Discontinuous finite volume element methods for the optimal control of Brinkman equations

Sarvesh Kumar, Ricardo Ruiz-Baier, Ruchi Sandilya

Abstract We introduce and analyse a family of hybrid discretisations based on lowest order discontinuous finite volume elements for the approximation of optimal control problems constrained by the Brinkman equations. The classical optimise-then-discretise approach is employed to handle the control problem leading to a non-symmetric discrete formulation. An a priori error estimate is derived for the control variable in the L^2-norm, and we exemplify the properties of the method with a numerical test in 3D.

Key words: Brinkman equations, optimal control problems, discontinuous finite volume element discretisation.

MSC (2010)**:** 49N05, 49K20, 65N30, 76D07, 76D55.

1 Introduction

The numerical solution of optimal control problems constrained by equations of viscous incompressible flow (Stokes and Navier-Stokes problems) is encountered in many application problems arising in science and engineering. An abundant body of relevant literature is available, mainly in the context of finite element methods (see e.g. [3, 6, 7, 14, 13] and the references therein). Most of these contributions employ conforming discretisations for state, co-state and control variables, which

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typically produce $\mathscr{O}(h)$ convergence rates for piecewise constant approximations of the control variables, where h is the meshsize. Here we propose a new discontinuous finite volume element (DFVE) method for the discretisation of optimal control problems constrained by the Brinkman equations. DFVE schemes are characterised by ability of writing local conservation equations as in classical finite volume methods, and through transformation maps between primal and dual meshes, they can be recast as discontinuous discretisations of Petrov-Galerkin type. A number of DFVE methods have been proposed for the primal formulation of Stokes and related flow problems in [8, 17] (see also their references). We consider the present method and its analysis as an extension of these contributions to the case of distributed optimal control, in combination with the ideas developed in [11, 12, 15, 16] for elliptic and parabolic optimal control problems. While here we will derive only an L^2-error bound for the control variable and motivate our findings with an example of optimal control in a porous cylinder, the corresponding error estimates in the energy norm for control, state, and co-state variables, as well as numerical verification of optimal convergence rates, will be presented in the forthcoming contribution [9].

2 The optimal control problem

Let us consider the following distributed optimal control problem

$$\min_{\mathbf{u}\in\mathbf{U}_{\mathrm{ad}}} J(\mathbf{u}) := \frac{1}{2}\,\|\mathbf{y}-\mathbf{y}_d\|^2_{0,\Omega} + \frac{\lambda}{2}\,\|\mathbf{u}\|^2_{0,\Omega}\,, \tag{1}$$

governed by the linear Brinkman equations

$$\mathbf{K}^{-1}\mathbf{y} - \mathbf{div}\,(\mu\varepsilon(\mathbf{y}) - p\mathbf{I}) = \mathbf{u}+\mathbf{f} \quad \text{in } \Omega, \tag{2}$$

$$\operatorname{div}\mathbf{y} = 0 \quad \text{in } \Omega, \tag{3}$$

$$\mathbf{y} = \mathbf{0} \quad \text{on } \partial\Omega, \tag{4}$$

where $\mathbf{U}_{\mathrm{ad}}$ is the set of feasible controls (defined for $-\infty \le a_j < b_j \le \infty,\ j = 1,2,3$)

$$\mathbf{U}_{\mathrm{ad}} = \{\mathbf{u}\in\mathbf{L}^2(\Omega) : a_j \le u_j \le b_j \text{ a.e. in } \Omega\}.$$

This model describes the motion of an incompressible viscous fluid within an array of porous particles, where $\mathbf{y}$ denotes the fluid velocity, p is the pressure field, $\mathbf{u}$ is the control variable, and $\lambda > 0$ is a given Tikhonov regularisation. The Cauchy stress is $\mu\varepsilon(\mathbf{y}) - p\mathbf{I}$, where $\varepsilon(\mathbf{y}) = \frac{1}{2}(\nabla\mathbf{y}+\nabla\mathbf{y}^T)$ is the infinitesimal rate of strain, $\mu = \mu(\mathbf{x})$ is the dynamic viscosity of the fluid, and $\mathbf{K} = \mathbf{K}(\mathbf{x})$ is for the permeability tensor of the medium (symmetric, uniformly bounded and positive definite). The desired velocity $\mathbf{y}_d$ and the applied body force $\mathbf{f}$ are known data in $\mathbf{L}^2(\Omega)$. The goal is to identify an additional force $\mathbf{u}$ giving rise to a velocity $\mathbf{y}$ in order to match a given target velocity $\mathbf{y}_d$.

The standard weak formulation of the state equations (2)-(4) is given by: find $(\mathbf{y}, p) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ such that

$$\begin{aligned} a(\mathbf{y},\mathbf{v}) + c(\mathbf{y},\mathbf{v}) + b(\mathbf{v},p) &= (\mathbf{f}+\mathbf{u},\mathbf{v})_{0,\Omega} \qquad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \\ b(\mathbf{y},q) &= 0 \qquad \forall q \in L_0^2(\Omega), \end{aligned} \tag{5}$$

where the bilinear forms $a(\cdot,\cdot) : \mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega) \to \mathbb{R}$, $c(\cdot,\cdot) : \mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega) \to \mathbb{R}$ and $b(\cdot,\cdot) : \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \to \mathbb{R}$ are defined as:

$$a(\mathbf{y},\mathbf{v}) = \int_\Omega \mathbf{K}^{-1}\mathbf{y}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x},\; c(\mathbf{y},\mathbf{v}) = \int_\Omega \mu\varepsilon(\mathbf{y}) : \varepsilon(\mathbf{v})\,\mathrm{d}\mathbf{x},\; b(\mathbf{v},q) = -\int_\Omega q \operatorname{div}\mathbf{v}\,\mathrm{d}\mathbf{x},$$

for all $\mathbf{y},\mathbf{v} \in \mathbf{H}_0^1(\Omega)$ and $q \in L_0^2(\Omega)$. Problem (5) satisfies the Babuška-Brezzi condition: there exists $\xi > 0$ such that

$$\inf_{q\in L_0^2(\Omega)} \sup_{\mathbf{0}\neq\mathbf{v}\in\mathbf{H}_0^1(\Omega)} \frac{b(\mathbf{v},q)}{\|\mathbf{v}\|_{1,\Omega}\,\|q\|_{0,\Omega}} \geq \xi,$$

and its unique solvability is therefore ensured. As the optimal control problem (1)-(4) is strictly convex, it admits a unique optimal solution [10], and the first order necessary conditions are also sufficient for optimality. Moreover, the optimality condition can be formulated as $J'(\mathbf{u})(\tilde{\mathbf{u}}-\mathbf{u}) \geq 0$ for all $\tilde{\mathbf{u}} \in \mathbf{U}_{\mathrm{ad}}$, or:

$$(\mathbf{w}+\lambda\mathbf{u},\tilde{\mathbf{u}}-\mathbf{u})_{0,\Omega} \geq 0 \quad \forall \tilde{\mathbf{u}} \in \mathbf{U}_{\mathrm{ad}}, \tag{6}$$

where $\mathbf{w}$ is the velocity associated to the adjoint equation

$$\mathbf{K}^{-1}\mathbf{w} - \mathbf{div}(\mu\varepsilon(\mathbf{w}) + r\mathbf{I}) = \mathbf{y} - \mathbf{y}_d \quad \text{in } \Omega, \tag{7}$$

$$\operatorname{div}\mathbf{w} = 0 \quad \text{in } \Omega, \tag{8}$$

$$\mathbf{w} = \mathbf{0} \quad \text{on } \partial\Omega. \tag{9}$$

The variational inequality (6) can be equivalently recast as

$$u_j(\mathbf{x}) = P_{[a_j,b_j]}\left(\frac{-1}{\lambda} w_j(\mathbf{x})\right) \quad \text{a.e. in } \Omega, j = 1,2,3,$$

where P denotes a projection defined for a generic scalar function f as

$$P_{[a,b]}(f(\mathbf{x})) = \max(a, \min(b, f(\mathbf{x}))), \quad \text{a.e. in } \Omega,$$

and if $f \in W^{1,\infty}(\Omega)$, it further satisfies $\left\|\nabla P_{[a,b]}(f)\right\|_{L^\infty(\Omega)} \leq \|\nabla f\|_{L^\infty(\Omega)}$.

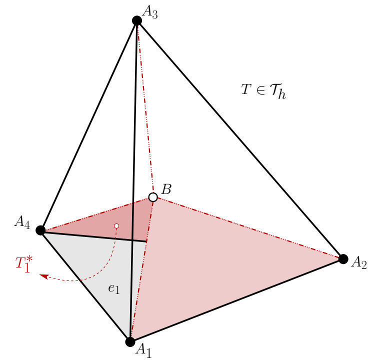

Fig. 1 Sketch of a single primal element T in $\mathscr{T}_h$, and sub-elements T_i^* belonging to the dual partition $\mathscr{T}_h^*$.

3 Discontinuous finite volume formulation

Let us consider a regular, quasi-uniform partition $\mathscr{T}_h$ of $\bar{\Omega}$ into closed tetrahedra, and referred to as *primal mesh*. By h_T we denote the diameter of a given element $T \in \mathscr{T}_h$, and the global meshsize by $h = \max_{T\in\mathscr{T}_h} h_T$; $\mathscr{E}_h$ and $\mathscr{E}_h^\Gamma$ will denote, respectively, the set of all faces and boundary faces in $\mathscr{T}_h$, and h_e is the area of the face e. In addition, each element $T \in \mathscr{T}_h$ is split into four sub-tetrahedra T_i^*, $i = 1,\ldots,4$, by connecting the barycentre of the element to its corner nodes (cf. Figure 1). The set of all these elements generated by barycentric subdivison will be denoted by $\mathscr{T}_h^*$ and will be called *dual partition* of Ω. The symbols $\{\cdot\}$ and $[\![\cdot]\!]$ will denote average and jump operators. A finite dimensional trial space (that will be used for the state and co-state velocity approximation) associated with $\mathscr{T}_h$ is

$$\mathbf{V}_h = \{\mathbf{v}_h \in \mathbf{L}^2(\Omega) : \mathbf{v}_h|_T \in \mathbf{P}_1(T),\ \forall T \in \mathscr{T}_h\},$$

the finite dimensional test space for velocities and corresponding to $\mathscr{T}_h^*$ is

$$\mathbf{V}_h^* = \{\mathbf{v}_h \in \mathbf{L}^2(\Omega) : \mathbf{v}_h|_{T^*} \in \mathbf{P}_0(T^*),\ \forall T^* \in \mathscr{T}_h^*\},$$

and the discrete space for state and co-state pressure approximation is defined as

$$Q_h = \{q_h \in L_0^2(\Omega) : q_h|_T \in P_0(T),\ \forall T \in \mathscr{T}_h\}.$$

In addition we define the higher-regularity space $\mathbf{V}(h) = \mathbf{V}_h + [\mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)]$, and the connection between discrete spaces associated to the two different meshes is characterised by $\gamma : \mathbf{V}(h) \to \mathbf{V}_h^*$, defined from $\gamma\mathbf{v}|_{T^*} = \frac{1}{h_e}\int_e \mathbf{v}|_{T^*}\,\mathrm{d}s, \quad \text{for } T^* \in \mathscr{T}_h^*$.

Let $\mathbf{v}_h \in \mathbf{V}_h$. We test (2) and (3) against $\gamma\mathbf{v}_h \in \mathbf{V}_h^*$ and $\phi_h \in Q_h$, respectively, and integrate by parts the momentum equation on each dual element and the mass equation on each primal element to obtain: find $(\mathbf{y}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

$$A_h(\mathbf{y}_h,\mathbf{v}_h) + c_h(\mathbf{y}_h,\mathbf{v}_h) + C_h(\mathbf{v}_h,p_h) = (\mathbf{u}_h + \mathbf{f}, \gamma\mathbf{v}_h)_{0,\Omega} \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{10}$$

$$B_h(\mathbf{y}_h,\phi_h) = 0 \quad \forall \phi_h \in Q_h, \tag{11}$$

where the discrete bilinear forms $A_h(\cdot,\cdot)$ and $B_h(\cdot,\cdot)$ are defined as (see also [4]):

$$A_h(\mathbf{w}_h,\mathbf{v}_h) = (\mathbf{K}^{-1}\mathbf{w}_h,\gamma\mathbf{v}_h)_{0,\Omega}, \quad B_h(\mathbf{v}_h,q_h) = b(\mathbf{v}_h,q_h) - \sum_{e\in\mathscr{E}_h}\int_e \{q_h\mathbf{n}\}_e\cdot[\![\gamma\mathbf{v}_h]\!]_e\,\mathrm{d}s,$$

$$\begin{aligned}c_h(\mathbf{w}_h,\mathbf{v}_h) = &-\sum_{T\in\mathscr{T}_h}\sum_{j=1}^{4}\int_{A_{j+1}BA_j}\mu\varepsilon(\mathbf{w}_h)\mathbf{n}\cdot\gamma\mathbf{v}_h\,\mathrm{d}s - \sum_{e\in\mathscr{E}_h}\int_e\{\mu\varepsilon(\mathbf{w}_h)\mathbf{n}\}_e\cdot[\![\gamma\mathbf{v}_h]\!]_e\,\mathrm{d}s\\ &-\sum_{e\in\mathscr{E}_h}\int_e\{\mu\varepsilon(\mathbf{v}_h)\mathbf{n}\}_e\cdot[\![\gamma\mathbf{w}_h]\!]_e\,\mathrm{d}s + \sum_{e\in\mathscr{E}_h}\int_e\frac{\alpha_d}{h_e^{\delta}}[\![\mathbf{w}_h]\!]_e\cdot[\![\mathbf{v}_h]\!]_e\,\mathrm{d}s,\end{aligned}$$

$$C_h(\mathbf{v}_h,q_h) = \sum_{T\in\mathscr{T}_h}\sum_{j=1}^{4}\int_{A_{j+1}BA_j} q_h\mathbf{n}\cdot\gamma\mathbf{v}_h\,\mathrm{d}s + \sum_{e\in\mathscr{E}_h}\int_e\{q_h\mathbf{n}\}_e\cdot[\![\gamma\mathbf{v}_h]\!]_e\,\mathrm{d}s,$$

for all $\mathbf{w}_h,\mathbf{v}_h\in\mathbf{V}_h$ and $q_h\in Q_h$. Here, α_d and δ are parameters independent of h. An appropriate inf-sup condition for B_h can be found in [17].

Analogously, we can state a DFVE formulation for the adjoint equation (7)-(9) as follows: find $(\mathbf{w}_h,r_h)\in\mathbf{V}_h\times Q_h$ such that

$$A_h(\mathbf{w}_h,\mathbf{z}_h)+c_h(\mathbf{w}_h,\mathbf{z}_h)-C_h(\mathbf{z}_h,r_h) = (\mathbf{y}_h-\mathbf{y}_d,\gamma\mathbf{z}_h) \quad \forall\mathbf{z}_h\in\mathbf{V}_h, \tag{12}$$

$$B_h(\mathbf{w}_h,\psi_h) = 0 \quad \forall\psi_h\in Q_h, \tag{13}$$

and introduce the following discrete norms in $\mathbf{V}(h)$:

$$|\!|\!|\mathbf{v}_h|\!|\!|_{1,h}^2 = \sum_{T\in\mathscr{T}_h}|\mathbf{v}_h|_{1,T}^2 + \sum_{e\in\mathscr{E}_h}h_e^{-\delta}\,\|[\![\mathbf{v}_h]\!]_e\|_{0,e}^2\,,\ |\!|\!|\mathbf{v}_h|\!|\!|_{2,h}^2 = |\!|\!|\mathbf{v}_h|\!|\!|_{1,h}^2 + \sum_{T\in\mathscr{T}_h}h_T^2|\mathbf{v}_h|_{2,T}^2,$$

which are equivalent on $\mathbf{V}_h$. Next, the discrete counterpart of (6) reads

$$(\mathbf{w}_h+\lambda\mathbf{u}_h,\tilde{\mathbf{u}}_h-\mathbf{u}_h)_{0,\Omega}\geq 0 \qquad \forall\tilde{\mathbf{u}}_h\in\mathbf{U}_{h,\mathrm{ad}}. \tag{14}$$

Lemma 1. *There exist suitable constants $C_i = C_i(\alpha_d)$ independent of h,δ, such that*

$$|A_h(\mathbf{v},\mathbf{w})|\leq C_1\|\mathbf{v}\|_{0,\Omega}\|\mathbf{w}\|_{0,\Omega}\,,\ \textit{and}\ |c_h(\mathbf{v},\mathbf{w})|\leq C_3\,|\!|\!|\mathbf{v}|\!|\!|_{2,h}\,|\!|\!|\mathbf{w}|\!|\!|_{2,h} \quad \forall\mathbf{v},\mathbf{w}\in\mathbf{V}(h),$$
$$A_h(\mathbf{v}_h,\mathbf{v}_h)\geq C_2\|\mathbf{v}_h\|_{0,\Omega}^2 \qquad \textit{and} \qquad c_h(\mathbf{v}_h,\mathbf{v}_h)\geq C_4\,|\!|\!|\mathbf{v}_h|\!|\!|_{2,h}^2 \quad \forall\mathbf{v}_h\in\mathbf{V}_h.$$

We now turn to the $\mathbf{L}^2-$error analysis for the control field under element-wise constant discretisation, where the discrete control space is defined as

$$\mathbf{U}_h^0 = \{\mathbf{u}_h\in\mathbf{L}^2(\Omega):\mathbf{u}_h|_T\in\mathbf{P}_0(T) \quad \forall T\in\mathscr{T}_h\}.$$

As in [5], the $\mathbf{L}^2-$projection $\Pi_0:\mathbf{L}^2(\Omega)\to\mathbf{U}_{h,0}$ is such that there exists a positive constant C independent of h satisfying

$$\|\mathbf{u}-\Pi_0\mathbf{u}\|_{0,\Omega}\leq Ch\|\mathbf{u}\|_{1,\Omega}\,, \qquad \mathbf{u}\in\mathbf{H}^1(\Omega). \tag{15}$$

Lemma 2. *Let* $\mathbf{u}$ *be the unique solution of* (1)-(4) *and* $\mathbf{u}_h$ *be the unique control solution of* (10)-(14) *under element-wise constant discretisation (to be verified in*

[9]). Then

$$\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega} = \mathcal{O}(h).$$

Proof. Since $\Pi_0\mathbf{U}_{\mathrm{ad}} \subset \mathbf{U}_{h,\mathrm{ad}} := \mathbf{U}_h \cap \mathbf{U}_{\mathrm{ad}}$, the continuous and discrete optimalities readily imply

$$(\mathbf{w}+\lambda\mathbf{u},\mathbf{u}_h-\mathbf{u})_{0,\Omega} + (\mathbf{w}_h+\lambda\mathbf{u}_h,\Pi_0\mathbf{u}-\mathbf{u}_h)_{0,\Omega} \geq 0.$$

Adding and subtracting $\mathbf{u}$ and rearranging terms we obtain

$$\lambda\,\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}^2 \leq (\mathbf{w}-\mathbf{w}_h,\mathbf{u}_h-\mathbf{u})_{0,\Omega} + (\mathbf{w}_h+\lambda\mathbf{u}_h,\Pi_0\mathbf{u}-\mathbf{u})_{0,\Omega},$$

and since Π_0 is an orthogonal projection and $\mathbf{u}_h \in \mathbf{U}_{h,\mathrm{ad}}$, then the term $\lambda(\mathbf{u}_h,\Pi_0\mathbf{u}-\mathbf{u})_{0,\Omega}$ vanishes to give

$$\lambda\,\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}^2 \leq (\mathbf{w}-\mathbf{w}_h,\mathbf{u}_h-\mathbf{u})_{0,\Omega} + (\mathbf{w}_h,\Pi_0\mathbf{u}-\mathbf{u})_{0,\Omega} =: I_1 + I_2. \tag{16}$$

For the first term, we use [11, Theorem 4.1] and arrive at

$$I_1 \leq Ch^2\,\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega} + Ch\,\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}^2\,,$$

whereas a bound for I_2 follows from the orthogonality of Π_0:

$$I_2 \leq \|\mathbf{w}_h-\Pi_0\mathbf{w}_h\|_{0,\Omega}\,\|\Pi_0\mathbf{u}-\mathbf{u}\|_{0,\Omega} \leq Ch\,|||\mathbf{w}_h|||_{2,h}\,\|\Pi_0\mathbf{u}-\mathbf{u}\|_{0,\Omega}\,.$$

It is left to show that $\mathbf{w}_h$ is uniformly bounded, which is a consequence of the coercivity of $A_h(\cdot,\cdot)$ and $c_h(\cdot,\cdot)$, and the uniform boundedness of $\mathbf{U}_{h,\mathrm{ad}}$:

$$|||\mathbf{w}_h|||_{2,h} \leq C\left(\|\mathbf{u}_h\|_{0,\Omega} + \|\mathbf{f}\|_{0,\Omega} + \|\mathbf{y}_d\|_{0,\Omega}\right) \leq C.$$

Substituting the bounds for I_1 and I_2 in (16), and using (15), the result follows. □

4 A numerical test

We close with the numerical solution of a three-dimensional optimal control problem. The domain consists of a cylinder of height 4 and radius 1, aligned with the x_2 axis. The anisotropic permeability field is characterised by the tensor $\mathbf{K} = \mathrm{diag}(0.1, 10^{-6}\chi_B + 0.1\chi_{B^c}, 0.1)$, where B is a ball of radius 1/4 located at the domain centre. A Poiseuille inflow profile is imposed for the state velocity at the bottom of the cylinder (i.e. on $x_2 = 0$): $\mathbf{y} = (0, 10(1-x_1^2-(x_3-1/2)^2), 0)^T$, a zero-pressure is considered on $x_2 = 4$, whereas homogeneous Dirichlet data are enforced on the remainder of $\partial\Omega$. The viscosity is constant $\mu = 0.01$, the Tikhonov regularisation parameter is $\lambda = 1/2$, the desired velocity is set to zero $\mathbf{y}_d = \mathbf{0}$, the bounds for the control are $a_j = a = -0.1$ and $b_j = b = 0.2$, and a smooth body force is considered as the one in [1]: $\mathbf{f} = \mathbf{K}^{-1}(\exp(-x_2x_3) + x_1\exp(-x_2^2), \cos(\pi x_1)\cos(\pi x_3) -$

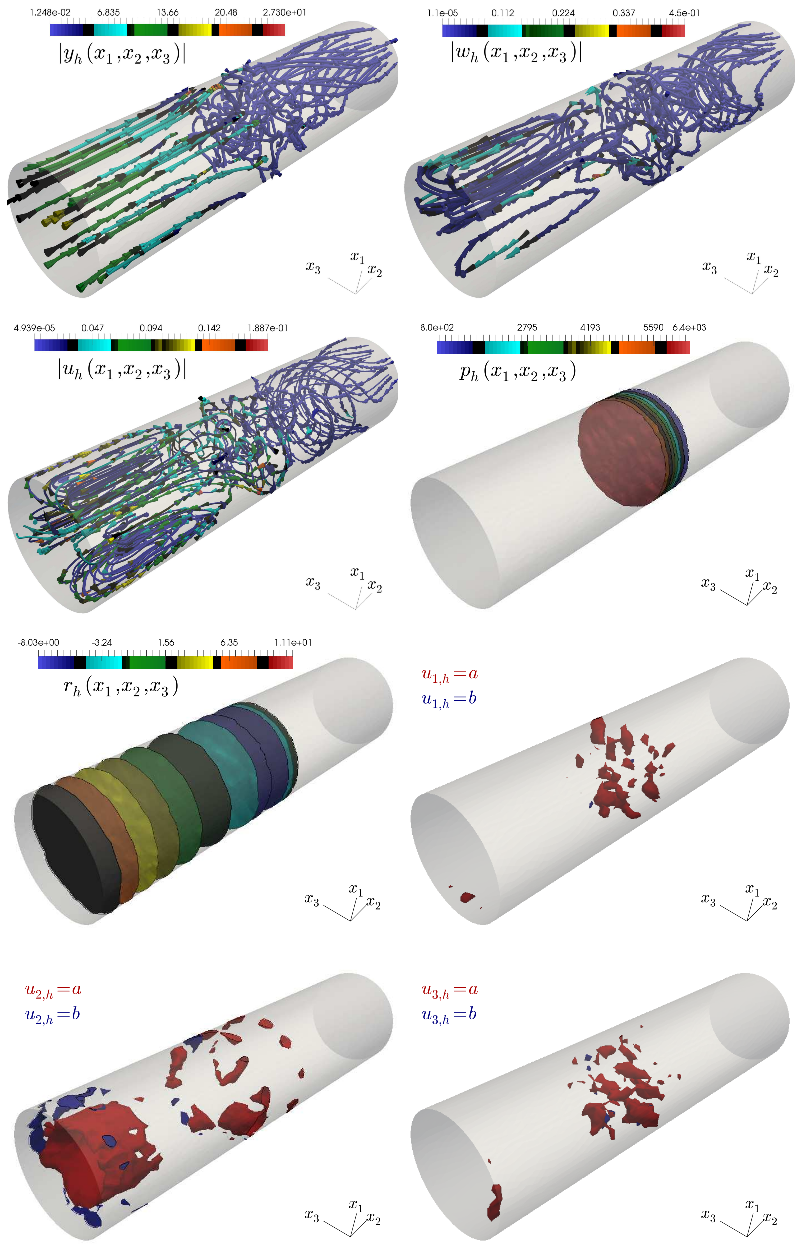

Fig. 2 Streamlines of the DFVE approximation of state and co-state velocities, along with control field, iso-surfaces of computed state and co-state pressures, and iso-surfaces of the control components associated to $a = a_1 = a_2 = a_3$ (in red) and $b = b_1 = b_2 = b_3$ (blue).

$x_2 \exp(-x_2^2), -x_1x_2x_3 - x_3\exp(-x_3^2))^T$. The primal mesh has 76766 internal tetrahedral elements and 13663 vertices. The solution is based on the active set strategy [2], involving primal and dual variables, and five iterations of that algorithm are required to reach an adequate stopping criterion. Snapshots of the resulting approximate fields are collected in Figure 2. The iso-surface of the u_2 component of the control indicates that most of the controlling occurs near the domain centre.

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