integrating Virtual Element Methods (VEM) and other general polytopal schemes provides significantly enhanced geometric



$\Omega_s, \omega_{h,1}, \text{dof}=128136$   $\Omega_f, \omega_{h,1}, \text{dof}=128136$   $\Omega_s, \omega_{h,1}, \text{dof}=1736770$   $\Omega_f, \omega_{h,1}, \text{dof}=1736770$



$\Omega_s, \omega_{h,3}, \text{dof}=331900$   $\Omega_f, \omega_{h,3}, \text{dof}=331900$   $\Omega_s, \omega_{h,3}, \text{dof}=1566206$   $\Omega_f, \omega_{h,3}, \text{dof}=1566206$



$\Omega_s, \omega_{h,4}, \text{dof}=213640$   $\Omega_f, \omega_{h,4}, \text{dof}=213640$   $\Omega_s, \omega_{h,4}, \text{dof}=1602525$   $\Omega_f, \omega_{h,4}, \text{dof}=1602525$

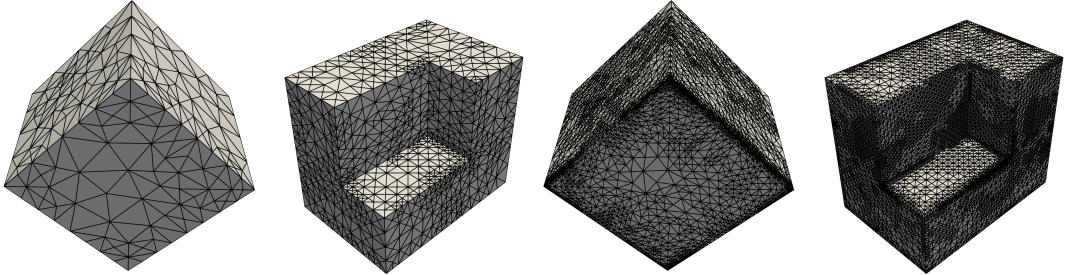


FIG. 6.10. Example 6.4. Comparison between intermediate meshes on the adaptive process for the first, second and fourth eigenfrequencies in  $\Omega_{CF}$  with  $\nu = 0.35$ .

flexibility. The straightforward treatment of on-conforming (non-matching) meshes, hanging nodes, and complex element geometries along the coupling interface is a flexibility especially appealing for large-scale industrial applications where generating matching grids across heterogeneous material interfaces is computationally prohibitive.

Finally, a theoretical aspect that we would like to explore involves relaxing the regularity assumptions required on the spectral solution operator. Standard convergence proofs rely on  $H^{1+\beta}(\Omega_s) \times H^\alpha(\Omega_f)$  bounds for the source problem. Formulating a generalized spectral framework capable of handling minimal regularity (e.g., under rough coefficient distributions) can draw upon foundational non-compact spectral theory [25], low-regularity Maxwell/spectral analyses [20], and non-compact mixed spectral approximations [14].

**Acknowledgements.** The constructive suggestions of the anonymous referees led to a substantial improvement to the clarity and correctness of the manuscript. Arbaz Khan was partially

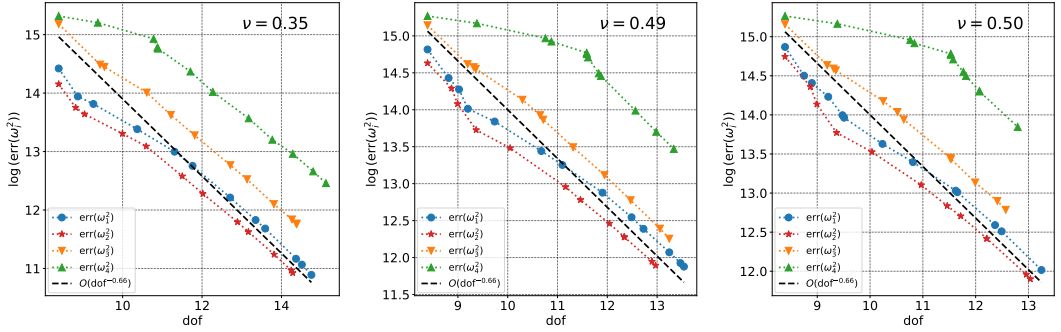


FIG. 6.11. Example 6.3. Error history for the first four lowest computed frequencies in the adaptive algorithm on  $\Omega_{CF}$  with different values of  $\nu$ .

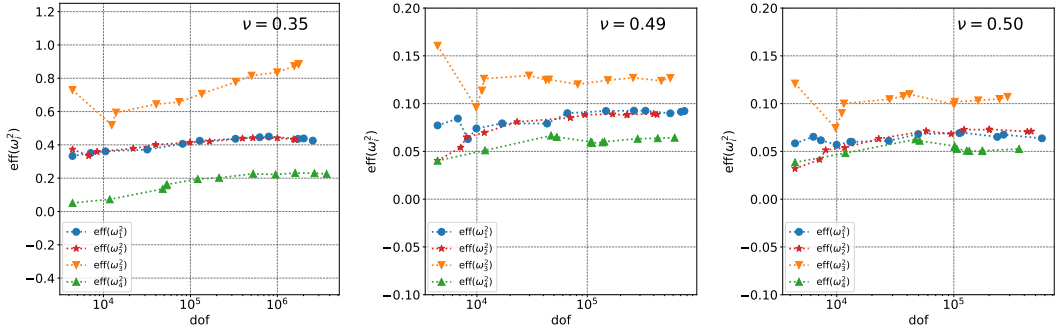


FIG. 6.12. Example 6.3. Effectivity indexes for the first four lowest computed frequencies in the adaptive algorithm on  $\Omega_{CF}$  with different values of  $\nu$ .

supported by SERB (ANRF) Core research grant CRG/2021/002569 and ANRF ARG-Matrices ANRF/ARGM/2025/001949/MTR.; Felipe Lepe was partially supported by Universidad del Bío-Bío through Proyecto Regular RE2514703; David Mora was partially supported by the National Agency for Research and Development, ANID-Chile through FONDECYT project 1261217, and by project Centro de Modelamiento Matemático (CMM), FB210005, BASAL funds for centers of excellence; Ricardo Ruiz-Baier was partially supported by the Australian Research Council through the FUTURE FELLOWSHIP grant FT220100496, by the Swedish Research Council under grant no. 2021-06594 while the author was in residence at Institut Mittag-Leffler in Djursholm, Sweden during the fall semester of 2025, and by the Center of Advanced Study (CAS) at the Norwegian Academy of Science and Letters under the program MATHEMATICAL CHALLENGES IN BRAIN MECHANICS; Jesus Vellojin was partially supported by the National Agency for Research and Development, ANID-Chile through FONDECYT Postdoctorado project 3230302.

**Declaration.** The authors declare that they have no conflict of interest.

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