A PRIORI AND A POSTERIORI ERROR BOUNDS FOR THE FULLY MIXED FEM FORMULATION OF POROELASTICITY WITH STRESS-DEPENDENT PERMEABILITY[*]

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Abstract. We develop a family of mixed finite element methods for a model of nonlinear poroelasticity where, thanks to a rewriting of the constitutive equations, the permeability depends on the total poroelastic stress and on the fluid pressure and therefore we can use the Hellinger–Reissner principle with weakly imposed stress symmetry for Biot's equations. The problem is adequately structured into a coupled system consisting of one saddle-point formulation, one linearised perturbed saddle-point formulation, and two off-diagonal perturbations. This system's unique solvability requires assumptions on regularity and Lipschitz continuity of the inverse permeability, and the analysis follows fixed-point arguments and the Babuška–Brezzi theory. The discrete problem is shown uniquely solvable by applying similar fixed-point and saddle-point techniques as for the continuous case. The method is based on the classical PEERS_k elements, it is exactly momentum and mass conservative, and it is robust with respect to the nearly incompressible as well as vanishing storativity limits. We derive a priori error estimates, we also propose fully computable residual-based a posteriori error indicators, and show that they are reliable and efficient with respect to the natural norms, and robust in the limit of near incompressibility. These a posteriori error estimates are used to drive adaptive mesh refinement. The theoretical analysis is supported and illustrated by several numerical examples in 2D and 3D.

Key words. Mixed finite elements, stress-based formulation, nonlinear poroelasticity, fixed-point operators, error estimates.

AMS subject classifications. 65N30, 65N15, 65J15, 76S05, 35Q74.

1. Introduction. Nonlinear interaction between flow and the mechanical response of saturated porous media is of a great importance in many applications in biophysics, geomechanics, and tissue engineering, for example. One of such models is the equations of nonlinear poroelasticity, whose mathematical properties were studied in great detail, for example, in the references [11, 13, 12]. In these works, it becomes clear that a distinctive property of nonlinear poroelasticity models targeted for, e.g., soft tissue (cartilage, trabecular meshwork, brain matter, etc.), is that the nonlinear permeability (the hydraulic conductivity, defined as how easily pore fluid escapes from the compacted pore spaces) that depends on the evolving total amount of fluid, does not entail a monotone operator, and therefore one cannot readily apply typical tools from monotone saddle-point problems.

Our interest is in deriving mixed finite element (FE) formulations (solving also for other variables of interest), and for this we can cite in particular [36, 38], where fully mixed formulations based on the Hu–Washizu principle are studied. Writing the poroelasticity equations in terms of the strain tensor was motivated in particular in [38] because the permeability – at least in the regime we focus here – depends nonlinearly on the total amount of fluid, which is a function of strain.

The upshot here compared to [38] is that we are able to rewrite the constitutive equation for permeability to depend on the total poroelastic stress and on the fluid pressure (similarly as in, e.g., [8]). This allows us to revert to the more popular Hellinger–Reissner type of mixed formulations for poroelasticity [7, 39, 40, 14, 46] (without solving explicitly for the strain). Consequently, another appealing advantage with respect to the formulation in [38] is that, as in the Hellinger–Reissner formulation, the model becomes robust with respect to the Lamé constants. Also in contrast to [38], in this work we use a mixed form for the fluid flow (adding the discharge flux as additional unknown), which gives the additional advantage of mass conservativity.

Regarding the well-posedness analysis, the aforementioned non-monotonicity of the permeability suggest, for example, to use a fixed-point argument. We opt for freezing the arguments of permeability, turning the double saddle-point structure with three perturbations coming from the stress trace operator and from the L^2 pressure blocks, into two decoupled saddle-point problems whose separate solvability can be established from the classical literature for weakly symmetric elasticity and mixed reaction-diffusion equations. Banach fixed-point theorem is then used to show well-posedness of the overall problem. This analysis needs to verify conditions of ball-mapping and contraction of the fixed-point map, and this imposes a small data assumption, which can be carried over to the external load, mass source, boundary displacement, and boundary fluid pressure. Compared to [38], these conditions are less restrictive and imply also a less restrictive discrete analysis (which follows closely the continuous one), due to the analysis being performed using the inverse of the Hooke tensor, which allows us to achieve robustness with respect to the first Lamé parameter λ.

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Note that at the discrete level we can simply use conforming FE spaces. Discrete inf-sup conditions are already well-known for the chosen FE families of PEERS$_k$ and Raviart–Thomas elements used for the solid and fluid sub-problems (but several other inf-sup stable spaces that satisfy a discrete kernel characterisation are also possible). We emphasise that, similarly to [39, 40], all estimates hold uniformly in the limit of nearly incompressibility (implying that the formulation is Poisson locking-free) as well as when the constrained storage coefficient vanishes (poroelastic locking-free), and therefore they are free of non-physical pressure oscillations.

An additional goal of this work is to derive efficient and reliable residual a posteriori error estimators for the nonlinear poroelasticity equations. The approach follows a similar treatment as that of [32] (which focuses on mixed formulations of stress-assisted diffusion equations), with the difference that here we do not need to include augmentation terms for the mixed form of the mixed diffusion problem. The main ingredients in the analysis of these estimates are Helmholtz decompositions and a global inf-sup (together with boundedness and Lipschitz continuity of coupling terms), local inverse and trace estimates, bubble-based localisation arguments, and properties of Clément and Raviart–Thomas interpolators. See also [3, 28] for estimators in a similar multi-physics context and, e.g., [19, 20, 41] for mixed linear elasticity. Note that for the reliability of the estimator the aforementioned Helmholtz decompositions – for both tensor-vector and vector-scalar cases – should be valid for mixed boundary conditions. For this we follow [3] and [25], from which we inherit an a convexity assumption on the Neumann sub-boundary (where we impose traction and flux boundary conditions).

Outline. The rest of the paper is organised as follows. The remainder of this Section has a collection of preliminary definitions and notational convention, as well as the statement of the governing partial differential equations. The weak formulation and proofs of the uniform boundedness of the bilinear forms and suitable inf-sup conditions are shown in Section 2. The fixed-point analysis of the coupled problem is carried out in Section 3. Section 4 then focuses on the Galerkin discretisation, including its well-posedness analysis and definition of specific FE subspaces that provide momentum and mass conservativity. In Section 5 we show a Céa estimate and using appropriate approximation properties we derive optimal a priori error bounds including also the higher order case. The definition of a residual a posteriori error estimator and the proofs of its reliability and efficiency are presented in Section 6. We conclude in Section 7 with some numerical tests that both validate and underline the theoretical properties of the proposed discretisations.

Notation and preliminaries. Let $\mathrm{L}^2(\Omega)$ be the set of all square-integrable functions in $\Omega \subset \mathbb{R}^d$ where $d \in \{2, 3\}$ is the spatial dimension, and denote by $\mathbf{L}^2(\Omega) = \mathrm{L}^2(\Omega)^d$ its vector-valued counterpart and by $\mathbb{L}^2(\Omega) = \mathrm{L}^2(\Omega)^{d \times d}$ its tensor-valued counterpart. We also write

$$\mathbb{L}^2_{\mathrm{skew}}(\Omega) := \{\boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \boldsymbol{\tau} = -\boldsymbol{\tau}^{\mathrm{t}}\},$$

to represent the skew-symmetric tensors in Ω with each component being square-integrable. Standard notation will be employed for Sobolev spaces $\mathrm{H}^m(\Omega)$ with $m \geq 0$ (and we note that $\mathrm{H}^0(\Omega) = \mathrm{L}^2(\Omega)$). Their norms and seminorms are denoted as $\|\cdot\|_{m,\Omega}$ and $|\cdot|_{m,\Omega}$, respectively (as well as for their vector and tensor-valued counterparts $\mathbf{H}^m(\Omega)$, $\mathbb{H}^m(\Omega)$) see, e.g., [15].

As usual $\mathbb{I}$ stands for the identity tensor in $\mathbb{R}^{d \times d}$, and $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$. Also, for any vector field $\boldsymbol{v} = (v_i)_{i=1,d}$ we set the gradient and divergence operators as $\boldsymbol{\nabla} \boldsymbol{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,d}$ and $\operatorname{div} \boldsymbol{v} := \sum_{j=1}^d \frac{\partial v_j}{\partial x_j}$. In addition, for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,d}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,d}$, we let $\mathbf{div}\, \boldsymbol{\tau}$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the tensor inner product, and the deviatoric tensor as $\boldsymbol{\tau}^{\mathrm{t}} := (\tau_{ji})_{i,j=1,d}$, $\operatorname{tr}(\boldsymbol{\tau}) := \sum_{i=1}^d \tau_{ii}$, $\boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^n \tau_{ij} \zeta_{ij}$, and $\boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{n} \operatorname{tr}(\boldsymbol{\tau})\, \mathbb{I}$, respectively. We also recall the Hilbert space

$$\mathbf{H}(\operatorname{div}; \Omega) := \{\boldsymbol{z} \in \mathbf{L}^2(\Omega) : \operatorname{div} \boldsymbol{z} \in \mathrm{L}^2(\Omega)\},$$

with norm $\|\boldsymbol{z}\|^2_{\operatorname{div};\Omega} := \|\boldsymbol{z}\|^2_{0,\Omega} + \|\operatorname{div} \boldsymbol{z}\|^2_{0,\Omega}$, and introduce its tensor-valued version

$$\mathbb{H}(\mathbf{div}; \Omega) := \{\boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \mathbf{div}\, \boldsymbol{\tau} \in \mathbf{L}^2(\Omega)\}.$$

Governing equations. Let us consider a fully-saturated poroelastic medium (consisting of a mechanically isotropic and homogeneous fluid-solid mixture) occupying the open and bounded domain Ω in $\mathbb{R}^d$, the Lipschitz boundary $\partial\Omega$ is partitioned into disjoint sub-boundaries $\partial\Omega := \overline{\Gamma_{\mathrm{D}}} \cup \overline{\Gamma_{\mathrm{N}}}$, and it is assumed for the sake of simplicity that both sub-boundaries are non-empty $|\Gamma_{\mathrm{D}}| \cdot |\Gamma_{\mathrm{N}}| > 0$. The symbol $\boldsymbol{n}$ will stand for the unit outward normal vector on the boundary. Let $\boldsymbol{f} \in \mathbf{L}^2(\Omega)$ be a prescribed body force per unit of volume (acting on the fluid-structure mixture) and let $g \in L^2(\Omega)$ be a net volumetric fluid production rate.

The balance of linear momentum for the solid-fluid mixture is written as

$$-\mathbf{div}\, \boldsymbol{\sigma} = \boldsymbol{f} \qquad \text{in } \Omega, \tag{1.1}$$

with $\boldsymbol{\sigma}$ being the total Cauchy stress tensor of the mixture (sum of the effective solid and fluid stresses), whose dependence on strain and on fluid pressure is given by the constitutive assumption (or effective stress principle)

$$\boldsymbol{\sigma} = \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \alpha p\mathbb{I} \qquad \text{in } \Omega. \tag{1.2}$$

Here the skeleton displacement vector $\boldsymbol{u}$ from the position $\boldsymbol{x} \in \Omega$ is an unknown, the tensor $\boldsymbol{\varepsilon}(\boldsymbol{u}) := \frac{1}{2}(\boldsymbol{\nabla u}+[\boldsymbol{\nabla u}]^{\mathrm{t}})$ is the infinitesimal strain, by $\mathcal{C}$ we denote the fourth-order elasticity tensor, also known as Hooke's tensor (symmetric and positive definite and characterised by $\mathcal{C}\boldsymbol{\tau} := \lambda(\mathrm{tr}\,\boldsymbol{\tau})\mathbb{I}+2\mu\,\boldsymbol{\tau}$), $\mathbb{I}$ is the identity second-order tensor, λ and μ are the Lamé parameters (assumed constant and positive), $0 \leq \alpha \leq 1$ is the Biot–Willis parameter, and p denotes the Darcy fluid pressure (positive in compression), which is an unknown in the system.

We also consider the balance of angular momentum, which in this context states that the total poroelastic stress is a symmetric tensor $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{t}}$. To weakly impose it, it is customary to use the rotation tensor

$$\boldsymbol{\rho} = \frac{1}{2}(\boldsymbol{\nabla u} - [\boldsymbol{\nabla u}]^{\mathrm{t}}) = \boldsymbol{\nabla u} - \boldsymbol{\varepsilon}(\boldsymbol{u}). \tag{1.3}$$

The fluid content (due to both fluid saturation and local volume dilation) is given by

$$\zeta = c_0 p + \alpha \, \mathrm{div}\,\boldsymbol{u}, \tag{1.4}$$

where $c_0 \geq 0$ is the constrained specific storage coefficient. Using Darcy's law to describe the discharge velocity in terms of the fluid pressure gradient, the balance of mass for the total amount of fluid is $\partial_t \zeta - \mathrm{div}(\kappa \nabla p) = g$ in $\Omega \times (0, t_{\mathrm{end}})$, where κ is the intrinsic permeability of the medium, a nonlinear function of the porosity. In turn, in the small strains limit the porosity can be approximated by a linear function of the fluid content ζ (see for example [45, Section 2.1]), and so, thanks to (1.4), we can simply write $\kappa(\boldsymbol{\varepsilon}(\boldsymbol{u}), p)$. Furthermore, after a backward Euler semi-discretisation in time with a constant time step and rescaling appropriately, we only consider the type of equations needed to solve at each time step and therefore we will concentrate on the form

$$c_0 p + \alpha \, \mathrm{tr}\,\boldsymbol{\varepsilon}(\boldsymbol{u}) - \mathrm{div}(\kappa(\boldsymbol{\varepsilon}(\boldsymbol{u}), p)\nabla p) = g \qquad \text{in } \Omega. \tag{1.5}$$

Typical constitutive relations for permeability are, e.g., exponential or Kozeny–Carman type (cf. [6])

$$\kappa(\boldsymbol{\varepsilon}(\boldsymbol{u}), p) = \frac{k_0}{\mu_f}\mathbb{I} + \frac{k_1}{\mu_f}\exp(k_2(c_0 p + \alpha\,\mathrm{tr}\,\boldsymbol{\varepsilon}(\boldsymbol{u})))\mathbb{I}, \quad \kappa(\boldsymbol{\varepsilon}(\boldsymbol{u}), p) = \frac{k_0}{\mu_f}\mathbb{I} + \frac{k_1(c_0 p + \alpha\,\mathrm{tr}\,\boldsymbol{\varepsilon}(\boldsymbol{u}))^3}{\mu_f(1 - (c_0 p + \alpha\,\mathrm{tr}\,\boldsymbol{\varepsilon}(\boldsymbol{u})))^2}\mathbb{I}, \tag{1.6}$$

where μ_f denotes the viscosity of the interstitial fluid and k_0, k_1, k_2 are model constants. We note that in the case of incompressible constituents one has $c_0 = 0$ and $\alpha = 1$, indicating that permeability depends only on the dilation $\mathrm{tr}\,\boldsymbol{\varepsilon}(\boldsymbol{u}) = \mathrm{div}\,\boldsymbol{u}$ (see, e.g., [11]). We also note that even in such a scenario (of incompressible phases) the overall mixture is not necessarily incompressible itself. More precise assumptions on the behaviour of the permeability are postponed to Section 2.2. Next we note that from (1.2) we can obtain

$$\mathrm{tr}\,\boldsymbol{\sigma} = (d\lambda + 2\mu)\,\mathrm{div}\,\boldsymbol{u} - d\alpha p \quad \text{and} \quad \mathcal{C}^{-1}\boldsymbol{\sigma} + \frac{\alpha}{d\lambda + 2\mu}p\mathbb{I} = \boldsymbol{\varepsilon}(\boldsymbol{u}) \qquad \text{in } \Omega. \tag{1.7}$$

Then, from the first equation in (1.7) we get

$$\mathrm{tr}\,\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{d\lambda + 2\mu}\,\mathrm{tr}\,\boldsymbol{\sigma} + \frac{d\alpha}{d\lambda + 2\mu}p,$$

and therefore the dependence of κ on $\boldsymbol{\varepsilon}(\boldsymbol{u})$ and p (cf. (1.6)) can be written in terms of $\boldsymbol{\sigma}$ and p as follows

$$\kappa(\boldsymbol{\sigma}, p) = \frac{k_0}{\mu_f}\mathbb{I} + \frac{k_1}{\mu_f}\exp\left(\frac{k_2}{d\lambda + 2\mu}\big((c_0(d\lambda + 2\mu) + d\alpha^2)\,p + \alpha\,\mathrm{tr}\,\boldsymbol{\sigma}\big)\right)\mathbb{I},$$
$$\kappa(\boldsymbol{\sigma}, p) = \frac{k_0}{\mu_f}\mathbb{I} + \frac{k_1\big((c_0(d\lambda + 2\mu) + d\alpha^2)\,p + \alpha\,\mathrm{tr}\,\boldsymbol{\sigma}\big)^3}{(d\lambda + 2\mu)\mu_f\big(d\lambda + 2\mu - ((c_0(d\lambda + 2\mu) + d\alpha^2)\,p + \alpha\,\mathrm{tr}\,\boldsymbol{\sigma})\big)^2}\mathbb{I}. \tag{1.8}$$

In addition, putting together the second equation in (1.7) and (1.3) we obtain:

$$\mathcal{C}^{-1}\boldsymbol{\sigma} + \frac{\alpha}{d\lambda + 2\mu}p\mathbb{I} = \boldsymbol{\nabla u} - \boldsymbol{\rho} \qquad \text{in } \Omega. \tag{1.9}$$

Finally, we introduce the discharge flux $\boldsymbol{\varphi}$ as an unknown defined by the constitutive relation

$$\kappa(\boldsymbol{\sigma}, p)^{-1}\boldsymbol{\varphi} = \nabla p, \tag{1.10}$$

and combining (1.7) and (1.5), we are able to rewrite the mass balance equation as

$$c_0 p + \frac{\alpha}{d\lambda + 2\mu}\,\mathrm{tr}\,\boldsymbol{\sigma} + \frac{d\alpha^2}{d\lambda + 2\mu}p - \mathrm{div}\,\boldsymbol{\varphi} = g \qquad \text{in } \Omega. \tag{1.11}$$

To close the system, we consider mixed boundary conditions for a given $\boldsymbol{u}_{\mathrm{D}} \in \mathbf{H}^{1/2}(\Gamma_{\mathrm{D}})$ and $p_{\mathrm{D}} \in \mathrm{H}^{1/2}(\Gamma_{\mathrm{D}})$:

$$\boldsymbol{u} = \boldsymbol{u}_{\mathrm{D}} \quad \text{and} \quad p = p_{\mathrm{D}} \quad \text{on } \Gamma_{\mathrm{D}}, \qquad \boldsymbol{\varphi} \cdot \boldsymbol{n} = 0 \quad \text{and} \quad \boldsymbol{\sigma n} = \boldsymbol{0} \quad \text{on } \Gamma_{\mathrm{N}}. \tag{1.12}$$

2. Weak formulation and preliminary properties.

2.1. Derivation of weak forms.
Let us define the following spaces

$$\mathbb{H}_N(\mathbf{div};\Omega) := \big\{\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\Omega) : \boldsymbol{\tau}\boldsymbol{n} = \mathbf{0} \quad \text{on} \quad \Gamma_N\big\}, \quad \mathbf{H}_N(\mathrm{div};\Omega) := \big\{\boldsymbol{\psi} \in \mathbf{H}(\mathrm{div};\Omega) : \boldsymbol{\psi}\cdot\boldsymbol{n} = 0 \quad \text{on} \quad \Gamma_N\big\}.$$

We test equation (1.1) against $\boldsymbol{v} \in \mathbf{L}^2(\Omega)$, equation (1.9) against $\boldsymbol{\tau} \in \mathbb{H}_N(\mathbf{div};\Omega)$, impose the symmetry of $\boldsymbol{\sigma}$ weakly, test equation (1.11) against $q \in \mathrm{L}^2(\Omega)$, equation (1.10) against $\boldsymbol{\psi} \in \mathbf{H}_N(\mathrm{div};\Omega)$, integrate by parts and using the boundary conditions (1.12) naturally, and then reorder the resulting equations. Then we arrive at

$$\int_\Omega \mathcal{C}^{-1}\boldsymbol{\sigma} : \boldsymbol{\tau} + \frac{\alpha}{d\lambda + 2\mu}\int_\Omega p\,\mathrm{tr}\,\boldsymbol{\tau} + \int_\Omega \boldsymbol{u}\cdot\mathbf{div}\,\boldsymbol{\tau} + \int_\Omega \boldsymbol{\rho} : \boldsymbol{\tau} = \langle\boldsymbol{\tau}\boldsymbol{n}, \boldsymbol{u}_D\rangle_{\Gamma_D} \qquad \forall\,\boldsymbol{\tau} \in \mathbb{H}_N(\mathbf{div};\Omega),$$

$$\int_\Omega \boldsymbol{v}\cdot\mathbf{div}\,\boldsymbol{\sigma} = -\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} \qquad \forall\,\boldsymbol{v} \in \mathbf{L}^2(\Omega),$$

$$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\eta} = 0 \qquad \forall\,\boldsymbol{\eta} \in \mathbb{L}^2_{\mathrm{skew}}(\Omega),$$

$$\int_\Omega \kappa(\boldsymbol{\sigma}, p)^{-1}\boldsymbol{\varphi}\cdot\boldsymbol{\psi} + \int_\Omega p\,\mathrm{div}\,\boldsymbol{\psi} = \langle\boldsymbol{\psi}\cdot\boldsymbol{n}, p_D\rangle_{\Gamma_D} \qquad \forall\,\boldsymbol{\psi} \in \mathbf{H}_N(\mathrm{div};\Omega),$$

$$\Big(c_0 + \frac{d\alpha^2}{d\lambda + 2\mu}\Big)\int_\Omega p\,q + \frac{\alpha}{d\lambda + 2\mu}\int_\Omega q\,\mathrm{tr}\,\boldsymbol{\sigma} - \int_\Omega q\,\mathrm{div}\,\boldsymbol{\varphi} = \int_\Omega g\,q \qquad \forall\,q \in \mathrm{L}^2(\Omega),$$

where $\langle\cdot,\cdot\rangle_{\Gamma_D}$ denotes the duality pairing between $\mathrm{H}^{-1/2}(\Gamma_D)$ and its dual $\mathrm{H}^{1/2}(\Gamma_D)$ with respect to the inner product in $L^2(\Gamma_D)$, and we use the same notation, $\langle\cdot,\cdot\rangle_{\Gamma_D}$, in the vector-valued case.

Introducing bilinear forms $a : \mathbb{H}_N(\mathbf{div};\Omega) \times \mathbb{H}_N(\mathbf{div};\Omega) \to \mathbb{R}$, $b : \mathbb{H}_N(\mathbf{div};\Omega) \times \mathbf{L}^2(\Omega) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega) \to \mathbb{R}$, $c : \mathbb{H}_N(\mathbf{div};\Omega) \times \mathrm{L}^2(\Omega) \to \mathbb{R}$, the nonlinear form $\widetilde{a}_{\widehat{\boldsymbol{\sigma}},\widehat{p}} : \mathbf{H}_N(\mathrm{div};\Omega) \times \mathbf{H}_N(\mathrm{div};\Omega) \to \mathbb{R}$, and the bilinear weak forms $\widetilde{b} : \mathbf{H}_N(\mathrm{div};\Omega) \times \mathrm{L}^2(\Omega) \to \mathbb{R}$ and $\widetilde{c} : \mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Omega) \to \mathbb{R}$, defined by

$$a(\boldsymbol{\sigma}, \boldsymbol{\tau}) := \int_\Omega \mathcal{C}^{-1}\boldsymbol{\sigma} : \boldsymbol{\tau}, \quad b(\boldsymbol{\tau}, (\boldsymbol{v}, \boldsymbol{\eta})) := \int_\Omega \boldsymbol{v}\cdot\mathbf{div}\,\boldsymbol{\tau} + \int_\Omega \boldsymbol{\tau} : \boldsymbol{\eta}, \quad c(\boldsymbol{\tau}, q) := \frac{\alpha}{d\lambda + 2\mu}\int_\Omega q\,\mathrm{tr}\,\boldsymbol{\tau},$$

$$\widetilde{a}_{\widehat{\boldsymbol{\sigma}},\widehat{p}}(\boldsymbol{\varphi}, \boldsymbol{\psi}) := \int_\Omega \kappa(\widehat{\boldsymbol{\sigma}}, \widehat{p})^{-1}\boldsymbol{\varphi}\cdot\boldsymbol{\psi}, \quad \widetilde{b}(\boldsymbol{\psi}, q) := \int_\Omega q\,\mathrm{div}\,\boldsymbol{\psi}, \quad \widetilde{c}(p, q) := \Big(c_0 + \frac{d\alpha^2}{d\lambda + 2\mu}\Big)\int_\Omega p\,q,$$

(2.1)

respectively, and linear functionals $H \in \mathbb{H}_N(\mathbf{div};\Omega)'$, $F \in (\mathbf{L}^2(\Omega) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega))'$, $\widetilde{H} \in \mathbf{H}_N(\mathrm{div};\Omega)'$, $G \in \mathrm{L}^2(\Omega)'$

$$H(\boldsymbol{\tau}) := \langle\boldsymbol{\tau}\boldsymbol{n}, \boldsymbol{u}_D\rangle_{\Gamma_D}, \quad F(\boldsymbol{v}, \boldsymbol{\eta}) := -\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}, \quad \widetilde{H}(\boldsymbol{\psi}) := \langle\boldsymbol{\psi}\cdot\boldsymbol{n}, p_D\rangle_{\Gamma_D}, \quad G(q) := -\int_\Omega g\,q,$$

we arrive at: find $(\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{\rho}, \boldsymbol{\varphi}, p) \in \mathbb{H}_N(\mathbf{div};\Omega) \times \mathbf{L}^2(\Omega) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega) \times \mathbf{H}_N(\mathrm{div};\Omega) \times \mathrm{L}^2(\Omega)$, such that:

$$a(\boldsymbol{\sigma}, \boldsymbol{\tau}) + \quad b(\boldsymbol{\tau}, (\boldsymbol{u}, \boldsymbol{\rho})) \quad + \quad c(\boldsymbol{\tau}, p) = H(\boldsymbol{\tau}) \qquad \forall\,\boldsymbol{\tau} \in \mathbb{H}_N(\mathbf{div};\Omega), \tag{2.2a}$$

$$b(\boldsymbol{\sigma}, (\boldsymbol{v}, \boldsymbol{\eta})) \qquad\qquad\qquad = F(\boldsymbol{v}, \boldsymbol{\eta}) \qquad \forall\,\boldsymbol{v} \in \mathbf{L}^2(\Omega), \forall\,\boldsymbol{\eta} \in \mathbb{L}^2_{\mathrm{skew}}(\Omega), \tag{2.2b}$$

$$\widetilde{a}_{\boldsymbol{\sigma},p}(\boldsymbol{\varphi}, \boldsymbol{\psi}) + \quad \widetilde{b}(\boldsymbol{\psi}, p) \qquad = \widetilde{H}(\boldsymbol{\psi}) \qquad \forall\,\boldsymbol{\psi} \in \mathbf{H}_N(\mathrm{div};\Omega), \tag{2.2c}$$

$$\widetilde{b}(\boldsymbol{\varphi}, q) - \quad \widetilde{c}(p, q) \quad - \quad c(\boldsymbol{\sigma}, q) = G(q) \qquad \forall\,q \in \mathrm{L}^2(\Omega). \tag{2.2d}$$

2.2. Stability properties and suitable inf-sup conditions.
For the sake of the analysis, we allow the permeability $\kappa(\boldsymbol{\sigma}, p)$ to be anisotropic but still require $\kappa(\boldsymbol{\sigma}, p)^{-1}$ to be uniformly positive definite in $\mathbb{L}^\infty(\Omega)$ and Lipschitz continuous with respect to $p \in \mathrm{L}^2(\Omega)$. That is, there exist positive constants κ_1, κ_2 such that

$$\kappa_1|\boldsymbol{v}|^2 \le \boldsymbol{v}^{\mathrm{t}}\kappa(\cdot,\cdot)^{-1}\boldsymbol{v}, \qquad \|\kappa(\cdot, p_1)^{-1} - \kappa(\cdot, p_2)^{-1}\|_{\mathbb{L}^\infty(\Omega)} \le \kappa_2\|p_1 - p_2\|_{0,\Omega}, \tag{2.3}$$

for all $\boldsymbol{v} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$, and for all $p_1, p_2 \in \mathrm{L}^2(\Omega)$.

We start by establishing the boundedness of the bilinear forms $a, b, c, \widetilde{b}, \widetilde{c}$:

$$\big|a(\boldsymbol{\sigma}, \boldsymbol{\tau})\big| \le \frac{1}{\mu}\|\boldsymbol{\sigma}\|_{\mathbf{div};\Omega}\|\boldsymbol{\tau}\|_{\mathbf{div};\Omega}, \qquad \big|b(\boldsymbol{\tau}, (\boldsymbol{v}, \boldsymbol{\eta}))\big| \le \|\boldsymbol{\tau}\|_{\mathbf{div};\Omega}(\|\boldsymbol{v}\|_{0,\Omega} + \|\boldsymbol{\eta}\|_{0,\Omega}), \tag{2.4a}$$

$$\big|c(\boldsymbol{\tau}, q)\big| \le \gamma\|\boldsymbol{\tau}\|_{\mathbf{div};\Omega}\|q\|_{0,\Omega}, \tag{2.4b}$$

$$\big|\widetilde{b}(\boldsymbol{\psi}, q)\big| \le \|\boldsymbol{\psi}\|_{\mathrm{div};\Omega}\|q\|_{0,\Omega}, \qquad \big|\widetilde{c}(p, q)\big| \le \widetilde{\gamma}\|p\|_{0,\Omega}\|q\|_{0,\Omega}, \tag{2.4c}$$

where

$$\gamma := \frac{\alpha\sqrt{d}}{d\lambda + 2\mu} \quad \text{and} \quad \widetilde{\gamma} := c_0 + \frac{d\alpha^2}{d\lambda + 2\mu}. \tag{2.5}$$

On the other hand, using Hölder's and trace inequalities we can readily observe that the right-hand side functionals are all bounded

$$\left|H(\boldsymbol{\tau})\right| \leq \|\boldsymbol{u}_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\|\boldsymbol{\tau}\|_{\mathbf{div};\Omega}, \qquad \left|F(\boldsymbol{v},\boldsymbol{\eta})\right| \leq \|\boldsymbol{f}\|_{0,\Omega}\|\boldsymbol{v}\|_{0,\Omega} \leq \|\boldsymbol{f}\|_{0,\Omega}(\|\boldsymbol{v}\|_{0,\Omega} + \|\boldsymbol{\eta}\|_{0,\Omega}),$$
$$\left|\widetilde{H}(\boldsymbol{\psi})\right| \leq \|p_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\|\boldsymbol{\psi}\|_{\mathrm{div};\Omega}, \qquad \left|G(q)\right| \leq \|g\|_{0,\Omega}\|q\|_{0,\Omega}.$$

Let us now denote by $\mathbb{V}$ and $\mathbf{V}$ the kernels of b and $\widetilde{b}$, respectively. They are characterised, respectively, as

$$\mathbb{V} = \left\{\boldsymbol{\tau} \in \mathbb{H}_{\mathrm{N}}(\mathbf{div};\Omega) : \quad \mathbf{div}\,\boldsymbol{\tau} = \mathbf{0} \quad \text{and} \quad \boldsymbol{\tau} = \boldsymbol{\tau}^{\mathrm{t}} \quad \text{in} \quad \Omega\right\}, \tag{2.6a}$$

$$\mathbf{V} = \left\{\boldsymbol{\psi} \in \mathbf{H}_{\mathrm{N}}(\mathrm{div};\Omega) : \quad \mathrm{div}\,\boldsymbol{\psi} = 0 \quad \text{in} \quad \Omega\right\}. \tag{2.6b}$$

From [2, Lemmas 3.1 and 3.2] we easily deduce that there exists $c_a > 0$ such that

$$a(\boldsymbol{\tau},\boldsymbol{\tau}) \geq c_a\|\boldsymbol{\tau}\|_{\mathbf{div};\Omega}^2 \quad \forall\,\boldsymbol{\tau} \in \mathbb{V}. \tag{2.7}$$

The following inf-sup conditions are well-known to hold (see, e.g., [18]):

$$\sup_{\mathbf{0}\neq\boldsymbol{\tau}\in\mathbb{H}_{\mathrm{N}}(\mathbf{div};\Omega)} \frac{b(\boldsymbol{\tau},(\boldsymbol{v},\boldsymbol{\eta}))}{\|\boldsymbol{\tau}\|_{\mathbf{div};\Omega}} \geq \beta(\|\boldsymbol{v}\|_{0,\Omega} + \|\boldsymbol{\eta}\|_{0,\Omega}) \quad \forall\,(\boldsymbol{v},\boldsymbol{\eta}) \in \mathbf{L}^2(\Omega) \times \mathbb{L}_{\mathrm{skew}}^2(\Omega), \tag{2.8a}$$

$$\sup_{\mathbf{0}\neq\boldsymbol{\psi}\in\mathbf{H}_{\mathrm{N}}(\mathrm{div};\Omega)} \frac{\widetilde{b}(\boldsymbol{\psi},q)}{\|\psi\|_{\mathrm{div};\Omega}} \geq \widetilde{\beta}\|q\|_{0,\Omega} \quad \forall\,q \in \mathrm{L}^2(\Omega). \tag{2.8b}$$

Finally, we observe that $\widetilde{c}$ is elliptic over $\mathrm{L}^2(\Omega)$

$$\widetilde{c}(q,q) \geq \widetilde{\gamma}\,\|q\|_{0,\Omega}^2. \tag{2.9}$$

3. Analysis of the coupled problem. We now use a combination of the classical Babuška–Brezzi and Banach fixed-point theorems to establish the well-posedness of (2.2) under appropriate assumptions on the data.

3.1. A fixed-point operator. We adopt a similar approach to, e.g., [24]. Firs, we define a closed ball of $\mathrm{L}^2(\Omega)$ centred at the origin and of given radius $r > 0$

$$\mathrm{W} := \{\widehat{p} \in \mathrm{L}^2(\Omega) : \quad \|\widehat{p}\|_{0,\Omega} \leq r\}. \tag{3.1}$$

Then, for a given $(\widehat{\boldsymbol{\sigma}},\widehat{p}) \in \mathbb{H}_{\mathrm{N}}(\mathbf{div};\Omega) \times \mathrm{W}$, thanks to the assumptions on the nonlinear permeability, we can infer that the form $\widetilde{a}_{\widehat{\boldsymbol{\sigma}},\widehat{p}}$ (cf. (2.1)) is continuous, as well as coercive over $\mathbf{V}$

$$\left|\widetilde{a}_{\widehat{\boldsymbol{\sigma}},\widehat{p}}(\boldsymbol{\varphi},\boldsymbol{\psi})\right| \leq C_{\widetilde{a}}\,\|\boldsymbol{\varphi}\|_{\mathrm{div};\Omega}\|\boldsymbol{\psi}\|_{\mathrm{div};\Omega}, \tag{3.2a}$$

$$\widetilde{a}_{\widehat{\boldsymbol{\sigma}},\widehat{p}}(\boldsymbol{\psi},\boldsymbol{\psi}) \geq \kappa_1\|\boldsymbol{\psi}\|_{\mathrm{div};\Omega}^2 \quad \forall\,\boldsymbol{\varphi},\,\boldsymbol{\psi} \in \mathbf{V}. \tag{3.2b}$$

Then, we define the auxiliary operators $\mathbf{R} : \mathrm{W} \subseteq \mathrm{L}^2(\Omega) \to \mathbb{H}_{\mathrm{N}}(\mathbf{div};\Omega) \times (\mathbf{L}^2(\Omega) \times \mathbb{L}_{\mathrm{skew}}^2(\Omega))$ and $\mathbf{S} : \mathbb{H}_{\mathrm{N}}(\mathbf{div};\Omega) \times \mathrm{W} \to \mathbf{H}_{\mathrm{N}}(\mathrm{div};\Omega) \times \mathrm{L}^2(\Omega)$, given by

$$\mathbf{R}(\widehat{p}) := \big(R_1(\widehat{p}),(R_2(\widehat{p}),R_3(\widehat{p}))\big) = (\boldsymbol{\sigma},(\boldsymbol{u},\boldsymbol{\rho})) \quad \forall\,\widehat{p} \in \mathrm{W},$$

with $(\boldsymbol{\sigma},(\boldsymbol{u},\boldsymbol{\rho})) \in \mathbb{H}_{\mathrm{N}}(\mathbf{div};\Omega) \times (\mathbf{L}^2(\Omega) \times \mathbb{L}_{\mathrm{skew}}^2(\Omega))$ satisfying

$$\begin{aligned} a(\boldsymbol{\sigma},\boldsymbol{\tau}) \quad + \quad b(\boldsymbol{\tau},(\boldsymbol{u},\boldsymbol{\rho})) \quad &= H(\boldsymbol{\tau}) - c(\boldsymbol{\tau},\widehat{p}) \quad &&\forall\,\boldsymbol{\tau} \in \mathbb{H}_{\mathrm{N}}(\mathbf{div};\Omega), \\ b(\boldsymbol{\sigma},(\boldsymbol{v},\boldsymbol{\eta})) \quad &= F(\boldsymbol{v},\boldsymbol{\eta}) \quad &&\forall\,(\boldsymbol{v},\boldsymbol{\eta}) \in \mathbf{L}^2(\Omega) \times \mathbb{L}_{\mathrm{skew}}^2(\Omega), \end{aligned} \tag{3.3}$$

and

$$\mathbf{S}(\widehat{\boldsymbol{\sigma}},\widehat{p}) := \big(S_1(\widehat{\boldsymbol{\sigma}},\widehat{p}),S_2(\widehat{\boldsymbol{\sigma}},\widehat{p})\big) = (\boldsymbol{\varphi},p) \quad \forall\,(\widehat{\boldsymbol{\sigma}},\widehat{p}) \in \mathbb{H}_{\mathrm{N}}(\mathbf{div};\Omega) \times \mathrm{W},$$

where $(\boldsymbol{\varphi},p)$ is such that

$$\begin{aligned} \widetilde{a}_{\widehat{\boldsymbol{\sigma}},\widehat{p}}(\boldsymbol{\varphi},\boldsymbol{\psi}) \quad + \quad \widetilde{b}(\boldsymbol{\psi},p) \quad &= \widetilde{H}(\boldsymbol{\psi}) \quad &&\forall\,\boldsymbol{\psi} \in \mathbf{H}_{\mathrm{N}}(\mathrm{div};\Omega), \\ \widetilde{b}(\boldsymbol{\varphi},q) \quad - \quad \widetilde{c}(p,q) \quad &= G(q) + c(\widehat{\boldsymbol{\sigma}},q) \quad &&\forall\,q \in \mathrm{L}^2(\Omega). \end{aligned} \tag{3.4}$$

By virtue of the above, by defining the operator $\mathbf{T} : \mathrm{W} \subseteq \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega)$ as

$$\mathbf{T}(\widehat{p}) := S_2(R_1(\widehat{p}),\widehat{p}), \tag{3.5}$$

it is clear that $(\boldsymbol{\sigma},\boldsymbol{u},\boldsymbol{\rho},\boldsymbol{\varphi},p)$ is a solution to (2.2) if and only if $p \in \mathrm{W}$ solves the fixed-point problem

$$\mathbf{T}(p) = p. \tag{3.6}$$

Thus, in what follows, we focus on proving the unique solvability of (3.6).

3.2. Well-definedness of T. From the definition of $\mathbf{T}$ in (3.5) it is evident that its well-definedness requires the well-posedness of problems (3.3) and (3.4). We begin by analysing that of (3.3).

LEMMA 3.1. *Let* $\widehat{p} \in W$ *(cf.* (3.1)*). Then, there exists a unique* $(\boldsymbol{\sigma}, (\boldsymbol{u}, \boldsymbol{\rho})) \in \mathbb{H}_N(\mathbf{div}; \Omega) \times \mathbf{L}^2(\Omega) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega)$ *solution to* (3.3)*. In addition, there exist* $C_1, C_2 > 0$*, such that*

$$\|\boldsymbol{\sigma}\|_{\mathbf{div};\Omega} \le C_1 \left(\|\boldsymbol{u}_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} + \|\boldsymbol{f}\|_{0,\Omega} \right) + \frac{1}{c_a} \gamma \|\widehat{p}\|_{0,\Omega},$$

$$\|\boldsymbol{u}\|_{0,\Omega} + \|\boldsymbol{\rho}\|_{0,\Omega} \le C_2 \left(\|\boldsymbol{u}_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} + \|\boldsymbol{f}\|_{0,\Omega} \right) + \frac{1}{\beta} \left(1 + \frac{1}{\mu c_a} \right) \gamma \|\widehat{p}\|_{0,\Omega}. \tag{3.7}$$

Proof. It is a direct consequence of the Babuška–Brezzi theory [29, Th. 2.34], using (2.7) and (2.8a) with

$$C_1 := \left(\frac{1}{c_a} + \frac{1}{\beta} \right) \left(1 + \frac{1}{\mu c_a} \right) \quad \text{and} \quad C_2 := \frac{1}{\beta} \left(1 + \frac{1}{\mu c_a} \right) \left(1 + \frac{1}{\mu \beta} \right); \tag{3.8}$$

we omit further details. □

Next, we provide the well-definedness of $\mathbf{S}$, or equivalently, the well-posedness of (3.4).

LEMMA 3.2. *Let* $(\widehat{\boldsymbol{\sigma}}, \widehat{p}) \in \mathbb{H}_N(\mathbf{div}; \Omega) \times W$*. Then, there exists a unique* $(\boldsymbol{\varphi}, p) \in \mathbf{H}_N(\mathrm{div}; \Omega) \times \mathrm{L}^2(\Omega)$ *solution to* (3.4)*. In addition, there exist* $\widetilde{C} > 0$ *such that*

$$\|\boldsymbol{\varphi}\|_{\mathrm{div};\Omega} + \|p\|_{0,\Omega} \le \widetilde{C} \left(\|g\|_{0,\Omega} + \|p_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} + \gamma \|\widehat{\boldsymbol{\sigma}}\|_{\mathbf{div};\Omega} \right). \tag{3.9}$$

Proof. The existence of a unique solution $(\boldsymbol{\varphi}, p)$ to (3.4) is straightforward given the properties of the forms $\widetilde{a}$, $\widetilde{b}$, and $\widetilde{c}$. By examining (3.2b), (2.9), and (2.8b), we can confirm that the assumptions of [26, Th. 3.4] are satisfied. In addition, $\boldsymbol{\varphi}$ and p satisfy the following bounds

$$\|\boldsymbol{\varphi}\|_{\mathrm{div};\Omega} \le \left(\frac{1}{\kappa_1} + \widetilde{C}_1 + \widetilde{C}_1 \sqrt{\widetilde{\gamma}} \right) \left(2 \max\{\widetilde{C}_2, \widetilde{C}_3\} \right)^{1/2} \left(\|g\|_{0,\Omega} + \|p_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} + \gamma \|\widehat{\boldsymbol{\sigma}}\|_{\mathbf{div};\Omega} \right),$$

$$\|p\|_{0,\Omega} \le \frac{1}{\widetilde{\beta}} \left(1 + \kappa_2 r \left(\frac{1}{\kappa_1} + \widetilde{C}_1 + \widetilde{C}_1 \sqrt{\widetilde{\gamma}} \right) \left(2 \max\{\widetilde{C}_2, \widetilde{C}_3\} \right)^{1/2} \right) \left(\|g\|_{0,\Omega} + \|p_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} + \gamma \|\widehat{\boldsymbol{\sigma}}\|_{\mathbf{div};\Omega} \right),$$

where

$$\widetilde{C}_1 := \frac{1}{\widetilde{\beta}} \left(1 + \frac{C_{\widetilde{a}}}{\kappa_1} \right), \quad \widetilde{C}_2 := \frac{1}{\kappa_1} + \widetilde{C}_1 + \widetilde{\gamma} \widetilde{C}_1^2 \quad \text{and} \quad \widetilde{C}_3 := \widetilde{C}_1 \left(1 + \frac{C_{\widetilde{a}}}{\widetilde{\beta}} + \widetilde{C}_1 \frac{C_{\widetilde{a}}^2 \widetilde{\gamma}}{\widetilde{\beta}^2} \right); \tag{3.10}$$

and the above implies (3.9). We leave out additional minor details. □

LEMMA 3.3. *Given* $r > 0$*, let us assume that*

$$\widetilde{C} \left(1 + \gamma C_1 \right) \left(\|g\|_{0,\Omega} + \|p_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} + \|\boldsymbol{u}_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} + \|\boldsymbol{f}\|_{0,\Omega} \right) + \frac{\widetilde{C}}{c_a} \gamma^2 r \le r, \tag{3.11}$$

where C_1*,* $\widetilde{C}_1$ *and* γ *are defined in* (3.8)*,* (3.10) *and* (2.5)*, respectively. Then, for a given* $\widehat{p} \in W$ *(cf.* (3.1)*), there exists a unique* $p \in W$ *such that* $\mathbf{T}(\widehat{p}) = p$*.*

Proof. From Lemmas 3.1 and 3.2, we ascertain that the operators $\mathbf{R}$ and $\mathbf{S}$, respectively, are well-defined, thereby ensuring the well-definition of $\mathbf{T}$. Furthermore, from (3.7) and (3.9), for each $\widehat{p} \in W$, we deduce that

$$\begin{aligned}
\|\mathbf{T}(\widehat{p})\|_{0,\Omega} &= \|S_2(R_1(\widehat{p}), \widehat{p})\|_{0,\Omega} \\
&\le \widetilde{C} \left(\|g\|_{0,\Omega} + \|p_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} \right) + \widetilde{C} \gamma \|(R_1(\widehat{p})\|_{\mathbf{div};\Omega} \\
&\le \widetilde{C} \left(\|g\|_{0,\Omega} + \|p_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} \right) + \widetilde{C} \gamma C_1 \left(\|\boldsymbol{u}_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} + \|\boldsymbol{f}\|_{0,\Omega} \right) + \frac{\widetilde{C}}{c_a} \gamma^2 \|\widehat{p}\|_{0,\Omega},
\end{aligned}$$

this, combined with assumption (3.11), implies $\mathbf{T}(W) \subseteq W$, which concludes the proof. □

REMARK 3.1. *Another option for defining the operator* $\mathbf{S}$ *(see* (3.4)*) is to introduce the perturbation* $\widetilde{c}$ *on the right-hand side of the system, given by*

$$\begin{aligned}
\widetilde{a}_{\widehat{\boldsymbol{\sigma}}, \widehat{p}}(\boldsymbol{\varphi}, \boldsymbol{\psi}) \quad + \quad \widetilde{b}(\boldsymbol{\psi}, p) \quad &= \widetilde{H}(\boldsymbol{\psi}) && \forall \, \boldsymbol{\psi} \in \mathbf{H}_N(\mathrm{div}; \Omega), \\
\widetilde{b}(\boldsymbol{\varphi}, q) \quad &= G(q) + c(\widehat{\boldsymbol{\sigma}}, q) + \widetilde{c}(\widehat{p}, q) && \forall \, q \in \mathrm{L}^2(\Omega).
\end{aligned}$$

But in this case, the assumption of small data in (3.11) *(as well as in other instances, later on) would also involve the storativity parameter* c_0*, making the analysis slightly more restrictive.*

3.3. Existence and uniqueness of weak solution. We begin by establishing two lemmas deriving conditions under which the operator $\mathbf{T}$ is a contraction.

LEMMA 3.4. *Given $\widehat{p}_1, \widehat{p}_2, \in \mathrm{W}$, the following estimate holds*

$$\|R_1(\widehat{p}_1) - R_1(\widehat{p}_2)\|_{\mathbf{div};\Omega} \leq \frac{1}{c_a}\,\gamma\,\|\widehat{p}_1 - \widehat{p}_2\|_{0,\Omega}. \tag{3.12}$$

Proof. Let $(\boldsymbol{\sigma}_1, (\boldsymbol{u}_1, \boldsymbol{\rho}_1)), (\boldsymbol{\sigma}_2, (\boldsymbol{u}_2, \boldsymbol{\rho}_2)) \in \mathbb{H}_{\mathrm{N}}(\mathbf{div};\Omega) \times (\mathbf{L}^2(\Omega) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega))$, such that $\mathbf{R}(\widehat{p}_1) = (\boldsymbol{\sigma}_1, (\boldsymbol{u}_1, \boldsymbol{\rho}_1))$ ad $\mathbf{R}(\widehat{p}_2) = (\boldsymbol{\sigma}_2, (\boldsymbol{u}_2, \boldsymbol{\rho}_2))$. Then, from the definition of $\mathbf{R}$ (cf. (3.3)), we have

$$\begin{aligned}
a(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2, \boldsymbol{\tau}) &+ b(\boldsymbol{\tau}, (\boldsymbol{u}_1 - \boldsymbol{u}_2, \boldsymbol{\rho}_1 - \boldsymbol{\rho}_2)) &&= -c(\boldsymbol{\tau}, \widehat{p}_1 - \widehat{p}_2) &&\forall \boldsymbol{\tau} \in \mathbb{H}_{\mathrm{N}}(\mathbf{div};\Omega), \\
b(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2, (\boldsymbol{v}, \boldsymbol{\eta})) && &&= 0 &&\forall (\boldsymbol{v}, \boldsymbol{\eta}) \in \mathbf{L}^2(\Omega) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega).
\end{aligned} \tag{3.13}$$

Since $\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2 \in \mathbb{V}$ (cf. (2.6a)), taking $\boldsymbol{\tau} = \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2$ in (3.13), and utilising the ellipticity of a on $\mathbb{V}$ (cf. (2.7)) along with the bound of c (cf. (2.4b)), we obtain:

$$c_a\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\|^2_{\mathbf{div};\Omega} \leq a(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2) = -c(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2, \widehat{p}_1 - \widehat{p}_2) \leq \gamma\,\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\|_{\mathbf{div};\Omega}\|\widehat{p}_1 - \widehat{p}_2\|_{0,\Omega},$$

which concludes the proof. □

LEMMA 3.5. *Given $(\widehat{\boldsymbol{\sigma}}_1, \widehat{p}_1), (\widehat{\boldsymbol{\sigma}}_2, \widehat{p}_2), \in \mathbb{H}_{\mathrm{N}}(\mathbf{div};\Omega) \times \mathrm{W}$, the following estimate holds*

$$\begin{aligned}
&\|S_2(\widehat{\boldsymbol{\sigma}}_1, \widehat{p}_1) - S_2(\widehat{\boldsymbol{\sigma}}_2, \widehat{p}_2)\|_{0,\Omega} \\
&\leq \frac{2\kappa_2\,\widetilde{C}}{\min\{\widetilde{\gamma},\,\kappa_1\}}\,(\|g\|_{0,\Omega} + \|p_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} + \gamma\|\widehat{\boldsymbol{\sigma}}_2\|_{\mathbf{div};\Omega})\|\widehat{p}_1 - \widehat{p}_2\|_{0,\Omega} + \frac{2}{\min\{\widetilde{\gamma},\,\kappa_1\}}\,\gamma\,\|\widehat{\boldsymbol{\sigma}}_1 - \widehat{\boldsymbol{\sigma}}_2\|_{\mathbf{div};\Omega}.
\end{aligned} \tag{3.14}$$

Proof. Let $(\boldsymbol{\varphi}_1, p_1), (\boldsymbol{\varphi}_2, p_2) \in \mathbf{H}_{\mathrm{N}}(\mathrm{div};\Omega) \times \mathrm{L}^2(\Omega)$, such that $\mathbf{S}(\widehat{\boldsymbol{\sigma}}_1, \widehat{p}_1) = (\boldsymbol{\varphi}_1, p_1)$ and $\mathbf{S}(\widehat{\boldsymbol{\sigma}}_2, \widehat{p}_2) = (\boldsymbol{\varphi}_2, p_2)$. Then, from the definition of $\mathbf{S}$ (cf. (3.4)), and employing similar arguments to those in Lemma 3.4, we have

$$\widetilde{a}_{\widehat{\boldsymbol{\sigma}}_1, \widehat{p}_1}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_1 - \boldsymbol{\varphi}_2) - \widetilde{a}_{\widehat{\boldsymbol{\sigma}}_2, \widehat{p}_2}(\boldsymbol{\varphi}_2, \boldsymbol{\varphi}_1 - \boldsymbol{\varphi}_2) + \widetilde{c}(p_1 - p_2, p_1 - p_2) = -c(\widehat{\boldsymbol{\sigma}}_1 - \widehat{\boldsymbol{\sigma}}_2, p_1 - p_2),$$

by adding $\pm\widetilde{a}_{\widehat{\boldsymbol{\sigma}}_1, \widehat{p}_1}(\boldsymbol{\varphi}_2, \boldsymbol{\varphi}_1 - \boldsymbol{\varphi}_2)$ in the last equation, we obtain

$$\widetilde{a}_{\widehat{\boldsymbol{\sigma}}_1, \widehat{p}_1}(\boldsymbol{\varphi}_1 - \boldsymbol{\varphi}_2, \boldsymbol{\varphi}_1 - \boldsymbol{\varphi}_2) + \widetilde{c}(p_1 - p_2, p_1 - p_2) = \widetilde{a}_{\widehat{\boldsymbol{\sigma}}_2, \widehat{p}_2}(\boldsymbol{\varphi}_2, \boldsymbol{\varphi}_1 - \boldsymbol{\varphi}_2) - \widetilde{a}_{\widehat{\boldsymbol{\sigma}}_1, \widehat{p}_1}(\boldsymbol{\varphi}_2, \boldsymbol{\varphi}_1 - \boldsymbol{\varphi}_2) - c(\widehat{\boldsymbol{\sigma}}_1 - \widehat{\boldsymbol{\sigma}}_2, p_1 - p_2).$$

Then, using the the first assumption for κ (cf. (2.3)), the ellipticity of $\widetilde{c}$ (see (2.9)), the definition of $\widetilde{a}_{\widehat{\boldsymbol{\sigma}}, \widehat{p}}$ (cf. (2.1)) and the continuity of the form c (see (2.4b)), we deduce

$$\begin{aligned}
\kappa_1\|\boldsymbol{\varphi}_1 - \boldsymbol{\varphi}_2\|^2_{0,\Omega} + \widetilde{\gamma}\|p_1 - p_2\|^2_{0,\Omega} &\leq \widetilde{a}_{\widehat{\boldsymbol{\sigma}}_1, \widehat{p}_1}(\boldsymbol{\varphi}_1 - \boldsymbol{\varphi}_2, \boldsymbol{\varphi}_1 - \boldsymbol{\varphi}_2) + \widetilde{c}(p_1 - p_2, p_1 - p_2) \\
&= \int_\Omega (\kappa(\widehat{\boldsymbol{\sigma}}_2, \widehat{p}_2)^{-1} - \kappa(\widehat{\boldsymbol{\sigma}}_1, \widehat{p}_1)^{-1})\,\boldsymbol{\varphi}_2 \cdot (\boldsymbol{\varphi}_1 - \boldsymbol{\varphi}_2) - c(\widehat{\boldsymbol{\sigma}}_1 - \widehat{\boldsymbol{\sigma}}_2, p_1 - p_2) \\
&\leq \|\kappa(\widehat{\boldsymbol{\sigma}}_2, \widehat{p}_2)^{-1} - \kappa(\widehat{\boldsymbol{\sigma}}_1, \widehat{p}_1)^{-1}\|_{\mathbb{L}^\infty(\Omega)}\,\|\boldsymbol{\varphi}_2\|_{0,\Omega}\|\boldsymbol{\varphi}_1 - \boldsymbol{\varphi}_2\|_{0,\Omega} + \gamma\,\|\widehat{\boldsymbol{\sigma}}_1 - \widehat{\boldsymbol{\sigma}}_2\|_{\mathbf{div};\Omega}\|p_1 - p_2\|_{0,\Omega}.
\end{aligned}$$

From the last equation, by utilising the second assumption regarding κ (see (2.3)), we obtain

$$\begin{aligned}
\tfrac{1}{2}\min\{\widetilde{\gamma}, \kappa_1\}(\|\boldsymbol{\varphi}_1 - \boldsymbol{\varphi}_2\|_{0,\Omega} + \|p_1 - p_2\|_{0,\Omega})^2 &\leq \kappa_1\|\boldsymbol{\varphi}_1 - \boldsymbol{\varphi}_2\|^2_{0,\Omega} + \widetilde{\gamma}\|p_1 - p_2\|^2_{0,\Omega} \\
&\leq \kappa_2\|\widehat{p}_2 - \widehat{p}_1\|_{0,\Omega}\|\boldsymbol{\varphi}_2\|_{0,\Omega}\|\boldsymbol{\varphi}_1 - \boldsymbol{\varphi}_2\|_{0,\Omega} + \gamma\,\|\widehat{\boldsymbol{\sigma}}_1 - \widehat{\boldsymbol{\sigma}}_2\|_{\mathbf{div};\Omega}\|p_1 - p_2\|_{0,\Omega} \\
&\leq \big(\kappa_2\|\widehat{p}_2 - \widehat{p}_1\|_{0,\Omega}\|\boldsymbol{\varphi}_2\|_{0,\Omega} + \gamma\,\|\widehat{\boldsymbol{\sigma}}_1 - \widehat{\boldsymbol{\sigma}}_2\|_{\mathbf{div};\Omega}\big)\big(\|\boldsymbol{\varphi}_1 - \boldsymbol{\varphi}_2\|_{0,\Omega} + \|p_1 - p_2\|_{0,\Omega}\big),
\end{aligned}$$

the last, together with the fact that $\boldsymbol{\varphi}_2$ satisfies (3.9), leads to the following bound

$$\begin{aligned}
\tfrac{1}{2}\min\{\widetilde{\gamma}, \kappa_1\}(\|\boldsymbol{\varphi}_1 - \boldsymbol{\varphi}_2\|_{0,\Omega} + \|p_1 - p_2\|_{0,\Omega}) &\leq \kappa_2\|\widehat{p}_2 - \widehat{p}_1\|_{0,\Omega}\|\boldsymbol{\varphi}_2\|_{0,\Omega} + \gamma\,\|\widehat{\boldsymbol{\sigma}}_1 - \widehat{\boldsymbol{\sigma}}_2\|_{\mathbf{div};\Omega} \\
&\leq \kappa_2\|\widehat{p}_2 - \widehat{p}_1\|_{0,\Omega}\widetilde{C}\,(\|g\|_{0,\Omega} + \|p_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} + \gamma\|\widehat{\boldsymbol{\sigma}}_2\|_{\mathbf{div};\Omega}) + \gamma\,\|\widehat{\boldsymbol{\sigma}}_1 - \widehat{\boldsymbol{\sigma}}_2\|_{\mathbf{div};\Omega},
\end{aligned}$$

and this yields (3.14), concluding the proof. □

The following theorem presents the main result of this section, establishing the existence and uniqueness of the solution to the fixed-point problem (3.6), or equivalently, the well-posedness of problem (2.2).

THEOREM 3.6. *Given $r > 0$, assume that $\boldsymbol{f} \in \mathbf{L}^2(\Omega)$, $g \in L^2(\Omega)$, $\boldsymbol{u}_{\mathrm{D}} \in \mathbf{H}^{1/2}(\Gamma_{\mathrm{D}})$, $p_{\mathrm{D}} \in \mathrm{H}^{1/2}(\Gamma_{\mathrm{D}})$ and γ satisfies*

$$\frac{2\max\{1, \kappa_2\}}{\min\{\widetilde{\gamma}, \kappa_1, r\}}\left\{\widetilde{C}(1 + C_1\gamma)(\|g\|_{0,\Omega} + \|p_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} + \|\boldsymbol{u}_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} + \|\boldsymbol{f}\|_{0,\Omega}) + \frac{\gamma^2}{c_a}\Big(\frac{1}{\kappa_2} + \widetilde{C}\,r\Big)\right\} < 1. \tag{3.15}$$

Then, $\mathbf{T}$ (cf. (3.5)) has a unique fixed point $p \in \mathrm{W}$. Equivalently, (2.2) has a unique solution $(\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{\rho}, \boldsymbol{\varphi}, p) \in \mathbb{H}_{\mathrm{N}}(\mathbf{div};\Omega) \times \mathbf{L}^2(\Omega) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega) \times \mathbf{H}_{\mathrm{N}}(\mathrm{div};\Omega) \times \mathrm{W}$. In addition, there exists $C > 0$, such that

$$\|\boldsymbol{\sigma}\|_{\mathbf{div};\Omega} + \|\boldsymbol{u}\|_{0,\Omega} + \|\boldsymbol{\rho}\|_{0,\Omega} + \|\boldsymbol{\varphi}\|_{\mathrm{div};\Omega} + \|p\|_{0,\Omega} \leq C(\|g\|_{0,\Omega} + \|p_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} + \|\boldsymbol{u}_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} + \|\boldsymbol{f}\|_{0,\Omega} + \gamma\,r). \tag{3.16}$$

Proof. Recall that (3.15) ensures the well-definedness of $\mathbf{T}$. Let $\widehat{p}_1, \widehat{p}_2, p_1, p_2 \in W$, such that $\mathbf{T}(\widehat{p}_1) = p_1$ and $\mathbf{T}(\widehat{p}_2) = p_2$. From the definition of $\mathbf{T}$ (see (3.5)), and the estimates (3.14) and (3.12), we deduce

$$\|p_1 - p_2\|_{0,\Omega} = \|\mathbf{T}(\widehat{p}_1) - \mathbf{T}(\widehat{p}_2)\|_{0,\Omega} = \|S_2(R_1(\widehat{p}_1), \widehat{p}_1) - S_2(R_1(\widehat{p}_2), \widehat{p}_2)\|_{0,\Omega}$$

$$\leq \frac{2\kappa_2 \widetilde{C}}{\min\{\widetilde{\gamma}, \kappa_1\}} \left(\|g\|_{0,\Omega} + \|p_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} + \gamma \|R_1(\widehat{p}_2)\|_{\mathbf{div};\Omega}\right)\|\widehat{p}_1 - \widehat{p}_2\|_{0,\Omega} + \frac{2}{\min\{\widetilde{\gamma}, \kappa_1\}} \gamma \|R_1(\widehat{p}_1) - R_1(\widehat{p}_2)\|_{\mathbf{div};\Omega}$$

$$\leq \frac{2\kappa_2 \widetilde{C}}{\min\{\widetilde{\gamma}, \kappa_1\}} \left(\|g\|_{0,\Omega} + \|p_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\right)\|\widehat{p}_1 - \widehat{p}_2\|_{0,\Omega} + \frac{2\kappa_2 \widetilde{C}}{\min\{\widetilde{\gamma}, \kappa_1\}}\gamma \|R_1(\widehat{p}_2)\|_{\mathbf{div};\Omega}\|\widehat{p}_1 - \widehat{p}_2\|_{0,\Omega}$$

$$+ \frac{2}{c_a \min\{\widetilde{\gamma}, \kappa_1\}} \gamma^2 \|\widehat{p}_1 - \widehat{p}_2\|_{0,\Omega},$$

the above, along with the fact that $R_1(\widehat{p}_2)$ satisfies (3.7) and $\widehat{p}_2 \in W$, implies

$$\|p_1 - p_2\|_{0,\Omega} \leq \frac{2\kappa_2 \widetilde{C}}{\min\{\widetilde{\gamma}, \kappa_1\}} \left(\|g\|_{0,\Omega} + \|p_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}\right)\|\widehat{p}_1 - \widehat{p}_2\|_{0,\Omega} + \frac{2}{c_a \min\{\widetilde{\gamma}, \kappa_1\}} \gamma^2 \|\widehat{p}_1 - \widehat{p}_2\|_{0,\Omega}$$

$$+ \frac{2\kappa_2 \widetilde{C}}{\min\{\widetilde{\gamma}, \kappa_1\}}\gamma \left(C_1 \left(\|\boldsymbol{u}_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} + \|\boldsymbol{f}\|_{0,\Omega}\right) + \frac{1}{c_a} \gamma r\right)\|\widehat{p}_1 - \widehat{p}_2\|_{0,\Omega}$$

$$\leq \frac{2}{\min\{\widetilde{\gamma}, \kappa_1\}} \left\{\kappa_2 \widetilde{C}(1 + C_1\gamma)(\|g\|_{0,\Omega} + \|p_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} + \|\boldsymbol{u}_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} + \|\boldsymbol{f}\|_{0,\Omega}) + \frac{\gamma^2}{c_a}(1 + \kappa_2 \widetilde{C} r)\right\}\|\widehat{p}_1 - \widehat{p}_2\|_{0,\Omega},$$

which together with (3.15) and the Banach fixed-point theorem yields that $\mathbf{T}$ has a unique fixed point in W. Finally, (3.16) is derived analogously to the estimates in (3.7) and (3.9), which completes the proof. □

REMARK 3.2. *The operator* $\mathbf{T}$ *(see (3.5)) could be also defined, for example* $\mathbf{T} : \mathbf{W} \to \mathbf{W}$, *with* $\mathbf{W} := \left\{(\widehat{\boldsymbol{\sigma}}, \widehat{p}) \in \mathbb{H}_{\mathrm{N}}(\mathbf{div};\Omega) \times \mathrm{L}^2(\Omega) : \quad \|\widehat{\boldsymbol{\sigma}}\|_{\mathbf{div};\Omega} + \|\widehat{p}\|_{0,\Omega} \leq r\right\}$ *and* $\mathbf{T}(\widehat{\boldsymbol{\sigma}}, \widehat{p}) := (R_1(\widehat{p}), S_2(\widehat{\boldsymbol{\sigma}}, \widehat{p})) = (\boldsymbol{\sigma}, p)$, *with* R_1 *and* S_2 *defined as in (3.3) and (3.4), respectively.*

4. Finite element discretisation.
In this section, we present and analyse the Galerkin scheme for problem (2.2). It is worth mentioning upfront that the well-posedness analysis can be straightforwardly extended from the continuous problem to the discrete case. Therefore, we omit many of the details.

4.1. Finite element spaces and Galerkin scheme.
Let us consider a regular partition $\mathcal{T}_h$ of $\bar{\Omega}$ made up of triangles K (in $\mathbb{R}^2$) or tetrahedra K (in $\mathbb{R}^3$) of diameter h_K, and denote the mesh size by $h := \max\{h_K : K \in \mathcal{T}_h\}$. Given an integer $\ell \geq 0$ and $K \in \mathcal{T}_h$, we first let $\mathrm{P}_\ell(K)$ be the space of polynomials of degree $\leq \ell$ defined on K, whose vector and tensor versions are denoted $\mathbf{P}_\ell(K) := [\mathrm{P}_\ell(K)]^d$ and $\mathbb{P}_\ell(K) = [\mathrm{P}_\ell(K)]^{d \times d}$, respectively. Also, we let $\mathbf{RT}_\ell(K) := \mathbf{P}_\ell(K) \oplus \mathrm{P}_\ell(K)\,\boldsymbol{x}$ be the local Raviart–Thomas space of order ℓ defined on K, where $\boldsymbol{x}$ stands for a generic vector in $\mathbb{R}^d$, and denote by $\mathbb{RT}_k(K)$ the tensor-valued counterpart of this space.

For each $K \in \mathcal{T}_h$ we consider the bubble space of order k, defined as

$$\mathbf{B}_k(K) := \begin{cases} \mathbf{curl}^{\mathrm{t}}(b_K \mathrm{P}_k(K)) & \text{in} \quad \mathbb{R}^2, \\ \nabla \times (b_K \mathbf{P}_k(K)) & \text{in} \quad \mathbb{R}^3, \end{cases}$$

where b_K is a suitably normalised cubic polynomial on K, which vanishes on the boundary of K (see [29]).

We recall the classical PEERS$_k$ elements (cf. [5]) to define the discrete subspaces for the stress tensor $\boldsymbol{\sigma}$, the displacement $\boldsymbol{u}$, and the rotation tensor $\boldsymbol{\rho}$

$$\begin{aligned} \mathbb{H}_h^{\boldsymbol{\sigma}} &:= \left\{\boldsymbol{\tau}_h \in \mathbb{H}_{\mathrm{N}}(\mathbf{div};\Omega) : \quad \boldsymbol{\tau}_h|_K \in \mathbb{RT}_k(K) \oplus [\mathbf{B}_k(K)]^d \quad \forall K \in \mathcal{T}_h\right\}, \\ \mathbf{H}_h^{\boldsymbol{u}} &:= \left\{\boldsymbol{v}_h \in \mathbf{L}^2(\Omega) : \quad \boldsymbol{v}_h|_K \in \mathbf{P}_k(K) \quad \forall K \in \mathcal{T}_h\right\}, \\ \mathbb{H}_h^{\boldsymbol{\rho}} &:= \left\{\boldsymbol{\eta}_h \in \mathbb{L}^2_{\mathrm{skew}}(\Omega) \cap \mathbb{C}(\overline{\Omega}) \quad \text{and} \quad \boldsymbol{\eta}_h|_K \in \mathbb{P}_{k+1}(K) \quad \forall K \in \mathcal{T}_h\right\}, \end{aligned} \tag{4.1}$$

and the following estimates are proven for the PEERS$_k$ elements (cf. [41, Remark 3.3])

$$\sup_{\mathbf{0} \neq \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}} \frac{b(\boldsymbol{\tau}_h, (\boldsymbol{v}_h, \boldsymbol{\eta}_h))}{\|\boldsymbol{\tau}_h\|_{\mathbf{div};\Omega}} \geq \beta^* (\|\boldsymbol{v}_h\|_{0,\Omega} + \|\boldsymbol{\eta}_h\|_{0,\Omega}) \quad \forall (\boldsymbol{v}_h, \boldsymbol{\eta}_h) \in \mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}}, \tag{4.2a}$$

$$a(\boldsymbol{\tau}_h, \boldsymbol{\tau}_h) \geq c_a \|\boldsymbol{\tau}_h\|_{\mathbf{div};\Omega}^2 \quad \forall \boldsymbol{\tau}_h \in \mathbb{V}_h, \tag{4.2b}$$

where $\mathbb{V}_h$ denotes the discrete kernel of b, that is

$$\mathbb{V}_h := \left\{\boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}} : \quad b(\boldsymbol{\tau}_h, (\boldsymbol{v}_h, \boldsymbol{\eta}_h)) = 0 \quad \forall (\boldsymbol{v}_h, \boldsymbol{\eta}_h) \in \mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}}\right\}. \tag{4.3}$$

Additionally, for $\boldsymbol{\varphi}$ and the pressure p, we define the FE subspaces

$$\mathbf{H}_h^{\boldsymbol{\varphi}} := \{\boldsymbol{\psi}_h \in \mathbf{H}_{\mathrm{N}}(\mathrm{div};\Omega): \ \boldsymbol{\psi}_h|_K \in \mathbf{RT}_k(K) \quad \forall K \in \mathcal{T}_h\},$$
$$\mathrm{H}_h^p := \{q_h \in \mathrm{L}^2(\Omega): \ q_h|_K \in \mathrm{P}_k(K) \quad \forall K \in \mathcal{T}_h\}, \tag{4.4}$$

and it is well known that $\widetilde{b}$ satisfies the inf-sup condition (see, e.g., [21, Lemma 4.6])

$$\sup_{\mathbf{0}\neq\boldsymbol{\psi}_h\in\mathbf{H}_h^{\boldsymbol{\varphi}}} \frac{\widetilde{b}(\boldsymbol{\psi}_h,q_h)}{\|\boldsymbol{\psi}_h\|_{\mathrm{div};\Omega}} \geq \widetilde{\beta}^*\|q_h\|_{0,\Omega} \quad \forall q_h \in \mathrm{H}_h^p. \tag{4.5}$$

Note that it is of course possible to consider other conforming and inf-sup stable spaces such as Arnold–Falk–Winther and Brezzi–Douglas–Marini instead of (4.1) and (4.4), respectively. The Galerkin scheme for (2.2) reads: find $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \boldsymbol{\rho}_h, \boldsymbol{\varphi}_h, p_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}} \times \mathbf{H}_h^{\boldsymbol{\varphi}} \times \mathrm{H}_h^p$, such that:

$$\begin{aligned}
a(\boldsymbol{\sigma}_h,\boldsymbol{\tau}_h) &+& b(\boldsymbol{\tau}_h,(\boldsymbol{u}_h,\boldsymbol{\rho}_h)) &+& c(\boldsymbol{\tau}_h,p_h) &= H(\boldsymbol{\tau}_h) && \forall\,\boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}, \\
b(\boldsymbol{\sigma}_h,(\boldsymbol{v}_h,\boldsymbol{\eta}_h)) &&&&&= F(\boldsymbol{v}_h,\boldsymbol{\eta}_h) && \forall(\boldsymbol{v}_h,\boldsymbol{\eta}_h) \in \mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}}, \\
\widetilde{a}_{\boldsymbol{\sigma}_h,p_h}(\boldsymbol{\varphi}_h,\boldsymbol{\psi}_h) &+& \widetilde{b}(\boldsymbol{\psi}_h,p_h) &&&= \widetilde{H}(\boldsymbol{\psi}_h) && \forall\,\boldsymbol{\psi}_h \in \mathbf{H}_h^{\boldsymbol{\varphi}}, \\
\widetilde{b}(\boldsymbol{\varphi}_h,q_h) &-& \widetilde{c}(p_h,q_h) &-& c(\boldsymbol{\sigma}_h,q_h) &= G(q_h) && \forall\,q_h \in \mathrm{H}_h^p.
\end{aligned} \tag{4.6}$$

4.2. Analysis of the discrete problem. In this section, we analyse the Galerkin scheme (4.6). It's worth noting that establishing well-posedness can be readily achieved by extending the results derived for the continuous problem to the discrete setting. Firstly, and similarly to the continuous case, we define the following set

$$\mathrm{W}_h := \left\{\widehat{p}_h \in \mathrm{H}_h^p: \quad \|\widehat{p}_h\|_{0,\Omega} \leq r\right\}.$$

Next, for a fixed $\widehat{p}_h$ in W_h, we have that the bilinear form $\widetilde{a}_{\boldsymbol{\sigma}_h,p_h}$ satisfies

$$\widetilde{a}_{\widehat{\boldsymbol{\sigma}}_h,\widehat{p}_h}(\boldsymbol{\psi}_h,\boldsymbol{\psi}_h) \geq \kappa_1 \|\boldsymbol{\psi}_h\|_{\mathrm{div};\Omega}^2 \quad \forall\,\boldsymbol{\psi}_h \in \mathbf{V}_h, \tag{4.7}$$

where $\mathbf{V}_h$ is the discrete kernel of $\widetilde{b}$

$$\mathbf{V}_h := \left\{\boldsymbol{\psi}_h \in \mathbf{H}_h^{\boldsymbol{\varphi}}: \quad \widetilde{b}(\boldsymbol{\psi}_h,q_h) = 0 \quad \forall\,q_h \in \mathrm{H}_h^p\right\}. \tag{4.8}$$

Additionally, we define the discrete operators $\mathbf{R}_h : \mathrm{W}_h \subseteq \mathrm{H}_h^p \to \mathbb{H}_h^{\boldsymbol{\sigma}} \times (\mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}})$ and $\mathbf{S}_h : \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathrm{W}_h \to \mathbf{H}_h^{\boldsymbol{\varphi}} \times \mathrm{H}_h^p$, respectively, by

$$\mathbf{R}_h(\widehat{p}_h) := \big(R_{1,h}(\widehat{p}_h), (R_{2,h}(\widehat{p}_h), R_{3,h}(\widehat{p}_h))\big) = (\boldsymbol{\sigma}_h, (\boldsymbol{u}_h, \boldsymbol{\rho}_h)) \quad \forall\,\widehat{p}_h \in \mathrm{W}_h,$$

where $(\boldsymbol{\sigma}_h, (\boldsymbol{u}_h, \boldsymbol{\rho}_h)) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times (\mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}})$ is the unique solution of

$$\begin{aligned}
a(\boldsymbol{\sigma}_h,\boldsymbol{\tau}_h) &+& b(\boldsymbol{\tau}_h,(\boldsymbol{u}_h,\boldsymbol{\rho}_h)) &= H(\boldsymbol{\tau}_h) - c(\boldsymbol{\tau}_h,\widehat{p}_h) && \forall\,\boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}, \\
b(\boldsymbol{\sigma}_h,(\boldsymbol{v}_h,\boldsymbol{\eta}_h)) &&&= F(\boldsymbol{v}_h,\boldsymbol{\eta}_h) && \forall\,(\boldsymbol{v}_h,\boldsymbol{\eta}_h) \in \mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}},
\end{aligned} \tag{4.9}$$

and

$$\mathbf{S}_h(\widehat{\boldsymbol{\sigma}}_h,\widehat{p}_h) := \big(S_{1,h}(\widehat{\boldsymbol{\sigma}}_h,\widehat{p}_h), S_{2,h}(\widehat{\boldsymbol{\sigma}}_h,\widehat{p}_h)\big) = (\boldsymbol{\varphi}_h, p_h) \quad \forall\,(\widehat{\boldsymbol{\sigma}}_h,\widehat{p}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathrm{W}_h,$$

where $(\boldsymbol{\varphi}_h, p_h)$ is the unique tuple in $\mathbf{H}_h^{\boldsymbol{\varphi}} \times \times\mathrm{H}_h^p$ such that

$$\begin{aligned}
\widetilde{a}_{\widehat{\boldsymbol{\sigma}}_h,\widehat{p}_h}(\boldsymbol{\varphi}_h,\boldsymbol{\psi}_h) &+& \widetilde{b}(\boldsymbol{\psi}_h,p_h) &= \widetilde{H}(\boldsymbol{\psi}_h) && \forall\,\boldsymbol{\psi}_h \in \mathbf{H}_h^{\boldsymbol{\varphi}}, \\
\widetilde{b}(\boldsymbol{\varphi}_h,q_h) &-& \widetilde{c}(p_h,q_h) &= G(q_h) + c(\widehat{\boldsymbol{\sigma}}_h,q_h) && \forall\,q_h \in \mathrm{H}_h^p.
\end{aligned}$$

Employing properties (4.2a), (4.2b), (4.5), (4.7) and (2.9) and proceeding exactly as for the continuous case (Lemmas 3.1 and 3.2), it can be easily deduced that both operators are well-defined. Then, analogously to the continuous case, we define the following fixed-point operator

$$\mathbf{T}_h : \mathrm{W}_h \subseteq \mathrm{H}_h^p \to \mathrm{H}_h^p, \quad \widehat{p}_h \mapsto \mathbf{T}_h(\widehat{p}_h) := S_{2,h}(R_{1,h}(\widehat{p}_h), \widehat{p}_h), \tag{4.10}$$

which is clearly well-defined (since R_h and S_h are). Further, it can be easily deduced that $\mathbf{T}_h(\mathrm{W}_h) \subseteq \mathrm{W}_h$ if

$$\widetilde{C}^*\big(1 + \gamma\,C_1^*\big)\big(\|g\|_{0,\Omega} + \|p_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} + \|\boldsymbol{u}_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} + \|\boldsymbol{f}\|_{0,\Omega}\big) + \frac{\widetilde{C}^*}{c_a}\gamma^2 r \leq r, \tag{4.11}$$

where $\widetilde{C}^*$ and C_1^* (depending on c_a, μ, κ_1, κ_2, $C_{\widetilde{a}}$, β^*, $\widetilde{\beta}^*$) are the discrete versions of the constants $\widetilde{C}$ and C_1 (cf. (3.7) and (3.9)). Finally, it is clear that $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \boldsymbol{\rho}_h, \boldsymbol{\varphi}_h, p_h)$ is a solution to (4.6) if and only if p_h satisfies

$$\mathbf{T}_h(p_h) = p_h. \tag{4.12}$$

The main outcome of this section is presented in the following theorem, establishing the existence and uniqueness of a solution to the fixed-point problem (4.12), equivalently proving the well-posedness of problem (4.6).

THEOREM 4.1. *Given $r > 0$, assume that the data and γ satisfy*

$$\frac{2\max\{1, \kappa_2\}}{\min\{\widetilde{\gamma}, \kappa_1, r\}} \left\{ \widetilde{C}^*(1 + C_1^*\gamma)(\|g\|_{0,\Omega} + \|p_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} + \|\boldsymbol{u}_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} + \|\boldsymbol{f}\|_{0,\Omega}) + \frac{\gamma^2}{c_a}\left(\frac{1}{\kappa_2} + \widetilde{C}^* r\right) \right\} < 1. \quad (4.13)$$

Then, $\mathbf{T}_h$ (cf. (4.10)) has a unique fixed point $p_h \in \mathrm{W}_h$. Equivalently, problem (4.6) has a unique solution $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \boldsymbol{\rho}_h, \boldsymbol{\varphi}_h, p_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}} \times \mathbf{H}_h^{\boldsymbol{\varphi}} \times \mathrm{W}_h$. In addition, there exists $C^* > 0$, such that

$$\|\boldsymbol{\sigma}_h\|_{\mathbf{div};\Omega} + \|\boldsymbol{u}_h\|_{0,\Omega} + \|\boldsymbol{\rho}_h\|_{0,\Omega} + \|\boldsymbol{\varphi}_h\|_{\mathrm{div};\Omega} + \|p_h\|_{0,\Omega} \leq C^*(\|g\|_{0,\Omega} + \|p_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} + \|\boldsymbol{u}_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} + \|\boldsymbol{f}\|_{0,\Omega} + \gamma\, r). \quad (4.14)$$

Proof. First, we observe that, similar to the continuous case (as seen in the proof of Theorem 3.6), assumption (4.13) ensures the well-definedness of $\mathbf{T}_h$ and that $\mathbf{T}_h(\mathrm{W}_h) \subseteq \mathrm{W}_h$. Now, by adapting the arguments used in Section 3.3 (cf. Lemmas 3.4 and 3.5), one can derive the following estimates

$$\|R_{1,h}(\widehat{p}_1) - R_{1,h}(\widehat{p}_2)\|_{\mathbf{div};\Omega} \leq \frac{1}{c_a}\,\gamma\,\|\widehat{p}_1 - \widehat{p}_2\|_{0,\Omega},$$

$$\|S_{2,h}(\widehat{\boldsymbol{\sigma}}_1, \widehat{p}_1) - S_{2,h}(\widehat{\boldsymbol{\sigma}}_2, \widehat{p}_2)\|_{0,\Omega} \leq \frac{2\kappa_2\,\widetilde{C}^*}{\min\{\widetilde{\gamma}, \kappa_1\}}\left(\|g\|_{0,\Omega} + \|p_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} + \gamma\|\widehat{\boldsymbol{\sigma}}_2\|_{\mathbf{div};\Omega}\right)\|\widehat{p}_1 - \widehat{p}_2\|_{0,\Omega}$$
$$+ \frac{2}{\min\{\widetilde{\gamma}, \kappa_1\}}\,\gamma\,\|\widehat{\boldsymbol{\sigma}}_1 - \widehat{\boldsymbol{\sigma}}_2\|_{\mathbf{div};\Omega},$$

for all $\widehat{p}_1, \widehat{p}_2 \in \mathrm{W}_h$ and $\widehat{\boldsymbol{\sigma}}_1, \widehat{\boldsymbol{\sigma}}_2 \in \mathbb{H}_h^{\boldsymbol{\sigma}}$, which together with the definition of $\mathbf{T}_h$ (see (4.10)), yield

$$\|\mathbf{T}_h(\widehat{p}_1) - \mathbf{T}_h(\widehat{p}_2)\|_{0,\Omega} \leq \frac{2}{\min\{\widetilde{\gamma}, \kappa_1\}}\left\{ \kappa_2\,\widetilde{C}^*(1 + C_1^*\gamma)(\|g\|_{0,\Omega} + \|p_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} + \|\boldsymbol{u}_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} + \|\boldsymbol{f}\|_{0,\Omega}) \right.$$
$$\left. + \frac{\gamma^2}{c_a}(1 + \kappa_2\,\widetilde{C}^* r) \right\}\|\widehat{p}_1 - \widehat{p}_2\|_{0,\Omega},$$

for all $\widehat{p}_1, \widehat{p}_2 \in \mathrm{W}_h$. In this way, using estimate (4.13) we obtain that $\mathbf{T}_h$ is a contraction mapping on W_h, thus problem (4.12), or equivalently (4.6) is well-posed. Finally, analogously to the proof of Theorem 3.6 (see also Lemmas 3.4 and 3.5) we can obtain (4.14), which concludes the proof. □

5. A priori error estimates. In this section, we aim to provide the convergence of the Galerkin scheme (4.6) and derive the corresponding rate of convergence. From now on we assume that the hypotheses of Theorem 3.6 and Theorem 4.1 hold.

5.1. Preliminaries. Let the tuples $(\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{\rho}, \boldsymbol{\varphi}, p) \in \mathbb{H}_{\mathrm{N}}(\mathbf{div};\Omega) \times \mathbf{L}^2(\Omega) \times \mathbb{L}_{\mathrm{skew}}^2(\Omega) \times \mathbf{H}_{\mathrm{N}}(\mathrm{div};\Omega) \times \mathrm{L}^2(\Omega)$ and $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \boldsymbol{\rho}_h, \boldsymbol{\varphi}_h, p_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}} \times \mathbf{H}_h^{\boldsymbol{\varphi}} \times \mathrm{H}_h^p$ be the unique solutions of (3.3) and (4.9), respectively.

Let us write $\mathbf{e}_{\boldsymbol{\sigma}} = \boldsymbol{\sigma} - \boldsymbol{\sigma}_h$, $\mathbf{e}_{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{u}_h$, $\mathbf{e}_{\boldsymbol{\rho}} = \boldsymbol{\rho} - \boldsymbol{\rho}_h$, $\mathbf{e}_{\boldsymbol{\varphi}} = \boldsymbol{\varphi} - \boldsymbol{\varphi}_h$ and $\mathbf{e}_p = p - p_h$. As usual, for a given $(\widehat{\boldsymbol{\tau}}_h, (\widehat{\boldsymbol{v}}_h, \widehat{\boldsymbol{\eta}}_h)) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times (\mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}})$ and $(\widehat{\boldsymbol{\psi}}_h, \widehat{q}_h) \in \mathbf{H}_h^{\boldsymbol{\varphi}} \times \mathrm{H}_h^p$, we shall then decompose these errors into

$$\mathbf{e}_{\boldsymbol{\sigma}} = \boldsymbol{\xi}_{\boldsymbol{\sigma}} + \boldsymbol{\chi}_{\boldsymbol{\sigma}}, \qquad \mathbf{e}_{\boldsymbol{u}} = \boldsymbol{\xi}_{\boldsymbol{u}} + \boldsymbol{\chi}_{\boldsymbol{u}}, \qquad \mathbf{e}_{\boldsymbol{\rho}} = \boldsymbol{\xi}_{\boldsymbol{\rho}} + \boldsymbol{\chi}_{\boldsymbol{\rho}}, \qquad \mathbf{e}_{\boldsymbol{\varphi}} = \boldsymbol{\xi}_{\boldsymbol{\varphi}} + \boldsymbol{\chi}_{\boldsymbol{\varphi}}, \qquad \mathbf{e}_p = \xi_p + \chi_p, \qquad (5.1)$$

with $\boldsymbol{\xi}_{\boldsymbol{\sigma}} := \boldsymbol{\sigma} - \widehat{\boldsymbol{\tau}}_h$, $\boldsymbol{\chi}_{\boldsymbol{\sigma}} := \widehat{\boldsymbol{\tau}}_h - \boldsymbol{\sigma}_h$, $\boldsymbol{\xi}_{\boldsymbol{u}} := \boldsymbol{u} - \widehat{\boldsymbol{v}}_h$, $\boldsymbol{\chi}_{\boldsymbol{u}} := \widehat{\boldsymbol{v}}_h - \boldsymbol{u}_h$, $\boldsymbol{\xi}_{\boldsymbol{\rho}} := \boldsymbol{\rho} - \widehat{\boldsymbol{\eta}}_h$, $\boldsymbol{\chi}_{\boldsymbol{\rho}} := \widehat{\boldsymbol{\eta}}_h - \boldsymbol{\rho}_h$, $\boldsymbol{\xi}_{\boldsymbol{\varphi}} := \boldsymbol{\varphi} - \widehat{\boldsymbol{\psi}}_h$, $\boldsymbol{\chi}_{\boldsymbol{\varphi}} := \widehat{\boldsymbol{\psi}}_h - \boldsymbol{\varphi}_h$, $\xi_p := p - \widehat{q}_h$, and $\chi_p := \widehat{q}_h - p_h$.

Considering the first two equations of problems (2.2) and (4.6), the following identities hold

$$\begin{aligned} a(\boldsymbol{\sigma}, \boldsymbol{\tau}) \quad + \quad b(\boldsymbol{\tau}, (\boldsymbol{u}, \boldsymbol{\rho})) &= H(\boldsymbol{\tau}) - c(\boldsymbol{\tau}, p) \quad \forall \boldsymbol{\tau} \in \mathbb{H}_{\mathrm{N}}(\mathbf{div};\Omega), \\ b(\boldsymbol{\sigma}, (\boldsymbol{v}, \boldsymbol{\eta})) &= F(\boldsymbol{v}, \boldsymbol{\eta}) \qquad \forall \boldsymbol{v} \in \mathbf{L}^2(\Omega), \, \forall \boldsymbol{\eta} \in \mathbb{L}_{\mathrm{skew}}^2(\Omega), \end{aligned}$$

and

$$\begin{aligned} a(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) \quad + \quad b(\boldsymbol{\tau}_h, (\boldsymbol{u}_h, \boldsymbol{\rho}_h)) &= H(\boldsymbol{\tau}_h) - c(\boldsymbol{\tau}_h, p_h) \quad \forall \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}, \\ b(\boldsymbol{\sigma}_h, (\boldsymbol{v}_h, \boldsymbol{\eta}_h)) &= F(\boldsymbol{v}_h, \boldsymbol{\eta}_h) \qquad \forall (\boldsymbol{v}_h, \boldsymbol{\eta}_h) \in \mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}}. \end{aligned}$$

From these relations we can obtain that for all $(\boldsymbol{\tau}_h, (\boldsymbol{v}_h, \boldsymbol{\rho}_h)) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times (\mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}})$, there holds

$$\begin{aligned} a(\mathbf{e}_{\boldsymbol{\sigma}}, \boldsymbol{\tau}_h) \quad + \quad b(\boldsymbol{\tau}_h, (\mathbf{e}_{\boldsymbol{u}}, \mathbf{e}_{\boldsymbol{\rho}})) &= -c(\boldsymbol{\tau}_h, \mathbf{e}_p), \\ b(\mathbf{e}_{\boldsymbol{\sigma}}, (\boldsymbol{v}_h, \boldsymbol{\eta}_h)) &= 0 \end{aligned}$$

which together with the definition of the errors in (5.1), implies that

$$\begin{aligned} a(\boldsymbol{\chi}_{\boldsymbol{\sigma}}, \boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h, (\boldsymbol{\chi}_{\boldsymbol{u}}, \boldsymbol{\chi}_{\boldsymbol{\rho}})) + b(\boldsymbol{\chi}_{\boldsymbol{\sigma}}, (\boldsymbol{v}_h, \boldsymbol{\eta}_h)) & \\ = -a(\boldsymbol{\xi}_{\boldsymbol{\sigma}}, \boldsymbol{\tau}_h) - b(\boldsymbol{\tau}_h, (\boldsymbol{\xi}_{\boldsymbol{u}}, \boldsymbol{\xi}_{\boldsymbol{\rho}})) - b(\boldsymbol{\xi}_{\boldsymbol{\sigma}}, (\boldsymbol{v}_h, \boldsymbol{\eta}_h)) - c(\boldsymbol{\tau}_h, \boldsymbol{\chi}_p) - c(\boldsymbol{\tau}_h, \boldsymbol{\xi}_p) \end{aligned} \quad (5.2)$$

for all $(\boldsymbol{\tau}_h, (\boldsymbol{v}_h, \boldsymbol{\rho}_h)) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times (\mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}})$.

Next, considering the last two equations of both problems (2.2) and (4.6), we obtain

$$\begin{aligned}
\widetilde{a}_{\boldsymbol{\sigma},p}(\boldsymbol{\varphi},\boldsymbol{\psi}) \quad + \quad \widetilde{b}(\boldsymbol{\psi},p) \quad &= \widetilde{H}(\boldsymbol{\psi}) && \forall \boldsymbol{\psi} \in \mathbf{H}_{\mathrm{N}}(\mathrm{div};\Omega), \\
\widetilde{b}(\boldsymbol{\varphi},q) \quad - \quad \widetilde{c}(p,q) \quad &= G(q) + c(\boldsymbol{\sigma},q) && \forall\, q \in \mathrm{L}^2(\Omega).
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{a}_{\boldsymbol{\sigma}_h,p_h}(\boldsymbol{\varphi}_h,\boldsymbol{\psi}_h) \quad + \quad \widetilde{b}(\boldsymbol{\psi}_h,p_h) \quad &= \widetilde{H}(\boldsymbol{\psi}_h) && \forall \boldsymbol{\psi}_h \in \mathbf{H}_h^{\boldsymbol{\varphi}}, \\
\widetilde{b}(\boldsymbol{\varphi}_h,q_h) \quad - \quad \widetilde{c}(p_h,q_h) \quad &= G(q_h) + c(\boldsymbol{\sigma}_h,q_h) && \forall\, q_h \in \mathrm{H}_h^p.
\end{aligned}$$

Then, using arguments similar to those in Lemma 3.5, by adding $\pm\widetilde{a}_{\boldsymbol{\sigma}_h,p_h}(\boldsymbol{\varphi},\boldsymbol{\psi}_h)$, we have

$$\widetilde{a}_{\boldsymbol{\sigma}_h,p_h}(\mathsf{e}_{\boldsymbol{\varphi}_h},\boldsymbol{\psi}_h) + \widetilde{b}(\boldsymbol{\psi}_h,\mathsf{e}_{p_h}) + \widetilde{b}(\mathsf{e}_{\boldsymbol{\varphi}_h},q_h) - \widetilde{c}(\mathsf{e}_{p_h},q_h) = -\int_\Omega \left(\kappa(\boldsymbol{\sigma},p)^{-1} - \kappa(\boldsymbol{\sigma}_h,p_h)^{-1}\right)\boldsymbol{\varphi}\cdot\boldsymbol{\psi}_h + c(\mathsf{e}_{\boldsymbol{\sigma}},q_h),$$

which together with (5.1), implies that

$$\begin{aligned}
&\widetilde{a}_{\boldsymbol{\sigma}_h,p_h}(\boldsymbol{\chi}_{\boldsymbol{\varphi}},\boldsymbol{\psi}_h) + \widetilde{b}(\boldsymbol{\psi}_h,\chi_{p_h}) + \widetilde{b}(\boldsymbol{\chi}_{\boldsymbol{\varphi}},q_h) - \widetilde{c}(\chi_p,q_h) + \widetilde{a}_{\boldsymbol{\sigma}_h,p_h}(\boldsymbol{\xi}_{\boldsymbol{\varphi}},\boldsymbol{\psi}_h) \\
&= -\widetilde{b}(\boldsymbol{\psi}_h,\xi_p) - \widetilde{b}(\boldsymbol{\xi}_{\boldsymbol{\varphi}},q_h) + \widetilde{c}(\xi_p,q_h) - \int_\Omega \left(\kappa(\boldsymbol{\sigma},p)^{-1} - \kappa(\boldsymbol{\sigma}_h,p_h)^{-1}\right)\boldsymbol{\varphi}\cdot\boldsymbol{\psi}_h + c(\mathsf{e}_{\boldsymbol{\sigma}},q_h).
\end{aligned} \tag{5.3}$$

5.2. Derivation of Céa estimates.

LEMMA 5.1. *There exist C_3^*, $C_4^* > 0$, independent of h, such that*

$$\|\boldsymbol{\chi}_{\boldsymbol{\sigma}}\|_{\mathbf{div};\Omega} + \|\boldsymbol{\chi}_{\boldsymbol{u}}\|_{0,\Omega} + \|\boldsymbol{\chi}_{\boldsymbol{\rho}}\|_{0,\Omega} \le C_3^*(\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\|_{\mathbf{div};\Omega} + \|\boldsymbol{\xi}_{\boldsymbol{u}}\|_{0,\Omega} + \|\boldsymbol{\xi}_{\boldsymbol{\rho}}\|_{0,\Omega} + \|\xi_p\|_{0,\Omega}) + C_4^*\gamma\|\chi_p\|_{0,\Omega}. \tag{5.4}$$

Proof. From the properties of a and b (refer to (4.2b) and (4.2a)), and [29, Proposition 2.36], we derive the following discrete global inf-sup condition

$$\|\boldsymbol{\chi}_{\boldsymbol{\sigma}}\|_{\mathbf{div};\Omega} + \|\boldsymbol{\chi}_{\boldsymbol{u}}\|_{0,\Omega} + \|\boldsymbol{\chi}_{\boldsymbol{\rho}}\|_{0,\Omega} \le (C_1^* + C_2^*) \sup_{\mathbf{0}\neq(\boldsymbol{\tau}_h,\boldsymbol{v}_h,\boldsymbol{\psi}_h)\in\mathbb{H}_h^{\boldsymbol{\sigma}}\times\mathbf{H}_h^{\boldsymbol{u}}\times\mathbb{H}_h^{\boldsymbol{\rho}}} \frac{a(\boldsymbol{\chi}_{\boldsymbol{\sigma}},\boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h,(\boldsymbol{\chi}_{\boldsymbol{u}},\boldsymbol{\chi}_{\boldsymbol{\rho}})) + b(\boldsymbol{\chi}_{\boldsymbol{\sigma}},(\boldsymbol{v}_h,\boldsymbol{\eta}_h))}{\|\boldsymbol{\tau}_h\|_{\mathbf{div};\Omega} + \|\boldsymbol{v}_h\|_{0,\Omega} + \|\boldsymbol{\eta}_h\|_{0,\Omega}},$$

with C_1^*, $C_2^* > 0$ independent of h, are the discrete version of the constants C_1, C_2 defined in (3.8). Then, combining the last inequality with (5.2), and the continuity properties of a and b (see (2.4a)), we obtain

$$\begin{aligned}
&\|\boldsymbol{\chi}_{\boldsymbol{\sigma}}\|_{\mathbf{div};\Omega} + \|\boldsymbol{\chi}_{\boldsymbol{u}}\|_{0,\Omega} + \|\boldsymbol{\chi}_{\boldsymbol{\rho}}\|_{0,\Omega} \\
&\le (C_1^* + C_2^*)(\frac{1}{\mu}\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\|_{\mathbf{div};\Omega} + \|\boldsymbol{\xi}_{\boldsymbol{u}}\|_{0,\Omega} + \|\boldsymbol{\xi}_{\boldsymbol{\rho}}\|_{0,\Omega} + \|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\|_{\mathbf{div};\Omega} + \gamma\|\chi_p\|_{0,\Omega} + \gamma\|\xi_p\|_{0,\Omega}),
\end{aligned}$$

which implies (5.4) with $C_3^* := (C_1^* + C_2^*)(\frac{1}{\mu} + 1 + \gamma)$ and $C_4^* := C_1^* + C_2^*$, and concludes the proof. $\square$

LEMMA 5.2. *There exist $\widetilde{C}_5^*$, $\widetilde{C}_6^* > 0$, independent of h, such that*

$$\begin{aligned}
\|\boldsymbol{\chi}_{\boldsymbol{\varphi}}\|_{\mathrm{div};\Omega} + \|\chi_p\|_{0,\Omega} &\le \widetilde{C}_5^* \left(\|\boldsymbol{\xi}_{\boldsymbol{\varphi}}\|_{\mathrm{div};\Omega} + \|\xi_p\|_{0,\Omega} + \|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\|_{\mathbf{div};\Omega}\right) \\
&\quad + \widetilde{C}_6^*\left((\|g\|_{0,\Omega} + \|p_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} + \|\boldsymbol{u}_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} + \|\boldsymbol{f}\|_{0,\Omega} + \gamma\,r)\|\chi_p\|_{0,\Omega} + \gamma\|\boldsymbol{\chi}_{\boldsymbol{\sigma}}\|_{\mathbf{div};\Omega}\right).
\end{aligned} \tag{5.5}$$

Proof. Similarly to Lemma 5.1, using the properties of $\widetilde{a}$, $\widetilde{b}$ and $\widetilde{c}$ (refer to (4.7), (4.5) and (2.9)), and [26, Th. 3.4], we derive the following discrete global inf-sup condition

$$\|\boldsymbol{\chi}_{\boldsymbol{\varphi}}\|_{\mathrm{div};\Omega} + \|\chi_p\|_{0,\Omega} \le 2\widetilde{C}^* \sup_{\mathbf{0}\neq(\boldsymbol{\psi}_h,q_h)\in\mathbf{H}_h^{\boldsymbol{\varphi}}\times\mathrm{H}_h^p} \frac{\widetilde{a}_{\boldsymbol{\sigma}_h,p_h}(\boldsymbol{\chi}_{\boldsymbol{\varphi}},\boldsymbol{\psi}_h) + b(\boldsymbol{\psi}_h,\chi_p) + b(\boldsymbol{\chi}_{\boldsymbol{\varphi}},q_h) - \widetilde{c}(\chi_p,q_h)}{\|\boldsymbol{\psi}_h\|_{\mathrm{div};\Omega} + \|q_h\|_{0,\Omega}},$$

with $\widetilde{C}^*$ defined as in (4.11). Then, using the equation (5.3), the bound properties of $\widetilde{a}$, $\widetilde{b}$ and $\widetilde{c}$ (see (3.2a) and (2.4c)), and the second assumption for κ (cf. (2.3)), we obtain

$$\begin{aligned}
&\|\boldsymbol{\chi}_{\boldsymbol{\varphi}}\|_{\mathrm{div};\Omega} + \|\chi_p\|_{0,\Omega} \\
&\le 2\widetilde{C}^*(C_{\widetilde{a}}\|\boldsymbol{\xi}_{\boldsymbol{\varphi}}\|_{\mathrm{div};\Omega} + \|\xi_p\|_{0,\Omega} + \|\boldsymbol{\xi}_{\boldsymbol{\varphi}}\|_{\mathrm{div};\Omega} + \widetilde{\gamma}\|\xi_p\|_{0,\Omega} + \|\kappa(\boldsymbol{\sigma},p)^{-1} - \kappa(\boldsymbol{\sigma}_h,p_h)^{-1}\|_{\mathbb{L}^\infty(\Omega)}\|\boldsymbol{\varphi}\|_{\mathrm{div};\Omega} + \gamma\|\mathsf{e}_{\boldsymbol{\sigma}}\|_{\mathbf{div};\Omega}) \\
&\le 2\widetilde{C}^*(C_{\widetilde{a}}\|\boldsymbol{\xi}_{\boldsymbol{\varphi}}\|_{\mathrm{div};\Omega} + \|\xi_p\|_{0,\Omega} + \|\boldsymbol{\xi}_{\boldsymbol{\varphi}}\|_{\mathrm{div};\Omega} + \widetilde{\gamma}\|\xi_p\|_{0,\Omega} + \kappa_2\|\mathsf{e}_p\|_{0,\Omega}\|\boldsymbol{\varphi}\|_{\mathrm{div};\Omega} + \gamma\|\mathsf{e}_{\boldsymbol{\sigma}}\|_{\mathbf{div};\Omega}),
\end{aligned}$$

hence, using the fact that $\boldsymbol{\varphi}$ satisfies (3.16) and the error decomposition (5.1), we have

$$\|\boldsymbol{\chi}_{\boldsymbol{\varphi}}\|_{\mathrm{div};\Omega} + \|\chi_p\|_{0,\Omega} \le 2\widetilde{C}^*(C_{\widetilde{a}}\|\boldsymbol{\xi}_{\boldsymbol{\varphi}}\|_{\mathrm{div};\Omega} + \|\xi_p\|_{0,\Omega} + \|\boldsymbol{\xi}_{\boldsymbol{\varphi}}\|_{\mathrm{div};\Omega} + \widetilde{\gamma}\|\xi_p\|_{0,\Omega} + \kappa_2\|\xi_p\|_{0,\Omega}\|\boldsymbol{\varphi}\|_{\mathrm{div};\Omega} + \gamma\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\|_{\mathbf{div};\Omega}$$

$$+ 2\widetilde{C}^*(\kappa_2 \|\chi_p\|_{0,\Omega} \|\boldsymbol{\varphi}\|_{\mathbf{div};\Omega} + \gamma \|\boldsymbol{\chi_\sigma}\|_{\mathbf{div};\Omega})$$

$$\leq \widetilde{C}_5^* (\|\boldsymbol{\xi_\varphi}\|_{\mathrm{div};\Omega} + \|\xi_p\|_{0,\Omega} + \|\boldsymbol{\xi_\sigma}\|_{\mathbf{div};\Omega})$$

$$+ 2\widetilde{C}^*(\kappa_2 C(\|g\|_{0,\Omega} + \|p_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} + \|\boldsymbol{u}_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} + \|\boldsymbol{f}\|_{0,\Omega} + \gamma\, r)\|\chi_p\|_{0,\Omega} + \gamma\|\boldsymbol{\chi_\sigma}\|_{\mathbf{div};\Omega}),$$

the last equation implies (5.5), with $\widetilde{C}_5^* := 2\widetilde{C}^*\big(C_{\widetilde{a}} + 1 + \widetilde{\gamma} + \gamma + \kappa_2 C(\|g\|_{0,\Omega} + \|p_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} + \|\boldsymbol{u}_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} + \|\boldsymbol{f}\|_{0,\Omega} + \gamma\, r)\big)$ and $\widetilde{C}_6^* := 2\widetilde{C}^*(\kappa_2 C + 1)$, and concludes the proof.   □

THEOREM 5.3. *Assume that*

$$(C_4^* + \widetilde{C}_6^* + \widetilde{C}_6^* r)\gamma + \widetilde{C}_6^*(\|g\|_{0,\Omega} + \|p_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} + \|\boldsymbol{u}_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} + \|\boldsymbol{f}\|_{0,\Omega}) \leq \frac{1}{2}, \tag{5.6}$$

with C_4^* and $\widetilde{C}_6^*$ being the constants in Lemmas 5.1 and 5.2. Furthermore, assume that the hypotheses of Theorem 3.6 and Theorem 4.1 hold. Let $(\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{\rho}, \boldsymbol{\varphi}, p) \in \mathbb{H}_{\mathrm{N}}(\mathbf{div};\Omega) \times \mathbf{L}^2(\Omega) \times \mathbb{L}_{\mathrm{skew}}^2(\Omega) \times \mathbf{H}_{\mathrm{N}}(\mathrm{div};\Omega) \times \mathrm{L}^2(\Omega)$ and $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \boldsymbol{\rho}_h, \boldsymbol{\varphi}_h, p_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}} \times \mathbf{H}_h^{\boldsymbol{\varphi}} \times \mathrm{H}_h^p$ be the unique solutions of (2.2) and (4.6), respectively. Then, there exists $C_{\mathrm{Céa}} > 0$, independent of h, such that

$$\|\mathrm{e}_{\boldsymbol{\sigma}}\|_{\mathbf{div};\Omega} + \|\mathrm{e}_{\boldsymbol{u}}\|_{0,\Omega} + \|\mathrm{e}_{\boldsymbol{\rho}}\|_{0,\Omega} + \|\mathrm{e}_{\boldsymbol{\varphi}}\|_{\mathrm{div};\Omega} + \|\mathrm{e}_p\|_{0,\Omega}$$
$$\leq C_{\mathrm{Céa}} \, \mathrm{dist}\,\big((\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{\rho}, \boldsymbol{\varphi}, p), \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}} \times \mathbf{H}_h^{\boldsymbol{\varphi}} \times \mathrm{H}_h^p\big). \tag{5.7}$$

Proof. Combining (5.4) and (5.5), and using the assumption (5.6), we deduce

$$\|\boldsymbol{\chi_\sigma}\|_{\mathbf{div};\Omega} + \|\boldsymbol{\chi_u}\|_{0,\Omega} + \|\boldsymbol{\chi_\rho}\|_{0,\Omega} + \|\boldsymbol{\chi_\varphi}\|_{\mathrm{div};\Omega} + \|\chi_p\|_{0,\Omega} \leq C_3^*(\|\boldsymbol{\xi_\sigma}\|_{\mathbf{div};\Omega} + \|\boldsymbol{\xi_u}\|_{0,\Omega} + \|\boldsymbol{\xi_\rho}\|_{0,\Omega} + \|\xi_p\|_{0,\Omega})$$

$$+ \widetilde{C}_5^*(\|\boldsymbol{\xi_\varphi}\|_{\mathrm{div};\Omega} + \|\xi_p\|_{0,\Omega} + \|\boldsymbol{\xi_\sigma}\|_{\mathbf{div};\Omega}) + \frac{1}{2}\|\chi_p\|_{0,\Omega} + \frac{1}{2}\|\boldsymbol{\chi_\sigma}\|_{\mathbf{div};\Omega}.$$

And from the latter inequality we obtain

$$\|\boldsymbol{\chi_\sigma}\|_{\mathbf{div};\Omega} + \|\boldsymbol{\chi_u}\|_{0,\Omega} + \|\boldsymbol{\chi_\rho}\|_{0,\Omega} + \|\boldsymbol{\chi_\varphi}\|_{\mathrm{div};\Omega} + \|\chi_p\|_{0,\Omega}$$
$$\leq 2(C_3^* + \widetilde{C}_5^*)(\|\boldsymbol{\xi_\sigma}\|_{\mathbf{div};\Omega} + \|\boldsymbol{\xi_u}\|_{0,\Omega} + \|\boldsymbol{\xi_\rho}\|_{0,\Omega} + \|\boldsymbol{\xi_\varphi}\|_{\mathrm{div};\Omega} + \|\xi_p\|_{0,\Omega}). \tag{5.8}$$

In this way, from (5.1), (5.8) and the triangle inequality we obtain

$$\|\mathrm{e}_{\boldsymbol{\sigma}}\|_{\mathbf{div};\Omega} + \|\mathrm{e}_{\boldsymbol{u}}\|_{0,\Omega} + \|\mathrm{e}_{\boldsymbol{\rho}}\|_{0,\Omega} + \|\mathrm{e}_{\boldsymbol{\varphi}}\|_{\mathrm{div};\Omega} + \|\mathrm{e}_p\|_{0,\Omega} \leq \|\boldsymbol{\chi_\sigma}\|_{\mathbf{div};\Omega} + \|\boldsymbol{\xi_\sigma}\|_{\mathbf{div};\Omega} + \|\boldsymbol{\chi_u}\|_{0,\Omega} + \|\boldsymbol{\xi_u}\|_{0,\Omega}$$

$$+ \|\boldsymbol{\chi_\rho}\|_{0,\Omega} + \|\boldsymbol{\xi_\rho}\|_{0,\Omega} + \|\boldsymbol{\chi_\varphi}\|_{\mathrm{div};\Omega} + \|\boldsymbol{\xi_\varphi}\|_{\mathrm{div};\Omega} + \|\chi_p\|_{0,\Omega} + \|\xi_p\|_{0,\Omega}$$

$$\leq (2C_3^* + 2\widetilde{C}_5^* + 1)(\|\boldsymbol{\xi_\sigma}\|_{\mathbf{div};\Omega} + \|\boldsymbol{\xi_u}\|_{0,\Omega} + \|\boldsymbol{\xi_\rho}\|_{0,\Omega} + \|\boldsymbol{\xi_\varphi}\|_{\mathrm{div};\Omega} + \|\xi_p\|_{0,\Omega}),$$

which combined with the fact that $(\widehat{\boldsymbol{\tau}}_h, (\widehat{\boldsymbol{v}}_h, \widehat{\boldsymbol{\eta}}_h)) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times (\mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}})$ and $(\widehat{\boldsymbol{\psi}}_h, \widehat{q}_h) \in \mathbf{H}_h^{\boldsymbol{\varphi}} \times \mathrm{H}_h^p$ are arbitrary (see (5.1)), concludes the proof.   □

5.3. Rates of convergence. In order to establish the rate of convergence of the Galerkin scheme (4.6), we first recall the following approximation properties associated with the FE spaces specified in Section 4.1.

For each $0 < m \leq k + 1$ and for each $\boldsymbol{\tau} \in \mathbb{H}^m(\Omega) \cap \mathbb{H}_{\mathrm{N}}(\mathbf{div};\Omega)$ with $\mathbf{div}\,\boldsymbol{\tau} \in \mathbf{H}^m(\Omega)$, $\boldsymbol{v} \in \mathbf{H}^m(\Omega)$, $\boldsymbol{\eta} \in \mathbb{H}^m(\Omega) \cap \mathbb{L}_{\mathrm{skew}}^2(\Omega)$, $\boldsymbol{\psi} \in \mathbf{H}^m(\Omega) \cap \mathbf{H}_{\mathrm{N}}(\mathrm{div};\Omega)$ with $\mathrm{div}\,\boldsymbol{v} \in \mathrm{H}^m(\Omega)$, and $q \in \mathrm{H}^m(\Omega)$, there holds

$$\mathrm{dist}\,\big(\boldsymbol{\tau}, \mathbb{H}_h^{\boldsymbol{\sigma}}\big) := \inf_{\boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}} \|\boldsymbol{\tau} - \boldsymbol{\tau}_h\|_{\mathbf{div};\Omega} \lesssim h^m \big\{\|\boldsymbol{\tau}\|_{m,\Omega} + \|\mathbf{div}\,\boldsymbol{\tau}\|_{m,\Omega}\big\}, \tag{5.9a}$$

$$\mathrm{dist}\,\big(\boldsymbol{v}, \mathbf{H}_h^{\boldsymbol{u}}\big) := \inf_{\boldsymbol{v}_h \in \mathbf{H}_h^{\boldsymbol{u}}} \|\boldsymbol{v} - \boldsymbol{v}_h\|_{0,\Omega} \lesssim h^m \|\boldsymbol{v}\|_{m,\Omega}, \tag{5.9b}$$

$$\mathrm{dist}\,\big(\boldsymbol{\eta}, \mathbb{H}_h^{\boldsymbol{\rho}}\big) := \inf_{\boldsymbol{\eta}_h \in \mathbb{H}_h^{\boldsymbol{\rho}}} \|\boldsymbol{\eta} - \boldsymbol{\eta}_h\|_{0,\Omega} \lesssim h^m \|\boldsymbol{\eta}\|_{m,\Omega}, \tag{5.9c}$$

$$\mathrm{dist}\,\big(\boldsymbol{\psi}, \mathbf{H}_h^{\boldsymbol{\varphi}}\big) := \inf_{\boldsymbol{\psi}_h \in \mathbf{H}_h^{\boldsymbol{\varphi}}} \|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{\mathrm{div};\Omega} \lesssim h^m \big\{\|\boldsymbol{\psi}\|_{m,\Omega} + \|\mathrm{div}\,\boldsymbol{\psi}\|_{m,\Omega}\big\}, \tag{5.9d}$$

$$\mathrm{dist}\,\big(q, \mathrm{H}_h^p\big) := \inf_{q_h \in \mathrm{H}_h^p} \|q - q_h\|_{0,\Omega} \lesssim h^m \|q\|_{m,\Omega}. \tag{5.9e}$$

For (5.9a), (5.9b) and (5.9c) we refer to [33, Th. 2.4], whereas (5.9d) and (5.9e) are provided in [30, Th. 3.6] and [29, Proposition 1.134], respectively. With these steps we are now in position to state the rates of convergence associated with the Galerkin scheme (4.6).

THEOREM 5.4. *Assume that the hypotheses of Theorem 5.3 hold and let $(\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{\rho}, \boldsymbol{\varphi}, p) \in \mathbb{H}_{\mathrm{N}}(\mathbf{div};\Omega) \times \mathbf{L}^2(\Omega) \times \mathbb{L}_{\mathrm{skew}}^2(\Omega) \times \mathbf{H}_{\mathrm{N}}(\mathrm{div};\Omega) \times \mathrm{L}^2(\Omega)$ and $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \boldsymbol{\rho}_h, \boldsymbol{\varphi}_h, p_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\boldsymbol{u}} \times \mathbb{H}_h^{\boldsymbol{\rho}} \times \mathbf{H}_h^{\boldsymbol{\varphi}} \times \mathrm{H}_h^p$ be the unique solutions of the continuous and discrete problems (2.2) and (4.6), respectively. Assume further that $\boldsymbol{\sigma} \in \mathbb{H}^m(\Omega)$, $\mathbf{div}\,\boldsymbol{\sigma} \in \mathbf{H}^m(\Omega)$,*

$\boldsymbol{u} \in \mathbf{H}^m(\Omega)$, $\boldsymbol{\rho} \in \mathbb{H}^m(\Omega)$, $\boldsymbol{\varphi} \in \mathbf{H}^m(\Omega)$, $\mathrm{div}\,\boldsymbol{\varphi} \in \mathrm{H}^m(\Omega)$ and $p \in \mathrm{H}^m(\Omega)$, for $1 \leq m \leq k+1$. Then there exists $C_{\mathrm{rate}} > 0$, independent of h, such that

$$\|e_{\boldsymbol{\sigma}}\|_{\mathbf{div};\Omega} + \|e_{\boldsymbol{u}}\|_{0,\Omega} + \|e_{\boldsymbol{\rho}}\|_{0,\Omega} + \|e_{\boldsymbol{\varphi}}\|_{\mathrm{div};\Omega} + \|e_p\|_{0,\Omega}$$
$$\leq C_{\mathrm{rate}}\, h^m \Big\{ \|\boldsymbol{\sigma}\|_{m,\Omega} + \|\mathbf{div}\,\boldsymbol{\sigma}\|_{m,\Omega} + \|\boldsymbol{u}\|_{m,\Omega} + \|\boldsymbol{\rho}\|_{m,\Omega} + \|\boldsymbol{\varphi}\|_{m,\Omega} + \|\mathrm{div}\,\boldsymbol{\varphi}\|_{m,\Omega} + \|p\|_{m,\Omega} \Big\}.$$

Proof. The result follows from Céa estimate (5.7) and the approximation properties (5.9). □

REMARK 5.1. *Similarly to [38], the analysis developed in Sections 2-5 can be adapted to a formulation without the variable $\boldsymbol{\rho}$ ($\boldsymbol{\rho}_h$ in the discrete problem), imposing symmetry of $\boldsymbol{\sigma}$ by taking $\boldsymbol{\sigma} \in \mathbb{H}_{\mathrm{sym}}(\mathbf{div};\Omega) := \big\{ \boldsymbol{\tau} \in \mathbb{L}^2_{\mathrm{sym}}(\Omega) : \mathbf{div}\,\boldsymbol{\tau} \in \mathbf{L}^2(\Omega) \big\}$ and $\mathbb{L}^2_{\mathrm{sym}}(\Omega) := \{ \boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \boldsymbol{\tau} = \boldsymbol{\tau}^{\mathtt{t}} \}$, utilizing results from [38, Section 2.2] ([38, Section 4.1] for the discrete problem), and adapting the strategy used in, e.g., [38, Sections 3 and 4].*

6. A posteriori error estimates. In this section we derive residual-based a posteriori error estimates and demonstrate the reliability and efficiency of the proposed estimators. Mainly due to notational convenience, we confine our analysis to the two-dimensional case. The extension to three-dimensional case should be quite straightforward (see, e.g., [22]). Similarly to [17, Section 4], we introduce additional notation. Let $\mathcal{E}_h$ be the set of edges of $\mathcal{T}_h$, whose corresponding diameters are denoted h_E, and define

$$\mathcal{E}_h(\Omega) := \big\{ E \in \mathcal{E}_h : E \subseteq \Omega \big\}, \quad \mathcal{E}_h(\Gamma_{\mathrm{D}}) := \big\{ E \in \mathcal{E}_h : E \subseteq \Gamma_{\mathrm{D}} \big\}, \quad \text{and} \quad \mathcal{E}_h(\Gamma_{\mathrm{N}}) := \big\{ E \in \mathcal{E}_h : E \subseteq \Gamma_{\mathrm{N}} \big\}.$$

On each $E \in \mathcal{E}_h$, we also define the unit normal vector $\boldsymbol{n}_E := (n_1, n_2)^{\mathtt{t}}$ and the tangential vector $\boldsymbol{s}_E := (-n_2, n_1)^{\mathtt{t}}$. However, when no confusion arises, we will simply write $\boldsymbol{n}$ and $\boldsymbol{s}$ instead of $\boldsymbol{n}_E$ and $\boldsymbol{s}_E$, respectively. Also, by $\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{s}}$ we denote the tangential derivative. The usual jump operator $[\![\cdot]\!]$ across internal edges are defined for piecewise continuous matrix, vector, or scalar-valued functions. For sufficiently smooth scalar ψ, vector $\boldsymbol{v} := (v_1, v_2)^{\mathtt{t}}$, and tensor fields $\boldsymbol{\tau} := (\tau_{ij})_{1 \leq i,j \leq 2}$, we let

$$\mathrm{curl}\,(\psi) := \Big(\frac{\partial \psi}{\partial x_2}, -\frac{\partial \psi}{\partial x_1} \Big)^{\mathtt{t}}, \quad \mathrm{rot}\,(\boldsymbol{v}) := \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}, \quad \mathbf{curl}\,(\boldsymbol{v}) = \begin{pmatrix} \mathrm{curl}\,(v_1)^{\mathtt{t}} \\ \mathrm{curl}\,(v_2)^{\mathtt{t}} \end{pmatrix} \quad \text{and} \quad \underline{\mathbf{curl}}(\boldsymbol{\tau}) = \begin{pmatrix} \mathrm{rot}\,(\boldsymbol{\tau}_1) \\ \mathrm{rot}\,(\boldsymbol{\tau}_2) \end{pmatrix}.$$

In addition, we denote by Π_h the Raviart–Thomas interpolator and by I_h the Clément operator (see, e.g., [3, Section 3] for their properties). In what follows, we denote by $\boldsymbol{\Pi}_h$ the tensor version of Π_h, which is defined row-wise by Π_h and by $\mathbf{I}_h$ the corresponding vector version of I_h which is defined componentwise by I_h.

In what follows, we will assume that the hypotheses of Theorems 3.6 and 4.1 are satisfied. Let $\boldsymbol{\sigma}_h$, $\boldsymbol{u}_h$, $\boldsymbol{\rho}_h$, $\boldsymbol{\varphi}_h$, p_h denote the FE solutions of (4.6). We define the residual-based and fully computable local contributions to the error estimator Ξ^2_K, defined as the sum of $\Xi^2_{s,K}$ and $\Xi^2_{f,K}$, where $\Xi_{s,K}$ and $\Xi_{f,K}$ pertain to the solid (mixed elasticity) and fluid (mixed Darcy) components, respectively:

$$\Xi^2_{s,K} := \|\boldsymbol{f} + \mathbf{div}\,\boldsymbol{\sigma}_h\|^2_{0,K} + \|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^{\mathtt{t}}\|^2_{0,K} + h_K^2 \|\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \frac{\alpha}{d\lambda+2\mu}p_h\mathbb{I} - \boldsymbol{\nabla}\boldsymbol{u}_h + \boldsymbol{\rho}_h\|^2_{0,K}$$
$$+ h_K^2 \|\underline{\mathbf{curl}}(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \frac{\alpha}{d\lambda+2\mu}p_h\mathbb{I} + \boldsymbol{\rho}_h)\|^2_{0,K} + \sum_{E \in \partial K \cap \mathcal{E}_h(\Omega)} h_E \|[\![(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \frac{\alpha}{d\lambda+2\mu}p_h\mathbb{I} + \boldsymbol{\rho}_h)\boldsymbol{s}]\!]\|^2_{0,E}$$
$$+ \sum_{E \in \partial K \cap \mathcal{E}_h(\Gamma_{\mathrm{D}})} h_E \|(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \frac{\alpha}{d\lambda+2\mu}p_h\mathbb{I} + \boldsymbol{\rho}_h)\boldsymbol{s} - \frac{\mathrm{d}\boldsymbol{u}_{\mathrm{D}}}{\mathrm{d}\boldsymbol{s}}\|^2_{0,E} + \sum_{E \in \partial K \cap \mathcal{E}_h(\Gamma_{\mathrm{D}})} h_E \|\boldsymbol{u}_{\mathrm{D}} - \boldsymbol{u}_h\|^2_{0,E}, \quad (6.1a)$$

$$\Xi^2_{f,K} := \| - g + c_0 p_h + \frac{\alpha}{d\lambda+2\mu}\mathrm{tr}\,\boldsymbol{\sigma}_h + \frac{d\alpha^2}{d\lambda+2\mu}p_h - \mathrm{div}\,\boldsymbol{\varphi}_h\|^2_{0,K} + h_K^2 \|\kappa(\boldsymbol{\sigma}_h, p_h)^{-1}\boldsymbol{\varphi}_h - \nabla p_h\|^2_{0,K}$$
$$+ h_K^2 \|\mathrm{rot}\,(\kappa(\boldsymbol{\sigma}_h, p_h)^{-1}\boldsymbol{\varphi}_h)\|^2_{0,K} + \sum_{E \in \partial K \cap \mathcal{E}_h(\Omega)} h_E \|[\![(\kappa(\boldsymbol{\sigma}_h, p_h)^{-1}\boldsymbol{\varphi}_h) \cdot \boldsymbol{s}]\!]\|^2_{0,E}$$
$$+ \sum_{E \in \partial K \cap \mathcal{E}_h(\Gamma_{\mathrm{D}})} h_E \|\kappa(\boldsymbol{\sigma}_h, p_h)^{-1}\boldsymbol{\varphi}_h \cdot \boldsymbol{s} - \frac{\mathrm{d}p_{\mathrm{D}}}{\mathrm{d}\boldsymbol{s}}\|^2_{0,E} + \sum_{E \in \partial K \cap \mathcal{E}_h(\Gamma_{\mathrm{D}})} h_E \|p_{\mathrm{D}} - p_h\|^2_{0,E}. \quad (6.1b)$$

Then, we define the global estimator

$$\Xi^2 := \sum_{K \in \mathcal{T}_h} \Xi^2_{s,K} + \Xi^2_{f,K}. \quad (6.2)$$

6.1. Reliability of the a posteriori error estimator. First we prove preliminary results that will be key in showing the reliability of the global estimator.

LEMMA 6.1. *There exists $C_1 > 0$, such that*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div};\Omega} + \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega} + \|\boldsymbol{\rho} - \boldsymbol{\rho}_h\|_{0,\Omega} \leq C_1 \big(\|\mathcal{R}_1\| + \|\boldsymbol{f} + \mathbf{div}(\boldsymbol{\sigma}_h)\|_{0,\Omega} + \|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^t\|_{0,\Omega} \big),$$

where

$$\mathcal{R}_1(\boldsymbol{\tau}) := a(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, (\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{\rho} - \boldsymbol{\rho}_h)), \tag{6.3}$$

with $\mathcal{R}_1(\boldsymbol{\tau}_h) = -c(\boldsymbol{\tau}_h, p - p_h)$ *for all* $\boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}$, *and* $||\mathcal{R}_1|| = \sup\limits_{\boldsymbol{0} \neq \boldsymbol{\tau} \in \mathbb{H}_{\mathrm{N}}(\mathbf{div};\Omega)} \dfrac{\mathcal{R}_1(\boldsymbol{\tau})}{||\boldsymbol{\tau}||_{\mathbf{div};\Omega}}.$

Proof. Using the properties of bilinear forms a and b, as outlined in equations (2.7) and (2.8a), along with the insight from [29, Proposition 2.36], there exists $C_1 > 0$ depending on μ, c_a, β such that

$$\|\mathsf{e}_{\boldsymbol{\sigma}}\|_{\mathbf{div};\Omega} + \|\mathsf{e}_{\boldsymbol{u}}\|_{0,\Omega} + \|\mathsf{e}_{\boldsymbol{\rho}}\|_{0,\Omega} \leq C_1 \sup_{\substack{\boldsymbol{0} \neq (\boldsymbol{\tau}, \boldsymbol{v}, \boldsymbol{\eta}) \\ \in \mathbb{H}_{\mathrm{N}}(\mathbf{div};\Omega) \times \mathbf{L}^2(\Omega) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega)}} \frac{a(\mathsf{e}_{\boldsymbol{\sigma}}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, (\mathsf{e}_{\boldsymbol{u}}, \mathsf{e}_{\boldsymbol{\rho}})) + b(\mathsf{e}_{\boldsymbol{\sigma}}, (\boldsymbol{v}, \boldsymbol{\eta}))}{\|\boldsymbol{\tau}\|_{\mathbf{div};\Omega} + \|\boldsymbol{v}\|_{0,\Omega} + \|\boldsymbol{\eta}\|_{0,\Omega}}$$

$$\leq C_1 \left(\sup_{\boldsymbol{0} \neq \boldsymbol{\tau} \in \mathbb{H}_{\mathrm{N}}(\mathbf{div};\Omega)} \frac{\mathcal{R}_1(\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\mathbf{div};\Omega}} + \sup_{\boldsymbol{0} \neq (\boldsymbol{v}, \boldsymbol{\eta}) \in \mathbf{L}^2(\Omega) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega)} \frac{b(\mathsf{e}_{\boldsymbol{\sigma}}, (\boldsymbol{v}, \boldsymbol{\eta}))}{\|\boldsymbol{v}\|_{0,\Omega} + \|\boldsymbol{\eta}\|_{0,\Omega}} \right).$$

Then, recalling the definitions of the bilinear form b (cf. (2.1)), using the equation (2.2b), along with the fact that $\int_\Omega \boldsymbol{\sigma}_h : \boldsymbol{\eta} = \frac{1}{2} \int_\Omega (\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^{\mathsf{t}}) : \boldsymbol{\eta}$ for $\boldsymbol{\eta} \in \mathbb{L}^2_{\mathrm{skew}}(\Omega)$, the following estimate holds

$$|b(\mathsf{e}_{\boldsymbol{\sigma}}, (\boldsymbol{v}, \boldsymbol{\eta}))| \leq (\|\boldsymbol{f} + \mathbf{div}\,\boldsymbol{\sigma}_h\|_{0,\Omega} + \|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^{\mathsf{t}}\|_{0,\Omega})(\|\boldsymbol{v}\|_{0,\Omega} + \|\boldsymbol{\eta}\|_{0,\Omega}),$$

and this gives the asserted inequality. $\quad\square$

LEMMA 6.2. *There exists* $C_2 > 0$ *such that*

$$\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{\mathrm{div};\Omega} + \|p - p_h\|_{0,\Omega} \leq C_2 \Big(\|\mathcal{R}_2\| + \|g - c_0 p_h - \frac{\alpha}{d\lambda + 2\mu} \mathrm{tr}\,\boldsymbol{\sigma}_h - \frac{d\alpha^2}{d\lambda + 2\mu} p_h + \mathrm{div}\,\boldsymbol{\varphi}_h\|_{0,\Omega} + \gamma \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div};\Omega} \Big),$$

where

$$\mathcal{R}_2(\boldsymbol{\psi}) := \widetilde{a}_{\boldsymbol{\sigma},p}(\boldsymbol{\varphi} - \boldsymbol{\varphi}_h, \boldsymbol{\psi}) + \widetilde{b}(\boldsymbol{\psi}, p - p_h), \tag{6.4}$$

satisfies $\mathcal{R}_2(\boldsymbol{\psi}_h) = 0$ *for all* $\boldsymbol{\psi}_h \in \mathbf{H}_h^{\boldsymbol{\varphi}}$, *and* $||\mathcal{R}_2|| = \sup\limits_{\boldsymbol{0} \neq \boldsymbol{\psi} \in \mathbf{H}_{\mathrm{N}}(\mathrm{div};\Omega)} \dfrac{\mathcal{R}_2(\boldsymbol{\psi})}{\|\boldsymbol{\psi}\|_{\mathrm{div};\Omega}}.$

Proof. Similarly to Lemma 5.1, using the properties of bilinear forms $\widetilde{a}_{\boldsymbol{\sigma},p}$ and $\widetilde{b}$ (as outlined in equations (3.2b), (2.8b) and (2.9)), along with the insight from [26, Th. 3.4], we establish that there exists $C_2 > 0$ depending on κ_1, κ_2, $C_{\widetilde{a}}$, γ, $\widetilde{\gamma}$, $\widetilde{\beta}$ such that

$$\|\mathsf{e}_{\boldsymbol{\varphi}}\|_{\mathrm{div};\Omega} + \|\mathsf{e}_p\|_{0,\Omega} \leq C_2 \sup_{\boldsymbol{0} \neq (\boldsymbol{\psi}, q) \in \mathbf{H}_{\mathrm{N}}(\mathrm{div};\Omega) \times \mathrm{L}^2(\Omega)} \frac{\widetilde{a}_{\boldsymbol{\sigma},p}(\mathsf{e}_{\boldsymbol{\varphi}}, \boldsymbol{\psi}) + \widetilde{b}(\boldsymbol{\psi}, \mathsf{e}_p) + \widetilde{b}(\mathsf{e}_{\boldsymbol{\varphi}}, q) - \widetilde{c}(\mathsf{e}_p, q)}{\|\boldsymbol{\psi}\|_{\mathrm{div};\Omega} + \|q\|_{0,\Omega}}$$

$$\leq C_2 \left(\sup_{\boldsymbol{0} \neq \boldsymbol{\psi} \in \mathbf{H}_{\mathrm{N}}(\mathrm{div};\Omega)} \frac{\mathcal{R}_2(\boldsymbol{\psi})}{\|\boldsymbol{\psi}\|_{\mathrm{div};\Omega}} + \sup_{0 \neq q \in \mathrm{L}^2(\Omega)} \frac{\widetilde{b}(\mathsf{e}_{\boldsymbol{\varphi}}, q) - \widetilde{c}(\mathsf{e}_p, q)}{\|q\|_{0,\Omega}} \right).$$

Hence, recalling the definitions of $\widetilde{b}$, $\widetilde{c}$, adding $\pm c(\boldsymbol{\sigma}_h, p)$, and using (2.2d), we arrive at

$$|\widetilde{b}(\mathsf{e}_{\boldsymbol{\varphi}}, q) - \widetilde{c}(\mathsf{e}_p, q)| \leq \|g - c_0 p_h - \frac{\alpha}{d\lambda + 2\mu} \mathrm{tr}\,\boldsymbol{\sigma}_h - \frac{d\alpha^2}{d\lambda + 2\mu} p_h + \mathrm{div}\,\boldsymbol{\varphi}_h\|_{0,\Omega} \|q\|_{0,\Omega} + \gamma \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div};\Omega} \|q\|_{0,\Omega},$$

and therefore, we obtain the desired result. $\quad\square$

Throughout the rest of this section, we provide suitable upper bounds for $\mathcal{R}_1$ and $\mathcal{R}_2$. We begin by establishing the corresponding estimates for $\mathcal{R}_1$, which are based on a suitable Helmholtz decomposition of the space $\mathbb{H}_{\mathrm{N}}(\mathbf{div};\Omega)$ (see [3, Lemma 3.9] for details), along with the following two technical results.

LEMMA 6.3. *There exists a positive constant* C_3, *independent of* h, *such that for each* $\boldsymbol{\xi} \in \mathbb{H}^1(\Omega)$ *there holds*

$$|\mathcal{R}_1(\boldsymbol{\xi} - \boldsymbol{\Pi}_h(\boldsymbol{\xi}))| \leq C_3 \gamma \|p - p_h\|_{0,\Omega} \|\boldsymbol{\xi}\|_{1,\Omega}$$

$$+ C_3 \left(\sum_{K \in \mathcal{T}_h} h_K^2 \|\mathcal{C}^{-1} \boldsymbol{\sigma}_h + \frac{\alpha}{d\lambda + 2\mu} p_h \mathbb{I} - \boldsymbol{\nabla} \boldsymbol{u}_h + \boldsymbol{\rho}_h\|_{0,K}^2 + \sum_{E \in \mathcal{E}_h(\Gamma_{\mathrm{D}})} h_E \|\boldsymbol{u}_{\mathrm{D}} - \boldsymbol{u}_h\|_{0,E}^2 \right)^{1/2} \|\boldsymbol{\xi}\|_{1,\Omega}.$$

Proof. From the definition of $\mathcal{R}_1$ (cf. (6.3)), adding $\pm c(\boldsymbol{\xi} - \boldsymbol{\Pi}_h(\boldsymbol{\xi}), p_h)$, and using equation (2.2a), we have

$$\mathcal{R}_1(\boldsymbol{\xi} - \boldsymbol{\Pi}_h(\boldsymbol{\xi})) = H(\boldsymbol{\xi} - \boldsymbol{\Pi}_h(\boldsymbol{\xi})) - c(\boldsymbol{\xi} - \boldsymbol{\Pi}_h(\boldsymbol{\xi}), p) - a(\boldsymbol{\sigma}_h, \boldsymbol{\xi} - \boldsymbol{\Pi}_h(\boldsymbol{\xi})) - b(\boldsymbol{\xi} - \boldsymbol{\Pi}_h(\boldsymbol{\xi}), (\boldsymbol{u}_h, \boldsymbol{\rho}_h))$$

$$= \langle (\boldsymbol{\xi} - \boldsymbol{\Pi}_h(\boldsymbol{\xi})) \boldsymbol{n}, \boldsymbol{u}_{\mathrm{D}} \rangle_{\Gamma_{\mathrm{D}}} - \frac{\alpha}{d\lambda + 2\mu} \int_\Omega (p - p_h) \mathrm{tr}(\boldsymbol{\xi} - \boldsymbol{\Pi}_h(\boldsymbol{\xi})) - \frac{\alpha}{d\lambda + 2\mu} \int_\Omega p_h \, \mathrm{tr}(\boldsymbol{\xi} - \boldsymbol{\Pi}_h(\boldsymbol{\xi}))$$

$$- \int_\Omega \mathcal{C}^{-1} \boldsymbol{\sigma}_h : (\boldsymbol{\xi} - \boldsymbol{\Pi}_h(\boldsymbol{\xi})) - \int_\Omega \boldsymbol{\rho}_h : (\boldsymbol{\xi} - \boldsymbol{\Pi}_h(\boldsymbol{\xi})) - \int_\Omega \boldsymbol{u}_h \cdot \mathbf{div}(\boldsymbol{\xi} - \boldsymbol{\Pi}_h(\boldsymbol{\xi})),$$

then, applying a local integration by parts to the last term above, using the identity $\int_E \mathbf{\Pi}_h(\boldsymbol{\tau})\boldsymbol{n} \cdot \boldsymbol{\xi} = \int_E \boldsymbol{\tau}\boldsymbol{n} \cdot \boldsymbol{\xi}$, for all $\boldsymbol{\xi} \in \mathbf{P}_k(E)$, for all edge E of $\mathcal{T}_h$, the fact that $\boldsymbol{u}_{\mathrm{D}} \in \mathbf{L}^2(\Gamma_{\mathrm{D}})$, and the Cauchy-Schwarz inequality, we obtain

$$
\begin{aligned}
\mathcal{R}_1(\boldsymbol{\xi} - \mathbf{\Pi}_h(\boldsymbol{\xi})) &= \sum_{K \in \mathcal{T}_h} \int_K (-\mathcal{C}^{-1}\boldsymbol{\sigma}_h - \frac{\alpha}{d\lambda + 2\mu} p_h \mathbb{I} + \boldsymbol{\nabla}\boldsymbol{u}_h - \boldsymbol{\rho}_h) : (\boldsymbol{\xi} - \mathbf{\Pi}_h(\boldsymbol{\xi})) \\
&+ \sum_{E \in \mathcal{E}_h(\Gamma_{\mathrm{D}})} \langle (\boldsymbol{\xi} - \mathbf{\Pi}_h(\boldsymbol{\xi}))\boldsymbol{n}, \boldsymbol{u}_{\mathrm{D}} - \boldsymbol{u}_h \rangle_E - \frac{\alpha}{d\lambda + 2\mu} \int_\Omega (p - p_h) \, \mathrm{tr}(\boldsymbol{\xi} - \mathbf{\Pi}_h(\boldsymbol{\xi})) \\
&\leq \sum_{K \in \mathcal{T}_h} \|\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \frac{\alpha}{d\lambda + 2\mu} p_h \mathbb{I} - \boldsymbol{\nabla}\boldsymbol{u}_h + \boldsymbol{\rho}_h\|_{0,K} \|\boldsymbol{\xi} - \mathbf{\Pi}_h(\boldsymbol{\xi})\|_{0,K} \\
&+ \sum_{E \in \mathcal{E}_h(\Gamma_{\mathrm{D}})} \|\boldsymbol{u}_{\mathrm{D}} - \boldsymbol{u}_h\|_{0,E} \|\boldsymbol{\xi} - \mathbf{\Pi}_h(\boldsymbol{\xi})\|_{0,E} + \gamma \|p - p_h\|_{0,\Omega} \|\boldsymbol{\xi} - \mathbf{\Pi}_h(\boldsymbol{\xi})\|_{0,\Omega},
\end{aligned}
$$

with γ defined as in (2.5). Therefore, using the approximations properties of $\mathbf{\Pi}_h$ (see, e.g., [3, Section 3]) and the Cauchy–Schwarz inequality, we obtain the desired result. $\square$

LEMMA 6.4. *Let* $\boldsymbol{\chi} \in \mathbf{H}^1_{\Gamma_{\mathrm{N}}}(\Omega) := \{\boldsymbol{w} \in \mathbf{H}^1(\Omega) : \boldsymbol{w} = \boldsymbol{0} \text{ on } \Gamma_{\mathrm{N}}\}$ *and assume that* $\boldsymbol{u}_{\mathrm{D}} \in \mathbf{H}^1(\Gamma_{\mathrm{D}})$. *Then, there exists* $C_4 > 0$, *independent of* h, *such that*

$$
|\mathcal{R}_1(\mathbf{curl}\,(\boldsymbol{\chi} - \mathbf{I}_h\boldsymbol{\chi}))| \leq C_4 \gamma \|p - p_h\|_{0,\Omega} \|\boldsymbol{\chi}\|_{1,\Omega}
$$

$$
+ C_4 \Bigg(\sum_{K \in \mathcal{T}_h} h_K^2 \|\underline{\mathbf{curl}}(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \frac{\alpha}{d\lambda + 2\mu} p_h \mathbb{I} + \boldsymbol{\rho}_h)\|_{0,K}^2 + \sum_{E \in \mathcal{E}_h(\Omega)} h_E \|[\![(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \frac{\alpha}{d\lambda + 2\mu} p_h \mathbb{I} + \boldsymbol{\rho}_h)\boldsymbol{s}]\!]\|_{0,E}^2
$$

$$
+ \sum_{E \in \mathcal{E}_h(\Gamma_{\mathrm{D}})} h_E \|(C^{-1}\boldsymbol{\sigma}_h + \frac{\alpha}{d\lambda + 2\mu} p_h \mathbb{I} + \boldsymbol{\rho}_h)\boldsymbol{s} - \frac{\mathrm{d}\boldsymbol{u}_{\mathrm{D}}}{\mathrm{d}\boldsymbol{s}}\|_{0,E}^2 \Bigg)^{1/2} \|\boldsymbol{\chi}\|_{1,\Omega}.
$$

Proof. Similarly to Lemma 6.3, adding $\pm c\,(\mathbf{curl}\,(\boldsymbol{\chi} - \mathbf{I}_h\boldsymbol{\chi}), p_h)$, we have

$$
\begin{aligned}
\mathcal{R}_1&(\mathbf{curl}\,(\boldsymbol{\chi} - \mathbf{I}_h\boldsymbol{\chi})) \\
&= H(\mathbf{curl}\,(\boldsymbol{\chi} - \mathbf{I}_h\boldsymbol{\chi})) - c(\mathbf{curl}\,(\boldsymbol{\chi} - \mathbf{I}_h\boldsymbol{\chi}), p) - a(\boldsymbol{\sigma}_h, \mathbf{curl}\,(\boldsymbol{\chi} - \mathbf{I}_h\boldsymbol{\chi})) - b(\mathbf{curl}\,(\boldsymbol{\chi} - \mathbf{I}_h\boldsymbol{\chi}), (\boldsymbol{u}_h, \boldsymbol{\rho}_h)) \\
&= \langle (\mathbf{curl}\,(\boldsymbol{\chi} - \mathbf{I}_h\boldsymbol{\chi}))\boldsymbol{n}, \boldsymbol{u}_{\mathrm{D}} \rangle_{\Gamma_{\mathrm{D}}} - \frac{\alpha}{d\lambda + 2\mu} \int_\Omega (p - p_h) \, \mathrm{tr}(\mathbf{curl}\,(\boldsymbol{\chi} - \mathbf{I}_h\boldsymbol{\chi})) \\
&- \int_\Omega \mathcal{C}^{-1}\boldsymbol{\sigma}_h : \mathbf{curl}\,(\boldsymbol{\chi} - \mathbf{I}_h\boldsymbol{\chi}) - \int_\Omega \boldsymbol{\rho}_h : \mathbf{curl}\,(\boldsymbol{\chi} - \mathbf{I}_h\boldsymbol{\chi}) - \frac{\alpha}{d\lambda + 2\mu} \int_\Omega p_h \, \mathrm{tr}(\mathbf{curl}\,(\boldsymbol{\chi} - \mathbf{I}_h\boldsymbol{\chi})).
\end{aligned}
\tag{6.5}
$$

Then, applying a local integration by parts, using that $\boldsymbol{u}_{\mathrm{D}} \in \mathbf{H}^1(\Gamma_{\mathrm{D}})$, the identity $\langle \mathbf{curl}\,(\boldsymbol{\chi} - \mathbf{I}_h\boldsymbol{\chi})\boldsymbol{n}, \boldsymbol{u}_{\mathrm{D}} \rangle_{\Gamma_{\mathrm{D}}} = -\langle \boldsymbol{\chi} - \mathbf{I}_h\boldsymbol{\chi}, \frac{\mathrm{d}\boldsymbol{u}_{\mathrm{D}}}{\mathrm{d}\boldsymbol{s}} \rangle_{\Gamma_{\mathrm{D}}}$, and the Cauchy–Schwarz inequality, we obtain

$$
\begin{aligned}
\mathcal{R}_1&(\mathbf{curl}\,(\boldsymbol{\chi} - \mathbf{I}_h\boldsymbol{\chi})) = - \sum_{K \in \mathcal{T}_h} \int_K \underline{\mathbf{curl}}(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \frac{\alpha}{d\lambda + 2\mu} p_h \mathbb{I} + \boldsymbol{\rho}_h) : (\boldsymbol{\chi} - \mathbf{I}_h\boldsymbol{\chi}) \\
&+ \sum_{E \in \mathcal{E}_h(\Omega)} \int_E [\![(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \frac{\alpha}{d\lambda + 2\mu} p_h \mathbb{I} + \boldsymbol{\rho}_h)\boldsymbol{s}]\!] \cdot (\boldsymbol{\chi} - \mathbf{I}_h\boldsymbol{\chi}) - \frac{\alpha}{d\lambda + 2\mu} \int_\Omega (p - p_h) \, \mathrm{tr}(\mathbf{curl}\,(\boldsymbol{\chi} - \mathbf{I}_h\boldsymbol{\chi})) \\
&+ \sum_{E \in \mathcal{E}_h(\Gamma_{\mathrm{D}})} \int_E (\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \frac{\alpha}{d\lambda + 2\mu} p_h \mathbb{I} + \boldsymbol{\rho}_h - \boldsymbol{\nabla}\boldsymbol{u}_{\mathrm{D}})\boldsymbol{s} \cdot (\boldsymbol{\chi} - \mathbf{I}_h\boldsymbol{\chi}) \\
&\leq \sum_{K \in \mathcal{T}_h} \|\underline{\mathbf{curl}}(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \frac{\alpha}{d\lambda + 2\mu} p_h \mathbb{I} + \boldsymbol{\rho}_h)\|_{0,K} \|\boldsymbol{\chi} - \mathbf{I}_h\boldsymbol{\chi}\|_{0,K} \\
&+ \sum_{E \in \mathcal{E}_h(\Omega)} \|[\![(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \frac{\alpha}{d\lambda + 2\mu} p_h \mathbb{I} + \boldsymbol{\rho}_h)\boldsymbol{s}]\!]\|_{0,E} \|\boldsymbol{\chi} - \mathbf{I}_h\boldsymbol{\chi}\|_{0,E} + \gamma \|p - p_h\|_{0,\Omega} \|\mathbf{curl}\,(\boldsymbol{\chi} - \mathbf{I}_h\boldsymbol{\chi})\|_{0,\Omega} \\
&+ \sum_{E \in \mathcal{E}_h(\Gamma_{\mathrm{D}})} \|(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \frac{\alpha}{d\lambda + 2\mu} p_h \mathbb{I} + \boldsymbol{\rho}_h)\boldsymbol{s} - \frac{\mathrm{d}\boldsymbol{u}_{\mathrm{D}}}{\mathrm{d}\boldsymbol{s}}\|_{0,E} \|\boldsymbol{\chi} - \mathbf{I}_h\boldsymbol{\chi}\|_{0,E}.
\end{aligned}
\tag{6.6}
$$

As in the previous result, the approximation properties of the Clément interpolation (see, e.g., [3, Section 3]) in conjunction with the Cauchy–Schwarz inequality, produces the desired result. $\square$

The following lemma establishes the desired estimate for $\mathcal{R}_1$.

LEMMA 6.5. *Assume that there exists a convex domain* B *such that* $\Omega \subseteq B$ *and* $\Gamma_{\mathrm{N}} \subseteq \partial B$, *and that* $\boldsymbol{u}_{\mathrm{D}} \in \mathbf{H}^1(\Gamma_{\mathrm{D}})$. *Then, there exists a constant* $C_5 > 0$, *independent of* h, *such that*

$$
\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div};\Omega} + \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega} + \|\boldsymbol{\rho} - \boldsymbol{\rho}_h\|_{0,\Omega} \leq C_5 \Bigg\{ \sum_{T \in \mathcal{T}_h} \Xi_{s,K}^2 \Bigg\}^{1/2} + C_5 \gamma \|p - p_h\|_{0,\Omega}.
$$

Proof. Let $\boldsymbol{\tau} \in \mathbb{H}_\mathrm{N}(\mathbf{div}; \Omega)$. From [3, Lemma 3.9], there exist $\boldsymbol{\xi} \in \mathbb{H}^1(\Omega)$ and $\boldsymbol{\chi} \in \mathbf{H}^1_{\Gamma_\mathrm{N}}(\Omega)$, such that

$$\boldsymbol{\tau} = \boldsymbol{\xi} + \mathbf{curl}\,\boldsymbol{\chi} \quad \text{and} \quad \|\boldsymbol{\xi}\|_{1,\Omega} + \|\boldsymbol{\chi}\|_{1,\Omega} \le C_{\mathrm{Helm}}\|\boldsymbol{\tau}\|_{\mathbf{div};\Omega}, \tag{6.7}$$

Using that $\mathcal{R}_1(\boldsymbol{\tau}_h) = c(\boldsymbol{\tau}_h, p_h - p)$ for all $\boldsymbol{\tau}_h \in \mathbb{H}^{\boldsymbol{\sigma}}_h$, we have

$$\mathcal{R}_1(\boldsymbol{\tau}) = \mathcal{R}_1(\boldsymbol{\tau} - \boldsymbol{\tau}_h) + c(\boldsymbol{\tau}_h, p - p_h) \quad \forall\, \boldsymbol{\tau}_h \in \mathbb{H}^{\boldsymbol{\sigma}}_h.$$

In particular, this holds for $\boldsymbol{\tau}_h$ defined as $\boldsymbol{\tau}_h = \boldsymbol{\Pi}_h\boldsymbol{\xi} + \mathbf{curl}\,(\mathbf{I}_h\boldsymbol{\chi})$, whence

$$\mathcal{R}_1(\boldsymbol{\tau}) = \mathcal{R}_1(\boldsymbol{\xi} - \boldsymbol{\Pi}_h\boldsymbol{\xi}) + \mathcal{R}_1(\mathbf{curl}\,(\boldsymbol{\chi} - \mathbf{I}_h\boldsymbol{\chi})) + c(\boldsymbol{\tau}_h, p - p_h).$$

Hence, the proof follows from Lemmas 6.1, 6.3 and 6.4, and estimate (6.7). $\quad\square$

The following lemma establishes the estimate for $\mathcal{R}_2$.

LEMMA 6.6. *Assume that there exists a convex domain B such that $\Omega \subseteq B$ and $\Gamma_\mathrm{N} \subseteq \partial B$, and that $p_\mathrm{D} \in \mathrm{H}^1(\Gamma_\mathrm{D})$. Then, there exists a constant $C_8 > 0$, independent of h, such that*

$$\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{\mathrm{div};\Omega} + \|p - p_h\|_{0,\Omega} \le C_8 \left\{ \sum_{T \in \mathcal{T}_h} \Xi^2_{f,K} \right\}^{1/2} + C_8 \big(\|\boldsymbol{\varphi}_h\|_{0,\Omega}\|p - p_h\|_{0,\Omega} + \gamma\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega} \big).$$

Proof. It follows the steps of Lemma 6.5. From [25, Lemma 4.4], we have that for all $\boldsymbol{\psi} \in \mathbf{H}_\mathrm{N}(\mathrm{div}; \Omega)$ there exist $\boldsymbol{z} \in \mathbf{H}^1(\Omega)$ and $\phi \in \mathrm{H}^1_{\Gamma_\mathrm{N}}(\Omega)$, such that $\boldsymbol{\psi} = \boldsymbol{z} + \mathrm{curl}\,\phi$ and $\|\boldsymbol{z}\|_{1,\Omega} + \|\phi\|_{1,\Omega} \le \widetilde{C}_{\mathrm{Helm}}\|\boldsymbol{\psi}\|_{\mathbf{div};\Omega}$. Thus, proceeding similarly to Lemmas 6.3 and 6.4, applying local integration by parts and the approximation properties of Π_h and I_h along with the Cauchy–Schwarz inequality, we can obtain the following estimates

$$\begin{aligned}
|\mathcal{R}_2(\boldsymbol{z} - \Pi_h(\boldsymbol{z}))| &\le C_6\,\|\boldsymbol{\varphi}_h\|_{0,\Omega}\|p - p_h\|_{0,\Omega}\|\boldsymbol{z}\|_{1,\Omega} \\
&\quad + C_6 \bigg(\sum_{K \in \mathcal{T}_h} h_K^2\|\kappa(\boldsymbol{\sigma}_h, p_h)^{-1}\boldsymbol{\varphi}_h - \nabla p_h\|^2_{0,K} + \sum_{E \in \mathcal{E}_h(\Gamma_\mathrm{D})} h_E\|p_\mathrm{D} - p_h\|^2_{0,E} \bigg)^{1/2} \|\boldsymbol{z}\|_{1,\Omega}, \\
|\mathcal{R}_2(\mathrm{curl}\,(\phi - I_h\phi))| &\le C_7\|\boldsymbol{\varphi}_h\|_{0,\Omega}\|p - p_h\|_{0,\Omega}\|\phi\|_{1,\Omega} \\
&\quad + C_7 \bigg(\sum_{K \in \mathcal{T}_h} h_K^2\|\mathrm{rot}\,(\kappa(\boldsymbol{\sigma}_h, p_h)^{-1}\boldsymbol{\varphi}_h)\|^2_{0,K} + \sum_{E \in \mathcal{E}_h(\Omega)} h_E\|[\![(\kappa(\boldsymbol{\sigma}_h, p_h)^{-1}\boldsymbol{\varphi}_h) \cdot \boldsymbol{s}]\!]\|^2_{0,E} \\
&\quad + \sum_{E \in \mathcal{E}_h(\Gamma_\mathrm{D})} h_E\|\kappa(\boldsymbol{\sigma}_h, p_h)^{-1}\boldsymbol{\varphi}_h \cdot \boldsymbol{s} - \frac{\mathrm{d}p_\mathrm{D}}{\mathrm{d}\boldsymbol{s}}\|^2_{0,E} \bigg)^{1/2} \|\phi\|_{1,\Omega}.
\end{aligned} \tag{6.8}$$

Then, noting that $\mathcal{R}_2(\boldsymbol{\psi}_h) = 0$ for all $\boldsymbol{\psi}_h \in \mathbf{H}^{\boldsymbol{\varphi}}_h$, and defining $\boldsymbol{\psi}_h$ as $\boldsymbol{\psi}_h = \Pi_h\boldsymbol{z} + \mathrm{curl}\,(I_h\phi)$, we have

$$\mathcal{R}_2(\boldsymbol{\psi}) = \mathcal{R}_2(\boldsymbol{\psi} - \boldsymbol{\psi}_h) = \mathcal{R}_2(\boldsymbol{z} - \Pi_h\boldsymbol{z}) + \mathcal{R}_2(\mathrm{curl}\,(\phi - I_h\phi)).$$

Hence, the proof follows from Lemma 6.2, estimates (6.8), and the Helmholtz decomposition of $\mathbf{H}_\mathrm{N}(\mathrm{div}; \Omega)$. $\quad\square$

Finally we state the main reliability bound for the proposed estimator.

THEOREM 6.7. *Assume that the hypotheses stated in Theorem 5.3 and Lemmas 6.5-6.6 are satisfied. Let $(\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{\rho}, \boldsymbol{\varphi}, p) \in \mathbb{H}_\mathrm{N}(\mathbf{div}; \Omega) \times \mathbf{L}^2(\Omega) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega) \times \mathbf{H}_\mathrm{N}(\mathrm{div}; \Omega) \times \mathrm{L}^2(\Omega)$ and $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \boldsymbol{\rho}_h, \boldsymbol{\varphi}_h, p_h) \in \mathbb{H}^{\boldsymbol{\sigma}}_h \times \mathbf{H}^{\boldsymbol{u}}_h \times \mathbb{H}^{\boldsymbol{\rho}}_h \times \mathbf{H}^{\boldsymbol{\varphi}}_h \times \mathrm{H}^p_h$ be the unique solutions of (2.2) and (4.6), respectively. Assume further that*

$$(C_5 + C_8)\gamma + C_8 C^* (\|g\|_{0,\Omega} + \|p_\mathrm{D}\|_{1/2,\Gamma_\mathrm{D}} + \|\boldsymbol{u}_\mathrm{D}\|_{1/2,\Gamma_\mathrm{D}} + \|\boldsymbol{f}\|_{0,\Omega} + \gamma\,r) \le \frac{1}{2}. \tag{6.9}$$

Then, there exists $C_{\mathrm{rel}} > 0$, independent of h, such that

$$\|e_{\boldsymbol{\sigma}}\|_{\mathbf{div};\Omega} + \|e_{\boldsymbol{u}}\|_{0,\Omega} + \|e_{\boldsymbol{\rho}}\|_{0,\Omega} + \|e_{\boldsymbol{\varphi}}\|_{\mathbf{div};\Omega} + \|e_p\|_{0,\Omega} \le C_{\mathrm{rel}}\,\Xi.$$

Proof. It follows directly from Lemmas 6.5 and 6.6, using the fact that $\boldsymbol{\varphi}_h$ satisfies the estimate (4.14), and the assumption (6.9). $\quad\square$

6.2. Efficiency of the a posteriori error estimator. In this section we derive the efficiency estimate of the estimator defined in (6.2). The main result of this section is stated as follows.

THEOREM 6.8. *There exists $C_{\mathrm{eff}} > 0$, independent of h, such that*

$$C_{\mathrm{eff}}\,\Xi \le \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div};\Omega} + \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega} + \|\boldsymbol{\rho} - \boldsymbol{\rho}_h\|_{0,\Omega} + \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{\mathbf{div};\Omega} + \|p - p_h\|_{0,\Omega} + \mathrm{h.o.t}, \tag{6.10}$$

where h.o.t. *stands for one or several terms of higher order.*

We begin with the estimates for the zero order terms appearing in the definition of $\Xi_{s,K}$ (cf. (6.1a)).

LEMMA 6.9. *For all $K \in \mathcal{T}_h$ there holds*

$$\|\boldsymbol{f} + \mathbf{div}\boldsymbol{\sigma}_h\|_{0,K} \lesssim \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div};K} \quad and \quad \|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^{\mathtt{t}}\|_{0,K} \lesssim \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div};K}.$$

Proof. By employing the same arguments as in [17, Theorem 3.2], we can conclude that $\boldsymbol{f} = -\mathbf{div}\,\boldsymbol{\sigma}$, which, together with the symmetry of $\boldsymbol{\sigma}$, implies the desired result. Further details are omitted. □

In order to derive the upper bounds for the remaining terms defining the error estimator $\Xi_{s,K}$, we use results from [19], inverse inequalities, and the localisation technique based on element-bubble and edge-bubble functions. The main properties that we will use can be found in [32, Lemmas 4.4-4.7].

LEMMA 6.10. *For all $K \in \mathcal{T}_h$ there holds*

$$h_K\|\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \frac{\alpha}{d\lambda + 2\mu}p_h\mathbb{I} - \boldsymbol{\nabla}\boldsymbol{u}_h + \boldsymbol{\rho}_h\|_{0,K}$$
$$\lesssim h_K\big(\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div};K} + \|\boldsymbol{\rho} - \boldsymbol{\rho}_h\|_{0,K} + \|p - p_h\|_{0,K}\big) + \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,K}.$$

Proof. It follows from an application of [32, Lemma 4.4] with $\boldsymbol{q} = \mathcal{C}^{-1}\boldsymbol{\sigma}_h + \dfrac{\alpha}{d\lambda + 2\mu}p_h\mathbb{I} - \boldsymbol{\nabla}\boldsymbol{u}_h + \boldsymbol{\rho}_h$, using that $\mathcal{C}^{-1}\boldsymbol{\sigma} + \dfrac{\alpha}{d\lambda + 2\mu}p\mathbb{I} = \boldsymbol{\nabla}\boldsymbol{u} - \boldsymbol{\rho}$ and [32, Lemma 4.5]. We refer to [19, Lemma 6.6] for further details. □

LEMMA 6.11. *For all $K \in \mathcal{T}_h$ and $E \in \mathcal{E}_h(\Omega)$, there holds*

$$h_K\|\underline{\mathbf{curl}}(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \frac{\alpha}{d\lambda + 2\mu}p_h\mathbb{I} + \boldsymbol{\rho}_h)\|_{0,K} \lesssim \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div};K} + \|\boldsymbol{\rho} - \boldsymbol{\rho}_h\|_{0,K} + \|p - p_h\|_{0,K},$$

$$h_E^{1/2}\|[\![(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \frac{\alpha}{d\lambda + 2\mu}p_h\mathbb{I} + \boldsymbol{\rho}_h)\boldsymbol{s}]\!]\|_{0,E} \lesssim \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div};\omega_E} + \|\boldsymbol{\rho} - \boldsymbol{\rho}_h\|_{0,\omega_E} + \|p - p_h\|_{0,\omega_E},$$

where the patch of elements sharing the edge E is denoted as $\omega_E := \cup\{K' \in \mathcal{T}_h \colon E \in \mathcal{E}_h(K')\}$.

Proof. It suffices to apply [32, Lemma 4.7] with $\boldsymbol{\xi} := \mathcal{C}^{-1}\boldsymbol{\sigma} + \dfrac{\alpha}{d\lambda + 2\mu}p\mathbb{I} + \boldsymbol{\rho} = \boldsymbol{\nabla}\boldsymbol{u}$ and $\boldsymbol{\xi}_h := \mathcal{C}^{-1}\boldsymbol{\sigma}_h + \dfrac{\alpha}{d\lambda + 2\mu}p_h\mathbb{I} + \boldsymbol{\rho}_h$. □

LEMMA 6.12. *Assume that $\boldsymbol{u}_{\mathrm{D}}$ is piecewise polynomial. Then, for all $E \in \mathcal{E}_h(\Gamma_{\mathrm{D}})$, there holds*

$$h_E^{1/2}\|(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \frac{\alpha}{d\lambda + 2\mu}p_h\mathbb{I} + \boldsymbol{\rho}_h)\boldsymbol{s} - \frac{\mathrm{d}\boldsymbol{u}_{\mathrm{D}}}{\mathrm{d}\boldsymbol{s}}\|_{0,E} \lesssim \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div};K_E} + \|\boldsymbol{\rho} - \boldsymbol{\rho}_h\|_{0,K_E} + \|p - p_h\|_{0,K_E},$$

$$h_E^{1/2}\|\boldsymbol{u}_{\mathrm{D}} - \boldsymbol{u}_h\|_{0,E} \lesssim h_{K_E}\big(\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div};K_E} + \|\boldsymbol{\rho} - \boldsymbol{\rho}_h\|_{0,K_E} + \|p - p_h\|_{0,K_E}\big) + \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,K_E},$$

where K_E is a triangle in $\mathcal{T}_h$ that contains E on its boundary.

Proof. The first estimate follows as in [32, Lemma 4.18], defining $\boldsymbol{\xi}$ and $\boldsymbol{\xi}_h$ as in Lemma 6.11. On the other hand, the second estimate follows from an application of the discrete trace inequality (see [32, Lemma 4.6]), using that $\mathcal{C}^{-1}\boldsymbol{\sigma} + \dfrac{\alpha}{d\lambda + 2\mu}p\mathbb{I} = \boldsymbol{\nabla}\boldsymbol{u} - \boldsymbol{\rho}$, and the fact that $\boldsymbol{u} = \boldsymbol{u}_{\mathrm{D}}$ on Γ_{D}. See also [34, Lemma 4.14]. □

A direct application of Lemmas 6.9-6.12 yields

$$\sum_{K \in \mathcal{T}_h} \Xi_{s,K} \lesssim \|\mathsf{e}_{\boldsymbol{\sigma}}\|_{\mathbf{div};\Omega} + \|\mathsf{e}_{\boldsymbol{u}}\|_{0,\Omega} + \|\mathsf{e}_{\boldsymbol{\rho}}\|_{0,\Omega} + \|\mathsf{e}_{\boldsymbol{\varphi}}\|_{\mathrm{div};\Omega} + \|\mathsf{e}_p\|_{0,\Omega}. \tag{6.11}$$

Similarly, using the same arguments as in Lemmas 6.9-6.12, along with algebraic manipulations as in Section 5, assuming that p_{D} is piecewise polynomial, together with the Lipschitz continuity of κ (cf. (2.3)), we can bound each of the terms that appear in the estimator $\Xi_{f,K}$ and obtain the following result

$$\sum_{K \in \mathcal{T}_h} \Xi_{f,K} \lesssim \|\mathsf{e}_{\boldsymbol{\sigma}}\|_{\mathbf{div};\Omega} + \|\mathsf{e}_{\boldsymbol{u}}\|_{0,\Omega} + \|\mathsf{e}_{\boldsymbol{\rho}}\|_{0,\Omega} + \|\mathsf{e}_{\boldsymbol{\varphi}}\|_{\mathrm{div};\Omega} + \|\mathsf{e}_p\|_{0,\Omega}. \tag{6.12}$$

We remark that the efficiency of Ξ (cf. (6.2)) in Theorem 6.8 is now a straightforward consequence of estimates (6.11) and (6.12). In turn, we emphasize that the resulting constant, denoted by $C_{\mathrm{eff}} > 0$ is independent of h.

REMARK 6.1. *For simplicity, we have assumed that $\boldsymbol{u}_{\mathrm{D}}$ and p_{D} are piecewise polynomial in the derivation of (6.11) and (6.12). However, similar estimates can also be obtained by assuming $\boldsymbol{u}_{\mathrm{D}}$ and p_{D} are sufficiently smooth (taking, for example, $\boldsymbol{u}_{\mathrm{D}} \in \mathbf{H}^1(\Gamma_{\mathrm{D}})$ and $p_{\mathrm{D}} \in \mathrm{H}^1(\Gamma_{\mathrm{D}})$, as in Lemmas 6.5-6.6), and proceeding as in [23, Section 6.2]. In such cases, higher-order terms, stemming from errors in the polynomial approximations, would appear in (6.11) and (6.12), accounting for the presence of h.o.t in (6.10).*

REMARK 6.2. *We conclude this section by noting that the a posteriori error estimation analysis developed here can be adapted to the three-dimensional case. In particular, in [31, Th 3.2] and [25, Lemma 4.4], one can find the suitable Helmholtz decompositions for the spaces $\mathbb{H}_{\mathrm{N}}(\mathbf{div};\Omega)$ and $\mathbf{H}_{\mathrm{N}}(\mathrm{div};\Omega)$, respectively.*

k	DoFs	h	$e(\boldsymbol{\sigma})$	rate	$e(\boldsymbol{u})$	rate	$e(\boldsymbol{\rho})$	rate	$e(\boldsymbol{\varphi})$	rate	$e(p)$	rate
0	98	0.7071	3.5e+0	$\star$	4.5e-02	$\star$	3.7e-01	$\star$	4.9e-01	$\star$	2.4e-01	$\star$
	354	0.3536	1.9e+0	0.86	2.0e-02	1.20	8.0e-02	2.19	2.6e-01	0.92	1.3e-01	0.92
	1346	0.1768	9.9e-01	0.96	1.0e-02	0.98	3.3e-02	1.30	1.3e-01	0.98	6.5e-02	0.98
	5250	0.0884	5.0e-01	0.99	5.0e-03	1.00	1.3e-02	1.37	6.6e-02	0.99	3.3e-02	1.00
	20738	0.0442	2.5e-01	1.00	2.5e-03	1.00	4.7e-03	1.43	3.3e-02	1.00	1.6e-02	1.00
	82434	0.0221	1.2e-01	1.00	1.3e-03	1.00	1.7e-03	1.47	1.6e-02	1.00	8.2e-03	1.00
1	290	0.7071	1.3e+0	$\star$	1.3e-02	$\star$	4.1e-02	$\star$	1.5e-01	$\star$	7.4e-02	$\star$
	1090	0.3536	4.1e-01	1.61	4.2e-03	1.68	1.4e-02	1.50	3.9e-02	1.92	2.0e-02	1.93
	4226	0.1768	1.1e-01	1.94	1.1e-03	1.95	5.8e-03	1.30	9.9e-03	1.98	5.0e-03	1.98
	16642	0.0884	2.7e-02	1.98	2.7e-04	1.99	2.0e-03	1.55	2.5e-03	1.99	1.2e-03	2.00
	66050	0.0442	6.8e-03	2.00	6.8e-05	2.00	5.8e-04	1.78	6.2e-04	2.00	3.1e-04	2.00
	263170	0.0221	1.7e-03	2.00	1.7e-05	2.00	1.5e-04	1.92	1.6e-04	2.00	7.8e-05	2.00

TABLE 7.1

Example 1. Error history (degrees of freedom, mesh size, individual errors and experimental rates of convergence) in 2D for the formulation using the two lowest-order FE families with $PEERS_k$ elements.

7. Numerical results.
The computational examples in this section verify the theoretical properties (optimal convergence, conservativity, and robustness) of the proposed schemes. The implementation has been carried out using the FE library FEniCS [1]. The nonlinear systems were solved with Newton–Raphson's method with a residual tolerance of 10^{-7}, and the linear systems were solved using the sparse LU factorisation of MUMPS [4].

7.1. Optimal convergence to smooth solutions and conservativity in 2D. We first consider a simple planar problem setup with manufactured exact solution. We take the unit square domain $\Omega = (0, 1)^2$, the bottom and left segments represent Γ_D and the top and right sides are Γ_N. We choose the body load $\boldsymbol{f}$, mass source g, boundary displacement $\boldsymbol{u}_D$, boundary pressure p_D, as well as (not necessarily homogeneous, but standard arguments can be used to extend the theory to the inhomogeneous case) boundary data $\boldsymbol{\varphi} \cdot \boldsymbol{n} = \varphi_N$ and $\boldsymbol{\sigma} \boldsymbol{n} = \sigma_N$, such that the exact displacement and fluid pressure are

$$\boldsymbol{u}(x,y) = \frac{1}{20} \begin{pmatrix} \cos\left[\frac{3\pi}{2}(x+y)\right] \\ \sin\left[\frac{3\pi}{2}(x-y)\right] \end{pmatrix}, \quad p(x,y) = \sin(\pi x) \sin(\pi y).$$

These exact primary variables are used to construct exact mixed variables of stress, rotation, and discharge flux. We choose the second constitutive relation for the permeability in (1.8) and choose the following arbitrary model parameters (all adimensional) $k_0 = k_1 = c_0 = \alpha = 0.1, \quad \lambda = \mu = \mu_f = 1$. These values indicate a mild permeability variation and it is expected that the nonlinear solver (in this case, Newton–Raphson) converges in only a few iterations. We construct six levels of uniform mesh refinement of the domain, on which we compute approximate solutions and the associated errors for each primal and mixed variable in their natural norms. Convergence rates are calculated as usual:

$$\text{rate} = \log(e/\widehat{e})[\log(h/\widehat{h})]^{-1},$$

where e and $\widehat{e}$ denote errors produced on two consecutive meshes of sizes h and $\widehat{h}$, respectively. Table 7.1 reports on this error history focusing on the methods defined by the $PEERS_k$ family with $k = 0$ and $k = 1$, showing a $O(h^{k+1})$ convergence for all unknowns as expected from the theoretical error bound of Theorem 5.4 (except for the rotation approximation that shows a slight superconvergence for the lowest-order case and only in 2D – a well-known phenomenon associated with $PEERS_k$ elements). With the purpose of illustrating the character of the chosen manufactured solution and the parameter regime, we show sample discrete solutions in Figure 7.1.

We also exemplify the momentum and mass conservativity of the formulation. To do so we represent the loss of momentum and mass as

$$\texttt{mom}_h := \left\| \boldsymbol{\mathcal{P}}_h [\mathbf{div}(\boldsymbol{\sigma}_h) + \boldsymbol{f}] \right\|_{\ell^\infty}, \quad \texttt{mass}_h := \left\| \mathcal{P}_h \left[(c_0 + \frac{d\alpha^2}{d\lambda + 2\mu}) p_h + \frac{\alpha}{d\lambda + 2\mu} \operatorname{tr} \boldsymbol{\sigma}_h + \operatorname{div}(\boldsymbol{\varphi}_h) + g \right] \right\|_{\ell^\infty},$$

where $\mathcal{P}_h : \mathrm{L}^2(\Omega) \to \mathrm{P}_k(\mathcal{T}_h)$ is the scalar version of $\boldsymbol{\mathcal{P}}_h$. They are computed at each refinement level and tabulated in Table 7.2 together with the total error $\texttt{e} := e(\boldsymbol{\sigma}) + e(\boldsymbol{u}) + e(\boldsymbol{\rho}) + e(\boldsymbol{\varphi}) + e(p)$, and its experimental convergence rate. We report on the nonlinear iteration count as well. The expected optimal convergence of the total error, and the announced local conservativity are confirmed. We also note that, at least for this parameter regime, for all the refinements and polynomial degrees the nonlinear solver takes three iterations to get a residual below the tolerance. In the last column of the same table we report on the efficiency of the global a posteriori error estimator designed in Section 6 $\texttt{eff}(\Xi) = \frac{\texttt{e}}{\Xi}$, which – in this case of smooth solutions – is asymptotically constant (approximately 0.98 for $k = 0$ and 1.52 for $k = 1$).
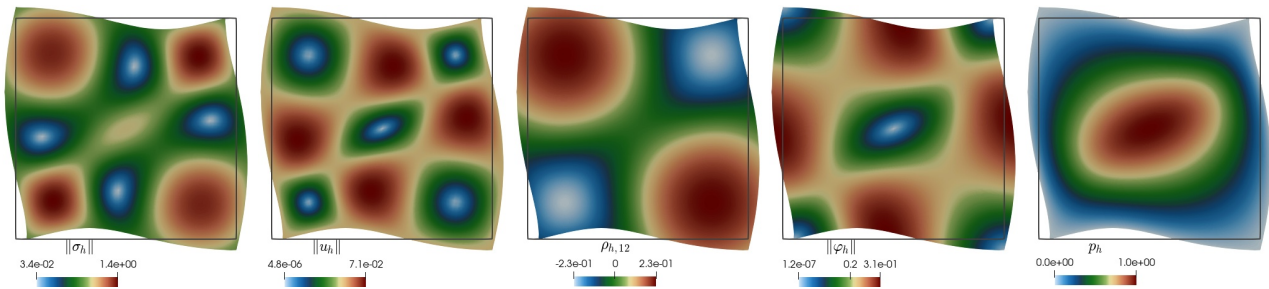
FIG. 7.1. *Example 1. Sample of approximate solutions (stress magnitude, displacement magnitude, non-zero entry of rotation, flux magnitude, and fluid pressure) computed with the second-order scheme and plotted on the deformed domain (for reference we also show the contour of the undeformed domain).*

DoFs	h	e	rate	mom_h	mass_h	iter	$\text{eff}(\Xi)$
			$k = 0$				
97	0.707	4.83e+0	$\star$	9.30e-16	1.94e-16	3	0.93
353	0.354	2.45e+0	0.98	9.44e-16	2.98e-16	3	0.94
1345	0.177	1.25e+0	0.98	2.64e-15	8.26e-16	3	0.96
5249	0.088	6.22e-01	1.00	5.58e-15	2.49e-15	3	0.97
20737	0.044	3.10e-01	1.00	1.46e-14	7.03e-15	3	0.98
82433	0.022	1.55e-01	1.00	1.72e-13	1.09e-13	3	0.98
328705	0.011	7.63e-02	1.00	4.25e-13	3.46e-13	3	0.98
			$k = 1$				
289	0.707	1.57e+0	$\star$	2.88e-15	1.20e-15	3	1.45
1089	0.354	5.08e-01	1.63	6.77e-15	2.25e-15	3	1.50
4225	0.177	1.34e-01	1.92	2.21e-14	6.60e-15	3	1.53
16641	0.088	3.42e-02	1.97	8.00e-14	1.31e-14	3	1.52
66049	0.044	8.62e-03	1.99	3.91e-13	3.83e-14	3	1.51
263169	0.022	2.16e-03	2.00	8.20e-12	7.61e-14	3	1.51
1050625	0.011	5.04e-04	2.00	2.17e-11	4.90e-13	3	1.52

TABLE 7.2

Example 1. Total error, experimental rates of convergence, ℓ^∞-norm of the projected residual of the momentum and mass balance equations, Newton–Raphson iteration count, and efficiency index of the global a posteriori error estimator. Tabulated results correspond to the two lowest-order polynomial degrees.

7.2. Convergence in 3D using physically relevant parameters. Next we investigate the behaviour of the proposed numerical methods in a 3D setting and taking model parameters more closely related to applications in tissue poroelasticity. We still use manufactured solutions to assess the accuracy of the formulation, but take an exact displacement that satisfies $\text{div}\, \boldsymbol{u} \to 0$ as $\lambda \to \infty$. The domain is the 3D box $\Omega = (0, L) \times (0, L) \times (0, 2L)$ with $L = 0.01\,\text{m}$, and mixed boundary conditions were taken analogously as before, separating the domain boundary between Γ_N defined as the planes $x = 0$, $y = 0$ and $z = 0$, and Γ_D as the remainder of the boundary. The manufactured displacement and pressure head are

$$\boldsymbol{u}(x, y, z) = \frac{L}{4} \begin{pmatrix} \sin(x/L)\cos(y/L)\sin(z/(2L)) + x^2/\lambda \\ -2\cos(x/L)\sin(y/L)\cos(z/(2L)) + y^2/\lambda \\ 2\cos(x/L)\cos(y/L)\sin(z/(2L)) - 2z^2/\lambda \end{pmatrix}, \quad p(x, y, z) = \sin(x/L)\cos(y/L)\sin(z/(2L)).$$

First we set again the model parameters to mild values $\lambda = \mu = c_0 = k_0 = \alpha = \mu_f = 1$, $k_1 = k_2 = 0.1$, and we compare them against the following values (from, e.g., [9, 43])

$$k_0 = 2.28 \times 10^{-11}\,\text{m}^3, \quad k_1 = 5 \times 10^{-12}\,\text{m}^3, \quad \lambda = 1.44 \times 10^6\,\text{Pa}, \quad \mu = 9.18 \times 10^3\,\text{Pa}, \quad \mu_f = 7.5 \times 10^{-4}\,\text{Pa} \cdot \text{s},$$

and $c_0 = 0$, $\alpha = 0.99$. Table 7.3 reports on the convergence of the method. While the magnitude of the stress errors is much higher for the second parameter regime, the discharge flux error magnitude is smaller than in the first case and the displacement, rotation, and fluid pressure errors remain roughly of the same magnitude. In any case, the table confirms that the optimal slope of the error decay is not affected by a vanishing storativity nor large Lamé constants. The numerical solutions are displayed in Figure 7.2.

7.3. Convergence in the case of adaptive mesh refinement. We continue with a test targeting the recovery of optimal convergence through adaptive mesh refinement guided by the a posteriori error estimator proposed in Section 6. We employ the well-known adaptive mesh refinement approach of solving, then computing

DoFs	h	$e(\boldsymbol{\sigma})$	rate	$e(\boldsymbol{u})$	rate	$e(\boldsymbol{\rho})$	rate	$e(\boldsymbol{\varphi})$	rate	$e(p)$	rate
		unity parameters									
328	0.0173	2.9e-02	$\star$	1.1e-06	$\star$	4.8e-04	$\star$	3.1e+0	$\star$	1.3e-04	$\star$
2311	0.0087	1.5e-02	0.94	5.7e-07	0.96	1.3e-04	1.90	1.6e+0	1.00	6.3e-05	1.00
17443	0.0043	7.5e-03	0.99	2.8e-07	1.00	4.4e-05	1.54	7.8e-01	1.00	3.2e-05	1.00
135715	0.0022	3.8e-03	1.00	1.4e-07	1.00	1.8e-05	1.34	3.9e-01	1.00	1.6e-05	1.00
1071043	0.0011	1.9e-03	1.00	7.0e-08	1.00	7.7e-06	1.19	1.9e-01	1.00	7.9e-06	1.00
		physically relevant parameters									
328	0.0173	2.1e+02	$\star$	1.1e-06	$\star$	4.2e-04	$\star$	4.8e-07	$\star$	1.9e-04	$\star$
2311	0.0087	1.1e+02	0.91	5.6e-07	0.99	1.1e-04	1.93	3.9e-07	0.55	9.1e-05	1.05
17443	0.0043	5.6e+01	0.98	2.8e-07	1.00	3.3e-05	1.74	2.2e-07	0.89	4.4e-05	1.04
135715	0.0022	2.8e+01	1.00	1.4e-07	1.00	1.2e-05	1.42	1.2e-07	0.93	2.2e-05	0.99
1071043	0.0011	1.4e+01	1.00	7.0e-08	1.00	5.3e-06	1.23	6.0e-08	0.97	1.1e-05	0.98

TABLE 7.3

Example 2. Error history (degrees of freedom, mesh size, individual errors and experimental rates of convergence) in 3D for the formulation using the lowest-order FE family with PEERS$_k$ elements and changing from unity (top) to physically relevant (bottom) parameters.
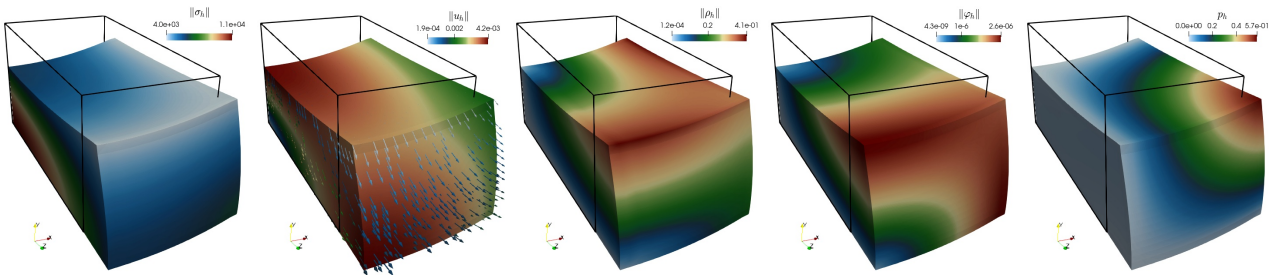

FIG. 7.2. *Example 2. Sample of approximate solutions (stress magnitude, displacement magnitude, rotation magnitude, flux magnitude, and fluid pressure) computed with the first-order scheme and plotted on the deformed domain (for reference we also show the outline of the undeformed domain).*

the estimator, marking, and refining. Marking is done as follows [27]: a given $K \in \mathcal{T}_h$ is added to the marking set $\mathcal{M}_h \subset \mathcal{T}_h$ whenever the local error indicator Ξ_K satisfies

$$\sum_{K \in \mathcal{M}_h} \Xi_K^2 \geq \zeta \sum_{K \in \mathcal{T}_h} \Xi_K^2,$$

where ζ is a user-defined bulk density. All elements in $\mathcal{M}_h$ are marked for refinement and also some neighbours are marked for the sake of closure. For convergence rates we use the alternative form

$$\text{rate} = -2\log(e/\widehat{e})[\log(\text{DoFs}/\widehat{\text{DoF}})]^{-1}.$$

Let us consider the non-convex rotated L-shaped domain $\Omega = (-1,1)^2 \setminus (-1,0)^2$ and use manufactured displacement and fluid pressure with sharp gradients near the domain re-entrant corner (see, e.g., [20] for the displacement and [16] for the fluid pressure)

$$\boldsymbol{u}(r,\theta) = \frac{r^\chi}{2\mu} \begin{pmatrix} -(\chi+1)\cos([\chi+1]\theta) + (M_2 - \chi - 1)M_1 \cos([\chi-1]\theta) \\ (\chi+1)\sin([\chi+1]\theta) + (M_2 + \chi - 1)M_1 \sin([\chi-1]\theta) \end{pmatrix}, \qquad p(r,\theta) = r^{1/3} \sin\left(\frac{1}{3}(\frac{\pi}{2} + \theta)\right),$$

with polar coordinates $r = \sqrt{x_1^2 + x_2^2}$, $\theta = \arctan(x_2, x_1)$, $\chi \approx 0.54448373$, $M_1 = -\cos([\chi+1]\omega)/\cos([\chi-1]\omega)$, and $M_2 = 2(\lambda + 2\mu)/(\mu + \lambda)$. The boundary conditions (taking as Γ_N the segments at $x = \pm 1$ and $y = \pm 1$ and Γ_D the remainder of the boundary) and forcing data are constructed from these solutions and the model parameters are $\lambda = 10^3$, $\mu = 10$, $k_0 = \frac{1}{2}$, $\mu_f = c_0 = k_1 = 0.1$, $\alpha = \frac{1}{4}$, where for this test we consider a Kozeny–Carman permeability form. As in [20], a sub-optimal rate of convergence is expected for the mixed elasticity sub-problem in its energy norm. Note that since the exact pressure is in $\text{H}^{4/3-\epsilon}(\Omega)$ for any $\epsilon > 0$ (cf. [37, Chapter 5]), it is still regular enough to have optimal convergence. However its gradient (and therefore also the exact discharge flux $\boldsymbol{\varphi}$) has a singularity located at the reentrant corner and therefore we expect an order of convergence of approximately $O(h^{1/3})$.

The numerical results of this test are reported in Table 7.4. We observe the expected sub-optimal convergence under an uniform mesh refinement while the optimal convergence in all variables is attained as the mesh is locally refined (the first three rows are very similar as most of the elements are refined in the first three steps. This can be controlled by the bulk density, here taken as $\zeta = 9.5 \cdot 10^{-5}$). We also note that the individual errors

DoFs	h	$e(\boldsymbol{\sigma})$	rate	$e(\boldsymbol{u})$	rate	$e(\boldsymbol{\rho})$	rate	$e(\boldsymbol{\varphi})$	rate	$e(p)$	rate	$\mathtt{eff}(\Xi)$
						uniform mesh refinement						
77	1.4142	1.7e+3	$\star$	1.7e+1	$\star$	7.6e+1	$\star$	9.1e+0	$\star$	1.6e+0	$\star$	2.67
273	0.7071	1.5e+3	0.27	7.8e+0	1.10	2.5e+1	1.61	5.8e+0	0.65	7.8e-01	1.06	2.65
1025	0.3536	1.2e+3	0.31	5.0e+0	0.63	2.1e+1	0.27	4.3e+0	0.44	3.9e-01	1.00	3.12
3969	0.1768	8.9e+2	0.40	3.6e+0	0.50	1.7e+1	0.26	3.3e+0	0.38	2.0e-01	0.99	2.82
15617	0.0884	6.6e+2	0.43	2.6e+0	0.45	1.4e+1	0.32	2.6e+0	0.35	9.9e-02	0.99	2.60
61953	0.0442	4.8e+2	0.45	1.9e+0	0.45	1.1e+1	0.35	2.0e+0	0.34	5.0e-02	1.00	2.42
246785	0.0221	3.5e+2	0.46	1.4e+0	0.46	8.4e+0	0.38	1.6e+0	0.34	2.5e-02	1.00	2.27
985089	0.0011	2.9e+2	0.45	1.1e+0	0.45	6.7e+0	0.38	1.2e+0	0.34	1.3e-02	1.00	2.40
						adaptive mesh refinement						
77	1.4142	1.7e+3	$\star$	1.7e+1	$\star$	7.6e+1	$\star$	9.1e+0	$\star$	1.6e+0	$\star$	2.67
273	0.7071	1.5e+3	0.29	7.8e+0	1.20	2.5e+1	1.76	5.8e+0	0.72	7.8e-01	1.16	2.65
1025	0.3536	1.2e+3	0.33	5.0e+0	0.66	2.1e+1	0.28	4.3e+0	0.47	3.9e-01	1.04	3.12
3813	0.3536	8.9e+2	0.42	3.6e+0	0.52	1.7e+1	0.27	3.3e+0	0.40	2.0e-01	1.04	2.82
7113	0.2500	6.7e+2	0.89	2.6e+0	0.98	1.4e+1	0.68	2.6e+0	0.77	1.1e-01	1.91	2.55
11013	0.2500	5.2e+2	1.20	2.0e+0	1.31	1.1e+1	0.99	2.0e+0	1.07	6.0e-02	1.74	2.58
17449	0.2500	4.0e+2	1.10	1.5e+0	1.18	9.1e+0	0.95	1.6e+0	1.01	3.7e-02	1.79	2.52
27225	0.1768	3.1e+2	1.14	1.2e+0	1.18	7.2e+0	1.05	1.3e+0	1.05	2.1e-02	1.80	2.52
38081	0.1768	2.5e+2	1.37	9.0e-01	1.53	5.7e+0	1.38	1.0e+0	1.37	1.5e-02	1.57	2.55
54797	0.1250	2.0e+2	1.18	7.2e-01	1.21	4.6e+0	1.22	8.1e-01	1.26	1.2e-02	1.48	2.53

TABLE 7.4

Example 3. Convergence history (degrees of freedom, mesh size, individual errors, experimental rates of convergence, and effectivity index) in a rotated L-shaped domain for the formulation using the lowest-order FE family with $PEERS_k$ elements and changing from uniform (top) to adaptive mesh refinement (bottom) guided by the a posteriori error indicator from (6.2).
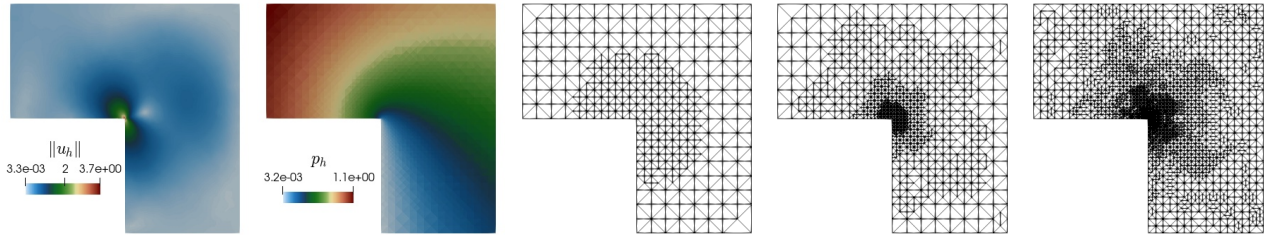

FIG. 7.3. *Example 3. Approximate primal variable solutions (solid displacement and fluid pressure) computed with the first-order scheme, and meshes generated after two, three, and four adaptive refinement steps.*

are approximately of the same magnitude in the last row of each section of the table, but for the adaptive case this is achieved using approximately 5.5% of the number of degrees of freedom needed in the uniformly refined case. The last column of the table again confirms the reliability and efficiency of the a posteriori error estimator. Note that for this case we compute the divergence part of the error norm in the stress and fluxes as projections of the momentum and mass residuals onto the displacement and pressure discrete spaces, respectively. We plot in Figure 7.3 the approximate displacement and pressure as well as sample triangulations obtained after a few adaptive refinement steps that confirm the expected agglomeration of vertices near the reentrant corner.

7.4. Adaptive computation of cross-sectional flow and deformation in a soft tissue specimen. Finally, we apply the proposed methods to simulate the localisation of stress, deformation, and flow patterns in a multi-layer cross-section of cervical spinal cord. We follow the setup in [42, 44]. The geometry and unstructured mesh have been generated using GMSH [35] from the images in [44]. The heterogeneous porous material consists of white and grey matter surrounded by the pia mater (a thin layer, also considered poroelastic. See Figure 7.4, top two left panels). All components are assumed fully saturated with cerebospinal fluid. The transversal cross-section has 1.3 cm in maximal diameter and the indentation region is a curved subset of the anterior part of the pia mater (a sub-boundary of length 0.4 cm). Boundary conditions are of mixed load-traction type, but slightly different than the ones analysed in the previous section. We conduct an indentation test applying a traction $\boldsymbol{\sigma n} = (0, -P)^{-\mathtt{t}}$, with P a constant solid pressure of 950 dyne/cm^2. The posterior part of the pia mater acts as a rigid posterior support where we prescribe zero displacement. The remainder of the boundary of the pia mater is stress-free. For the fluid phase we impose a constant inflow pressure of cerebrospinal fluid of 1.1 dyne/cm^2 and zero outflow pressure at the stress-free sub-boundary, as well as zero normal discharge flux at the posterior support. For the three different layers of the domain we use the following values for Young modulus, Poisson ratio, and lower bound for permeability (some values from [10, 42, 44]) $E^{\mathrm{pia}} = 23'000$ dyne/cm^2, $\nu^{\mathrm{pia}} = 0.3$, $E^{\mathrm{white}} = 8'400$ dyne/cm^2, $\nu^{\mathrm{white}} = 0.479$, $E^{\mathrm{grey}} = 16'000$ dyne/cm^2, $\nu^{\mathrm{grey}} = 0.49$, $k_0^{\mathrm{grey}} = 1.4 \cdot 10^{-9}$ dyne/cm^2,
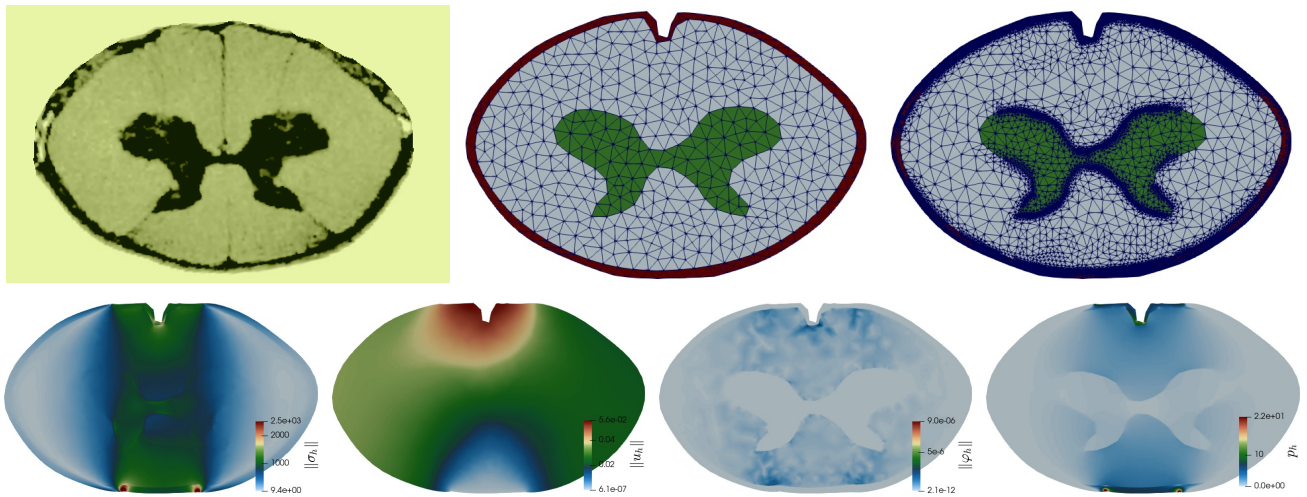
Fig. 7.4. *Example 4. Cross sectional area of cervical spinal cord segmented from sheep imaging data in [44], initial coarse mesh indicating subdomains with distinct material properties (outer pia mater in red, mid white matter in grey, inner grey matter in green), final adapted mesh after six refinement steps, and sample of stress, displacement, fluid flux and fluid pressure at the indentation test (bottom row figures are obtained with the lowest-order scheme and rendered on the deformed configuration).*

$k_0^{\text{white}} = 1.4 \cdot 10^{-6} \, \text{dyne/cm}^2$, $k_0^{\text{pia}} = 3.9 \cdot 10^{-10} \, \text{dyne/cm}^2$. Further, we take $\boldsymbol{f} = \boldsymbol{0}$, $g = 0$, $\mu_f = 70 \, \text{dyne/cm}^2 \cdot \text{s}$ (for cerebrospinal fluid at 37°), $k_1 = \frac{1}{2} k_0$, $\alpha = \frac{1}{4}$, and $c_0 = 10^{-3}$.

The initial and the final adapted mesh, together with samples of solutions are shown in Figure 7.4, where we have used the mesh density parameter $\zeta = 5.5 \cdot 10^{-4}$. After each adaptation iteration guided by the a posteriori error indicator (6.2), a mesh smoothing step was included. The figure indicates that most of the refinement occurs near the interface between the heterogeneous components of the porous media, and the plots also confirm a flow pattern moving slowly from top to bottom, consistently with a typical indentation test.

REFERENCES

[1] M. S. ALNÆS, J. BLECHTA, J. HAKE, A. JOHANSSON, B. KEHLET, A. LOGG, C. RICHARDSON, J. RING, M. E. ROGNES, AND G. N. WELLS, *The FEniCS project version 1.5*, Arch. Numer. Softw., 3 (2015), pp. 9–23.

[2] M. ÁLVAREZ, G. N. GATICA, AND R. RUIZ-BAIER, *An augmented mixed-primal finite element method for a coupled flow-transport problem*, ESAIM: Math. Model. Numer. Anal., 49, 5 (2015), pp. 1399–1427.

[3] ———, *A posteriori error analysis for a viscous flow–transport problem*, ESAIM: Math. Model. Numer. Anal., 50 (2016), pp. 1789–1816.

[4] P. R. AMESTOY, I. S. DUFF, AND J.-Y. L'EXCELLENT, *Multifrontal parallel distributed symmetric and unsymmetric solvers*, Comput. Methods Appl. Mech. Engrg., 184 (2000), pp. 501–520.

[5] D. N. ARNOLD, F. BREZZI, AND J. DOUGLAS, *PEERS: A new mixed finite element method for plane elasticity*, Japan J. Appl. Math., 1 (1984), pp. 347–367.

[6] G. A. ATESHIAN AND J. A. WEISS, *Anisotropic hydraulic permeability under finite deformation*, J. Biomech. Engrg., 132 (2010), p. 111004(7).

[7] T. BÆRLAND, J. J. LEE, K.-A. MARDAL, AND R. WINTHER, *Weakly imposed symmetry and robust preconditioners for Biot's consolidation model*, Comput. Methods Appl. Math., 17 (2017), pp. 377–396.

[8] S. BARBEIRO AND M. F. WHEELER, *A priori error estimates for the numerical solution of a coupled geomechanics and reservoir flow model with stress-dependent permeability*, Comput. Geosci., 14 (2010), pp. 755–768.

[9] N. BARNAFI, B. GOMEZ-VARGAS, W. D. J. LOURENCO, R. F. REIS, B. M. ROCHA, M. LOBOSCO, R. RUIZ-BAIER, AND R. WEBER DOS SANTOS, *Finite element methods for large-strain poroelasticity/chemotaxis models simulating the formation of myocardial oedema*, J. Sci. Comput., 92 (2022), pp. e92(1–40).

[10] I. BLOOMFIELD, I. JOHNSTON, AND L. BILSTON, *Effects of proteins, blood cells and glucose on the viscosity of cerebrospinal fluid*, Pediatr. Neurosurg., 28 (1998), pp. 246–251.

[11] L. BOCIU, G. GUIDOBONI, R. SACCO, AND J. T. WEBSTER, *Analysis of nonlinear poro-elastic and poro-visco-elastic models*, Arch. Rational Mech. Anal., 222 (2016), pp. 1445–1519.

[12] L. BOCIU, B. MUHA, AND J. T. WEBSTER, *Weak solutions in nonlinear poroelasticity with incompressible constituents*, Nonl. Anal.: Real World Appl., 67 (2022), p. 103563.

[13] L. BOCIU AND J. T. WEBSTER, *Nonlinear quasi-static poroelasticity*, J. Diff. Eqns., 296 (2021), pp. 242–278.

[14] M. A. BORREGALES REVERÓN, K. KUMAR, J. M. NORDBOTTEN, AND F. A. RADU, *Iterative solvers for Biot model under small and large deformations*, Comput. Geosci., 25 (2021), pp. 687–699.

[15] S. BRENNER AND L. SCOTT, *The Mathematical Theory of Finite Element Methods*, Springer–Verlag, New York, 1994.

[16] R. BULLE, J. S. HALE, A. LOZINSKI, S. P. BORDAS, AND F. CHOULY, *Hierarchical a posteriori error estimation of Bank–Weiser type in the FEniCS project*, Comput. Math. Appl., 131 (2023), pp. 103–123.

[17] J. CAMAÑO, S. CAUCAO, R. OYARZÚA, AND S. VILLA-FUENTES, *A posteriori error analysis of a momentum conservative banach-spaces based mixed-FEM for the navier–stokes problem*, Applied Numerical Mathematics, 176 (2022), pp. 134–158.

[18] J. CAMAÑO, G. GATICA, R. OYARZÚA, R. RUIZ-BAIER, AND P. VENEGAS, *New fully-mixed finite element methods for the Stokes–Darcy coupling*, Comput. Methods Appl. Mech. Engrg., 295 (2015), pp. 362–395.

[19] C. CARSTENSEN AND G. DOLZMANN, *A posteriori error estimates for mixed FEM in elasticity*, Numer. Math., 81 (1998),

pp. 187–209.

[20] C. CARSTENSEN, G. DOLZMANN, S. A. FUNKEN, AND D. S. HELM, *Locking-free adaptive mixed finite element methods in linear elasticity*, Comput. Methods Appl. Mech. Engrg., 190 (2000), pp. 1701–1718.

[21] S. CAUCAO, M. DISCACCIATI, G. GATICA, AND R. OYARZÚA, *A conforming mixed finite element method for the Navier–Stokes/Darcy–Forchheimer coupled problem*, ESAIM: Math. Model. Numer. Anal., 54, 5 (2020), pp. 1689–1723.

[22] S. CAUCAO, G. N. GATICA, AND J. P. ORTEGA, *A posteriori error analysis of a banach spaces-based fully mixed fem for double-diffusive convection in a fluid-saturated porous medium*, Computational Geosciences, 27, 2 (2023), pp. 289–316.

[23] S. CAUCAO, D. MORA, AND R. OYARZÚA, *A priori and a posteriori error analysis of a pseudostress-based mixed formulation of the stokes problem with varying density*, IMA Journal of Numerical Analysis, 36, 2 (2016), pp. 947–983.

[24] S. CAUCAO, R. OYARZÚA, AND S. VILLA-FUENTES, *A new mixed-FEM for steady-state natural convection models allowing conservation of momentum and thermal energy*, Calcolo, 57, 4, article: 36 (2020).

[25] S. CAUCAO, R. OYARZÚA, AND S. VILLA-FUENTES, *A posteriori error analysis of a momentum and thermal energy conservative mixed-fem for the boussinesq equations*, Calcolo, 59, 4, article: 45 (2022).

[26] C. CORREA AND G. GATICA, *On the continuous and discrete well-posedness of perturbed saddle-point formulations in Banach spaces*, Comput. Math. Appl., 117 (2022), pp. 14–23.

[27] W. DÖRFLER, *A convergent adaptive algorithm for Poisson's equation*, SIAM J. Numer. Anal., 33 (1996), pp. 1106–1124.

[28] A. ELYES, F. RADU, AND J. NORDBOTTEN, *Adaptive poromechanics computations based on a posteriori error estimates for fully mixed formulations of Biot's consolidation model*, Comput. Methods Appl. Mech. Engrg., 347 (2019), pp. 264–294.

[29] A. ERN AND J.-L. GUERMOND, *Theory and Practice of Finite Elements*, Applied Mathematical Sciences, 159. Springer-Verlag, New York, 2004.

[30] G. N. GATICA, *A Simple Introduction to the Mixed Finite Element Method. Theory and Applications*, Springer-Verlag, Berlin, 2014.

[31] ———, *A note on stable helmholtz decompositions in 3d*, Applicable Analysis, 99 (2020), pp. 1110–1121.

[32] G. N. GATICA, B. GÓMEZ-VARGAS, AND R. RUIZ-BAIER, *A posteriori error analysis of mixed finite element methods for stress-assisted diffusion problems*, J. Comput. Appl. Math., 409 (2022), pp. e114144(1–23).

[33] G. N. GATICA, A. MÁRQUEZ, AND W. RUDOLPH, *A priori and a posteriori error analyses of augmented twofold saddle point formulations for nonlinear elasticity problems*, Comput. Methods Appl. Mech. Engrg., 264 (2013), pp. 23–48.

[34] G. N. GATICA, A. MÁRQUEZ, AND M. SÁNCHEZ, *Analysis of a velocity-pressure-pseudostress formulation for the stationary stokes equations*, Comput. Methods Appl. Mech. Engrg., 199 (2010), pp. 1064–1079.

[35] C. GEUZAINE AND J.-F. REMACLE, *Gmsh: A 3-D finite element mesh generator with built-in pre-and post-processing facilities*, International Journal for Numerical Methods in Engineering, 79 (2009), pp. 1309–1331.

[36] B. GÓMEZ-VARGAS, K.-A. MARDAL, R. RUIZ-BAIER, AND V. VINJE, *Twofold saddle-point formulation of Biot poroelasticity with stress-dependent diffusion*, SIAM J. Numer. Anal., 63 (2023), pp. 1449–1481.

[37] P. GRISVARD, *Elliptic problems in nonsmooth domains*, SIAM, 2011.

[38] B. P. LAMICHHANE, R. RUIZ-BAIER, AND S. VILLA-FUENTES, *New twofold saddle-point formulations for Biot poroelasticity with porosity-dependent permeability*, Results in Applied Mathematics, 21 (2024), pp. e100438(1–22).

[39] J. J. LEE, *Robust error analysis of coupled mixed methods for Biot's consolidation model*, J. Sci. Comput., 69 (2016), pp. 610–632.

[40] T. LI AND I. YOTOV, *A mixed elasticity formulation for fluid-poroelastic structure interaction*, ESAIM: Math. Model. Numer. Anal., 56 (2022), pp. 1–40.

[41] M. LONSING AND R. VERFÜRTH, *On the stability of BDMS and PEERS elements*, Numer. Math., 99 (2004), pp. 131–140.

[42] S. MEDDAHI AND R. RUIZ-BAIER, *A mixed discontinuous Galerkin method for a linear viscoelasticity problem with strongly imposed symmetry*, SIAM J. Sci. Comput., 45 (2023), pp. B27–B56.

[43] R. RUIZ-BAIER, M. TAFFETANI, H. D. WESTERMEYER, AND I. YOTOV, *The Biot–Stokes coupling using total pressure: formulation, analysis and application to interfacial flow in the eye*, Comput. Methods Appl. Mech. Engrg., 389 (2022), pp. e114384(1–30).

[44] K. H. STØVERUD, M. ALNÆS, H. P. LANGTANGEN, V. HAUGHTON, AND K.-A. MARDAL, *Poro-elastic modeling of Syringomyelia–a systematic study of the effects of pia mater, central canal, median fissure, white and gray matter on pressure wave propagation and fluid movement within the cervical spinal cord*, Comput. Methods Biomech. Biomed. Engrg., 19 (2016), pp. 686–698.

[45] C. VAN DUIJN AND A. MIKELIĆ, *Mathematical theory of nonlinear single-phase poroelasticity*, J. Nonl. Sci., 33 (2023), p. 44.

[46] S.-Y. YI, *Convergence analysis of a new mixed finite element method for Biot's consolidation model*, Numer. Methods Partial Differential Equations, 30 (2014), pp. 1189–1210.